Pacific Journal of Mathematics

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Volume 256 No. 2 April 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 256, No. 2, 2012

ENTIRE SOLUTIONS OF DONALDSON'S EQUATION

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We construct infinitely many special entire solutions to Donaldson's equation. We also prove a Liouville type theorem for entire solutions of Donaldson's equation. We believe that all entire solutions of Donaldson's equation have the form of the examples constructed in the paper.

1. Introduction

Donaldson [2010] introduced an interesting differential operator when he set up a geometric structure for the space of volume forms on compact Riemannian manifolds. The Dirichlet problems for Donaldson's operator are considered in [He 2008; Chen and He 2011]. In this note we shall consider this operator on Euclidean spaces.

For $(t, x) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$ $(n \geq 1)$, let $u(t, x)$ be a smooth function such that $\triangle u > 0$, $u_{tt} > 0$. We use ∇u, $\triangle u$ to denote derivatives with respect to x and $u_t = \partial_t u$, $u_{tt} = \partial_t^2 u$ to denote derivatives with respect to t. Define a differential operator Q by

$$Q(D^2u) = u_{tt}\triangle u - |\nabla u_t|^2.$$

This operator is strictly elliptic when $u_{tt} > 0$, $\triangle u > 0$ and $Q(D^2u) > 0$. When $n = 1$, then

$$Q(D^2u) = u_{tt}u_{xx} - u_{xt}^2$$

is a real Monge–Ampère operator. When $n = 2$, Q can be viewed as a special case of the complex Monge–Ampère operator. In the x direction, we identify $\mathbb{R}^2 = \mathbb{C}$ with a coordinate w. In the t direction, we take a product by $\mathbb{R}$ with a coordinate s and let $z = t + \sqrt{-1}s$. We extend u on $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^4 = \mathbb{C}^2$ by $u(z, w) = u(t, x)$. Then

$$Q(D^2u) = 4(u_{z\bar{z}}u_{w\overline{w}} - u_{z\overline{w}}u_{w\bar{z}})$$

is a complex Monge–Ampère operator.

The author is supported in part by a start-up grant of the University of Oregon and by NSF grant DMS 1005392.
MSC2010: 35J60, 35J96.
Keywords: Donaldson's equation, entire solution, Liouville type theorem.

In this paper we shall consider entire solutions $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of

$$Q(D^2u) = 1. \tag{1-1}$$

One celebrated result, proved by Jörgens [1954] in dimension 2 and by Calabi [1958] and Pogorelov [1978] in higher dimensions, is that the only convex solutions of the real Monge–Ampère equation

$$\det(f_{ij}) = 1 \tag{1-2}$$

on the whole of $\mathbb{R}^n$ are the obvious ones: quadratic functions.

Theorem 1.1 (Calabi, Jörgens, Pogorelov). *Let f be a global convex viscosity solution of* (1-2) *on the whole of* $\mathbb{R}^n$. *Then f has to be a quadratic function.*

One can also ask similar questions for the complex Monge–Ampère equations for plurisubharmonic functions. Let $v : \mathbb{C}^n \to \mathbb{R}$ be a strictly plurisubharmonic function such that $(v_{i\bar{j}}) > 0$, which satisfies

$$\det(v_{i\bar{j}}) = 1. \tag{1-3}$$

The analogous results to Theorem 1.1 for the complex Monge–Ampère equation (1-3) or Donaldson's equation (1-1) ($n > 1$) are not known. For the complex Monge–Ampère equation, LeBrun [1991] investigated the Euclidean Taub–NUT metric constructed by Hawking [1977] and proved that it is a Kähler Ricci-flat metric on $\mathbb{C}^2$ but a nonflat metric. His example provides a nontrivial entire solution of the complex Monge–Ampère equation. We shall construct infinitely many solutions for Donaldson's equation (1-1), which are nontrivial solutions in the sense that u_{tt} is constant, but $\triangle u$, ∇u_t are both not constant. However, when $n = 2$, the Kähler metrics corresponding to these examples are the Euclidean metric on $\mathbb{C}^2$. We shall prove a Liouville type theorem for Donaldson's equation (1-1), which says u_{tt} has to be constant provided some restrictions on u_{tt}. Our proof relies on a transformation introduced by Donaldson [2010]. We then ask if all solutions of (1-1) satisfy that u_{tt} is constant; this would characterize all entire solutions of (1-1) if confirmed.

2. Examples of entire solutions

In this section we shall construct infinitely many nontrivial solutions of (1-1) and (1-3). First we consider (1-1). Let $u_{tt} = 2a$ for some $a > 0$; also let $u(0, x) = g(x)$ and $u_t(0, x) = b(x)$. Then

$$u(t, x) = at^2 + tb(x) + g(x). \tag{2-1}$$

If u solves (1-1), then

$$2a(t\triangle b + \triangle g) - |\nabla b|^2 = 1.$$

It follows that

$$\triangle b = 0 \quad \text{and} \quad \triangle g = \frac{1}{2a}\big(1 + |\nabla b|^2\big).$$

So we shall construct the examples as follows. Let $b = b(x_1, x_2, \ldots, x_n)$ be a harmonic function in $\mathbb{R}^n$. Define

$$h(x) = \frac{1 + |\nabla b|^2}{2a}.$$

Consider the following equation for $g(x)$:

$$\triangle g = h(x). \tag{2-2}$$

We can write $g = b^2(x)/4a + f$ for some function f such that $\triangle f = 1/2a$. We can summarize our results above as follows.

Theorem 2.1. *Let u be the form of* (2-1) *such that b is a harmonic function and g satisfies* (2-2). *Then u is an entire solution of* (1-1). *Moreover, any entire solution of* (1-1) *with u_{tt} = constant has the form of* (2-1).

When $n = 2$, these examples also provide solutions of the complex Monge–Ampère equation (1-3). Actually, let $u(z, w) : \mathbb{C}^2 \to \mathbb{R}$ be a solution of (1-3). If $u_{z\bar{z}} = a$ for some constant $a > 0$, it is not hard to derive that

$$u(z, w) = az\bar{z} + f(z, \bar{z}) + zb(w, \overline{w}) + \bar{z}\bar{b}(w, \overline{w}) + g(w, \overline{w}) \tag{2-3}$$

such that

$$\frac{\partial^2 f}{\partial z\, \partial \bar{z}} = \frac{\partial^2 b}{\partial w\, \partial \overline{w}} = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial w\, \partial \overline{w}} = \frac{1}{a}\left(1 + \left|\frac{\partial b}{\partial \overline{w}}\right|^2\right).$$

However these examples are all trivial solutions of the complex Monge–Ampère equation in the sense that the corresponding Kähler metrics are flat. For simplicity, we can assume $a = 1$. Since $\partial^2 b/\partial w\, \partial \overline{w} = 0$, we can assume that b is holomorphic or antiholomorphic. If b is holomorphic, then the corresponding Kähler metric is just $dz \otimes d\bar{z} + dw \otimes d\overline{w}$. If b is antiholomorphic, we can set $b(w, \overline{w}) = c(\overline{w})$ and $\bar{b}(w, \overline{w}) = c(w)$. The corresponding Kähler metric is given by

$$\begin{aligned} &dz \otimes d\bar{z} + c_{\overline{w}}\, dz \otimes d\overline{w} + c_w\, d\bar{z} \otimes dw + g_{w\overline{w}}\, dw \otimes d\overline{w} \\ &\qquad = d(z + c(w)) \otimes d(\bar{z} + c(\overline{w})) + dw \otimes d\overline{w}. \end{aligned}$$

Then under the holomorphic transformation $(z, w) \to (z + c(w), w)$ it is clear that the Kähler metric is actually flat.

3. A theorem of Liouville type

In this section we shall prove a Liouville type result for solutions of (1-1). We shall describe a transformation introduced by Donaldson [2010], which relates the solutions of (1-1) with harmonic functions. Using this transformation, Theorem 3.1 follows from the standard Liouville theorem for positive harmonic functions.

Theorem 3.1. *Let u be a solution of* (1-1) *with* $u_{tt} > 0$. *For any* $x \in \mathbb{R}^n$, *if* $u_{tt}(t,x)\,dt^2$ *defines a complete metric on* $\mathbb{R}\times\{x\}$, *then* u_{tt} *is constant. In particular, it has the form of* (2-1) *such that* b *is a harmonic function and* g *satisfies* (2-2).

Proof. For any x fixed, let $z = u_t(t,x)$. Then $\Phi : (t,x) \to (z,x)$ gives a transformation since $u_{tt} > 0$ and the Jacobian of Φ is always positive. In particular, $\Phi : \mathbb{R}\times\mathbb{R}^n \to \operatorname{Image}\Phi \subset \mathbb{R}\times\mathbb{R}^n$ is a diffeomorphism. When $u_{tt}(x,t)\,dt^2$ is a complete metric on $\mathbb{R}\times\{x\}$ for all x, then $\operatorname{Image}\Phi = \mathbb{R}\times\mathbb{R}^n$. To see this, we note that for any x fixed, then

$$z(t,x) = u_t(0,x) + \int_0^t u_{ss}(s,x)\,ds.$$

Hence if $u_{tt}(t,x)\,dt^2$ is complete, the map $z : t \to z(t,x)$ satisfies $z(\mathbb{R}) = \mathbb{R}$. For x fixed, there exists a unique $t = t(z,x)$ such that $z = u_t(t,x)$. Define a function $\theta(z,x) = t(z,x)$. We claim that θ is a harmonic function in $\mathbb{R}\times\mathbb{R}^n$. The identity $z = u_t(\theta,x)$ implies

$$\frac{\partial\theta}{\partial x_i}u_{tt} + u_{tx_i} = 0 \quad\text{and}\quad u_{tt}\frac{\partial\theta}{\partial z} = 1.$$

It then follows that

$$u_{tt}\frac{\partial^2\theta}{\partial x_i^2} + 2u_{ttx_i}\frac{\partial\theta}{\partial x_i} + u_{ttt}\left(\frac{\partial\theta}{\partial x_i}\right)^2 + u_{tx_ix_i} = 0 \quad\text{and}\quad u_{tt}\frac{\partial^2\theta}{\partial z^2} + \frac{u_{ttt}}{u_{tt}^2} = 0.$$

We compute, if u solves (1-1),

$$\begin{aligned}
\triangle_{(z,x)}\theta &= \frac{\partial^2\theta}{\partial z^2} + \sum_i \frac{\partial^2\theta}{\partial x_i^2}\\
&= \frac{1}{u_{tt}}\left(-\frac{u_{ttt}}{u_{tt}^2} - \triangle u_t + 2\sum_i \frac{u_{ttx_i}u_{tx_i}}{u_{tt}} - \sum_i \frac{u_{ttt}u_{tx_i}^2}{u_{tt}^2}\right)\\
&= \frac{1}{u_{tt}}\left(-\frac{u_{ttt}}{u_{tt}^2}\Big(1+\sum u_{tx_i}^2\Big) - \triangle u_t + 2\sum_i \frac{u_{ttx_i}u_{tx_i}}{u_{tt}}\right)\\
&= \frac{-1}{u_{tt}}\left(\frac{u_{ttt}\triangle u}{u_{tt}} + \triangle u_t - 2\sum_i \frac{u_{ttx_i}u_{tx_i}}{u_{tt}}\right) = \frac{-1}{u_{tt}^2}\partial_t\big(\triangle u u_{tt} - |\nabla u_t|^2\big) = 0.
\end{aligned}$$

On the other hand, $\partial\theta/\partial z = 1/u_{tt} > 0$. Hence $\partial\theta/\partial z$ is a positive harmonic function on $\mathbb{R} \times \mathbb{R}^n$. It follows that $\partial\theta/\partial z$ is constant, and so u_{tt} is constant. □

One could classify all solutions of (1-1) if one could prove that u_{tt} does not decay too fast to zero when $|t| \to \infty$, such that $u_{tt}\,dt^2$ defines a complete metric on a line. This motivates the following:

Problem 3.2. *Do all solutions of* (1-1) *with* $u_{tt} > 0$ *satisfy* $u_{tt} = constant$?

Acknowledgement

I am grateful to Professors Xiuxiong Chen and Jingyi Chen for constant support and encouragements. I benefitted from conversations with Professor Pengfei Guan about the complex Monge–Ampère equations and I would like to thank him. I also thank the referee for valuable comments and suggestions.

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Received November 1, 2011. Revised January 27, 2012.

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PACIFIC JOURNAL OF MATHEMATICS

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Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

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0030-8730(201204)256:2;1-8