Pacific Journal of Mathematics

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Volume 256 No. 2 April 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 256, No. 2, 2012

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Let G be a reductive algebraic group over $\mathbb{Q}$, and suppose that $\Gamma \subset G(\mathbb{R})$ is an arithmetic subgroup defined by congruence conditions. A basic problem in arithmetic is to determine the multiplicities of discrete series representations in $L^2(\Gamma\backslash G(\mathbb{R}))$, and in general to determine the traces of Hecke operators on these spaces. In this paper we give a conjectural formula for the traces of Hecke operators, in terms of stable distributions. It is based on a stable version of Arthur's formula for L^2-Lefschetz numbers, which is due to Kottwitz. We reduce this formula to the computation of elliptic p-adic orbital integrals and the theory of endoscopic transfer. As evidence for this conjecture, we demonstrate the agreement of the central terms of this formula with the unipotent contributions to the multiplicity coming from Selberg's trace formula of Wakatsuki, in the case $G = \mathrm{GSp}_4$ and $\Gamma = \mathrm{GSp}_4(\mathbb{Z})$.

1. Introduction

Let G be a reductive algebraic group over $\mathbb{Q}$, and Γ an arithmetic subgroup of $G(\mathbb{R})$ defined by congruence conditions. Then $G(\mathbb{R})$ acts on $L^2(\Gamma\backslash G(\mathbb{R}))$ via right translation; let us write R for this representation. A fundamental problem in arithmetic is to understand R. As a first step, we may decompose R as

$$R = R_{\mathrm{disc}} \oplus R_{\mathrm{cont}},$$

where R_{disc} is a direct sum of irreducible representations, and R_{cont} decomposes continuously. The continuous part may be understood inductively through Levi subgroups of G as in [Langlands 1976], leaving us with the study of R_{disc}. Given an irreducible representation π of $G(\mathbb{R})$, write $R_{\mathrm{disc}}(\pi)$ for the π-isotypic subspace of R_{disc}. Then

$$R_{\mathrm{disc}}(\pi) \cong \pi^{\oplus m_{\mathrm{disc}}(\pi)}$$

MSC2010: 11F46, 11F72, 22E55, 32N10.
Keywords: discrete series, Hecke operators, orbital integrals, Shimura varieties, endoscopy, fundamental lemma, stable trace formula.

for some integer $m_{\text{disc}}(\pi)$. (We may also write $m_{\text{disc}}(\pi, \Gamma)$.) A basic problem is to compute these integers.

There is more structure than simply these dimensions, however. Arithmetic provides us with a multitude of Hecke operators h on $L^2(\Gamma\backslash G(\mathbb{R}))$ that commute with R. Write $R_{\text{disc}}(\pi, h)$ for the restriction of h to $R_{\text{disc}}(\pi)$. The general problem is to find a formula for the trace of $R_{\text{disc}}(\pi, h)$.

We focus on discrete series representations π. These are representations that behave like representations of compact or finite groups, in the sense that their associated matrix coefficients are square integrable. Like other smooth representations, they have a theory of characters developed by Harish-Chandra. They separate naturally into finite sets called L-packets. For an irreducible finite-dimensional algebraic representation E of $G(\mathbb{C})$, there is a corresponding L-packet Π_E of discrete series representations, consisting of those with the same infinitesimal and central characters as E.

We follow the tradition of computing $\operatorname{tr} R_{\text{disc}}(\pi, h)$ through trace formulas. This method has gone through several incarnations, beginning with Selberg [1956] for GL_2, in which he also investigated the continuous Eisenstein series. A goal was to compute dimensions of spaces of modular forms, and traces of Hecke operators on these spaces. These spaces of modular forms correspond to the spaces $R_{\text{disc}}(\pi)$ we are discussing in this case. His trace formula is an integral, over the quotient of the upper half space X by Γ, of a sum of functions H_γ, one for each element of Γ. Let us write it roughly as

$$\dim_{\mathbb{C}} S(\Gamma) = \int_{\Gamma\backslash X} \sum_{\gamma\in\Gamma} H_\gamma(Z)\,dZ,$$

for some space $S(\Gamma)$ of cusp forms with a suitable Γ-invariance condition.

Here dZ is a $G(\mathbb{R})$-invariant measure on X. When the quotient $\Gamma\backslash X$ is compact, the sum and integral may be interchanged, leading to a simple expression for the dimensions in terms of orbital integrals. The interference of the Eisenstein series precludes this approach in the noncompact quotient case. Here there are several convergence difficulties, which Selberg overcomes by employing a truncation process. Unfortunately the truncation process leads to notoriously complicated expressions, which are far from being in closed form. This study of $R_{\text{disc}}(\pi)$ has been expanded to other reductive groups using what is called the Arthur–Selberg trace formula. See [Arthur 2005].

Generally, a trace formula is an equality of distributions on $G(\mathbb{R})$, or on the adelic group $G(\mathbb{A})$. One distribution is called the geometric side; it is a sum of terms corresponding to conjugacy classes of G. Given a test function f, the formula is essentially made up of combinations $I_M(\gamma, f)$ of weighted integrals of f over the conjugacy classes of elements γ. (Here M is a Levi subgroup of G.)

The other distribution is called the spectral side, involving the Harish-Chandra transforms $\operatorname{tr}\pi(f)$ for various representations π. Here, the operator $\pi(f)$ is given by weighting the representation π by f. The geometric and spectral sides agree, and in applications we can learn much about the latter from the former. Some of the art is in picking test functions to extract information about both sides.

The best general result using the trace formula to study $\operatorname{tr} R_{\mathrm{disc}}(\pi, h)$ seems to be Arthur's [1989]. He produces a formula for

$$\sum_{\pi\in\Pi} \operatorname{tr} R_{\mathrm{disc}}(\pi, h), \tag{1-1}$$

where Π is a given discrete series L-packet for $G(\mathbb{R})$. He uses test functions f which he calls "stable cuspidal". Their Fourier transforms $\pi \mapsto \operatorname{tr}\pi(f)$ are "stable" in that they are constant on L-packets, and "cuspidal" in that, considered as a function defined on tempered representations, they are supported on discrete series. (Tempered representations are those that appear in the Plancherel formula for $G(\mathbb{R})$.) Using his invariant trace formula, Arthur [1988a; 1988b] obtains (1-1) as the spectral side. The geometric side is a combination of orbital integrals for h and values of Arthur's Φ-function, which describes the asymptotic values of discrete series characters averaged over an L-packet.

In particular, he produces a formula for

$$\sum_{\pi\in\Pi} m_{\mathrm{disc}}(\pi), \tag{1-2}$$

for an L-packet Π of (suitably regular) discrete series representations.

In the case of $G = \mathrm{GL}_2$, there is a discrete series representation π_k for each integer $k \geq 1$. In this case $m_{\mathrm{disc}}(\pi_k)$ is the dimension of the space $S_k(\Gamma)$ of Γ-cusp forms of weight k on the upper half plane. Restriction to $\mathrm{SL}_2(\mathbb{R})$ gives two discrete series $\{\pi_k^+, \pi_k^-\}$ in each L-packet. However we may still use Arthur's formula here since $m_{\mathrm{disc}}(\pi_k^+, \Gamma) = m_{\mathrm{disc}}(\pi_k^-, \Gamma)$ for every arithmetic subgroup Γ. (Endoscopy does not play a role.)

For the group $\mathrm{GSp}_4(\mathbb{R})$ there are two discrete series representations in each L-packet: one "holomorphic" and one "large" discrete series. Let π be a holomorphic discrete series, and write π' for the large discrete series representation in the same L-packet as π. The multiplicity $m_{\mathrm{disc}}(\pi, \Gamma)$ is also the dimension of a certain space of vector-valued Siegel cusp forms (see [Wallach 1984]) on the Siegel upper half space, an analogue of the usual cusp forms on the upper half plane. For $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$, the dimensions of these spaces of cusp forms were calculated by Tsushima [1983; 1997] by using the Riemann–Roch–Hirzebruch formula, and later by Wakatsuki [2012] by using the Selberg trace formula and the properties of prehomogeneous vector spaces. In [$\geq$ 2012], Wakatsuki then evaluated Arthur's formula to compute $m_{\text{disc}}(\pi, \Gamma) + m_{\text{disc}}(\pi', \Gamma)$, thereby deducing a formula for $m_{\text{disc}}(\pi', \Gamma)$.

A natural approach to isolating the individual $m_{\text{disc}}(\pi)$, or generally the individual $\operatorname{tr} R_{\text{disc}}(\pi, h)$, is to apply a trace formula to a matrix coefficient, or more properly, a pseudocoefficient f. This means that f is a test function whose Fourier transform picks out π rather than the entire packet Π containing π; see Definition 6 below. Such a function will not be stable cuspidal, but merely cuspidal. Arthur [1989] (see also [2005]) showed that $I_M(\gamma, f)$ vanishes when f is stable cuspidal and the unipotent part of γ is nontrivial. If we examine the geometric side of Arthur's formula for a pseudocoefficient f, we must evaluate the more complicated terms $I_M(\gamma, f)$ for elements γ with nontrivial unipotent part. At the time of this writing, such calculations have not been made in general; we take another approach.

Distinguishing the individual representations π from others in its L-packet leads to the theory of endoscopy, and stable trace formulas. The grouping of representations π into packets Π on the spectral side mirrors the fusion of conjugacy classes that occurs when one extends the group $G(\mathbb{R})$ to the larger group $G(\mathbb{C})$. If F is a local or global field, then a stable conjugacy class in $G(F)$ is, roughly, the union of classes which become conjugate in $G(\bar{F})$. (See [Langlands 1979] for a precise definition.)

The distribution that takes a test function to its integral over a regular semisimple stable conjugacy class is a basic example of a stable distribution. Indeed, a stable distribution is defined to be a closure of the span of such distributions; see [Langlands 1983; 1979]. A distribution on $G(F)$ is stabilized if it can be written as a sum of stable distributions, the sum being over smaller subgroups H related to G. These groups H are called endoscopic groups for G; they are tethered to G not as subgroups but through their Langlands dual groups. As part of a series of techniques called endoscopy, one writes unstable distributions on G as combinations of stable distributions on the groups H. Part of this process is the theory of transfer, associating suitable test functions f^H on $H(F)$ to test functions f on $G(F)$ that yield a matching of orbital integrals. Indeed this was the drive for [Ngô 2010]. As the name suggests, the theory of endoscopy, while laborious, leads to an intimate understanding of G.

There has been much work in stabilizing Arthur's formula. See for example [Langlands 1983; Arthur 2002; 2001; 2003]. In Kottwitz's preprint [$\geq$ 2012], he defines a stable version of Arthur's Lefschetz formula, which we review below. (See also [Morel 2010].) It is a combination $\mathcal{K}(f) = \sum_H \iota(G, H)\, ST_g(f^H)$ of distributions $f \mapsto f^H \mapsto ST_g(f^H)$ over endoscopic groups H for G. Here the distributions ST_g, defined for each H, are stable. (See Section 5.1 for the definition of the rational numbers $\iota(G, H)$.) Each ST_g is a sum of terms corresponding to stable conjugacy classes of elliptic elements $\gamma \in H(\mathbb{Q})$. Kottwitz's main result is that $\mathcal{K}$ agrees with Arthur's distribution, at least for functions f that are stable cuspidal at the real place.

As part of the author's thesis [Spallone 2004], the identity terms of $\mathcal{K}$ were evaluated for the group $G = \mathrm{SO}_5$ at a function f that was a pseudocoefficient for a discrete series representation at the real place. Later, Wakatsuki noted that the resulting expressions match up with the terms in his multiplicity formulas for $m_{\mathrm{disc}}(\pi, \Gamma)$ and $m_{\mathrm{disc}}(\pi', \Gamma)$ corresponding to unipotent elements. Moreover, the contribution in [Spallone 2004] from the endoscopic group accounted for the difference in these multiplicity formulas, while the stable part corresponded to the sum. After further investigation, we conjecture simply that Kottwitz's distribution evaluated at a function $f = f_{\pi,\Gamma}$ suitably adapted to π and Γ is equal to $m_{\mathrm{disc}}(\pi, \Gamma)$, under a regularity condition on π. (See Section 5.3 for the precise statement.) Of course this is compatible with Arthur's results in [1989].

In this paper we give some computational evidence for this conjecture. We also reduce the computation of each $ST(f^H_{\pi,\Gamma})$ to evaluating elliptic orbital p-adic integrals for the transfer $f^{\infty H}$ at the finite places. The rest breaks naturally into a problem at the real points and a global volume computation.

The main ingredient at the archimedean place is the Φ-function $\Phi_M(\gamma, \Theta^E)$ of Arthur, which we review. This quantity gives the contribution from the real place to the trace formulas in [Arthur 1989] and [Goresky et al. 1997]. It also plays a prominent role in Kottwitz's formula. This function, originally defined by the asymptotic behavior of a stable character near a singular element γ, was expressed in closed form in many cases by the author in [Spallone 2009].

There are two volume-related constants that enter into any explicit computation of ST_g. The first is $\bar{v}(G)$, which is essentially the volume of an inner form of G over $\mathbb{R}$. It depends on the choice of local measure dg_∞. The second comes about from orbital integrals at the finite adeles, and depends on the choice of local measure dg_f. These integrals may frequently be written in terms of the volumes of open compact subgroups K_f of $G(\mathbb{A}_f)$. In practice, one is left computing expressions such as $\bar{v}(G)^{-1}\,\mathrm{vol}_{dg_f}(K_f)^{-1}$, which are independent of the choice of local measures. More specifically, we define

$$\chi_{K_f}(G) = \bar{v}(G)^{-1}\,\mathrm{vol}_{dg_f}(K_f)^{-1}\tau(G)d(G).$$

Here $\tau(G)$ is the Tamagawa number of G and $d(G)$ is the index of the real Weyl group in the complex Weyl group. A main general result of this paper, Theorem 2, interprets $\chi_{K_f}(G)$ via Euler characteristics of arithmetic subgroups. It extends a computation of Harder [1971], which was for semisimple simply connected groups, to the case of reductive groups, under some mild hypotheses on G.

We work out two examples in this paper, one for SL_2 and another for GSp_4. It is easy to verify our conjecture for $G = \mathrm{SL}_2$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ using the classic dimension formula for cusp forms. In this case endoscopy does not appear. The calculations for GSp_4 are more complex; we content ourselves with working out the central terms of Kottwitz's formula.

If π is a holomorphic discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$, write H_1^π for the central-unipotent terms of the Selberg trace formula, as evaluated in [Wakatsuki $\geq$ 2012] to compute $m_{\mathrm{disc}}(\pi, \Gamma)$. Here $\Gamma = \mathrm{GSp}_4(\mathbb{Z})$. If π is a large discrete series representation, write H_1^π for the central-unipotent terms in [Wakatsuki $\geq$ 2012] contributing to $m_{\mathrm{disc}}(\pi, \Gamma)$. In both cases, write $f = f_{\pi,\Gamma} = f_\infty f^\infty$, with f_∞ a pseudocoefficient for π, and f^∞ the (normalized) characteristic function of the integer adelic points of G. Write $\mathcal{K}(f, \pm 1)$ for the central terms of Kottwitz's formula applied to f.

As evidence for our conjecture, we show this:

Theorem 1. *For each regular discrete series representation π of $G(\mathbb{R})$, we have*

$$\mathcal{K}(f_{\pi,\Gamma}, \pm 1) = H_1^\pi.$$

We believe that the $\mathcal{K}(f_{\pi,\Gamma}, \pm 1)$ terms will generally match up with the difficult central-unipotent terms of the Arthur–Selberg formula, as in this case.

Our conjecture reduces the computation of discrete series multiplicities to the computation of stable elliptic orbital integrals of various transfers f_p^H, written for functions on $G(\mathbb{Q}_p)$. Let us write this as $SO_{\gamma_H}(f_p^H)$. Here f_p are characteristic functions of congruence subgroups of $G(\mathbb{Q}_p)$ related to Γ. Certainly at suitably regular elements, $SO_{\gamma_H}(f_p^H)$ is an unstable combination of orbital integrals of f_p; however there are also contributions from elliptic singular γ_H, notably $\gamma_H = 1$. At present, there are expressions for f_p^H in the parahoric case and of course for $G(\mathbb{Z}_p)$, but less seems to be known for smaller congruence subgroups. On the other hand, there are many formulas for dimensions of Siegel cusp forms and discrete series multiplicities for these cases (for example, [Wakatsuki $\geq$ 2012]). This suggests that one could predict stable singular elliptic orbital integrals $SO_{\gamma_H}(f_p^H)$ for the transfer f_p^H of characteristic functions of congruence subgroups (see for example Klingen, Iwahori and Siegel), by comparing our formulas.

Finally, we refer the casual reader to our survey [Spallone 2011] of the present approach to discrete series multiplicities.

In Section 2, we set up the conventions for this study. We explain how we are setting up the orbital integrals, and indicate our main computational tools. We also review the Langlands correspondence for real groups.

The theory of Arthur's Φ-function is reviewed in Section 4. In Section 5, we review Kottwitz's stable version of Arthur's formula from [Kottwitz $\geq$ 2012]. We also state our conjecture here. The heart of the volume computations in this paper is in Section 6, where we determine $\chi_K(G)$. As a warm up, we work out the classic case of SL_2, with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in Section 7.

The case of $G = \mathrm{GSp}_4$ is considerably more difficult. We must work out several isomorphisms of real tori. These are described in Section 8. The basic structure of G and its Langlands dual $\hat{G}$ is set up in Section 9. In Section 10 we work out the Langlands parameters for discrete series of $G(\mathbb{R})$. There is only one elliptic endoscopic group H for G. We describe H in Section 11. In Section 12, we describe the Langlands parameters for discrete series of $H(\mathbb{R})$ and describe the transfer of discrete series in this case. In Section 13, we describe the Levi subgroups of G and H and compute various constants that occur in Kottwitz's formula for these groups. In Section 14, we compute explicitly Arthur's Φ-function for Levi subgroups of G, and we do this for Levi subgroups of H in Section 15. In Section 16, we write out the terms of Kottwitz's formula corresponding to central elements of G and H, for a general arithmetic subgroup Γ. In Section 17, we specialize to the case of $\Gamma = \mathrm{GSp}_4(\mathbb{Z})$, and in Section 18 we gather our results to demonstrate Theorem 1.

2. Preliminaries and notation

If F is a field, write Γ_F for the absolute Galois group of F. Suppose G is an algebraic group over F. If E is an extension field of F, we write G_E for G viewed as an algebraic group over E (by restriction). If γ is an element of $G(F)$, we denote by G_γ the centralizer of γ in G. By G° we denote the identity component of G (with the Zariski topology). Write G_{der} for the derived group of G. If G is a reductive group, write G_{sc} for the simply connected cover of G_{der}. Let $X^*(G) = \mathrm{Hom}(G_{\overline{F}}, \mathbb{G}_m)$ and $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G_{\overline{F}})$. These are abelian groups. Write $X^*(G)_{\mathbb{C}}$ and $X_*(G)_{\mathbb{C}}$ for the tensor product of these groups over $\mathbb{Z}$ with $\mathbb{C}$. Similarly with the subscript $\mathbb{R}$. Write A_G for the maximal F-split torus in the center of G.

We denote by $\mathbb{A}$ the ring of adeles over $\mathbb{Q}$. We denote by $\mathbb{A}_f$ the ring of finite adeles over $\mathbb{Q}$, so that $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$. Write $\mathbb{O}_f$ for the integral points of $\mathbb{A}_f$.

If G is a real Lie group, we write G^+ for the connected component of G (using the classical topology rather than any Zariski topology).

Let G be a connected reductive group over $\mathbb{R}$. A torus T in G is elliptic if T/A_G is anisotropic (as an $\mathbb{R}$-torus). Say that G is cuspidal if it contains a maximal torus T that is elliptic. An element of $G(\mathbb{R})$ is elliptic if it is contained in an elliptic maximal torus of G. Having fixed an elliptic maximal torus T, the absolute Weyl group Ω_G of T in G is the quotient of the normalizer of $T(\mathbb{C})$ in $G(\mathbb{C})$ by $T(\mathbb{C})$. The real Weyl group $\Omega_{G,\mathbb{R}}$ of T in G is the quotient of the normalizer of $T(\mathbb{R})$ in $G(\mathbb{R})$ by $T(\mathbb{R})$. We may drop the subscript G if it is clear from context. Also fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$.

Write $q(G)$ for half the dimension of $G(\mathbb{R})/K_{\mathbb{R}}Z(\mathbb{R})$. If we write R for the roots of G, with a set of positive roots R^+, then

$$q(G) = \tfrac{1}{2}(|R^+| + \dim(X)),$$

where X is the span of R.

If G is an algebraic group over $\mathbb{Q}$, let $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$.

2.1. ***Endoscopy.*** Here we review the theory of based root data and endoscopy in the form we will use in this paper.

The notion of a based root datum is defined in [Springer 1979]. First, a root datum is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$, where

- X and $X^\vee$ are free, finitely generated abelian groups, in duality by a pairing
$$\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z};$$
- R and $R^\vee$ are finite subsets of X and $X^\vee$, respectively;
- there is a bijection $\alpha \mapsto \alpha^\vee$ from R onto $R^\vee$;
- we have $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$;
- $s_\alpha(R) = R$ if s_α is the reflection of X determined by α, and similarly with α replaced by $\alpha^\vee$ and R by $R^\vee$.

A based root datum is a quadruple $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$, where Δ and $\Delta^\vee$ are sets of simple roots of root system R and $R^\vee$ respectively, so that $(X, R, X^\vee, R^\vee)$ is a root datum. The dual of $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$ is given simply by $\Psi_0^\vee = (X^\vee, \Delta^\vee, X, \Delta)$.

Let $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$ and $\Psi_0' = (X', \Delta', X'^\vee, \Delta'^\vee)$ be two based root data. Then an isomorphism between Ψ and Ψ' is an isomorphism of groups $f : X \to X'$ so that f induces a bijection of Δ onto Δ' and so that the transpose of f induces a bijection of $\Delta^\vee$ onto $\Delta'^\vee$.

Let G be a connected reductive group over an algebraically closed field F. Fix a maximal torus T and a Borel subgroup B of G with $T \subseteq B$. We say in this situation that (T, B) is a pair (for G). The choice of pair determines a based root datum

$$\Psi_0(G, T, B) = (X^*(T), \Delta(T, B), X_*(T), \Delta^\vee(T, B))$$

for G. Here $\Delta(T, B)$ is the set of simple B-positive roots of T, and $\Delta^\vee(T, B)$ is the set of simple B-positive coroots of T. If another pair $T' \subseteq B'$ is chosen, the new based root datum obtained is canonically isomorphic to the original via an inner automorphism α of G. We have $\alpha(T') = T$ and $\alpha(B') = B$. Although the inner automorphism α need not be unique, its restriction to an isomorphism $T' \xrightarrow{\sim} T$ is unique.

We may remove the dependence of the based root datum on the choice of pair as follows. Write X^*, Δ, X_*, and $\Delta^\vee$ for the inverse limit over the set of pairs (T, B) of $X^*(T)$, $\Delta(T, B)$, $X_*(T)$ and $\Delta^\vee(T, B)$, respectively. Then we simply define the based root datum of G to be

$$\Psi_0(G) = (X^*, \Delta, X_*, \Delta^\vee).$$

Let G be a connected reductive group over a field F, and $\Psi_0(G)$ a based root datum of $G_{\overline{F}}$. Then Γ_F acts naturally (via isomorphisms) on $\Psi_0(G)$. The action of Γ_F on G is said to be an L-action if it fixes some splitting of G; see [Kottwitz 1984, Section 1.3].

Definition 1. A dual group for G is the following data:

(i) A connected complex reductive group with a based root datum $\Psi_0(\hat{G})$. We write its complex points as $\hat{G}$.

(ii) An L-action of Γ_F on $\hat{G}$.

(iii) A Γ_F-isomorphism from $\Psi_0(\hat{G})$ to the dual of $\Psi_0(G)$.

To specify the isomorphism for (iii) above, one typically fixes pairs (T_0, B_0) of G and $(\hat{S}_0, \hat{B}_0)$ of a dual group $\hat{G}$ and an isomorphism from $\Psi_0(\hat{G}, \hat{S}_0, \hat{B}_0)$ to the dual of $\Psi_0(G, T_0, B_0)$.

In the case that G is a torus T, the dual group $\hat{T}$ is simply given by

$$\hat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times, \tag{2-1}$$

with the Γ_F-action induced from $X^*(T)$. There are canonical Γ_F-isomorphisms $X^*(\hat{T}) \xrightarrow{\sim} X_*(T)$ and $X_*(\hat{T}) \xrightarrow{\sim} X^*(T)$.

The formalism for dual groups encodes canonical isomorphisms between tori. If T and T' are tori, and $\varphi : T \to T'$ is a homomorphism, it induces a homomorphism $\hat{T}' \to \hat{T}$ in the evident way.

Suppose that (T, B) is a pair for G and $(\hat{S}, \hat{B})$ is a pair for $\hat{G}$. By (iii) above, one has in particular a fixed isomorphism from $\Psi_0(G, T, B)$ to the dual of $\Psi_0(\hat{G}, \hat{S}, \hat{B})$. In particular this yields an isomorphism from $X^*(T)$ to $X_*(\hat{S})$, which induces an isomorphism

$$\hat{T} \xrightarrow{\sim} \hat{S}. \tag{2-2}$$

Next, let G be a connected reductive group over a field F, which is either local or global.

Definition 2. An endoscopic group for G is a triple (H, s, η) as follows:

- H is a quasisplit connected group, with a fixed dual group $\hat{H}$ as above;
- $s \in Z(\hat{H})$.

- $\eta : \hat{H} \to \hat{G}$ is an embedding.
- The image of η is $(\hat{G})^\circ_{\eta(s)}$, the connected component of the centralizer in $\hat{G}$ of $\eta(s)$.
- The $\hat{G}$-conjugacy class of η is fixed by Γ_F.
  Cohomology of Γ_F-modules then yields a boundary map
$$[Z(\hat{H})/Z(\hat{G})]^{\Gamma_F} \to H^1(F, Z(\hat{G})).$$
- The image of s in $Z(\hat{H})/Z(\hat{G})$ is fixed by Γ, and its image under the boundary map above is trivial if F is local and locally trivial if F is global.

An endoscopic group is elliptic if the identity components of $Z(\hat{G})^{\Gamma_F}$ and $Z(\hat{H})^{\Gamma_F}$ agree.

Isomorphism of endoscopic groups is defined in [Kottwitz 1984, Section 7.5]; we do not review it here.

2.2. ***Langlands correspondence.*** Let G be a connected reductive group over $\mathbb{R}$. In this section we review elliptic Langlands parameters for G and the corresponding L-packets for discrete series representations of $G(\mathbb{R})$. Our main references are [Borel 1979] and [Kottwitz 1990]. Write $W_\mathbb{R}$ for the Weil group of $\mathbb{R}$, and $W_\mathbb{C}$ for the canonical image of $\mathbb{C}^\times$ in $W_\mathbb{R}$. There is an exact sequence
$$1 \to W_\mathbb{C} \to W_\mathbb{R} \to \Gamma_\mathbb{R} \to 1.$$
The Weil group $W_\mathbb{R}$ is generated by $W_\mathbb{C}$ and a fixed element τ satisfying $\tau^2 = -1$ and $\tau z \tau^{-1} = \bar{z}$ for $z \in W_\mathbb{C}$. The action of $\Gamma_\mathbb{R}$ on $\hat{G}$ inflates to an action of $W_\mathbb{R}$ on $\hat{G}$, and through this action we form the L-group ${}^LG = \hat{G} \rtimes W_\mathbb{R}$.

A Langlands parameter φ for G is an equivalence class of continuous homomorphisms $\varphi : W_\mathbb{R} \to {}^LG$ commuting with projection to $\Gamma_\mathbb{R}$, satisfying a mild hypothesis on the image; see [Borel 1979]. The equivalence relation is via inner automorphisms from $\hat{G}$. One associates to a Langlands parameter φ an L-packet $\Pi(\varphi)$ of irreducible admissible representations of G.

Suppose that G is cuspidal, so that there is a discrete series representation of $G(\mathbb{R})$. This implies that the longest element w_0 of the Weyl group Ω acts as -1 on $X_*(T)$. If φ is a Langlands parameter, write C_φ for the centralizer of $\varphi(W_\mathbb{R})$ in $\hat{G}$ and $\hat{S}$ for the centralizer of $\varphi(W_\mathbb{C})$ in $\hat{G}$. Write S_φ for the product $C_\varphi Z(\hat{G})$. We say φ is elliptic if $S_\varphi/Z(\hat{G})$ is finite, and describe the L-packet $\Pi(\varphi)$ in this case.

Since φ is elliptic, the centralizer $\hat{S}$ is a maximal torus in $\hat{G}$. Since φ commutes with the projection to $\Gamma_\mathbb{R}$, it restricts to a homomorphism
$$W_\mathbb{C} \to \hat{S} \times \{1\}.$$

We may view this restriction as a continuous homomorphism $\varphi : \mathbb{C}^\times \to \hat{S}$, which may be written in exponential form

$$\varphi(z) = z^\mu \bar{z}^\nu$$

with μ and ν regular elements of $X_*(\hat{T})_\mathbb{C}$. Write $\hat{B}$ for the unique Borel subgroup of $\hat{G}$ containing $\hat{S}$ so that $\langle \mu, \alpha \rangle$ is positive for every root α of $\hat{S}$ that is positive for $\hat{B}$. We say that φ determines the pair $(\hat{S}, \hat{B})$, at least up to conjugacy in $\hat{G}$.

Let B be a Borel subgroup of $G_\mathbb{C}$ containing T. Then φ and B determine a quasicharacter $\chi_B = \chi(\varphi, B)$, as follows. There is a canonical (up to $\hat{G}$-conjugacy) homomorphism $\eta_B : {}^LT \to {}^LG$ described in [Kottwitz 1990] such that

$$\eta_B(z) = z^\rho \bar{z}^{-\rho} \times z \in \hat{G} \rtimes W_\mathbb{R} \quad \text{for } z \in W_\mathbb{C}.$$

Here $\rho = \rho_G$ is the half sum of the B-positive roots for T. Then a Langlands parameter φ_B for T may be chosen so that $\varphi = \eta_B \circ \varphi_B$. Finally χ_B is the quasi-character associated to φ_B by the Langlands correspondence for T (as described in [Borel 1979, Section 9.4]).

Write $\mathcal{B}$ for the set of Borels of $G_\mathbb{C}$ containing T. The L-packet associated to φ is indexed by $\Omega_\mathbb{R} \backslash \mathcal{B}$. For $B \in \Omega_\mathbb{R} \backslash \mathcal{B}$, a representation $\pi(\varphi, B)$ in the L-packet is given by the irreducible discrete series representation of $G(\mathbb{R})$ whose character Θ_π is given on regular elements γ of $T(\mathbb{R})$ by

$$(-1)^{q(G)} \sum_{\omega \in \Omega_\mathbb{R}} \chi_{\omega(B)}(\gamma) \cdot \Delta_{\omega(B)}(\gamma)^{-1}.$$

Here Δ_B is the usual discriminant

$$\Delta_B(\gamma) = \prod_{\alpha > 0 \text{ for } B} (1 - \alpha(\gamma)^{-1}).$$

Finally, let

$$\Pi(\varphi) = \{\pi(\varphi, B) \mid B \in \Omega_\mathbb{R} \backslash \mathcal{B}\}.$$

It has order $d(G) = |\Omega / \Omega_\mathbb{R}|$. There is a unique irreducible finite-dimensional algebraic complex representation E of $G(\mathbb{C})$ with the same infinitesimal character and central character as the representations in this L-packet. It has highest weight $\mu - \rho \in X^*(T)$ with respect to B. The isomorphism classes of such E are in one-to-one correspondence with elliptic Langlands parameters φ, and we often write Π_E for $\Pi(\varphi)$.

Definition 3. We say that a discrete series representation $\pi \in \Pi_E$ is regular if the highest weight of E is regular.

2.3. ***Measures and orbital integrals.*** Let G be a locally compact group with Haar measure dg. If f is a continuous function on G, write $f\,dg$ for the measure on G given by

$$\varphi \mapsto \int_G \varphi(g) f(g) dg,$$

for φ continuous and compactly supported in G. We will refer to the measures obtained in this way simply as "measures". If G is a p-adic, real, or adelic Lie group, we require that f be suitably smooth.

In this paper, we will view orbital integrals and Fourier transforms as distributions defined on measures, rather than on functions. This approach eases their dependence on choices of local measures, choices that do not matter in the end.

For K an open compact subset of G, write e_K for the measure given by $f\,dg$, where f is the characteristic function of K divided by $\mathrm{vol}_{dg}(K)$. Note that the measure e_K is independent of the choice of Haar measure dg.

Let G be a reductive group defined over a local field F. Fix a Haar measure dg on $G(F)$. Let $f\,dg$ be a measure on $G(F)$, and take a semisimple element $\gamma \in G(F)$. Fix a Haar measure dt of $G(F)^\circ_\gamma$. Then we write $O_\gamma(f\,dg; dt)$ for the usual orbital integral

$$O_\gamma(f\,dg; dt) = \int_{G_\gamma{}^\circ(F)\backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dt}.$$

Many cases of finite orbital integrals are easy to compute by the following result, a special case extracted from [Kottwitz 1986, Section 7].

Proposition 1. *Let F be a p-adic field with ring of integers $\mathbb{O}$. Let G be a split connected reductive group defined over $\mathbb{O}$, and let $K = G(\mathbb{O})$. Suppose that $\gamma \in K$ is semisimple, and that $1 - \alpha(\gamma)$ is either 0 or a unit for every root α of G. Let γ' be stably conjugate to γ. Then $O_{\gamma'}(e_K; dt)$ vanishes unless γ' is conjugate to γ, in which case*

$$O_{\gamma'}(e_K; dt) = \mathrm{vol}_{dt}(G_\gamma{}^\circ(F) \cap K)^{-1}.$$

Now let G be a reductive group defined over $\mathbb{Q}$.

Let $f^\infty dg_f$ be a measure on $G(\mathbb{A}_f)$ and take a semisimple element $\gamma \in G(\mathbb{A}_f)$. Fix a Haar measure dt_f of $G^\circ_\gamma(\mathbb{A}_f)$. Write $O_\gamma(f^\infty dg_f; dt_f)$ for the orbital integral

$$O_\gamma(f^\infty dg_f; dt_f) = \int_{G_\gamma{}^\circ(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} f^\infty(g^{-1}\gamma g) \frac{dg_f}{dt_f}.$$

We also have the stable orbital integrals

$$SO_\gamma(f^\infty dg_f; dt_f) = \sum_i e(\gamma_i) O_{\gamma_i}(f^\infty dg_f; dt_{i,f}),$$

the sum being over $\gamma_i \in G(\mathbb{A}_f)$ (up to $G(\mathbb{A}_f)$-conjugacy) whose local components are stably conjugate to γ. The centralizers of γ and a given γ_i are inner forms of each other, and we use corresponding measures dt_f and $dt_{i,f}$. The number $e(\gamma_i)$ is defined as follows: For a reductive group A over a local field, Kottwitz [1983] has defined an invariant $e(A)$. It is equal to 1 if A is quasisplit. For each place v of $\mathbb{Q}$, write $\gamma_{i,v}$ for the vth component of γ_i. Let

$$e(\gamma_{i,v}) = e(G^\circ_{\gamma_{i,v}}(\mathbb{Q}_v)).$$

Finally, let

$$e(\gamma_i) = \prod_v e(\gamma_{i,v}).$$

Definition 4. Let M be a Levi component of a parabolic subgroup P of G, and dm_f a Haar measure on $M(\mathbb{A}_f)$. Given a measure $f^\infty dg_f$, its *M-constant term* is the measure $f^\infty_M dm_f$, where f^∞_M is defined via

$$f^\infty_M(m) = \delta^{-1/2}_{P(\mathbb{A}_f)}(m) \int_{N(\mathbb{A}_f)} \int_{K_f} f^\infty(k^{-1}nmk)dk_f dn_f.$$

Here we fix the Haar measure dk_f on K_f giving it mass one, and the Haar measure dn_f on $N(\mathbb{A}_f)$ is chosen so that $dg_f = dk_f dn_f dm_f$. The function $\delta_{P(\mathbb{A}_f)}$ is the modulus function on $P(\mathbb{A}_f)$.

It is independent of the choice of parabolic subgroup P.

Proposition 2. *Let G be a split group defined over $\mathbb{Z}$ and let $K_f = G(\mathbb{O}_f)$. Then*

$$(e_{K_f})_M = e_{M(\mathbb{A}_f)\cap K_f}.$$

Proof. Write $e_{K_f} = f^\infty dg_f$. Then it is easy to see that $f^\infty_M(m) = 0$ unless $m \in K_f$. If $m \in K_f$, we compute that

$$f^\infty_M(m) = \frac{\mathrm{vol}_{dk_f}(K_f)\,\mathrm{vol}_{dn_f}(K_f \cap N(\mathbb{A}_f))}{\mathrm{vol}_{dg_f}(K_f)}.$$

The result follows since

$$\mathrm{vol}_{dg_f}(K_f) = \mathrm{vol}_{dm_f}(M(\mathbb{A}_f)\cap K_f)\,\mathrm{vol}_{dn_f}(N(\mathbb{A}_f)\cap K_f)\,\mathrm{vol}_{dk_f}(K_f). \qquad \square$$

2.4. *Pseudocoefficients.* We continue with a connected reductive group G over $\mathbb{Q}$, and adopt some terminology from [Arthur 1989]. Fix a maximal compact subgroup $K_\mathbb{R}$ of $G(\mathbb{R})$. We put $K'_\mathbb{R} = K_\mathbb{R} A_G(\mathbb{R})^+$. Given a quasicharacter (smooth homomorphism to $\mathbb{C}^\times$) ξ on $A_G(\mathbb{R})^+$, write $\mathcal{H}_{\mathrm{ac}}(G(\mathbb{R}), \xi)$ for the space of smooth, $K'_\mathbb{R}$-finite functions on $G(\mathbb{R})$ that are compactly supported modulo $A_G(\mathbb{R})^+$, and transform under $A_G(\mathbb{R})^+$ according to ξ. Write $\Pi(G(\mathbb{R}),\xi)$ for the set of irreducible representations of $G(\mathbb{R})$ whose central character restricted to $A_G(\mathbb{R})^+$ is equal to ξ.

Given a function $f \in \mathcal{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi^{-1})$, a representation $\pi \in \Pi(G(\mathbb{R}),\xi)$, and a Haar measure dg_∞ on $G(\mathbb{R})$, write $\pi(f\,dg_\infty)$ for the operator on the space of π given by the formula

$$\pi(f\,dg_\infty) = \int_{G(\mathbb{R})/A_G(\mathbb{R})^+} f(x)\pi(x)\,dg_\infty.$$

Here we give $A_G(\mathbb{R})^+$ the measure corresponding to Lebesgue measure on $\mathbb{R}^n$, if A_G is n-dimensional. The operator is of trace class.

Write $\Pi_{\mathrm{temp}}(G(\mathbb{R}),\xi)$ (respectively $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi)$) for the subset of tempered (respectively discrete series) representations in $\Pi(G(\mathbb{R}),\xi)$.

Definition 5. Suppose that $f \in \mathcal{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi^{-1})$. We say that the measure $f\,dg_\infty$ is cuspidal if $\operatorname{tr}\pi(f\,dg_\infty)$, viewed as a function on $\Pi_{\mathrm{temp}}(G(\mathbb{R}),\xi)$, is supported on $\Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi)$.

Write $\tilde{E}$ for the contragredient of the representation E. Arthur [1989] employs functions $f_E \in \mathcal{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi^{-1})$ with $f_E\,dg_\infty$ cuspidal, whose defining property is that, for all $\pi \in \Pi_{\mathrm{temp}}(G(\mathbb{R}),\xi)$,

$$\operatorname{tr}\pi(f_E\,dg_\infty) = \begin{cases} (-1)^{q(G)} & \text{if } \pi \in \Pi_{\tilde{E}}, \\ 0 & \text{otherwise.} \end{cases} \tag{2-3}$$

Such measures can be broken down further.

Definition 6. Fix a representation $\pi_0 \in \Pi_{\mathrm{disc}}(G(\mathbb{R}),\xi^{-1})$, and suppose that $f_0 \in \mathcal{H}_{\mathrm{ac}}(G(\mathbb{R}),\xi^{-1})$. Suppose the measure $f_0\,dg_\infty$ satisfies, for all $\pi \in \Pi_{\mathrm{temp}}(G(\mathbb{R}),\xi)$,

$$\operatorname{tr}\pi(f_0\,dg_\infty) = \begin{cases} (-1)^{q(G)} & \text{if } \pi \cong \tilde{\pi}_0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the corollary in [Clozel and Delorme 1984, Section 5.2] that such functions exist. Pick such a function f_0, and put $e_{\pi_0} = f_0\,dg_\infty$.

Suppose that for each $\pi \in \Pi_E$ we fix measures e_π as above. Let

$$f_E\,dg_\infty = \sum_\pi e_\pi,$$

the sum being over $\pi \in \Pi_E$. Then clearly $f_E\,dg_\infty$ satisfies Arthur's condition (2-3).

We remark that the measure $(-1)^{q(G)}e_\pi$ is called a pseudocoefficient of $\tilde{\pi}$.

3. Transfer

We sketch the important theory of transfer in the form that we will use in this paper.

Suppose that G is a real connected reductive group, and that (H, s, η) is an elliptic endoscopic group for G. Fix an elliptic maximal torus T_H of H, an elliptic maximal torus T of G, and an isomorphism $j : T_H \xrightarrow{\sim} T$ between them. Also fix a Borel subgroup B of $G_{\mathbb{C}}$ containing T and a Borel subgroup B_H of $H_{\mathbb{C}}$ containing T_H.

Suppose that ξ is a quasicharacter on $A_G(\mathbb{R})$, and that $f_\infty \in \mathscr{H}_{\mathrm{ac}}(G(\mathbb{R}), \xi^{-1})$, with $f_\infty dg_\infty$ cuspidal. There is a corresponding quasicharacter ξ_H on $A_H(\mathbb{R})$ described in [Kottwitz $\geq$ 2012, Section 5.5].

There is also a measure $f_\infty^H dh_\infty$ on $H(\mathbb{R})$ with $f_\infty^H \in \mathscr{H}_{\mathrm{ac}}(H(\mathbb{R}), \xi_H^{-1})$, having matching character values. See [Shelstad 1982; Clozel and Delorme 1984; 1990; Langlands and Shelstad 1987]. More specifically, let φ_H be a tempered Langlands parameter for $H_{\mathbb{R}}$, and write $\Pi_H = \Pi(\varphi_H)$ for the corresponding L-packet of discrete series representations of $H(\mathbb{R})$. Transport φ_H via η to a tempered Langlands parameter φ_G for G. The parameters φ_G and φ_H determine pairs $(\hat{S}, \hat{B})$ and $(\hat{S}_H, \hat{B}_H)$ as in Section 2.2.

Then

$$\operatorname{tr} \Pi_H(f_\infty^H dh_\infty) = \sum_{\pi \in \Pi} \Delta_\infty(\varphi_H, \pi) \cdot \operatorname{tr} \pi(f_\infty dg_\infty), \tag{3-1}$$

using Shelstad's transfer factors $\Delta_\infty(\varphi_H, \pi)$. Both sides of (3-1) vanish unless Π_H is a discrete series packet. In particular, $f_\infty^H dh_\infty$ is cuspidal, and it may be characterized by (3-1). (The transfer $f_\infty^H dh_\infty$ is only defined up to the kernel of stable distributions.) We may use this formula to identify it as a combination of pseudocoefficients.

It is a delicate matter to specify the transfer factors. We will use a formula for $\Delta_\infty(\varphi_H, \pi)$ from [Kottwitz 1990], which is itself a reformulates a formula from [Shelstad 1982]. One must carefully specify the duality between G and $\hat{G}$, and between H and $\hat{H}$, because this factor depends on precisely how this is done. It also depends on the isomorphism $j : T_H \xrightarrow{\sim} T$, which must be compatible with correspondences of tori determined by the Langlands parameters, as specified below.

Definition 7. The triple (j, B_T, B_{T_H}) is aligned with φ_H if the following diagram commutes:

$$\begin{array}{ccc} \hat{T} & \longrightarrow & \hat{S} \\ {\scriptstyle \hat{j}}\downarrow & & \uparrow{\scriptstyle \eta} \\ \hat{T}_H & \longrightarrow & \hat{S}_H. \end{array} \tag{3-2}$$

Here the isomorphisms $\hat{T} \to \hat{S}$ and $\hat{T}_H \to \hat{S}_H$ are determined, as in (2-2), by $(B, \hat{B})$ and $(B_H, \hat{B}_H)$, respectively. The map $\hat{j}$ is the map dual to j using the identification (2-1) of the dual tori.

For each $\omega \in \Omega$, there is a character

$$a_\omega : (\hat{T}/Z(\hat{G}))^{\Gamma_{\mathbb{R}}} \to \{\pm 1\}$$

described in [Kottwitz 1990].

If the triple (j, B_T, B_{T_H}) is aligned with φ_H, then we may take as transfer factors

$$\Delta_\infty(\varphi_H, \pi(\varphi, \omega^{-1}(B))) = \langle a_\omega, \hat{j}^{-1}(s)\rangle.$$

Next, let G be a connected reductive algebraic group over $\mathbb{Q}$, and let (H, s, η) be an endoscopic group for G. Given a measure $f^\infty dg_f$ on $G(\mathbb{A}_f)$, there is a measure $f^{\infty H} dh_f$ on $H(\mathbb{A}_f)$ such that for all $\gamma_H \in H(\mathbb{A}_f)$ suitably regular, one has

$$SO_{\gamma_H}(f^{\infty H} dh_f) = \sum_\gamma \Delta^\infty(\gamma_H, \gamma) O_\gamma(f^\infty dg_f).$$

The sum is taken over $G(\mathbb{A}_f)$-conjugacy classes of "images" $\gamma \in G(\mathbb{A}_f)$ of γ_H. We have written $\Delta^\infty(\gamma_H, \gamma)$ for the Langlands–Shelstad transfer factors. One takes matching measures on the centralizers of γ_H and the various γ in forming the quotient measures for the orbital integrals. We have left out many details; please see [Langlands and Shelstad 1987] and [Kottwitz and Shelstad 1999] for definitions, and [Ngô 2010] for the celebrated proof.

4. Arthur's Φ-function

In this section we consider a reductive group G defined over $\mathbb{R}$. Let T be a maximal torus contained in a Borel subgroup B of $G_{\mathbb{C}}$. Let A be the split part of T, let T_c be the maximal compact subtorus of T, and let M be the centralizer of A in G. It is a Levi subgroup of G containing T. Let E be an irreducible finite-dimensional (algebraic) representation of $G(\mathbb{C})$, and consider the L-packet Π_E of discrete series representations π of $G(\mathbb{R})$ that have the same infinitesimal and central characters as E. Write Θ_π for the character of π, and put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi_E} \Theta_\pi.$$

Note that $\Theta^E(\gamma)$ will not extend continuously to all elements $\gamma \in T(\mathbb{R})$, and in particular not to $\gamma = 1$. Define the function D_M^G on T by

$$D_M^G(\gamma) = \det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(G)/\mathrm{Lie}(M)).$$

Then a result of Arthur and Shelstad [Arthur 1989] states that the function

$$\gamma \mapsto |D_M^G(\gamma)|^{1/2} \Theta^E(\gamma),$$

defined on the set of regular elements $T_{\mathrm{reg}}(\mathbb{R})$, extends continuously to $T(\mathbb{R})$. We denote this extension by $\Phi_M(\gamma, \Theta^E)$. The following closed expression for $\Phi_M(\gamma, \Theta^E)$ when $\gamma \in T_c$ is given in [Spallone 2009].

Proposition 3. *If $\gamma \in T_c(\mathbb{R})$, then*

$$\Phi_M(\gamma, \Theta^E) = (-1)^{q(L)} |\Omega_L| \sum_{\omega \in \Omega^{LM}} \varepsilon(\omega) \operatorname{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{4-1}$$

In particular,

(i) *if T is compact, then $M = G$ and $\Phi_G(\gamma, \Theta^E) = \operatorname{tr}(\gamma; E)$;*

(ii) *if T is split, then $M = A$ and $\Phi_A(1, \Theta^E) = (-1)^{q(G)} |\Omega_G|$.*

The notation needs to be explained. Here L is the centralizer of T_c in G. The roots of T in L and M are the real and imaginary roots, respectively, of T in G. Write Ω_L and Ω_M for the respective Weyl groups. Write Ω^{LM} for the set of elements that are simultaneously Kostant representatives for both L and M, relative to B. We write ε for the sign character of Ω_G. Finally by $V^M_{\omega(\lambda_B+\rho_B)-\rho_B}$ we denote the irreducible finite-dimensional representation of $M(\mathbb{C})$ with highest weight $\omega(\lambda_B + \rho_B) - \rho_B$, where λ_B is the B-dominant highest weight of E.

If $z \in G(\mathbb{R})$ is central, it is easy to see that $\Phi_M(\gamma z, \Theta^E) = \lambda_E(z)\Phi_M(\gamma, \Theta^E)$, where λ_E is the central character of E. Thus, for the case of central $\gamma = z$, computing $\Phi_M(z, \Theta^E)$ amounts to computing the dimensions of finite-dimensional representations of $M(\mathbb{C})$ with various highest weights. For this we use the Weyl dimension formula, in the following form.

Proposition 4 (Weyl dimension formula). *Let G be a complex reductive group and T a maximal torus in G, contained in a Borel subgroup B. Write ρ_B for the half-sum of the positive roots for T in G (with respect to B). Let $\lambda_B \in X^*(T)$ be a positive weight. Then there is a unique irreducible representation V_{λ_B} of G with highest weight λ_B. Its dimension is given by*

$$\dim_{\mathbb{C}} V_{\lambda_B} = \prod_{\alpha > 0} \frac{\langle \alpha, \lambda_B + \rho_B \rangle}{\langle \alpha, \rho_B \rangle}.$$

Here $\langle \cdot , \cdot \rangle$ is a nondegenerate Ω_G-invariant inner product on $X^*(T)_{\mathbb{R}}$, which is unique up to a scalar.

5. Kottwitz's formula

5.1. ***Various invariants.*** In this section we introduce some invariants involved in Kottwitz's formula.

By $\overline{G}$ we generally denote an inner form of $G_{\mathbb{R}}$ such that $\overline{G}/A_G$ is anisotropic over $\mathbb{R}$.

Definition 8. Let G be a cuspidal reductive group over $\mathbb{R}$, and dg_∞ a Haar measure on $G(\mathbb{R})$. Let

$$\bar{v}(G; dg_\infty) = e(\overline{G}) \operatorname{vol}(\overline{G}(\mathbb{R})/A_G(\mathbb{R})^+).$$

This is a stable version of the constant $v(G)$ that appears in [Arthur 1989]. As before, $e(\overline{G})$ is the sign defined in [Kottwitz 1983]. (Note that $e(\overline{G}) = (-1)^{q(G)}$ when G is quasisplit.) In both cases the Haar measure on $\overline{G}(\mathbb{R})$ is transported from dg_∞ on $G(\mathbb{R})$ in the usual way, and the measure on $A_G(\mathbb{R})^+$ is the standard Lebesgue measure.

Definition 9. Let G be a cuspidal connected reductive group over $\mathbb{Q}$. Then G contains a maximal torus T such that T/A_G is anisotropic over $\mathbb{R}$. Write T_{sc} for the inverse image in G_{sc} of T. Then $k(G)$ is the cardinality of the image of $H^1(\mathbb{R}, T_{\mathrm{sc}}) \to H^1(\mathbb{R}, T)$.

Definition 10. If G is a reductive group over $\mathbb{Q}$, write $\tau(G)$ for the Tamagawa number of G, as defined in [Ono 1966].

By [Kottwitz 1988] or [Kottwitz $\geq$ 2012], the Tamagawa numbers $\tau(G)$ for a reductive group G over $\mathbb{Q}$ may be computed using the formula

$$\tau(G) = |\pi_0(Z(\hat{G})^{\Gamma_{\mathbb{Q}}})| \cdot |\ker^1(\mathbb{Q}, Z(\hat{G}))|^{-1}.$$

Here π_0 denotes the topological connected component.

Definition 11. Let M be a Levi subgroup of G. Then put

$$n_M^G = [N_G(M)(\mathbb{Q}) : M(\mathbb{Q})].$$

Here $N_G(M)$ denotes the normalizer of M in G.

Definition 12. Let $\gamma \in M(\mathbb{Q})$ be semisimple. Then put

$$\bar{\iota}^M(\gamma) = |(M_\gamma/M_\gamma^\circ)(\mathbb{Q})| \quad \text{and} \quad \iota^M(\gamma) = [M_\gamma(\mathbb{Q}) : M_\gamma^\circ(\mathbb{Q})].$$

Let (H, s, η) be an endoscopic triple for G, and write $\operatorname{Out}(H, s, \eta)$ for its outer automorphisms. Put

$$\iota(G, H) = \tau(G)\tau(H)^{-1}|\operatorname{Out}(H, s, \eta)|^{-1}.$$

5.2. ***The formula.*** In this section we give Kottwitz's formula [≥ 2012].

Our G will now be a cuspidal connected reductive group over $\mathbb{Q}$. Let $f^\infty \in C_c^\infty(G(\mathbb{A}_f))$ and $f_\infty \in \mathcal{H}_{\text{ac}}(G(\mathbb{R}), \xi)$ for some ξ. We consider measures $f\,dg$ of the form $f\,dg = f^\infty dg_f \cdot f_\infty dg_\infty \in C_c^\infty(G(\mathbb{A}))$, for some decomposition $dg = dg_f dg_\infty$ of the Tamagawa measure on $G(\mathbb{A}_f)$. Also choose such decompositions for every cuspidal Levi subgroup M of G.

First we define the stable distribution $S\Phi_M$ at the archimedean place:

Definition 13. Let M be a cuspidal Levi subgroup of G. Let $\gamma \in M(\mathbb{Q})$ be elliptic, and pick a Haar measure dt_∞ of $M_\gamma^\circ(\mathbb{R})$. Then $S\Phi_M(\gamma, f_\infty dg_\infty; dt_\infty)$ is defined to be

$$(-1)^{\dim(A_M/A_G)} k(M) k(G)^{-1} \bar{v}(M_\gamma{}^\circ; dt_\infty)^{-1} \sum_\Pi \Phi_M(\gamma^{-1}, \Theta_\Pi) \operatorname{tr} \Pi(f_\infty dg_\infty),$$

the sum being taken over L-packets of discrete series representations.

Here is the basic building block of Kottwitz's formula:

Definition 14. Let M be a cuspidal Levi subgroup of G, and $\gamma \in M(\mathbb{Q})$ an elliptic element. Pick Haar measures dt_f on $M_\gamma^\circ(\mathbb{A}_f)$ and dt_∞ on $M_\gamma^\circ(\mathbb{R})$ whose product is the Tamagawa measure dt on $M_\gamma^\circ(\mathbb{A})$.

We define

$$\begin{aligned} &ST_g(f\,dg, \gamma, M) \\ &\qquad = (n_M^G)^{-1} \tau(M) \bar{\iota}^M(\gamma)^{-1}\, SO_\gamma(f_M^\infty dm_f; dt_f)\, S\Phi_M(\gamma, f_\infty dg_\infty; dt_\infty). \end{aligned}$$

Here $f_M^\infty dm_f$ is the M-constant term of $f^\infty dg_f$. The product

$$SO_\gamma(f_M^\infty dm_f; dt_f) \bar{v}(M; dt_\infty)$$

is independent of the decompositions of dt and dg. We will therefore often write this simply as $SO_\gamma(f_M^\infty dm_f)\bar{v}(M)$, and similarly for other such products.

Kottwitz defines

$$ST_g(f\,dg) = \sum_M \sum_{\gamma \in M} ST_g(f\,dg, \gamma, M).$$

Here M runs over $G(\mathbb{Q})$-conjugacy classes of cuspidal Levi subgroups in G, and the second sum runs over stable $M(\mathbb{Q})$-conjugacy classes of semisimple elements $\gamma \in M(\mathbb{Q})$ that are elliptic in $M(\mathbb{R})$.

For convenience we also define, for $\gamma \in G(\mathbb{Q})$ semisimple,

$$ST_g(f\,dg, \gamma) = \sum_M ST_g(f\,dg, \gamma, M),$$

the sum being taken over cuspidal Levi subgroups of G with semisimple $\gamma \in M(\mathbb{Q})$ that are elliptic in $M(\mathbb{R})$.

Kottwitz's stable version of Arthur's trace formula is given by

$$\mathcal{K}(f\,dg) = \sum_{(H,s,\eta)\in\mathcal{E}_0} \iota(G,H)\, ST_g(f^H dh),$$

where $\mathcal{E}_0$ is the set of (equivalence classes of) elliptic endoscopic groups for G.

We record here the simpler form of $ST_g(f\,dg, \gamma, M)$ when $\gamma = z$ is in the rational points $Z(\mathbb{Q})$ of the center of G. We have

$$\begin{aligned} &ST_g(f\,dg, z, M)\\ &\quad= (-1)^{\dim(A_M/A_G)} \frac{k(M)}{k(G)} (n_M^G)^{-1} \tau(M) f_M^\infty(z) \bar{v}(M; dm_\infty)^{-1} \Phi_M(z^{-1}, \Theta_\Pi). \end{aligned}$$

5.3. ***Conjecture.*** Recall the stable cuspidal measure $f_E dg_\infty$ from Section 2.4. Fix any test function $f^\infty dg_f$ and put $f = f^\infty f_E dg$.

Let

$$T_g(f\,dg) = \sum_M (n_M^G)^{-1} \sum_\gamma \iota^M(\gamma)^{-1} \tau(M_\gamma) O_\gamma(f_M^\infty dm_f) \Phi_M(\gamma, f_E dg_\infty).$$

Again, the sum is over cuspidal Levi subgroups M and semisimple $\gamma \in M(\mathbb{Q})$ that are elliptic in $M(\mathbb{R})$. Here as in [Arthur 1989], $\Phi_M(\gamma, \cdot)$ is the unnormalized form of the distribution I_M defined in [Arthur 1988a].

Now suppose that $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi)$, and let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Write

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \xi)$$

for the space of functions on this double coset space that transform by $A_G(\mathbb{R})^+$ according to ξ and are square integrable modulo center. Write $R_{\mathrm{disc}}(\pi, K_f)$ for the π-isotypical subspace of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \xi)$; it is finite-dimensional. If $f^\infty dg_f$ is K_f-biinvariant, then convolution gives an operator $R_{\mathrm{disc}}(\pi, f^\infty dg_f)$ on $R_{\mathrm{disc}}(\pi, K_f)$. According to [Arthur 1989, Corollary 6.2], if the highest weight of E is regular, then

$$\sum_{\pi\in\Pi_E} \operatorname{tr} R_{\mathrm{disc}}(\pi, f^\infty dg_f) = T_g(f\,dg).$$

The main result of [Kottwitz $\geq$ 2012] is that if $f_\infty dg_\infty$ is stable cuspidal, then $T_g(f\,dg) = \mathcal{K}(f\,dg)$. Since we may assume $f_E dg_\infty = \sum_{\pi\in\Pi_E} e_\pi$, the following conjecture is plausible:

Conjecture 1. *Fix a regular discrete series representation π of $G(\mathbb{R})$. As in Section 2.4, let $f_\infty dg_\infty = e_\pi$. Pick a measure $f^\infty dg_f$ with $f^\infty \in C_c(G(\mathbb{A}_f))$, and $dg_f dg_\infty = dg$ the Tamagawa measure on $G(\mathbb{A})$. Put $f = f^\infty f_\infty$. Then*

$$\mathcal{K}(f\,dg) = \operatorname{tr} R_{\mathrm{disc}}(\pi, f^\infty dg_f).$$

In particular, if we choose a compact open subgroup K_f of $G(\mathbb{A}_f)$, and put $f^\infty dg_f = e_{K_f}$, we obtain

$$m_{\text{disc}}(\pi, K_f) = \mathcal{K}(e_\pi e_{K_f}).$$

In this paper we give some evidence for this conjecture. Moreover, we will see that $\mathcal{K}(f\,dg)$ is given by a closed algebraic expression, which is straightforward to evaluate, so long as one can compute the transfers e_π^H at the real place, and evaluate the semisimple orbital integrals of $f^{\infty H} dh_f$ at the finite adeles.

6. Euler characteristics

We have finished our discussion of Kottwitz's formula, and now solve the arithmetic volume problem mentioned in the introduction. For simplicity we will write K rather than K_f for open compact subgroups of $G(\mathbb{A}_f)$ in this section.

Definition 15. For K a compact open subgroup of $G(\mathbb{A}_f)$, we define

$$\chi_K(G) = \bar{v}(G; dg_\infty)^{-1} \operatorname{vol}_{dg_f}(K)^{-1} \tau(G) d(G)$$

if G is cuspidal. If G is not cuspidal, then $\chi_K(G) = 0$.

Note that if K_0 is another compact open subgroup of $G(\mathbb{A}_f)$, with $K \subseteq K_0$ of finite index, then $\chi_K(G) = [K_0 : K]\chi_{K_0}(G)$. In this section we compute the quantities $\chi_K(G)$ under some mild hypotheses on G.

6.1. ***Statement of theorem.*** Before getting embroiled in details, let us sketch the idea of the computation of $\chi_K(G)$. The computation is considerably easier if K is sufficiently small. In this case, $\chi_K(G)$ is the classical Euler characteristic of a Shimura variety. This in turn may be written in terms of Euler characteristics of an arithmetic subgroup of $G_{\text{ad}}(\mathbb{R})$. For G a semisimple and simply connected Chevalley group, such Euler characteristics were computed in [Harder 1971].

Our work is to reduce to this case. Given a compact open subgroup K_0 of $G(\mathbb{A}_f)$, we will pick a sufficiently small subgroup K of K_0. By the above we know the analogue of $\chi_K(G)$ for G^{sc}. To compute $\chi_{K_0}(G)$ we have two tasks: to change between G and G^{sc}, and to change between K and K_0.

The resulting formula entails several standard definitions:

Definition 16. Write $G(\mathbb{R})_+ \subseteq G(\mathbb{R})$ for the inverse image of $G_{\text{ad}}(\mathbb{R})^+$. Let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Write $\nu : G \twoheadrightarrow C$ for the quotient of G by G_{der}. Let $C(\mathbb{R})^\dagger = \nu(Z(\mathbb{R}))$, and $C(\mathbb{Q})^\dagger = C(\mathbb{Q}) \cap C(\mathbb{R})^\dagger$. Write $\rho : G_{\text{sc}} \to G_{\text{der}}$ for the usual covering of G_{der} by G_{sc}. For K a compact open subgroup of $G(\mathbb{A}_f)$, let $K^{\text{der}} = G_{\text{der}}(\mathbb{A}_f) \cap K$, and let K^{sc} be the preimage of K in $G_{\text{sc}}(\mathbb{A}_f)$. Let $\Gamma_K = G(\mathbb{Q})_+ \cap K$, let $\Gamma_K^{\text{der}} = G_{\text{der}}(\mathbb{Q})_+ \cap K$, let $\Gamma_K^{\text{sc}} = K^{\text{sc}} \cap G_{\text{sc}}(\mathbb{Q})_+$, and write Γ_K^{ad} for the image of Γ_K in $G_{\text{ad}}(\mathbb{Q})$.

In this section we avoid certain awkward tori for simplicity, preferring the following kind:

Definition 17. A torus T over $\mathbb{Q}$ is $\mathbb{Q}\mathbb{R}$-equitropic if the largest $\mathbb{Q}$-anisotropic torus in T is $\mathbb{R}$-anisotropic.

Here are some basic facts about $\mathbb{Q}\mathbb{R}$-equitropic tori.

Proposition 5. *If T is a $\mathbb{Q}\mathbb{R}$-equitropic torus, then $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$. If G is a reductive group, and the connected component Z° of the center of G is $\mathbb{Q}\mathbb{R}$-equitropic, then its derived quotient C is also $\mathbb{Q}\mathbb{R}$-equitropic.*

Proof. The first statement follows from [Milne 2005, Theorem 5.26]. The second is straightforward. □

Serre [1971] introduces an Euler characteristic $\chi_{\mathrm{alg}}(\Gamma) \in \mathbb{Q}$ applicable to any group Γ with a finite index subgroup Γ_0 that is torsion-free and has finite cohomological dimension. In particular, it applies to our congruence subgroups $\Gamma = \Gamma_K$. Here are some simple properties of χ_{alg}:

- For an exact sequence of the form
$$1 \to A \to B \to C \to 1,$$
with A, B and C groups as above, we have $\chi_{\mathrm{alg}}(B) = \chi_{\mathrm{alg}}(A) \cdot \chi_{\mathrm{alg}}(C)$.
- If Γ is a finite group, then $\chi_{\mathrm{alg}}(\Gamma) = |\Gamma|^{-1}$.

The theorem of this section relates $\chi_K(G)$ to $\chi_{\mathrm{alg}}(\Gamma_K^{\mathrm{sc}})$. More precisely:

Theorem 2. *Let G be a reductive group over $\mathbb{Q}$. Assume that G_{sc} has no compact factors and that the connected component Z° of the center of G is $\mathbb{Q}\mathbb{R}$-equitropic. Let $K_0 \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then $\chi_{K_0}(G)$ is equal to*
$$\frac{\begin{array}{c}|\ker(\rho(\mathbb{Q}))|[G_{\mathrm{der}}(\mathbb{A}_f) : G_{\mathrm{der}}(\mathbb{Q})_+ K_0^{\mathrm{der}}] \\ \cdot [\Gamma_{K_0}^{\mathrm{der}} : G_{\mathrm{der}}(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})][C(\mathbb{A}_f) : C(\mathbb{Q})^\dagger \nu(K_0)]\end{array}}{[G(\mathbb{R}) : G(\mathbb{R})_+]|\nu(K_0) \cap C(\mathbb{Q})^\dagger|} \chi_{\mathrm{alg}}(\Gamma_{K_0}^{\mathrm{sc}}).$$

Here $\rho(\mathbb{Q})$ denotes the map $\rho(\mathbb{Q}) : G_{\mathrm{sc}}(\mathbb{Q}) \to G(\mathbb{Q})$ on $\mathbb{Q}$-points. The assumption on the absence of compact factors is needed for strong approximation, and is discussed in [Milne 2005].

When G_{sc} is a Chevalley group and $\Gamma_{K_0}^{\mathrm{sc}} = G_{\mathrm{sc}}(\mathbb{Z})$, this reduces the problem to the calculation of Harder [1971]:

Proposition 6. *Let G be a simply connected, semisimple Chevalley group over $\mathbb{Z}$. Write $m_1, \dots, m_r$ for the exponents of its Weyl group Ω, and put $\Gamma = G(\mathbb{Z})$. We have*
$$\chi_{\mathrm{alg}}(\Gamma) = (-\tfrac{1}{2})^r |\Omega_{\mathbb{R}}|^{-1} \prod_{i=1}^{r} B_{m_i+1}.$$

Here B_n denotes the n-th Bernoulli number. Recall that $\Omega_{\mathbb{R}}$ is the real Weyl group of G.

6.2. *Shimura varieties.* To prove Theorem 2, we will use some basic Shimura variety theory, which may be found in [Deligne 1979] or [Milne 2005]. Much of the theory holds only for K sufficiently small. For simplicity, we will say "K is small" rather than "K is a sufficiently small finite index subgroup of K_0".

For convenience, we gather here many simplifying properties of small K, which we will often use without comment. For the rest of this section assume that $Z(G)^\circ$ is $\mathbb{QR}$-equitropic, and that G_{sc} has no compact factors.

Proposition 7. *Let K be small.*

(i) $K \cap Z(\mathbb{Q}) = \{1\}$.

(ii) $\nu(K) \cap C(\mathbb{Q}) = \{1\}$.

(iii) $G(\mathbb{Q}) \cap K G_{\mathrm{der}}(\mathbb{A}_f) \subseteq G_{\mathrm{der}}(\mathbb{Q})$.

(iv) $G_{\mathrm{der}}(\mathbb{A}_f) \cap G(\mathbb{Q})K = G_{\mathrm{der}}(\mathbb{Q})K_{\mathrm{der}}$.

(v) $K \cap G_{\mathrm{der}}(\mathbb{Q}) \subseteq \rho(G_{\mathrm{sc}}(\mathbb{Q}))$.

(vi) $K \cap G(\mathbb{Q}) \subseteq G(\mathbb{Q})^+$.

Proof. The first two items follow because Z° and thus C are $\mathbb{QR}$-equitropic. Item (iii) follows from [Deligne 1979, Corollaire 2.0.12], and the next is a corollary. Items (v) and (vi) follow from [Deligne 1979, Corollaire 2.0.5 and 2.0.14], respectively. □

Recall that we have chosen a maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$.

Definition 18. Let

$$X = G(\mathbb{R})/K_{\mathbb{R}}^+ Z(\mathbb{R}), \quad \overline{X} = G(\mathbb{R})/K_{\mathbb{R}} Z(\mathbb{R}), \quad S_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

be the double coset space obtained through the action $q(x, g)k = (qx, qgk)$ of $q \in G(\mathbb{Q})$ and $k \in K$.

Similarly, let

$$\overline{S}_K = G(\mathbb{Q})\backslash \overline{X} \times G(\mathbb{A}_f)/K,$$

with the action of $G(\mathbb{Q}) \times K$ defined in the same way.

The component group of S_K is finite and given (see [Deligne 1979, 2.1.3]) by

$$\pi_0(S_K) = G(\mathbb{A}_f)/G(\mathbb{Q})_+ K. \tag{6-1}$$

There is some variation in the literature regarding the use of X versus $\overline{X}$. Deligne [1979] and Milne [2005] implicitly use X (in light of Deligne's [Proposition 1.2.7]). Harder [1971] uses $\overline{X}$. Arthur [1989] uses

$$G(\mathbb{R})/K'_{\mathbb{R}}.$$

(Recall that $K'_{\mathbb{R}} = A_G(\mathbb{R})^+ K_{\mathbb{R}}$.) Since for us Z° is $\mathbb{QR}$-equitropic, we have

$$K'_{\mathbb{R}} = Z(\mathbb{R}) K_{\mathbb{R}},$$

and so this quotient is equal to $\overline{X}$.

Since we would like to combine results stated in terms of X with others stated in terms of $\overline{X}$, we must understand the precise relationship between the two. This is the purpose of Proposition 8 below.

Definition 19. Let G be a real group, and Z its center. Write

$$\mathrm{ad} : G(\mathbb{R}) \to G(\mathbb{R})/Z(\mathbb{R}) \tag{6-2}$$

for the quotient map.

Note that $\mathrm{ad}(G(\mathbb{R}))$ has finite index in $G_{\mathrm{ad}}(\mathbb{R})$.

Lemma 1. *For this lemma, let G be a Zariski-connected reductive real group, and $K_{\mathbb{R}}$ a maximal compact subgroup of $G(\mathbb{R})$. Let $L_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathrm{ad}}(\mathbb{R})$ containing* $\mathrm{ad}(K_{\mathbb{R}})$. *Then the following hold*:

(i) *$K_{\mathbb{R}}$ meets all the connected components of $G(\mathbb{R})$.*

(ii) $K_{\mathbb{R}} \cap G(\mathbb{R})^+ = K_{\mathbb{R}}^+$.

(iii) $\mathrm{ad}(K_{\mathbb{R}})$ *is a maximal compact subgroup of* $\mathrm{ad}(G(\mathbb{R}))$.

(iv) $\mathrm{ad}(K_{\mathbb{R}}^+) = L_{\mathbb{R}}^+$.

(v) $K_{\mathbb{R}} Z(\mathbb{R}) \cap G(\mathbb{R})_+ = K_{\mathbb{R}}^+ Z(\mathbb{R})$.

Proof. The first two statements follow from the Cartan decomposition [Satake 1980, Corollary 4.5].

For (iii), suppose that C is a subgroup of $G(\mathbb{R})$ with $\mathrm{ad}(K_{\mathbb{R}}) \subseteq \mathrm{ad}(C)$ and $\mathrm{ad}(C)$ compact. If $\mathrm{ad}(K_{\mathbb{R}}) \neq \mathrm{ad}(C)$, there is an element $a \in CZ(\mathbb{R}) - K_{\mathbb{R}} Z(\mathbb{R})$. By the Cartan decomposition, we may assume that $a = \exp(H)$, with H a semisimple element of $\mathrm{Lie}(G)$, and $\alpha(H)$ real and nonnegative for every root α of G. Since $a \notin Z(\mathbb{R})$, we have $\alpha(H) > 0$ for some root α. Thus $\mathrm{ad}(C)$ is not compact, a contradiction. Thus $\mathrm{ad}(K_{\mathbb{R}}) = \mathrm{ad}(C)$, and statement (iii) follows.

For (iv), note that $L_{\mathbb{R}} \cap \mathrm{ad}(G) = \mathrm{ad}(K_{\mathbb{R}})$, and therefore $L_{\mathbb{R}}/\mathrm{ad}(K_{\mathbb{R}})$ injects into $G_{\mathrm{ad}}(\mathbb{R})/\mathrm{ad}(G(\mathbb{R}))$. It follows that $\mathrm{ad}(K_{\mathbb{R}}^+)$ has finite index in $L_{\mathbb{R}}$. Since it is connected, statement (iv) follows.

For (v), let $g \in K_{\mathbb{R}} Z(\mathbb{R}) \cap G(\mathbb{R})_+$. Then $\mathrm{ad}(g) \in L_{\mathbb{R}} \cap G_{\mathrm{ad}}(\mathbb{R})^+$, so by statement (ii), we see $\mathrm{ad}(g) \in L_{\mathbb{R}}^+ = \mathrm{ad}(K_{\mathbb{R}}^+)$. Thus $g \in K_{\mathbb{R}}^+ Z(\mathbb{R})$. The other inclusion is obvious. □

Proposition 8.

(i) *The natural projection $p_X : X \to \overline{X}$ has fibers of order* $[G(\mathbb{R}) : G(\mathbb{R})_+]$.

(ii) *Let X^+ be a connected component of X. It is stabilized by $G(\mathbb{R})_+$, and the restriction of p_X to X^+ is a $G(\mathbb{R})_+$-isomorphism onto $\overline{X}$.*

(iii) *Let K be small. Then the natural projection $p_S : S_K \to \overline{S}_K$ has fibers of order $[G(\mathbb{R}) : G(\mathbb{R})_+]$.*

Proof. Consider the natural map

$$K_{\mathbb{R}} Z(\mathbb{R})/K_{\mathbb{R}}^+ Z(\mathbb{R}) \to G(\mathbb{R})/G(\mathbb{R})_+. \tag{6-3}$$

It is surjective because $K_{\mathbb{R}}$ meets every connected component of $G(\mathbb{R})$. It is injective because $K_{\mathbb{R}} Z(\mathbb{R}) \cap G(\mathbb{R})_+ \subseteq K_{\mathbb{R}}^+ Z(\mathbb{R})$. It follows that (6-3) is an isomorphism, and the first statement follows.

We now prove the second statement. Note that p_X is both an open and closed map, so that $p_X(X^+)$ is a component of $\overline{X}$. Since $K_{\mathbb{R}}$ meets every connected component of $G(\mathbb{R})$, the set $\overline{X}$ is connected. Therefore $p_X(X^+) = \overline{X}$. By [Milne 2005, Proposition 5.7], there are $[G(\mathbb{R}) : G(\mathbb{R})_+]$ connected components of X, each stabilized by $G(\mathbb{R})_+$. Thus the fiber over a point in $\overline{X}$ is composed of exactly one point from each component of X. So p_X restricted to X^+ is an isomorphism; it is clear that it respects the $G(\mathbb{R})_+$-action.

To prove the third statement, we require K to be sufficiently small, in the following way. Suppose K_* is an open compact subgroup of $G(\mathbb{A}_f)$ satisfying $K_* \cap G(\mathbb{Q}) \subseteq G(\mathbb{Q})^+$. Let $g_1, \dots, g_r$ be representatives of the finite quotient group $G(\mathbb{Q})K_* \backslash G(\mathbb{A}_f)$. Then we require that

$$K \subseteq \bigcap_{i=1}^{r} g_i^{-1} K_* g_i. \tag{6-4}$$

Now for $x \in X$, let $\mathrm{Fib}(x)$ be the fiber of p_X containing x. If we further fix $g \in G(\mathbb{A}_f)$, let $\mathrm{Fib}(x, g)$ be the fiber of p_S containing (x, g). (Here we understand (x, g) as an element of S_K.) We claim that for all such x and g, the map

$$\mathrm{Fib}(x) \to \mathrm{Fib}(x, g) \tag{6-5}$$

given by $x' \mapsto (x', g)$ is a bijection. This will imply the third statement.

For surjectivity of (6-5), pick $(x', g') \in \mathrm{Fib}(x, g)$. Then there are $q \in G(\mathbb{Q})$ and $k \in G(\mathbb{A}_f)$ such that $qp_X(x') = p_X(x)$ and $qg'k = g$. Let $x'' = qx'$. Then $x'' \in \mathrm{Fib}(x)$ and $(x'', g) = (x', g')$.

For injectivity of (6-5), suppose that $(x_1, g) = (x_2, g)$ in S_K with $x_1, x_2 \in \mathrm{Fib}(x)$. Then in particular, there is an element $q \in G(\mathbb{Q})$ and $k \in K$ such that $qgk = g$ and $qx_1 = x_2$. Write $g = q_0 k_0 g_i$ with $q_0 \in G(\mathbb{Q})$ and $k_0 \in K_*$. Then we have

$$q(q_0 k_0 g_i)k = q_0 k_0 g_i,$$

which we rewrite as

$$q_0^{-1} q q_0 = k_0 g_i k^{-1} g_i^{-1} k_0^{-1}.$$

Using this and (6-4) we see that $q_0^{-1} q q_0 \in G(\mathbb{Q}) \cap K_* \subseteq G(\mathbb{Q})^+$. Since $G(\mathbb{Q})^+$ is normal in $G(\mathbb{Q})$, in fact $q \in G(\mathbb{Q})^+$.

Meanwhile, pick $\xi_1, \xi_2 \in G(\mathbb{R})$ representing x_1 and x_2, respectively. Since $x_1, x_2 \in \mathrm{Fib}(x)$ we have $\xi_1^{-1}\xi_2 \in K_{\mathbb{R}} Z(\mathbb{R})$. Write $\xi_2 = \xi_1 k z$, with $k \in K_{\mathbb{R}}$ and $z \in Z(\mathbb{R})$. Since $q x_1 = x_2$, we have $\xi_2^{-1} q \xi_1 \in K_{\mathbb{R}}^+ Z(\mathbb{R})$, and thus $z^{-1} k^{-1} \xi_1^{-1} q \xi_1 \in K_{\mathbb{R}}^+ Z(\mathbb{R})$. Using the fact that q is in the normal subgroup $G(\mathbb{R})_+$ of $G(\mathbb{R})$, it follows that $k \in G(\mathbb{R})_+ \cap K_{\mathbb{R}} \subseteq K_{\mathbb{R}}^+ Z(\mathbb{R})$. Thus $x_1 = x_2$, as desired. □

Proposition 9 (Harder; see [Harder 1971; Serre 1971]). *If G is semisimple and K is small, then* $\chi_{\mathrm{top}}(\Gamma_K \backslash \bar{X}) = \chi_{\mathrm{alg}}(\Gamma_K)$.

Proposition 10 [Arthur 1989; Goresky et al. 1997]. *If K is small, then we have* $\chi_K(G) = \chi_{\mathrm{top}}(\bar{S}_K)$.

6.3. *Computations.* The next three lemmas will allow us to convert our computation for K_0 to a computation for K.

Lemma 2. *If K is small, then*

$$|C(\mathbb{Q})^\dagger \backslash C(\mathbb{A}_f)/\nu(K)| = [\nu(K_0) : \nu(K)] |\nu(K_0) \cap C(\mathbb{Q})^\dagger|^{-1} |C(\mathbb{Q})^\dagger \backslash C(\mathbb{A}_f)/\nu(K_0)|.$$

Proof. This follows from the exactness of the sequence

$$1 \to \nu(K_0) \cap C(\mathbb{Q})^\dagger \to \nu(K_0)/\nu(K) \to C(\mathbb{Q})^\dagger \backslash C(\mathbb{A}_f)/\nu(K) \to C(\mathbb{Q})^\dagger \backslash C(\mathbb{A}_f)/\nu(K_0) \to 1. \quad \square$$

Lemma 3. *If $K \subseteq K_0$ is small, then*

$$[\Gamma_{K_0}^{\mathrm{ad}} : \Gamma_K^{\mathrm{ad}}] = \frac{[\Gamma_{K_0} : \rho(\Gamma_{K_0}^{\mathrm{sc}})][K_0 : K]}{|K_0 \cap Z(\mathbb{Q})| [\nu(K_0) : \nu(K)] [K_0^{\mathrm{der}} : K^{\mathrm{der}} \rho(K_0^{\mathrm{sc}})]}. \tag{6-6}$$

In the proof we refer to conditions of Proposition 7.

Proof. Consider the map $\Gamma_{K_0}^{\mathrm{der}} / \Gamma_K^{\mathrm{der}} \to \Gamma_{K_0}^{\mathrm{ad}} / \Gamma_K^{\mathrm{ad}}$.

The kernel of this map sits in the middle of the exact sequence

$$1 \to \Gamma_{K_0}^{\mathrm{der}} \cap Z(\mathbb{Q}) \to (\Gamma_K Z(\mathbb{Q}) \cap \Gamma_{K_0}^{\mathrm{der}}) / \Gamma_K^{\mathrm{der}} \to (\Gamma_K Z(\mathbb{Q}) \cap \Gamma_{K_0}^{\mathrm{der}}) / \Gamma_K^{\mathrm{der}} (\Gamma_{K_0}^{\mathrm{der}} \cap Z(\mathbb{Q})) \to 1,$$

using condition (i). This last quotient is trivial, because actually $\Gamma_K = \Gamma_K^{\mathrm{der}}$ by condition (iii).

We have established the exactness of the sequence

$$1 \to \Gamma_{K_0}^{\mathrm{der}} \cap Z(\mathbb{Q}) \to \Gamma_{K_0}^{\mathrm{der}} / \Gamma_K^{\mathrm{der}} \to \Gamma_{K_0}^{\mathrm{ad}} / \Gamma_K^{\mathrm{ad}} \to \Gamma_{K_0} Z(\mathbb{Q}) / \Gamma_{K_0}^{\mathrm{der}} Z(\mathbb{Q}) \to 1.$$

The last quotient is isomorphic to $\Gamma_{K_0}/(Z(\mathbb{Q}) \cap K_0)\Gamma_{K_0}^{\mathrm{der}}$, which itself sits inside the exact sequence

$$1 \to K_0 \cap Z(\mathbb{Q}) / \Gamma_{K_0}^{\mathrm{der}} \cap Z(\mathbb{Q}) \to \Gamma_{K_0} / \Gamma_{K_0}^{\mathrm{der}} \to \Gamma_{K_0}/(Z(\mathbb{Q}) \cap K_0)\Gamma_{K_0}^{\mathrm{der}} \to 1.$$

The quantity $|\Gamma_{K_0}^{\mathrm{der}} \cap Z(\mathbb{Q})|$ cancels, and it follows that

$$[\Gamma_{K_0}^{\mathrm{ad}} : \Gamma_K^{\mathrm{ad}}] = \frac{[\Gamma_{K_0}^{\mathrm{der}} : \Gamma_K^{\mathrm{der}}] \cdot [\Gamma_{K_0} : \Gamma_{K_0}^{\mathrm{der}}]}{|K_0 \cap Z(\mathbb{Q})|}. \tag{6-7}$$

By condition (v) we have

$$1 \to \rho(\Gamma_{K_0}^{\mathrm{sc}})/\rho(\Gamma_K^{\mathrm{sc}}) \to \Gamma_{K_0}^{\mathrm{der}} / \Gamma_K^{\mathrm{der}} \to \Gamma_{K_0}^{\mathrm{der}}/\rho(\Gamma_{K_0}^{\mathrm{sc}}) \to 1.$$

Strong approximation tells us that $G_{\mathrm{sc}}(\mathbb{Q})$ is dense in $G_{\mathrm{sc}}(\mathbb{A}_f)$. Therefore we have isomorphisms

$$\rho(\Gamma_{K_0}^{\mathrm{sc}})/\rho(\Gamma_K^{\mathrm{sc}}) \xrightarrow{\sim} \Gamma_{K_0}^{\mathrm{sc}} / \Gamma_K^{\mathrm{sc}} \xrightarrow{\sim} K_0^{\mathrm{sc}}/K^{\mathrm{sc}} \xrightarrow{\sim} \rho(K_0^{\mathrm{sc}})/\rho(K^{\mathrm{sc}}).$$

Combining this with the exact sequences

$$1 \to K_0^{\mathrm{der}}/K^{\mathrm{der}} \to K_0/K \to \nu(K_0)/\nu(K) \to 1$$

and

$$1 \to \rho(K_0^{\mathrm{sc}})/\rho(K^{\mathrm{sc}}) \to K_0^{\mathrm{der}}/K^{\mathrm{der}} \to K_0^{\mathrm{der}}/K^{\mathrm{der}}\rho(K_0^{\mathrm{sc}}) \to 1, \tag{6-8}$$

we obtain

$$[\Gamma_{K_0}^{\mathrm{der}} : \Gamma_K^{\mathrm{der}}] = \frac{[\Gamma_{K_0}^{\mathrm{der}} : \rho(\Gamma_{K_0}^{\mathrm{sc}})][K_0 : K]}{[K_0^{\mathrm{der}} : K^{\mathrm{der}}\rho(K_0^{\mathrm{sc}})][\nu(K_0) : \nu(K)]}.$$

Plugging this into (6-7) gives the lemma. □

Corollary 1. *Suppose that $K \subseteq K_0$ is small, and $g \in G(\mathbb{A}_f)$ with $gKg^{-1} \subseteq K_0$ also small. Then*

$$[\Gamma_{K_0}^{\mathrm{ad}} : \Gamma_{gKg^{-1}}^{\mathrm{ad}}] = [\Gamma_{K_0}^{\mathrm{ad}} : \Gamma_K^{\mathrm{ad}}].$$

Proof. We show that the expression (6-6) does not change when K is replaced with gKg^{-1}. Clearly $\nu(K) = \nu(gKg^{-1})$. Since

$$[K_0 : K] = \mathrm{vol}_{dg_f}(K_0)/\mathrm{vol}_{dg_f}(K),$$

we have $[K_0 : gKg^{-1}] = [K_0 : K]$. Finally, we claim that

$$[K_0^{\mathrm{der}} : (gKg^{-1})^{\mathrm{der}}\rho(K_0^{\mathrm{sc}})] = [K_0^{\mathrm{der}} : K^{\mathrm{der}}\rho(K_0^{\mathrm{sc}})].$$

From the exact sequence (6-8), it is enough to show that $[K_0^{\mathrm{der}} : (gKg^{-1})^{\mathrm{der}}] = [K_0^{\mathrm{der}} : K^{\mathrm{der}}]$ and $[\rho(K_0^{\mathrm{sc}}) : \rho((gKg^{-1})^{\mathrm{sc}})] = [\rho(K_0^{\mathrm{sc}}) : \rho(K^{\mathrm{sc}})]$. These hold because $(gKg^{-1})^{\mathrm{der}} = gK^{\mathrm{der}}g^{-1}$ and $\rho((gKg^{-1})^{\mathrm{sc}}) = g\rho(K^{\mathrm{sc}})g^{-1}$. □

Lemma 4. *If G is semisimple and K is small, then*

$$|\pi_0(S_K)| = [K_0 : K\rho(K_0^{\mathrm{sc}})][\Gamma_{K_0} : G(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})]|\pi_0(S_{K_0})|.$$

Proof. The kernel of the projection $\pi_0(S_K) \twoheadrightarrow \pi_0(S_{K_0})$ is isomorphic to

$$K_0/(KG(\mathbb{Q})_+ \cap K_0).$$

By [Deligne 1979, Section 2.1.3], we have $\rho(G_{\mathrm{sc}}(\mathbb{A}_f)) \subseteq KG(\mathbb{Q})_+$. Using the exact sequence

$$1 \to (K_0 \cap KG(\mathbb{Q})_+)/K\rho(K_0^{\mathrm{sc}}) \to K_0/K\rho(K_0^{\mathrm{sc}}) \to K_0/(KG(\mathbb{Q})_+ \cap K_0) \to 1,$$

we are reduced to computing the order of

$$(K_0 \cap KG(\mathbb{Q})_+)/K\rho(K_0^{\mathrm{sc}}) \xrightarrow{\sim} \Gamma_{K_0}/(K\rho(K_0^{\mathrm{sc}}) \cap G(\mathbb{Q})_+).$$

This group sits in the sequence

$$\begin{aligned} 1 \to (G(\mathbb{Q})_+ \cap K\rho(K_0^{\mathrm{sc}}))/(G(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})) \\ \to \Gamma_{K_0}/(G(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})) \to \Gamma_{K_0}/(K\rho(K_0^{\mathrm{sc}}) \cap G(\mathbb{Q})_+) \to 1. \end{aligned}$$

We claim the kernel is trivial. Note that $K\rho(K_0^{\mathrm{sc}}) \subseteq K\rho(G_{\mathrm{sc}}(\mathbb{Q})K^{\mathrm{sc}})$ by strong approximation. So

$$\begin{aligned} G(\mathbb{Q})_+ \cap K\rho(K_0^{\mathrm{sc}}) &\subseteq G(\mathbb{Q})_+ \cap K\rho(G_{\mathrm{sc}}(\mathbb{Q})) \\ &= G(\mathbb{Q})_+ \cap (K \cap G(\mathbb{Q}))\rho(G_{\mathrm{sc}}(\mathbb{Q})). \end{aligned}$$

Since $K \cap G(\mathbb{Q}) \subseteq \rho(G_{\mathrm{sc}}(\mathbb{Q}))$ by Proposition 7(v), we have $G(\mathbb{Q})_+ \cap K\rho(K_0^{\mathrm{sc}}) \subseteq G(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})$. This proves the claim, and the lemma follows. □

In the course of proving the theorem, we will pass to the adjoint group to apply Harder's theorem (Proposition 9), but lift to G_{sc} to apply Harder's calculation (Proposition 6). We must record the difference between Serre's Euler characteristic at G_{ad} and G_{sc}.

Lemma 5. *We have*

$$\chi_{\mathrm{alg}}(\Gamma_{K_0}^{\mathrm{ad}}) = \frac{|\ker(\rho(\mathbb{Q}))||K_0 \cap Z(\mathbb{Q})|}{[\Gamma_{K_0}^{\mathrm{der}} : \rho(\Gamma_{K_0}^{\mathrm{sc}})][\Gamma_{K_0} : \Gamma_{K_0}^{\mathrm{der}}]}\chi_{\mathrm{alg}}(\Gamma_{K_0}^{\mathrm{sc}}).$$

Proof. This follows from the properties of χ_{alg} mentioned earlier. □

Proof of Theorem 2. Pick a set $g_1, \ldots, g_r$ of representatives of $\pi_0(S_{K_0})$, viewed as a quotient of $G(\mathbb{A}_f)$ as in (6-1).

Let K be small subgroup of finite index in K_0. Possibly by intersecting finitely many conjugates of K, we may assume that

- K is normal in K_0 and
- $g_i K g_i^{-1}$ is a small subgroup of K_0 for all i.

By Proposition 10, $\chi_K(G) = \chi_{\text{top}}(\overline{S}_K)$. By Proposition 8, this is equal to $[G(\mathbb{R}) : G(\mathbb{R})_+]^{-1}\chi_{\text{top}}(S_K)$. Write Γ_g for $\Gamma^{\text{ad}}_{gKg^{-1}}$. By [Deligne 1979, 2.1.2], the components of S_K are each isomorphic to $\Gamma_g\backslash X^+$, where X^+ is a component of X. Here g runs over $\pi_0(S_K)$.

By Proposition 8, the topological spaces $\Gamma_g\backslash X^+$ and $\Gamma_g\backslash\overline{X}$ are isomorphic. Therefore we have $\chi_{\text{top}}(\Gamma_g\backslash X^+) = \chi_{\text{top}}(\Gamma_g\backslash\overline{X})$.

Applying Proposition 9 to G_{ad}, this is equal to $\chi_{\text{alg}}(\Gamma_g)$. Therefore

$$\chi_K(G) = [G(\mathbb{R}) : G(\mathbb{R})_+]^{-1} \sum_{g\in\pi_0(S_K)} \chi_{\text{alg}}(\Gamma_g).$$

Every element in $\pi_0(S_K)$ may be written as the product of an element of $\pi_0(S_{K_0})$ with an element of K_0. Since K is normal in K_0, the groups Γ_{gk_0} and Γ_g are equal for $k_0 \in K_0$. It follows that

$$\chi_K(G) = \frac{|\pi_0(S_K)|}{[G(\mathbb{R}) : G(\mathbb{R})_+]|\pi_0(S_{K_0})|} \sum_{i=1}^{r} \chi_{\text{alg}}(\Gamma_{g_i}).$$

By Corollary 1 we have

$$\chi_{\text{alg}}(\Gamma_{g_i}) = [\Gamma^{\text{ad}}_{K_0} : \Gamma_{g_i}]\chi_{\text{alg}}(\Gamma^{\text{ad}}_{K_0}) = [\Gamma^{\text{ad}}_{K_0} : \Gamma^{\text{ad}}_{K}]\chi_{\text{alg}}(\Gamma^{\text{ad}}_{K_0}).$$

This gives

$$\chi_K(G) = [G(\mathbb{R}) : G(\mathbb{R})_+]^{-1}[\Gamma^{\text{ad}}_{K_0} : \Gamma^{\text{ad}}_{K}]|\pi_0(S_K)|\chi_{\text{alg}}(\Gamma^{\text{ad}}_{K_0}).$$

The component group $\pi_0(S_K)$ fits into the exact sequence

$$1 \to G_{\text{der}}(\mathbb{A}_f)/(G_{\text{der}}(\mathbb{A}_f) \cap G(\mathbb{Q})_+K) \to \pi_0(S_K) \to C(\mathbb{Q})^{\dagger}\backslash C(\mathbb{A}_f)/\nu(K) \to 1$$

This gives

$$\chi_K(G) = [G(\mathbb{R}) : G(\mathbb{R})_+]^{-1}|\pi_0(S_{K^{\text{der}}})||C(\mathbb{Q})^{\dagger}\backslash C(\mathbb{A}_f)/\nu(K)|[\Gamma^{\text{ad}}_{K_0} : \Gamma^{\text{ad}}_{K}]\chi_{\text{alg}}(\Gamma^{\text{ad}}_{K_0}).$$

where here $\pi_0(S_{K^{\text{der}}}) = G_{\text{der}}(\mathbb{A}_f)/G_{\text{der}}(\mathbb{Q})_+K^{\text{der}}$.

Using $\chi_{K_0}(G) = [K_0 : K]^{-1}\chi_K(G)$ together with Lemma 2 gives

$$\chi_{K_0}(G) = \frac{|\pi_0(S_{K^{\text{der}}})|[\nu(K_0) : \nu(K)]|C(\mathbb{Q})^{\dagger}\backslash C(\mathbb{A}_f)/\nu(K_0)|[\Gamma^{\text{ad}}_{K_0} : \Gamma^{\text{ad}}_{K}]}{[G(\mathbb{R}) : G(\mathbb{R})_+]|\nu(K_0) \cap C(\mathbb{Q})^{\dagger}|[K_0 : K]} \chi_{\text{alg}}(\Gamma^{\text{ad}}_{K_0}).$$

By Lemmas 3 and 5,

$$\chi_{K_0}(G) = \frac{|\ker(\rho(\mathbb{Q}))||\pi_0(S_{K^{\mathrm{der}}})||C(\mathbb{Q})^{\dagger}\backslash C(\mathbb{A}_f)/\nu(K_0)|}{[G(\mathbb{R}):G(\mathbb{R})_+]|\nu(K_0)\cap C(\mathbb{Q})^{\dagger}|[K_0^{\mathrm{der}}:K^{\mathrm{der}}\rho(K_0^{\mathrm{sc}})]}\chi_{\mathrm{alg}}(\Gamma_{K_0}^{\mathrm{sc}}).$$

The theorem then follows from Lemma 4. □

6.4. ***Examples.*** We now use Theorem 2 and Proposition 6 to explicitly compute some cases of $\chi_{K_0}(G)$. Recall that we write $\mathbb{O}_f$ for the integer points of $\mathbb{A}_f$.

Corollary 2. *If T is a torus and $K_0 \subset T(\mathbb{A}_f)$ is a compact open subgroup, then*

$$\chi_{K_0}(T) = |T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_0|\cdot|K_0\cap T(\mathbb{Q})|^{-1}.$$

Let $T=\mathbb{G}_m$, and $K_0 = T(\mathbb{O}_f)$. Then $\chi_{K_0}(T) = 1/2$.

Let T be the norm-one subgroup of an imaginary quadratic extension E of $\mathbb{Q}$. Let $K_0 = T(\mathbb{O}_f)$. Write $\mathbb{O}(E)$ for the integer points of the adeles $\mathbb{A}_E$ over E. Then $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_0$ injects into $E^{\times}\backslash\mathbb{A}_{E,f}^{\times}/\mathbb{O}(E)^{\times}$, which is in bijection with the class group. If the class number of E is trivial, it follows that $\chi_{K_0}(T) = |T(\mathbb{Z})|^{-1}$.

Corollary 3. *If G is semisimple and simply connected, then*

$$\chi_{K_0}(G) = [G(\mathbb{R}):G(\mathbb{R})_+]^{-1}\chi_{\mathrm{alg}}(\Gamma_{K_0}).$$

Let $G=\mathrm{SL}_2$ and $K_0 = G(\mathbb{O}_f)$. Then

$$\chi_{K_0}(G) = \chi_{\mathrm{alg}}(\mathrm{SL}_2(\mathbb{Z})) = -\tfrac{1}{2}B_2 = -2^{-2}3^{-1}.$$

Let $G=\mathrm{Sp}_4$ and $K_0 = G(\mathbb{O}_f)$. Then

$$\chi_{K_0}(G) = \chi_{\mathrm{alg}}(\mathrm{Sp}_4(\mathbb{Z})) = -\tfrac{1}{8}B_2B_4 = -2^{-5}3^{-2}5^{-1}.$$

When the derived group is simply connected the calculation is not much harder.

Corollary 4. *If G_{der} is simply connected, then*

$$\chi_{K_0}(G) = \frac{|C(\mathbb{Q})^{\dagger}\backslash C(\mathbb{A}_f)/\nu(K_0)|}{[G(\mathbb{R}):G(\mathbb{R})_+]|\nu(K_0)\cap C(\mathbb{Q})^{\dagger}|}\chi_{\mathrm{alg}}(\Gamma_{K_0}^{\mathrm{der}}).$$

Let $G=\mathrm{GL}_2$ and $K_0 = G(\mathbb{O}_f)$. Then $\chi_{K_0}(G) = \frac{1}{2}\chi_{\mathrm{alg}}(\mathrm{SL}_2(\mathbb{Z})) = -2^{-3}3^{-1}$.

Let $G=\mathrm{GSp}_4$ and $K_0 = G(\mathbb{O}_f)$. Then $\chi_{K_0}(G) = \frac{1}{2}\chi_{\mathrm{alg}}(\mathrm{Sp}_4(\mathbb{Z})) = -2^{-6}3^{-2}5^{-1}$.

Lemma 6. *If all the points of* $\ker\rho$ *are $\mathbb{Q}$-rational, then*

$$[\Gamma_{K_0}^{\mathrm{der}} : G_{\mathrm{der}}(\mathbb{Q})_+\cap\rho(K_0^{\mathrm{sc}})] = 1.$$

Proof. By [Deligne 1979, Section 2.0.3], we have an injection

$$G_{\mathrm{der}}(\mathbb{Q})/\rho(G_{\mathrm{sc}}(\mathbb{Q})) \hookrightarrow H^1(\mathrm{im}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})), (\ker\rho)(\overline{\mathbb{Q}})),$$

using the cohomology group defined in that paper. We also have an injection

$$\Gamma_{K_0}^{\mathrm{der}}/(G_{\mathrm{der}}(\mathbb{Q})_+ \cap \rho(K_0^{\mathrm{sc}})) \hookrightarrow G_{\mathrm{der}}(\mathbb{Q})/\rho(G_{\mathrm{sc}}(\mathbb{Q})).$$

Since all the points of $\ker\rho$ are $\mathbb{Q}$-rational, all these groups are trivial. □

Let $G = \mathrm{PGL}_2$ and $K_0 = G(\mathbb{O}_f)$. The only nontrivial factors in the formula are $[G(\mathbb{R}) : G(\mathbb{R})_+] = 2$, $|\ker\rho(\mathbb{Q})| = 2$, and $\chi_{\mathrm{alg}}(\mathrm{SL}_2(\mathbb{Z})) = -2^{-2}3^{-1}$. Thus $\chi_{K_0}(G) = -2^{-2}3^{-1}$.

7. The case of SL_2

Let $G = \mathrm{SL}_2$, defined over $\mathbb{Q}$. Let A be the subgroup of diagonal matrices in G, and let T be the maximal elliptic torus of G given by matrices

$$\gamma_{a,b} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{7-1}$$

with $a^2 + b^2 = 1$.

The characters and cocharacters of T are both isomorphic to $\mathbb{Z}$. We identify $\mathbb{Z} \xrightarrow{\sim} X^*(T)$ via $n \mapsto \chi_n$, where $\chi_n(\gamma_{a,b}) = (a+bi)^n$. We specify $\mathbb{Z} \xrightarrow{\sim} X_*(T)$ by identifying n with the cocharacter taking α to $\mathrm{diag}(\alpha, \alpha^{-1})$. The roots of T in G are then $\{\pm 2\}$, and the coroots of T in G are $\{\pm 1\}$. The Weyl group Ω of these systems has order 2 and the compact Weyl group $\Omega_{\mathbb{R}}$ is trivial. Thus each L-packet of discrete series has order 2. The group dual to G is $\hat{G} = \mathrm{PGL}_2(\mathbb{C})$ in the usual way.

Pick an element $\xi \in G(\mathbb{C})$ such that

$$\mathrm{Ad}(\xi)\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a+ib & \\ & a-ib \end{pmatrix},$$

and put $B_T = \mathrm{Ad}(\xi^{-1})B_A$. Then B_T is a Borel subgroup of $G(\mathbb{C})$ containing T.

Consider the Langlands parameter $\varphi_G : W_{\mathbb{R}} \to \hat{G}$ given by $\varphi_G(\tau) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \times 1$, and

$$\varphi_G(z) = \mathrm{diag}(z^n, \bar{z}^n) \times z = z^\mu \bar{z}^\nu \times z,$$

where μ corresponds to $n \in X_*(\hat{T}) \xrightarrow{\sim} X^*(T)$ and ν corresponds to $-n$. The corresponding representation E of $G(\mathbb{C})$ has highest weight $\lambda_B = n-1 \in X^*(T)$. It is the $(n-1)$-st symmetric power of the standard representation. Its central character is $\lambda_E(z) = z^{n-1}$, where $z = \pm 1$.

We put $\pi_G = \pi(\varphi_G, B_T)$, in the notation from Section 2.2. Write π_G' for the other discrete series representation in Π_E. Thus the L-packet determined by φ_G is

$$\Pi_E = \{\pi_G, \pi_G'\}.$$

We will put $f_\infty dg_\infty = e_{\pi_G}$ as in Section 2.4.

7.1. ***Main term.*** First we consider the terms $ST_g(f\,dg, \pm 1)$.

We have $S\Phi_G(1, e_{\pi_G}) = -n\bar{v}(G; dg_\infty)^{-1}$, and so

$$ST_g(f\,dg, \pm 1, G) = (\pm 1)^n n\bar{v}(G; dg_\infty)^{-1} f^\infty(\pm 1).$$

We have $S\Phi_A(1, e_{\pi_G}) = -\bar{v}(G; dg_\infty)^{-1}$, and so

$$ST_g(f\,dg, \pm 1, A) = (\pm 1)^n \tfrac{1}{2}\bar{v}(G; dg_\infty)^{-1} f_A^\infty(\pm 1).$$

If γ is a regular semisimple element of $G(\mathbb{C})$ with eigenvalues α and α^{-1}, then according to the Weyl character formula,

$$\mathrm{tr}(\gamma; E) = \frac{\alpha^n - \alpha^{-n}}{\alpha - \alpha^{-1}}.$$

Define $t_4(n) = \mathrm{tr}(\mathrm{diag}(i, -i); E)$, where i is a fourth root of unity. Then $t_4(n) = 0$ if n is even, and $t_4(n) = (-1)^{(n-1)/2}$ if n is odd.

Similarly, define $t_3(n) = \mathrm{tr}(\mathrm{diag}(\zeta, \zeta^2); E)$, where ζ is a third root of unity. Then $t_3(n) = [0, 1, -1; 3]_n$, meaning that

$$t_3(n) = \begin{cases} 0 & \text{if } n \equiv 0, \\ 1 & \text{if } n \equiv 1, \\ -1 & \text{if } n \equiv 2. \end{cases}$$

Here the congruence is modulo 3.

There are three stable conjugacy classes of elliptic $\gamma \in G(\mathbb{Q})$, which we represent by

$$\gamma_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Note that $-\gamma_4 \sim \gamma_4$, $\gamma_6^2 = \gamma_3$, and $-\gamma_3 \sim \gamma_6$.

Write T_3 for the elliptic torus consisting of elements

$$\begin{pmatrix} a & a-b \\ b-a & b \end{pmatrix}, \quad \text{with } a^2 - ab + b^2 = 1.$$

We have $S\Phi_G(\gamma_3, e_{\pi_G}) = -\bar{v}(T_3)^{-1} t_3(n)$, and so

$$ST_g(f\,dg, \gamma_3, G) = -\bar{v}(T_3)^{-1}\, SO_{\gamma_3}(f^\infty dg_f) t_3(n).$$

We have $S\Phi_G(\gamma_4, e_{\pi_G}) = -\bar{v}(T)^{-1} t_4(n)$, and so

$$ST_g(f\,dg, \gamma_4, G) = -\bar{v}(T)^{-1}\, SO_{\gamma_4}(f^\infty dg_f) t_4(n).$$

Finally $S\Phi_G(\gamma_6, e_{\pi_G}) = -\bar{v}(T_3) t_3(n)(-1)^{n-1}$, and so

$$ST_g(f\,dg, \gamma_6, G) = -\bar{v}(T_3)^{-1}\, SO_{-\gamma_3}(f^\infty dg_f) t_3(n)(-1)^{n-1}.$$

Thus, $ST_g(f\,dg)$ is equal to the sum

$$-n\bar{v}(G;dg_\infty)^{-1}f^\infty(1)+n\bar{v}(G;dg_\infty)^{-1}f^\infty(-1)(-1)^n-\tfrac{1}{2}\bar{v}(A;da_\infty)^{-1}f_A^\infty(1)\\ +\tfrac{1}{2}\bar{v}(A;da_\infty)^{-1}f_A^\infty(-1)(-1)^n-\bar{v}(T_3)^{-1}\,SO_{\gamma_3}(f^\infty dg_f)t_3(n)\\ -\bar{v}(T)^{-1}\,SO_{\gamma_4}(f^\infty dg_f)t_4(n)+\bar{v}(T_3)^{-1}\,SO_{-\gamma_3}(f^\infty dg_f)t_3(n)(-1)^n.$$

7.2. *Endoscopic terms.*

Definition 20. Let E be an imaginary quadratic extension of $\mathbb{Q}$. Write H_E for the kernel of the norm map $\mathrm{Res}^E_{\mathbb{Q}}\,\mathbb{G}_m\to\mathbb{G}_m$.

The H_E comprise the (proper) elliptic endoscopic groups for $G=\mathrm{SL}_2$. For each $H=H_E$ we have $\tau(H)=2$ and $|\mathrm{Out}(H,s,\eta)|=1$; see [Kottwitz 1984, Section 7]. Therefore $\iota(G,H)=\frac{1}{2}$. The character identities of Shelstad [1982] give $e^H_{\pi_G}=e_{\chi_n}+e_{\chi_n^{-1}}$.

Write $f^H dh=f^{\infty H}dh_f e^H_{\pi_G}$, where $f^{\infty H}dh_f$ is the transfer of $f^\infty dg_f$. Choose dh_∞ so that $dh_f dh_\infty$ is the Tamagawa measure on H. Then we obtain

$$ST_g(f^H dh)=2\bar{v}(H;dh_\infty)\sum_{\gamma_H}f^{\infty,H}(\gamma_H)\,\mathrm{Tr}^E_{\mathbb{Q}}(\gamma_H^n),$$

the sum being taken over $\gamma_H\in H(\mathbb{Q})$.

Remark. Consider the local transfer, where $f_p dg_p$ is a spherical (that is, invariant under $G(\mathbb{Z}_p)$) measure on $G(\mathbb{Q}_p)$. Then if H ramifies over p, a representation π_p in one of the L-packets transferring from H will also be ramified. This means that $\mathrm{tr}\,\pi_p(f_p dg_p)=0$. So we take $f_p^H=0$ in this case. Thus

$$\mathcal{K}(f\,dg)=ST_g(f\,dg);$$

there is no (proper) endoscopic contribution. This is compatible with the fact that m_{disc} is constant on L-packets in this case.

7.3. *Case of* $\Gamma=\mathrm{SL}_2(\mathbb{Z})$.

We take $K_f=K_0$ to be the integral points of $G(\mathbb{A}_f)$. Also let $K_A=K_0\cap A(\mathbb{A}_f)$ and $K_T=K_0\cap T(\mathbb{A}_f)$. Each of these breaks into a product of local groups $K_{0,p}$, etc.

We put $f^\infty dg_f=e_{K_0}$. Note that $f^\infty(g)=f^\infty(-g)$ for all $g\in G(\mathbb{A}_f)$ and $f_A^\infty(a)=f_A^\infty(-a)$ for all $a\in A(\mathbb{A}_f)$. Therefore, if n is even, then $ST_g(f\,dg)=0$. So assume henceforth that n is odd. Then our expression is equal to

$$-2n\bar{v}(G;dg_\infty)^{-1}f^\infty(1)-\bar{v}(A;da_\infty)^{-1}f_A^\infty(1)\\ -2\bar{v}(T_3)^{-1}\,SO_{\gamma_3}(f^\infty dg_f)t_3(n)+\bar{v}(T)^{-1}\,SO_{\gamma_4}(f^\infty dg_f)(-1)^{(n+1)/2}.$$

We have

$$\begin{aligned} -2n\bar{v}(G; dg_\infty)^{-1} f^\infty(1) &= -2n\bar{v}(G; dg_\infty)^{-1} \operatorname{vol}_{dg_f}(K_0)^{-1} \\ &= -2n\tau(G)^{-1} d(G)^{-1} \chi_{K_0}(G) = \tfrac{1}{12}n, \\ -\bar{v}(A; da_\infty)^{-1} f_A^\infty(1) &= -\bar{v}(A; da_\infty)^{-1} \operatorname{vol}_{da_f}(K_A)^{-1} \\ &= -\tau(A)^{-1} d(A)^{-1} \chi_{K_A}(A) = -\tfrac{1}{2}. \end{aligned}$$

Now we consider $SO_{\gamma_4}(f^\infty dg_f; dt_f)$. We have $1-\alpha(\gamma_4) = 2$ for the positive root α of G. Therefore by Proposition 1, the local orbital integrals are equal to $\operatorname{vol}_{dt_p}(K_{T,2})^{-1}$ for $p \neq 2$. At $p = 2$, one has two stable conjugacy classes γ_4 and γ_4' in the conjugacy class of γ_4, where $\gamma_4' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

It follows that

$$SO_{\gamma_4}(f^\infty dg_f; dt_f) = \big(O_{\gamma_4}(e_{K_2}; dt_2) + O_{\gamma_4'}(e_{K_2}; dt_2)\big) \prod_{p \neq 2} \operatorname{vol}_{dt_p}(T(\mathbb{Q}_p) \cap K_p)^{-1}.$$

To compute the local integral at $p = 2$, we reduce to a GL_2-computation by the following lemma. Its proof is straightforward.

Lemma 7. *Let F be a p-adic local field with ring of integers $\mathbb{O}$. Put $G = \mathrm{SL}_2$, $\tilde{G} = \mathrm{GL}_2$, and Z for the center of $\tilde{G}$. Pick Haar measures dg on $G(F)$, $d\tilde{g}$ on $\tilde{G}(F)$, and dz on $Z(F)$. Let $f \in C_c(Z(F)\backslash \tilde{G}(F))$. Then*

$$\frac{\operatorname{vol}_{dz}(Z(\mathbb{O}))}{\operatorname{vol}_{d\tilde{g}}(\tilde{G}(\mathbb{O}))} \int_{Z(F)\backslash \tilde{G}(F)} f(g) \frac{d\tilde{g}}{dz} = \operatorname{vol}_{dg}(G(\mathbb{O}))^{-1} |\mathbb{O}^\times / \mathbb{O}^{\times 2}|^{-1} \sum_\alpha \int_{G(F)} f(t_\alpha g) dg.$$

Here α runs over the square classes in $F^\times$, and $t_\alpha = \operatorname{diag}(\alpha, 1)$.

Proposition 11. *We have*

$$O_{\gamma_4}(e_{K_2}; dt_2) + O_{\gamma_4'}(e_{K_2}; dt_2) = 2 \operatorname{vol}_{dt_2}(K_{T,2})^{-1}.$$

Proof. Write $\tilde{f}_2$ for the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_2) Z(\mathbb{Q}_2)$. By the lemma,

$$\int_{Z(\mathbb{Q}_2)\backslash \mathrm{GL}_2(\mathbb{Q}_2)} \tilde{f}_2(g^{-1}\gamma_4 g) \frac{d\tilde{g}}{dz} = \operatorname{vol}_{dt_2}(K_{T,2}) |\mathbb{Z}_2^\times / \mathbb{Z}_2^{\times 2}|^{-1} \sum_\alpha O_{\mathrm{Ad}(t_\alpha)(\gamma_4)}(e_{K_0}; dt_2).$$

Here we are normalizing $d\tilde{g}$ and dz so that $\operatorname{vol}_{dz}(Z(\mathbb{Z}_2)) = \operatorname{vol}_{d\tilde{g}}(\mathrm{GL}_2(\mathbb{Z}_2)) = 1$.

In fact, $\mathrm{Ad}(t_\alpha)(\gamma_4)$ is conjugate in $G(\mathbb{Q}_2)$ to γ_4 if and only if α is a norm from $\mathbb{Q}_2(\sqrt{-1})$, and in the contrary case, it is conjugate to γ_4'. It follows that

$$\int_{Z(\mathbb{Q}_2)\backslash \mathrm{GL}_2(\mathbb{Q}_2)} \tilde{f}_2(g^{-1}\gamma_4 g) \frac{d\tilde{g}}{dz} = \big(O_{\gamma_4}(e_{K_2}; dt_2) + O_{\gamma_4'}(e_{K_2}; dt_2)\big) \operatorname{vol}_{dt_2}(K_{T,2}).$$

By an elliptic orbital integral computation in [Kottwitz 2005], the left hand side is equal to 2. □

We conclude that

$$SO_{\gamma_4}(f^{\infty}dg_f; dt_f) = 2\,\mathrm{vol}_{dt_f}(T(\mathbb{A}_f)\cap K_0)^{-1},$$

and so

$$\begin{aligned}-\bar{v}(T)^{-1}\,SO_{\gamma_4}(f^{\infty}dg_f)t_4(n) &= -2\bar{v}(T)^{-1}\,\mathrm{vol}_{dt_f}(T(\mathbb{A}_f)\cap K_0)^{-1}t_4(n)\\ &= -2\tau(T)^{-1}\chi_{K_T}(T)t_4(n) = 2^{-2}(-1)^{(n+1)/2}.\end{aligned}$$

Similarly, we find that

$$SO_{\gamma_3}(f^{\infty}dg_f) = 2\,\mathrm{vol}_{dt_{3,f}}(T_3(\mathbb{A}_f)\cap K_0)^{-1},$$

and so

$$-2\bar{v}(T_3)^{-1}\,SO_{\gamma_3}(f^{\infty}dg_f)t_3(n) = -3^{-1}t_3(n).$$

We conclude that in this case,

$$ST_g(f\,dg) = \tfrac{1}{12}n - \tfrac{1}{2} + \tfrac{1}{4}(-1)^{(n+1)/2} - \tfrac{1}{3}t_3(n).$$

Note that for $n > 1$ this agrees precisely with the discrete series multiplicities. For $n = 1$, this expression is equal to -1, but of course in this case π is not regular.

8. Real tori

We have finished our discussion of SL_2. Starting with this section, we begin to work out the example of GSp_4. Various isomorphisms of tori must be written carefully, so we begin by explicitly working out their parametrizations.

8.1. ***The real tori $\mathbb{G}_m$, S, and T_1.*** We identify the group of characters of $\mathbb{G}_m$ with $\mathbb{Z}$ in the usual way, via $(a \mapsto a^n) \leftrightarrow n$.

Let $A_0 = \mathbb{G}_m \times \mathbb{G}_m$, viewed as a maximal torus in GL_2 in the usual way. Via the identification above we obtain $X^*(A_0) \cong \mathbb{Z}^2$ and $X_*(A_0) \cong \mathbb{Z}^2$.

Let $S = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\,\mathbb{G}_m$. Recall that $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\,\mathbb{G}_m$ denotes the algebraic group over $\mathbb{R}$ whose $\mathscr{A}$-points are $(\mathscr{A}\otimes\mathbb{C})^{\times}$ for an $\mathbb{R}$-algebra $\mathscr{A}$. By choosing the basis $\{1, i\}$ of $\mathbb{C}$ over $\mathbb{R}$, we have an injection $(\mathscr{A}\times\mathbb{C})^{\times} \to \mathrm{GL}(\mathscr{A}\otimes\mathbb{C}) \cong \mathrm{GL}_2(\mathscr{A})$. Thus we have an embedding $\iota_S : S \to \mathrm{GL}_2$ as an elliptic maximal torus.

There is a ring isomorphism $\varphi : \mathbb{C}\otimes\mathbb{C} \xrightarrow{\sim} \mathbb{C}\times\mathbb{C}$ such that $\varphi(z_1\otimes z_2) = (z_1z_2, z_1\bar{z}_2)$, which restricts to an isomorphism $\varphi : S(\mathbb{C}) \xrightarrow{\sim} \mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$. This isomorphism is also actualized by conjugation within $\mathrm{GL}_2(\mathbb{C})$. Fix $x \in \mathrm{GL}_2(\mathbb{C})$ so that

$$\mathrm{Ad}(x)\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a+ib & \\ & a-ib \end{pmatrix};$$

then $\mathrm{Ad}(x) : S(\mathbb{C}) \xrightarrow{\sim} A_0(\mathbb{C})$ is identical to φ, viewing these two tori under the embeddings above.

We fix the isomorphism from $\mathbb{Z}^2$ to $X^*(S)$ that sends $(1, 0)$ and $(0, 1)$ to the character φ composed with projection to the first and, respectively, second component of $\mathbb{G}_m \times \mathbb{G}_m$. Similarly we fix the isomorphism from $\mathbb{Z}^2$ to $X_*(S)$ that sends $(1, 0)$ and $(0, 1)$ to the cocharacters $a \mapsto \varphi^{-1}(a, 1)$ and $a \mapsto \varphi^{-1}(1, a)$, respectively.

Write $\hat{S}$ for the Langlands dual torus to S. It is isomorphic to $\mathbb{C}^\times \times \mathbb{C}^\times$ as a group, with $\Gamma_\mathbb{R}$-action defined by $\sigma(\alpha, \beta) = (\beta, \alpha)$. We fix the isomorphism $X^*(S) \xrightarrow{\sim} X_*(\hat{S})$ given by $(a, b) \mapsto (z \mapsto (z^a, z^b))$.

We have an inclusion $\iota_S : \mathbb{G}_m \to S$ given on $\mathcal{A}$-points by $a \mapsto a \otimes 1$. Write σ_S for the automorphism of S given by $1 \otimes \sigma$ on $\mathcal{A}$-points. Note that the fixed point set of σ_S is precisely the image of ι_S.

Write $\mathrm{Nm} : S \to \mathbb{G}_m$ for the norm map given by $s \mapsto s \cdot \sigma_S(s)$. Note that the product $s \cdot \sigma_S(s)$ is in $\iota_S(\mathbb{G}_m)$, which we identify here with $\mathbb{G}_m$. One computes that the norm map induces the map $n \mapsto (n, n)$ from $X^*(\mathbb{G}_m)$ to $X^*(S)$ with the identifications above.

Write T_1 for the kernel of this norm map. Its group of characters fits into the exact sequence

$$0 \to X^*(\mathbb{G}_m) \to X^*(S) \to X^*(T_1) \to 0.$$

We identify $X^*(T_1)$ with $\mathbb{Z}$ so that the restriction map $X^*(S) \to X^*(T_1)$ is given by $(a, b) \mapsto a - b$. The corresponding map $\hat{S} \to \hat{T}$ is given by $(\alpha, \beta) \mapsto \alpha\beta^{-1}$.

8.2. *The kernel and cokernel tori.*

Definition 21. We define A_{ker} to be the kernel of the map from ${\mathbb{G}_m}^4 \to \mathbb{G}_m$ given by $(a, b, c, d) \mapsto (ab)/(cd)$. We define A_{cok} to be the cokernel of the map from $\mathbb{G}_m$ to ${\mathbb{G}_m}^4$ given by $x \mapsto (x, x, x^{-1}, x^{-1})$. Write T_{ker} for the kernel of the map

$$S \times S \to \mathbb{G}_m, \quad (\alpha, \beta) \mapsto \mathrm{Nm}(\alpha/\beta),$$

and T_{cok} for the cokernel of the map

$$\mathbb{G}_m \to S \times S, \quad x \mapsto (\iota_S(x), \iota_S(x^{-1})).$$

Identifying $X_*(\mathbb{G}_m)$ and $X^*(\mathbb{G}_m)$ with $\mathbb{Z}$ as before, we obtain exact sequences

$$\begin{aligned} &0 \to X_*(A_{\mathrm{ker}}) \to \mathbb{Z}^4 \to \mathbb{Z} \to 0, \\ &0 \to \mathbb{Z} \to \mathbb{Z}^4 \to X^*(A_{\mathrm{ker}}) \to 0, \\ &0 \to \mathbb{Z} \to \mathbb{Z}^4 \to X_*(A_{\mathrm{cok}}) \to 0, \\ &0 \to X^*(A_{\mathrm{cok}}) \to \mathbb{Z}^4 \to \mathbb{Z} \to 0. \end{aligned}$$

Here the maps from $\mathbb{Z} \to \mathbb{Z}^4$ are both $n \mapsto (n, n, -n, -n)$, and the maps from $\mathbb{Z}^4 \to \mathbb{Z}$ are both $(n_1, n_2, n_3, n_4) \mapsto n_1 + n_2 - n_3 - n_4$.

Thus we obtain isomorphisms

$$g_{\mathrm{kc}} : X^*(A_{\mathrm{ker}}) \xrightarrow{\sim} X_*(A_{\mathrm{cok}}) \quad \text{and} \quad g_{\mathrm{ck}} : X^*(A_{\mathrm{cok}}) \xrightarrow{\sim} X_*(A_{\mathrm{ker}}),$$

obtained from the exact sequences defining A_{ker} and A_{cok}. In this way we view $A_{\mathrm{cok}}(\mathbb{C})$ and $A_{\mathrm{ker}}(\mathbb{C})$ as the dual tori $\hat{A}_{\mathrm{ker}}$ and $\hat{A}_{\mathrm{cok}}$, respectively.

The isomorphism $\varphi \times \varphi : S(\mathbb{C}) \times S(\mathbb{C}) \xrightarrow{\sim} (\mathbb{C}^\times)^4$ gives isomorphisms $\Phi_{\mathrm{ker}} : T_{\mathrm{ker}}(\mathbb{C}) \xrightarrow{\sim} A_{\mathrm{ker}}(\mathbb{C})$ and $\Phi_{\mathrm{cok}} : T_{\mathrm{cok}}(\mathbb{C}) \xrightarrow{\sim} A_{\mathrm{cok}}(\mathbb{C})$.

Consider the map from $S \times S$ to $S \times S$ given by $(a, b) \mapsto (ab, a\sigma_S(b))$. This fits together with the previous maps to form an exact sequence

$$1 \to \mathbb{G}_m \to S \times S \to S \times S \to \mathbb{G}_m \to 1,$$

and yields an isomorphism $\Psi_T : T_{\mathrm{cok}} \xrightarrow{\sim} T_{\mathrm{ker}}$.

Consider the map from ${\mathbb{G}_m}^4$ to ${\mathbb{G}_m}^4$ given by $(a, b, c, d) \mapsto (ac, bd, ad, bc)$. This fits together with the previous maps to form an exact sequence

$$1 \to \mathbb{G}_m \to {\mathbb{G}_m}^4 \to {\mathbb{G}_m}^4 \to \mathbb{G}_m \to 1$$

and yields an isomorphism $\Psi_A : A_{\mathrm{cok}} \xrightarrow{\sim} A_{\mathrm{ker}}$. On $\mathbb{C}$-points we have

$$\Phi_{\mathrm{ker}} \circ \Psi_T(\mathbb{C}) = \Psi_A(\mathbb{C}) \circ \Phi_{\mathrm{cok}}. \tag{8-1}$$

9. Structure of $\mathrm{GSp}_4(F)$

9.1. ***The general symplectic group.*** Let F be a field of characteristic 0. Put

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

Take G to be the algebraic group $\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 \mid gJg^t = \mu J,\ \text{some } \mu = \mu(g) \in \mathbb{G}_m\}$. It is closely related to the group $G' = \mathrm{Sp}_4 = \{g \in \mathrm{GSp}_4 \mid \mu(g) = 1\}$. Write A for the subgroup of diagonal matrices in G, and Z for the subgroup of scalar matrices in G.

We fix the isomorphism $\iota_A : A_{\mathrm{ker}} \xrightarrow{\sim} A$ given by

$$(a, b, c, d) \mapsto \mathrm{diag}(a, c, d, b). \tag{9-1}$$

Let B_A be the Borel subgroup of upper triangular matrices in G.

9.2. ***Root data.*** Although A and A_{ker} are isomorphic tori, we prefer to parametrize their character and cocharacter groups differently, since the isomorphism ι_A permutes the order of the components.

So we express $X^*(A) = \mathrm{Hom}(A, \mathbb{G}_m)$ as the cokernel of the map

$$i : \mathbb{Z} \to \mathbb{Z}^4, \tag{9-2}$$

given by $i(n) = (n, -n, -n, n)$.

We write $e_1, \dots, e_4$ for the images in $X^*(A)$ of $(1, 0, 0, 0), \dots, (0, 0, 0, 1)$. Thus $e_1 + e_4 = e_2 + e_3$. The basis Δ_G of simple roots corresponding to B_A is $\{e_1 - e_2, e_2 - e_3\}$, with corresponding positive roots $\{e_1 - e_2, e_1 - e_4, e_2 - e_3, e_1 - e_3\}$. The half-sum of the positive roots is then $\rho_B = \frac{1}{2}(4e_1 - e_2 - 3e_3) \in X^*(A)$.

Definition 22. Write Ω for the Weyl group of A in G. Write w_0, w_1, w_2 for the elements of Ω that conjugate $\mathrm{diag}(a, b, c, d) \in A$ to

$$\mathrm{diag}(d, c, b, a), \quad \mathrm{diag}(a, c, b, d), \quad \mathrm{diag}(b, a, d, c),$$

respectively.

Ω has order 8 and is generated by w_0, w_1, and w_2.

Express $X_*(A)$ as the kernel of the map

$$p : \mathbb{Z}^4 \to \mathbb{Z}, \quad (a, b, c, d) \mapsto a - b - c + d. \tag{9-3}$$

Let $\vartheta_1 = (1, 0, 0, -1)$ and $\vartheta_2 = (0, 1, -1, 0) \in X_*(A)$. Then the coroots of A in G are given by $R^\vee = R^\vee(A, G) = \{\pm\vartheta_1 \pm \vartheta_2, \pm\vartheta_1, \pm\vartheta_2\}$. The basis $\Delta_G^\vee$ of simple coroots dual to Δ_G is $\{\vartheta_1 - \vartheta_2, \vartheta_2\}$. Then $(X^*(A), \Delta_G, X_*(A), \Delta_G^\vee)$ is a based root datum for G.

9.3. ***The dual group $\hat{G}$.*** We will take $\hat{G}$ to be $\mathrm{GSp}_4(\mathbb{C})$, with trivial L-action, and the same based root data as already discussed for G. The isomorphism

$$X^*(A) \xrightarrow{(\iota_A)^*} X^*(A_{\mathrm{ker}}) \xrightarrow{(\Psi_A)^*} X^*(A_{\mathrm{cok}}) \xrightarrow{g_{ck}} X_*(A_{\mathrm{ker}}) \xrightarrow{(\iota_A)_*} X_*(A) \tag{9-4}$$

(and its inverse) furnish the required isomorphism of based root data. Let us write this out more explicitly. Note that $(\iota_A)_*$ and $(\iota_A)^*$ are given by

$$(\iota_A)_*(a, b, c, d) = (a, c, d, b) \quad \text{and} \quad (\iota_A)^*(a, b, c, d) = (a, d, b, c).$$

The isomorphism in (9-4) is induced from the linear transformation $\Sigma : \mathbb{Z}^4 \to \mathbb{Z}^4$ represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which gives the exact sequence $0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}^4 \xrightarrow{\Sigma} \mathbb{Z}^4 \xrightarrow{p} \mathbb{Z} \to 0$, and thus an isomorphism

$$X^*(A) \xrightarrow[\sim]{\Sigma} X_*(A). \tag{9-5}$$

This agrees with the isomorphism used in [Roberts and Schmidt 2007, Section 2.3].

We have $\Sigma(e_1 - e_2) = \vartheta_2$ and $\Sigma(e_2 - e_3) = \vartheta_1 - \vartheta_2$. Thus the based root datum above is self-dual. Note that $\Sigma(\rho) = \frac{3}{2}\vartheta_1 + \frac{1}{2}\vartheta_2$. Write $\hat{A}$ for $A(\mathbb{C})$; it is the torus dual to A via the isomorphism in (9-5).

10. Discrete series for $\mathbf{GSp_4}(\mathbb{R})$

10.1. ***The maximal elliptic torus T of G.*** Consider the map $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} \mapsto \begin{pmatrix} a & & & b \\ & e & f & \\ & g & h & \\ c & & & d \end{pmatrix}.$$

The composition of this with the natural inclusion $S \times S \to \mathrm{GL}_2 \times \mathrm{GL}_2$ gives an embedding of $S \times S$ into GL_4. This restricts to an embedding of T_{ker} into G, whose image is an elliptic maximal torus T of G. Thus we have $\iota_T : T_{\mathrm{ker}} \xrightarrow{\sim} T$.

$T(\mathbb{R})$ is the subgroup of matrices of the form

$$\gamma_{r,\theta_1,\theta_2} = \begin{pmatrix} r\cos(\theta_1) & & & -r\sin(\theta_1) \\ & r\cos(\theta_2) & -r\sin(\theta_2) & \\ & r\sin(\theta_2) & r\cos(\theta_2) & \\ r\sin(\theta_1) & & & r\cos(\theta_1) \end{pmatrix} \tag{10-1}$$

for $r > 0$ and angles θ_1, θ_2.

Pick an element $\xi \in G(\mathbb{C})$ so that

$$\mathrm{Ad}(\xi) \begin{pmatrix} a & & & -b \\ & c & -d & \\ & d & c & \\ b & & & a \end{pmatrix} = \begin{pmatrix} a+ib & & & \\ & c+id & & \\ & & c-id & \\ & & & a-ib \end{pmatrix},$$

and put $B_T = \mathrm{Ad}(\xi^{-1})B_A$. Then B_T is a Borel subgroup of $G_{\mathbb{C}}$ containing T, and $\mathrm{Ad}(\xi) : T(\mathbb{C}) \xrightarrow{\sim} A(\mathbb{C})$ is the canonical isomorphism associated to the pairs (T, B_T) and (A, B_A). The definitions have been set up so that

$$\iota_A \circ \Phi_{\mathrm{ker}} = \mathrm{Ad}(\xi) \circ \iota_T.$$

We identify $A(\mathbb{C})$ as the torus dual $\hat{T}$ to T via the isomorphisms

$$(10\text{-}2)\quad X^*(T) \xrightarrow{(\iota_T)^*} X^*(T_{\mathrm{ker}}) \xrightarrow{\Phi^*_{\mathrm{ker}}} X^*(A_{\mathrm{ker}}) \xrightarrow{(\Psi_A)^*} X^*(A_{\mathrm{cok}}) \xrightarrow{g_{ck}} X_*(A_{\mathrm{ker}}) \xrightarrow{(\iota_A)_*} X_*(A).$$

10.2. ***Real Weyl group.*** We use $\mathrm{Ad}(\xi)$ to identify Ω with the Weyl group of $T(\mathbb{C})$ in $G(\mathbb{C})$. Recall that $\Omega_{\mathbb{R}}$ denotes the Weyl group of $T(\mathbb{R})$ in $G(\mathbb{R})$. By [Warner 1972, Proposition 1.4.2.1], we have

$$\Omega_{\mathbb{R}} = N_{K_{\mathbb{R}}}(T(\mathbb{R}))/(T(\mathbb{R}) \cap K_{\mathbb{R}}).$$

When discussing maximal compact subgroups of $\mathrm{GSp}_4(\mathbb{R})$, it is convenient to use a different realization of these symplectic groups. Following [Pitale and Schmidt 2009], take for J the symplectic matrix

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

Take for $K_{\mathbb{R}}$ the standard maximal compact subgroup of $\mathrm{GSp}_4(\mathbb{R})$ (the intersection of $G(\mathbb{R})$ with the orthogonal group), and $SK_{\mathbb{R}}$ the intersection of $K_{\mathbb{R}}$ with $\mathrm{Sp}_4(\mathbb{R})$. One finds that $SK_{\mathbb{R}}$ is isomorphic to the compact unitary group $U_2(\mathbb{R})$, and yields the Weyl group element w_2. The element $\mathrm{diag}(1, 1, -1, -1) \in N_{G(\mathbb{R})}(T(\mathbb{R})) \cap K_{\mathbb{R}}$ gives $w_0 \in \Omega_{\mathbb{R}}$, and these two elements generate $\Omega_{\mathbb{R}}$. This subgroup has index 2 in Ω, and does not contain the element w_1.

10.3. ***Admissible embeddings.*** Consider the admissible embedding $\eta_B : {}^LT \to {}^LG$. Write $\theta(z) = z/|z|$ for $z \in \mathbb{C}^\times$. We have ${}^LT = \hat{T} \rtimes W_{\mathbb{R}}$, with τ acting as the longest Weyl group element on $\hat{T}$.

Writing ${}^LT = \hat{T} \times W_{\mathbb{R}}$, we put

$$\begin{aligned} \eta_B(1 \times z) &= \mathrm{diag}(\theta(z)^3, \theta(z), \theta(z)^{-1}, \theta(z)^{-3}) \times z && \text{for } z \in \mathbb{C}^\times \cong W_{\mathbb{C}}, \\ \eta_B(\hat{t} \times 1) &= \hat{t} \times 1 && \text{for } \hat{t} \in \hat{T}, \\ \eta_B(1 \times \tau) &= J \times \tau. \end{aligned}$$

10.4. ***Elliptic Langlands parameters.*** Let a, b be odd integers with $a > b > 0$. Let t be an even integer. Put

$$\mu = \tfrac{1}{2}[(t, t, t, t) + (a, b, -b, -a)] \quad \text{and} \quad \nu = \tfrac{1}{2}[(t, t, t, t) + (-a, -b, b, a)],$$

viewed in $X_*(\hat{T})_{\mathbb{C}}$. Then we may define a Langlands parameter $\varphi_G : W_{\mathbb{R}} \to {}^LG$ by

$$\varphi_G(z) = z^{\mu}\bar{z}^{\nu} \times z = |z|^t\, \mathrm{diag}(\theta(z)^a, \theta(z)^b, \theta(z)^{-b}, \theta(z)^{-a}) \times z,$$

and $\varphi_G(\tau) = J \times \tau$.

Note that the centralizer of $\varphi_G(W_{\mathbb{C}})$ in $\hat{G}$ is simply $\hat{A}$, and that $\langle \mu, \alpha \rangle$ is positive for every root of A that is positive for $B_A(\mathbb{C})$. Thus φ_G determines the pair $(\hat{A}, \hat{B}_A)$, where $\hat{B}_A$ is simply $B_A(\mathbb{C})$.

Define a Langlands parameter $\varphi_B : W_{\mathbb{R}} \to {}^L T$ by

$$\varphi_B(z) = |z|^t \operatorname{diag}(\theta(z)^{a-3}, \theta(z)^{b-1}, \theta(z)^{1-b}, \theta(z)^{3-a}) \times z,$$

and $\varphi_B(\tau) = 1 \times \tau$. Then $\varphi_G = \eta_B \circ \varphi_B$.

Let $\pi_G = \pi(\varphi_G, B_T)$ and $\pi'_G = \pi(\varphi_G, w_1(B_T))$, with notation from Section 2.2. The L-packet determined by φ_G is $\Pi = \{\pi_G, \pi'_G\}$. Here π_G is called a holomorphic discrete series representation, and π'_G is called a large discrete series representation.

The highest weight for the associated representation E of $G(\mathbb{C})$ is

$$\lambda_B = \tfrac{1}{2}(a + b - 4, t - b + 1, t - a + 3, 0) \in X^*(A).$$

From this we may read off the central character $\lambda_E(zI) = z^t$ for $zI \in A_G(\mathbb{C})$.

11. The elliptic endoscopic group H

11.1. ***Root data.*** Let H be the cokernel of the map $\mathbb{G}_m \to \mathrm{GL}_2 \times \mathrm{GL}_2$ given by $t \mapsto tI \times t^{-1}I$. Write A^H for the diagonal matrices in H, and B_H for the pairs of upper triangular matrices in H. Fix $\iota_{A^H} : A_{\text{cok}} \xrightarrow{\sim} A^H$ given by

$$(a, b, c, d) \mapsto \operatorname{diag}(a, b) \times \operatorname{diag}(d, c).$$

Write T_H for the image of $S \times S$ in H. It is an elliptic maximal torus in H. Fix $\iota_{T_H} : T_{\text{cok}} \xrightarrow{\sim} T_H$ obtained from the map $S \times S \to \mathrm{GL}_2 \times \mathrm{GL}_2$, $\alpha \mapsto (\iota_S(\alpha), \iota_S(\alpha))$. Put $B_{T_H} = \mathrm{Ad}(x \times x)^{-1} B_H$, a Borel subgroup of $H_{\mathbb{C}}$ containing T_H. Then $\mathrm{Ad}(x \times x)$ is the canonical isomorphism $T_H(\mathbb{C}) \xrightarrow{\sim} A^H(\mathbb{C})$ associated to the pairs (T_H, B_{T_H}) and (A^H, B_H). We view $X^*(T_H)$ as the kernel of the map $p : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ given by $(a, b) \times (c, d) \mapsto a + b - c - d$. We have a basis of roots Δ_H given by

$$\Delta_H = \{(1, -1) \times (0, 0), (0, 0) \times (1, -1)\}, \tag{11-1}$$

and $\rho_H = \frac{1}{2}(1, -1) \times \frac{1}{2}(1, -1)$.

Furthermore, $X_*(T_H)$ is the cokernel of the map $\iota : \mathbb{Z} \to \mathbb{Z}^2 \times \mathbb{Z}^2$ given by $a \mapsto (a, a) \times (-a, -a)$. We have a basis of coroots $\Delta_H^\vee$ given by

$$\Delta_H^\vee = \{(1, -1) \times (0, 0), (0, 0) \times (1, -1)\}, \tag{11-2}$$

viewed in the quotient $X_*(T_H)$.

11.2. ***Dual group $\hat{H}$.*** Let $\hat{H} = \{(g, h) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \mid \det(g) = \det(h)\}$. We have an inclusion $A_{\mathrm{ker}}(\mathbb{C}) \to \hat{H}$ given by

$$(a, b, c, d) \mapsto \mathrm{diag}(a, b) \times \mathrm{diag}(d, c).$$

Write $\hat{A}^H \subset \hat{H}$ for the image. We thus have an isomorphism $\iota_{\hat{A}^H} : A_{\mathrm{ker}}(\mathbb{C}) \xrightarrow{\sim} \hat{A}^H$.

Also write $\hat{B}_H$ for the subgroup of upper triangular matrices in $\hat{H}$. This Borel subgroup determines a based root datum for $\hat{H}$.

Giving $\hat{H}$ the trivial L-action, we view it as a dual group to H via the isomorphisms

$$X^*(A^H) \xrightarrow{(\iota_{A^H})^*} X^*(A_{\mathrm{cok}}) \xrightarrow{g_{\mathrm{ck}}} X_*(A_{\mathrm{ker}}) \xrightarrow{(\iota_{\hat{A}^H})_*} X_*(\hat{A}^H),$$

$$X^*(\hat{A}^H) \xrightarrow{(\iota_{\hat{A}^H})^*} X^*(A_{\mathrm{ker}}) \xrightarrow{g_{\mathrm{kc}}} X_*(A_{\mathrm{cok}}) \xrightarrow{(\iota_{A^H})_*} X_*(A^H).$$

We identify $\hat{A}^H$ as the torus $\hat{T}_H$ dual to T_H via the isomorphisms

$$X^*(T_H) \xrightarrow{(\iota_{T_H})^*} X^*(T_{\mathrm{cok}}) \xrightarrow{\Phi^*_{\mathrm{cok}}} X^*(A_{\mathrm{cok}}) \xrightarrow{g_{ck}} X_*(A_{\mathrm{ker}}) \xrightarrow{(\iota_{\hat{A}^H})^*} X_*(\hat{A}^H). \tag{11-3}$$

Let $\eta : {}^L H \to {}^L G$ be given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} \times w \mapsto \begin{pmatrix} a & & & b \\ & e & f & \\ & g & h & \\ c & & & d \end{pmatrix} \times w. \tag{11-4}$$

Let $s = \mathrm{diag}(1, 1) \times \mathrm{diag}(-1, -1) \in \hat{H}$.

The image $\eta(\hat{H})$ is the connected centralizer in $\hat{G}$ of $\eta(s)$. Thus, (H, s, η) is an elliptic endoscopic triple for G. In fact it is the only one, up to isomorphism.

Moreover note that η restricted to $\hat{A}^H$ is given by

$$\eta|_{\hat{A}^H} = \iota_A \circ (\iota_{\hat{A}^H})^{-1}. \tag{11-5}$$

(Recall that $\hat{A} = A(\mathbb{C})$.)

12. Transfer for $H(\mathbb{R})$

The goal of this section is Proposition 12, in which we identify $e^H_{\pi_G}$ and $e^H_{\pi'_G}$. This is part of the global transfer $f^H dh$ that is to be entered into ST_g for the endoscopic group H. We will recognize it using the character theory of transfer reviewed in Section 3.

12.1. ***Parametrization of discrete series.*** First we must set up the Langlands parameters for discrete series representations of $H(\mathbb{R})$, and describe how they transfer to L-packets in $G(\mathbb{R})$. Recall that we have fixed three integers a, b, t, with a, b odd,

t even, and $a > b > 0$. Define the Langlands parameter $\varphi_H : W_{\mathbb{R}} \to {}^L H = \hat{H} \times W_{\mathbb{R}}$ by

$$\varphi_H(z) = |z|^t \operatorname{diag}(\theta(z)^a, \theta(z)^{-a}) \times |z|^t \operatorname{diag}(\theta(z)^b, \theta(z)^{-b}) \times z$$

for $z \in W_{\mathbb{C}}$, and

$$\varphi_H(\tau) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \times \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \times \tau.$$

Then φ_H determines the pair $(\hat{A}_H, \hat{B}_H)$. The L-packet is a singleton $\{\pi_H\}$. The corresponding representation E_H of $H(\mathbb{C})$ has highest weight

$$\lambda_H = \tfrac{1}{2}(t + a - 1, t - a + 1) \times \tfrac{1}{2}(t + b - 1, t - b + 1)$$

and central character $\lambda_{E_H}(z_1, z_2) = (z_1 z_2)^t$. Most importantly, we have $\varphi_G = \eta \circ \varphi_H$.

There is another Langlands parameter φ'_H given by

$$\varphi'_H(z) = |z|^t \operatorname{diag}(\theta(z)^b, \theta(z)^{-b}) \times |z|^t \operatorname{diag}(\theta(z)^a, \theta(z)^{-a}) \times z,$$

and by $\varphi'_H(\tau) = \varphi_H(\tau)$ as above.

Again the L-packet is a singleton $\{\pi'_H\}$. The corresponding representation E'_H has highest weight

$$\lambda'_H = \tfrac{1}{2}(t + b - 1, t - b + 1) \times \tfrac{1}{2}(t + a - 1, t - a + 1),$$

and central character $\lambda_{E'_H} = \lambda_{E_H}$ above.

Let $\varphi'_G = \eta \circ \varphi'_H$. Then $\varphi'_G = \operatorname{Int}(w_2) \circ \varphi_G$, so it is equivalent to φ_G. In particular, both L-packets $\{\pi_H\}$ and $\{\pi'_H\}$ transfer to $\Pi = \{\pi_G, \pi'_G\}$.

12.2. *Alignment.* Recall the definition of alignment from Section 3.

Lemma 8. *Define $j : T_H \xrightarrow{\sim} T$ by $j = \iota_T \circ \Psi_T \circ (\iota_{T_H})^{-1}$. Then (j, B_T, B_{T_H}) is aligned with φ_H, and $(j, w_1 B_T, B_{T_H})$ is aligned with φ'_H.*

Proof. Since the parameter φ_G gives the pair $(\hat{A}, \hat{B})$, the parameter φ'_G gives the pair $(\hat{A}, w_1 \hat{B})$, and because φ_H and φ'_H both give $(\hat{A}, \hat{B}_H)$, the horizontal maps in (3-2) are identities. The map $\hat{j} : \hat{T} \to \hat{T}_H$ may be computed by composing the isomorphism $X_*(\hat{T}) \xrightarrow{\sim} X^*(T)$ in (10-2) with the induced map $j^* : X^*(T) \xrightarrow{\sim} X^*(T_H)$ and finally with the inverse of the isomorphism $X_*(\hat{T}_H) \xrightarrow{\sim} X^*(T_H)$ in (11-3). Using equations (8-1) and (11-5), one finds that $\hat{j} = \iota_{\hat{A}_H} \circ (\iota_A)^{-1} = \eta^{-1}$, as desired. ☐

12.3. *Transfer for $H_{\mathbb{R}}$.*

Proposition 12. *Let $\pi_G = \pi(\varphi_G, B_T)$ and $\pi'_G = \pi(\varphi_G, \omega^{-1}(B_T))$ as described in Section 10.4. Then (using notation from Section 2.4) we may take $e^H_{\pi_G} = e_{\pi_H} + e_{\pi'_H}$, where π_H and π'_H are the discrete series representation determined by φ_H and φ'_H, respectively, as above. Furthermore, we may take $e^H_{\pi'_G} = -e^H_{\pi_G}$.*

Proof. By Lemma 8, we may use

$$\Delta_\infty(\varphi_H, \pi(\varphi_G, \omega^{-1}(B_T))) = \langle a_\omega, \hat{j}^{-1}(s)\rangle,$$
$$\Delta_\infty(\varphi'_H, \pi(\varphi_G, \omega^{-1}(w_1 B_T))) = \langle a_{w_1\omega}, \hat{j}^{-1}(s)\rangle$$

for $\omega \in \Omega$. In both cases, this is given by

$$\langle a_\omega, s\rangle = \begin{cases} 1 & \text{if } \omega \in \Omega_{\mathbb{R}}, \\ -1 & \text{if } \omega \notin \Omega_{\mathbb{R}}. \end{cases}$$

Note that $\langle a_{w_1\omega}, \hat{j}^{-1}(s)\rangle = -\langle a_\omega, \hat{j}^{-1}(s)\rangle$. Therefore the characterization (3-1) becomes, for a general measure $f_\infty dg_\infty$ at the real place,

$$\begin{aligned}\Theta_{\pi_H}(f_\infty^H dh_\infty) &= \sum_{\pi \in \Pi(\varphi_G)} \Delta_\infty(\varphi_H, \pi)\Theta_\pi(f_\infty dg_\infty) \\ &= \Theta_{\pi_G}(f_\infty dg_\infty) - \Theta_{\pi'_G}(f_\infty dg_\infty)\end{aligned}$$

and similarly

$$\Theta_{\pi'_H}(f_\infty^H dh_\infty) = \Theta_{\pi_G}(f_\infty dg_\infty) - \Theta_{\pi'_G}(f_\infty dg_\infty).$$

In our case, we obtain

$$\Theta_{\pi_H}(e^H_{\pi_G}) = \Theta_{\pi'_H}(e^H_{\pi_G}) = (-1)^{q(G)} \quad \text{and} \quad \Theta_{\pi_H}(e^H_{\pi'_G}) = \Theta_{\pi'_H}(e^H_{\pi'_G}) = -(-1)^{q(G)}.$$

The proposition follows. □

13. Levi subgroups

13.1. ***Levi subgroups.*** We give the standard Levi subgroups of G, which are those of the parabolic subgroups containing B_A. We have the group A, the group G itself, and the following two Levi subgroups:

$$M_1 = \{\text{diag}(g, \lambda g) \mid g \in \text{GL}_2,\ \lambda \in \mathbb{G}_m\},$$
$$M_2 = \{\text{diag}(a, g, b) \mid g \in \text{GL}_2,\ a, b \in \mathbb{G}_m,\ \det(g) = ab\}.$$

Note that both M_1 and M_2 are isomorphic to $\mathbb{G}_m \times \text{GL}_2$.

The group H also has four Levi subgroups, namely A^H, the group H itself, the image M_1^H of $\text{GL}_2 \times A_0$ in H, and the image M_2^H of $A_0 \times \text{GL}_2$ in H. Note that both M_1^H and M_2^H are isomorphic to $\text{GL}_2 \times \mathbb{G}_m$.

13.2. ***Miscellaneous constants.*** We now compute the invariants from Section 5.1 for the Levi subgroups of G and H.

First, we compute the various $k(M)$. When M is the split torus A its derived group is trivial and so $k(A) = 1$. For $i = 1, 2$, the Levi subgroup M_i is isomorphic to $\mathrm{GL}_2 \times \mathbb{G}_m$, and the torus is isomorphic to $S \times \mathbb{G}_m$. Since S and $\mathbb{G}_m$ have trivial first cohomology, again $k(M_1) = 1$.

Lemma 9. *We have $k(G) = 2$.*

Write T as before for the elliptic torus of G.

Proof. Recall that T_1 is the kernel of Nm and $H^1(\mathbb{R}, T_1)$ has order 2.

Recall that the torus T is isomorphic to the kernel of the map

$$S \times S \to \mathbb{G}_m, \quad (\alpha, \beta) \mapsto \mathrm{Nm}(\alpha/\beta).$$

Projection to the first (or second) component followed by Nm gives an exact sequence

$$1 \to T_1 \times T_1 \to T \to \mathbb{G}_m \to 1. \tag{13-1}$$

We have that $G_{\mathrm{sc}} = G_{\mathrm{der}}$ and the inclusion $T_{\mathrm{sc}} = G_{\mathrm{der}} \cap T \subset T$ may be identified with the map $T_1 \times T_1 \to T$ in the sequence above. In particular, $H^1(\mathbb{R}, T_{\mathrm{sc}})$ has order 4.

Taking the cohomology of (13-1) gives the exact sequence

$$1 \to \mathbb{R}^\times/\mathbb{R}^{\times 2} \to H^1(\mathbb{R}, T_{\mathrm{sc}}) \to H^1(\mathbb{R}, T) \to 1,$$

from which we conclude that $H^1(\mathbb{R}, T_{\mathrm{sc}}) \to H^1(\mathbb{R}, T)$ is surjective and $H^1(\mathbb{R}, T)$ has order 2. □

One must also compute $k(M_H)$ for Levi subgroups M_H of H. The intermediate Levi subgroups are again isomorphic to $\mathrm{GL}(2) \times \mathbb{G}_m$, and for A_H the derived group is trivial. So $k(M_H) = 1$ for each of these.

Lemma 10. *We have $k(H) = 1$.*

Proof. We have $T = P(S \times S)$, $H_{\mathrm{sc}} = \mathrm{SL}_2 \times \mathrm{SL}_2$, and $T_{\mathrm{sc}} = T_1 \times T_1$. The map $T_{\mathrm{sc}} \to T$ factors through $T_1 \times T_1 \to S \times S$. As above we conclude that $k(H) = 1$. □

Secondly, we compute the Tamagawa numbers. Recall that

$$\tau(G) = |\pi_0(Z(\hat{G})^{\Gamma_{\mathbb{Q}}})| \cdot |\ker^1(\mathbb{Q}, Z(\hat{G}))|^{-1}.$$

Proposition 13. *We have $\tau(M) = 1$ for all Levi subgroups of G and for all proper Levi subgroups of H, and $\tau(H) = 2$.*

Proof. For each of these groups, $Z(\hat{M})$ is either the group $\mathbb{C}^\times$ with trivial $\Gamma_{\mathbb{Q}}$-action, or a product of such groups. By the Chebotarev density theorem, the homomorphism

$$\mathrm{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{C}^\times) \to \prod_v \mathrm{Hom}(\Gamma_{\mathbb{Q}_v}, \mathbb{C}^\times)$$

is injective. So $|\ker^1(\mathbb{Q}, Z(\hat{G}))|$ is trivial for our examples. Computing the component group of each $Z(\hat{M})$ is straightforward. □

The quantities n_M^G are easy to compute using $N_G(M) \subseteq N_G(Z(M))$. If M is a maximal torus, n_M^G is of course the order of the Weyl group. For the intermediate cases, one finds that $n_{M_i}^G = n_{M_i^H}^H = 2$.

If $\gamma = 1$, then $\bar{\iota}^M(\gamma) = 1$ for each M, since each M is connected. Note that for Levi subgroups M of G, all proper Levi subgroups M of H, and all semisimple elements γ in G or H, we have $\bar{\iota}^M(\gamma) = 1$ since in all these cases the derived groups are simply connected.

Finally, we compute $\iota(G, H)$, which we recall is given by

$$\iota(G, H) = \tau(G)\tau(H)^{-1}|\mathrm{Out}(H, s, \eta)|^{-1}.$$

One may compute the order of $\mathrm{Out}(H, s, \eta)$ through [Kottwitz 1984, Section 7.6], which shows that this set is in bijection with $\bigwedge(\eta(s), \rho)$, in the notation of that paper. This last set is represented by $\{1, g\}$, where

$$g = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

The conclusion is that $\iota(G, H) = \frac{1}{4}$.

14. Computing $S\Phi_M$ for Levi subgroups of G

Recall from Proposition 3 the formula

$$\Phi_M(\gamma, \Theta^E) = (-1)^{q(L)}|\Omega_L| \sum_{\omega \in \Omega^{LM}} \varepsilon(\omega)\,\mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}) \quad \text{for } \gamma \in T_e(\mathbb{R}).$$

In this section, the maximal torus will be conjugate to A, and the character group will be identified with $X^*(A)$. We specify an inner product we use on $X^*(A)_{\mathbb{R}}$ for the Weyl dimension formula (Proposition 4).

Definition 23. The usual dot product gives an inner product $(\cdot\,,\cdot)$ on $X_*(A)_{\mathbb{R}}$, viewing it as a hypersurface in $\mathbb{R}^4$.

Consider the isomorphism

$$\mathrm{pr} : X^*(A)_{\mathbb{R}} \xrightarrow{\sim} X_*(A)_{\mathbb{R}}$$

given by

$$\mathrm{pr}(a, b, c, d) = (a, b, c, d) - \tfrac{1}{4}(a + d - b - c)(1, -1, -1, 1),$$

and let $\langle \lambda, \mu \rangle = (\mathrm{pr}(\lambda), \mathrm{pr}(\mu))$.

For instance,

$$\mathrm{pr}(\lambda_B) = \tfrac{1}{4}(a+b+t-4, a-b+t-2, -a+b+t+2, -a-b+t+4).$$

It will also be necessary to compute Ω^{LM} for each example. Recall that this is the set of $w \in \Omega$ such that $w^{-1}\alpha > 0$ for positive roots α that are either real or imaginary.

14.1. ***The term Φ_G.*** By (4-1) we have $\Phi_G(\gamma, \Theta^E) = \mathrm{tr}(\gamma; E)$. Using the Weyl dimension formula, we compute

$$S\Phi_G(1, e_{\pi_G}) = -\tfrac{1}{24}ab(a+b)(a-b)\bar{v}(G)^{-1}.$$

14.2. ***The term $S\Phi_{M_1}$.*** Consider the torus T_{M_1} given by

$$\begin{pmatrix} a & b & & \\ -b & a & & \\ & & \lambda a & \lambda b \\ & & -\lambda b & \lambda a \end{pmatrix},$$

with $a^2+b^2 \neq 0$ and $\lambda \neq 0$. This is an elliptic torus in M_1.

There is one positive real root $e_1 - e_3$ and one positive imaginary root $\alpha_{M_1} = e_1 - e_2$. We have $\Omega^{LM} = \{1, w_1\}$, $q(L) = 1$, and $|\Omega_L| = 2$. This gives

$$\Phi_{M_1}(1, \Theta^E) = (-2)\big(\dim_{\mathbb{C}} V^{M_1}_{\lambda_B} - \dim_{\mathbb{C}} V^{M_1}_{\lambda'_B}\big),$$

where $\lambda'_B = \frac{1}{2}(a+b-4, t-a+1, t-b+3, 0) \in X^*(T)$.

Note that $\langle \alpha_{M_1}, \lambda_B\rangle = \frac{1}{2}(b-1)$. The Weyl dimension formula yields

$$\dim_{\mathbb{C}} V^{M_1}_{\lambda_B} = b \quad \text{and} \quad \dim_{\mathbb{C}} V^{M_1}_{\lambda'_B} = a.$$

Thus

$$S\Phi_{M_1}(1, e_{\pi_G}) = -(b-a)\bar{v}(M_1)^{-1}.$$

14.3. ***The term $S\Phi_{M_2}$.*** Consider the torus T_{M_2} given by

$$\begin{pmatrix} s & & & \\ & a & -b & \\ & b & a & \\ & & & t \end{pmatrix},$$

with $st = a^2+b^2 \neq 0$. This is an elliptic torus in M_2.

We may conjugate this in $G(\mathbb{C})$ to matrices of the form

$$\gamma = \mathrm{diag}(s, a+ib, a-ib, t)$$

in $A(\mathbb{C})$. Composing the roots of A with this composition, we determine the positive imaginary root $\alpha_{M_2} = e_2 - e_3$. We have $\Omega^{LM} = \{1, w_2\}$.

This gives

$$\Phi_{M_2}(1, \Theta^E) = (-2)\bigl(\dim_{\mathbb{C}} V^{M_2}_{\lambda_B} - \dim_{\mathbb{C}} V^{M_2}_{\lambda''_B}\bigr),$$

where $\lambda''_B = \frac{1}{2}(t-b-1, a+b-2, 0, t-a+3) \in X^*(T)$. Note that

$$\mathrm{pr}(\lambda''_B) = \tfrac{1}{4}(t+a-b-4, t+a+b-2, t-a-b+2, t-a+b+4).$$

The Weyl dimension formula yields

$$\dim_{\mathbb{C}} V^{M_2}_{\lambda_B} = \tfrac{1}{2}(a-b) \quad \text{and} \quad \dim_{\mathbb{C}} V^{M_2}_{\lambda''_B} = \tfrac{1}{2}(a+b),$$

and so

$$S\Phi_{M_2}(1, e_{\pi_G}) = b \cdot \bar{v}(M_2)^{-1}.$$

14.4. ***The term $S\Phi_A$.*** By (4-1), we have $\Phi_A(1, \Theta^E) = (-1)^{q(G)}|\Omega_G| = -8$, and so

$$S\Phi_A(1, e_{\pi_G}) = 4\bar{v}(A)^{-1}.$$

15. Computing $S\Phi_{M_H}$ for Levi subgroups of H

Since $e^H_{\pi_G} = e_{\pi_H} + e_{\pi'_H}$, we have

$$S\Phi_{M_H}(1, e^H_{\pi_G}) = (-1)^{q(G)}(-1)^{\dim(A_{M_H}/A_H)}\bar{v}(M_H)^{-1}\bigl(\Phi_{M_H}(1, \Theta_{\pi_H}) + \Phi_{M_H}(1, \Theta_{\pi'_H})\bigr).$$

15.1. ***The term $S\Phi_H(1, e^H_{\pi_G})$.*** In this case H has the elliptic torus T_H.

From (4-1), we obtain $\Phi_H(1, \Theta_{\pi_H}) = \dim_{\mathbb{C}} E_H$. To apply the dimension formula, we compute for instance $\langle \alpha_1, \lambda_H \rangle = a-1$, $\langle \alpha_2, \lambda_H \rangle = b-1$, and $\langle \alpha_i, \rho_H \rangle = 1$.

We find that

$$\Phi_H(1, \Theta^{E_H}) = \Phi_H(1, \Theta^{E'_H}) = ab.$$

Therefore

$$S\Phi_H(1, e^H_{\pi_G}) = -2\bar{v}(H)^{-1}ab.$$

15.2. ***The term $S\Phi_{A^H}(1, e^H_{\pi_G})$.*** From (4-1), we obtain

$$\Phi_{A^H}(1, \Theta^{E_H}) = \Phi_{A^H}(1, \Theta^{E'_H}) = 4.$$

Therefore

$$S\Phi_{A^H}(1, e^H_{\pi_G}) = -8\bar{v}(A^H)^{-1}.$$

15.3. ***The terms $S\Phi_{M_H}(1, e^H_{\pi_G})$ for the intermediate Levi subgroups.*** For both $M = M_H^1$ and $M = M_H^2$, we have $\Omega_G = \Omega_L \Omega_M$, and so formula (4-1) becomes simply $\Phi_{M_H}(1, \Theta^{E_H}) = (-2) \dim_{\mathbb{C}} V_{\lambda_H}^{M_H}$ for both of these Levi subgroups.

We obtain

$$\Phi_{M_H^1}(1, \Theta^{E_H}) = \Phi_{M_H^2}(1, \Theta^{E'_H}) = -2a$$

and

$$\Phi_{M_H^2}(1, \Theta^{E_H}) = \Phi_{M_H^1}(1, \Theta^{E'_H}) = -2b.$$

Therefore

$$S\Phi_{M_H^1}(1, e^H_{\pi_G}) = S\Phi_{M_H^2}(1, e^H_{\pi_G}) = -2\bar{v}(M_H^1)^{-1}(a+b).$$

16. Final form: γ central

Recall that $G = \mathrm{GSp}_4$. For the convenience of the reader, we recall the setup.

Let a and b be odd integers with $a > b > 0$, and t an even integer. Consider the Langlands parameter $\varphi_G : W_{\mathbb{R}} \to {}^L G$ given by

$$\varphi_G(z) = |z|^t \operatorname{diag}(\theta(z)^a, \theta(z)^b, \theta(z)^{-b}, \theta(z)^{-a}) \times z \quad \text{and} \quad \varphi_G(\tau) = J \times \tau.$$

Let π_G be the discrete series representation $\pi(\varphi_G, B_T)$ of $G(\mathbb{R})$ as in Section 2.2. Write π'_G for the other representation in $\Pi(\varphi_G)$.

Put $f_\infty dg_\infty = e_{\pi_G}$ as in Section 2.4 for π_G and any measure $f^\infty dg_f$ on $G(\mathbb{A}_f)$. Let $f dg = e_{\pi_G} f^\infty dg_f$, a measure on $G(\mathbb{A})$. By the theory of endoscopic transfer there is a matching measure $f^H dh$ on $H(\mathbb{A})$, where H is the elliptic endoscopic group $P(\mathrm{GL}_2 \times \mathrm{GL}_2)$ discussed above.

If $z \in A_G(\mathbb{Q})$, then $\sum_M ST_g(f dg, z, M)$ is given by the product of $\lambda_E(z) = z^t$ with

$$-\tfrac{1}{24}ab(a+b)(a-b)\bar{v}(G)^{-1} f^\infty(z) + \tfrac{1}{2}(a-b)\bar{v}(M_1)^{-1} f^\infty_{M_1}(z) + \tfrac{1}{2}b\bar{v}(M_2)^{-1} f^\infty_{M_2}(z) + \tfrac{1}{2}\bar{v}(A)^{-1} f^\infty_A(z).$$

If $z = (z_1, z_2) \in A_H(\mathbb{Q})$, then $\sum_{M_H} ST_g(f^H dh, z, M_H)$ is given by the product of $\lambda_{E_H}(z) = (z_1 z_2)^t$ with

$$-4ab\bar{v}(H)^{-1} f^{H,\infty}(z) - 2(a+b)\bar{v}(M_H^1)^{-1} f^\infty_{M_2}(z) - 2\bar{v}(A^H)^{-1} f^\infty_{A^H}(z).$$

17. The case $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$

Let $f^\infty dg_f = e_{K_0}$, where $K_0 = G(\mathbb{O}_f)$. Here dg_f is an arbitrary Haar measure on $G(\mathbb{A}_f)$, so that $dg = dg_f dg_\infty$ is the Tamagawa measure on $G(\mathbb{A})$.

17.1. ***Central terms in G.*** Note that $f_M^\infty(z) = 0$ for all $z \in Z(\mathbb{Q})$ unless $z = \pm 1$, and that $f_M^\infty(1) = f_M^\infty(-1)$ for all Levi subgroups M.

First we compute $ST_g(f\,dg, \pm 1, G)$. We have

$$\begin{aligned} -\frac{1}{2^3 3} ab(a+b)(a-b)\bar{v}(G)^{-1} f^\infty(\pm 1) \\ &= -\frac{1}{2^3 3} ab(a+b)(a-b)\tau(G)^{-1} d(G)^{-1} \chi_{K_0}(G) \\ &= 2^{-10} 3^{-3} 5^{-1} ab(a+b)(a-b). \end{aligned}$$

Next we treat the $\pm 1 \in M_i$ terms, for the intermediate Levi subgroups. We have

$$\begin{aligned} ST_g(f\,dg, \pm 1, M_1) &= \tfrac{1}{2}(a-b)\bar{v}(M_1)^{-1} f_{M_1}^\infty(\pm 1) = -2^{-5} 3^{-1}(a-b), \\ ST_g(f\,dg, \pm 1, M_2) &= \tfrac{1}{2} b\bar{v}(M_2)^{-1} f_{M_2}^\infty(\pm 1) = -2^{-5} 3^{-1} b. \end{aligned}$$

Next we treat the $\pm 1 \in A$ terms. We have $f_A(1) = \mathrm{vol}_{da_f}(K \cap A(\mathbb{A}_f))^{-1}$, which is 1. Moreover we take Lebesgue measure on $A(\mathbb{R})$ so that $\bar{v}(A) = 8$. It follows that

$$ST_g(f\,dg, \pm 1, A) = \tfrac{1}{2}\bar{v}(A)^{-1} f_A^\infty(\pm 1) = 2^{-4}.$$

Doubling these terms to account for both central elements, we compute

$$\begin{aligned} \text{(17-1)} \quad \sum_{z,M} ST_g(f\,dg, z, M) \\ &= 2^{-9} 3^{-3} 5^{-1} ab(a+b)(a-b) - 2^{-4} 3^{-1}(a-b) - 2^{-4} 3^{-1} b + 2^{-3}. \end{aligned}$$

17.2. ***Central terms in H.*** By the fundamental lemma ([Hales 1997; Weissauer 2009] for GSp_4, and of course [Ngô 2010] in general), we may write $(e_{K_0})^H = e_{K_H}$, where $K_H = H(\mathbb{O}_f)$. Thus $(f^\infty)_M^H(z) = 0$ for all $z \in H(\mathbb{Q})$ unless $z = (1, \pm 1)$, and

$$f_M^{H\infty}(1, 1) = f_M^{H\infty}(1, -1)$$

for all Levi subgroups $M = M_H$ of H.

The only nontrivial factors in the formula of Theorem 2 are $|\ker \rho(\mathbb{Q})| = 2$, $[H(\mathbb{R}) : H(\mathbb{R})_+] = 4$, and $\chi_{\mathrm{alg}}(H^{\mathrm{sc}}(\mathbb{Z}))$. Note that $H^{\mathrm{sc}} = \mathrm{SL}_2 \times \mathrm{SL}_2$.

Therefore

$$\chi_{K_H}(H) = 2^{-1} \chi_{\mathrm{alg}}(\mathrm{SL}_2(\mathbb{Z}))^2 = 2^{-5} 3^{-2}.$$

We conclude that

$$ST_g(f^H dh, (1, \pm 1), H) = -4ab\bar{v}(H)^{-1} \mathrm{vol}(K_H)^{-1} = -2^{-4} 3^{-2} ab.$$

Next we find that

$$\sum_{i=1}^{2} ST_g(f^H dh, (1, \pm 1), M_i^H) = -2(a+b)\bar{v}(M_1^H)^{-1}\operatorname{vol}(K_M)^{-1}$$
$$= 2^{-3}3^{-1}(a+b).$$

Finally, we have

$$ST_g(f^H dh, (1, \pm 1), A^H) = -2\bar{v}(A)^{-1}\operatorname{vol}(K_A)^{-1} = -2^{-2}.$$

Multiplying by $\iota(G, H) = 4^{-1}$ and then doubling to account for both central elements, we compute

$$\text{(17-2)}\quad \iota(G, H)\sum_{z, M_H} ST_g(f^H dh, z, M_H) = -2^{-5}3^{-2}ab + 2^{-4}3^{-1}(a+b) - 2^{-3}.$$

18. Comparison

As mentioned in the introduction, Wakatsuki [≥ 2012; 2012] has used the Selberg trace formula and Arthur's L^2-Lefschetz number formula to compute the discrete series multiplicities $m_{\text{disc}}(\pi, \Gamma)$ for π both holomorphic and large discrete series representations for $\mathrm{Sp}_4(\mathbb{R})$, and for many cases of arithmetic subgroups Γ. We will compare our formula to his when Γ is the full modular group. (Note that if π is a discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$ with trivial central character, and π_1 is its restriction to $\mathrm{Sp}_4(\mathbb{R})$, then $m_{\text{disc}}(\pi, \Gamma) = m_{\text{disc}}(\pi_1, \Gamma_1)$, where $\Gamma_1 = \mathrm{Sp}_4(\mathbb{Z})$.) Since he is using the Selberg trace formula, his formula breaks into contributions from each conjugacy class in Γ. In particular, he identifies the central-unipotent contributions H_1^{Hol} and H_1^{Large} to $m_{\text{disc}}(\pi_G)$ and $m_{\text{disc}}(\pi'_G)$, respectively. Namely,

$$H_1^{\text{Hol}} = 2^{-9}3^{-3}5^{-1}ab(a-b)(a+b) - 2^{-5}3^{-2}ab + 2^{-4}3^{-1}b,$$
$$H_1^{\text{Large}} = 2^{-9}3^{-3}5^{-1}ab(a-b)(a+b) + 2^{-5}3^{-2}ab - 2^{-3}3^{-1}b + 2^{-2}.$$

(To translate from his notation to ours, use $j = b-1$ and $k = \frac{1}{2}(a-b)+2$.)

Comparing these formulas to our formulas above, we observe

$$H_1^{\text{Hol}} = \sum_M ST_g(f dg, \pm 1, M) + \iota(G, H)\sum_{M_H} ST_g(f^H dh, (1, \pm 1), M_H)$$

when $f dg = e_{\pi_G} e_{K_0}$ and

$$H_1^{\text{Large}} = \sum_M ST_g(f dg, \pm 1, M) + \iota(G, H)\sum_{M_H} ST_g(f^H dh, (1, \pm 1), M_H).$$

when $f dg = e_{\pi'_G} e_{K_0}$.

This proves Theorem 1. □

Acknowledgments

This paper is founded on my thesis under the direction of Robert Kottwitz. I would like to thank him for his continual help with this project. I am also indebted to Satoshi Wakatsuki for predicting Theorem 1, and for much useful correspondence. I would also like to thank Ralf Schmidt for helpful conversations.

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Received April 26, 2011. Revised December 5, 2011.

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E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 256 No. 2 April 2012

$\mathcal{O}$-operators on associative algebras and associative Yang–Baxter equations 257
CHENGMING BAI, LI GUO and XIANG NI

Botany of irreducible automorphisms of free groups 291
THIERRY COULBOIS and ARNAUD HILION

A note on inverse curvature flows in asymptotically Robertson–Walker spacetimes 309
CLAUS GERHARDT

Total curvature of graphs after Milnor and Euler 317
ROBERT GULLIVER and SUMIO YAMADA

Entire solutions of Donaldson's equation 359
WEIYONG HE

Energy identity and removable singularities of maps from a Riemann surface with tension field unbounded in L^2 365
YONG LUO

Quotients by actions of the derived group of a maximal unipotent subgroup 381
DMITRI I. PANYUSHEV

Invariants of totally real Lefschetz fibrations 407
NERMIN SALEPCI

Stable trace formulas and discrete series multiplicities 435
STEVEN SPALLONE

Small covers and the Halperin–Carlsson conjecture 489
LI YU

Acknowledgement 509

0030-8730(201204)256:2;1-8