STRONG SOLUTIONS TO THE COMPRESSIBLE LIQUID CRYSTAL SYSTEM

YU-MING CHU, XIAN-GAO LIU AND XIAO LIU
STRONG SOLUTIONS TO THE COMPRESSIBLE LIQUID CRYSTAL SYSTEM

YU-MING CHU, XIANG-GAO LIU AND XIAO LIU

We prove the existence of local strong solutions of the compressible liquid crystal system.

1. Introduction

We consider the following simplified system of Ericksen–Leslie equations:

\begin{align*}
\rho_t + \text{div} (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla p - \mu \Delta u + \lambda \left( \text{div} (\nabla n \otimes \nabla n) - \nabla \frac{|\nabla n|^2}{2} \right) &= 0, \\
\frac{\partial n}{\partial t} + u \cdot \nabla n - \nu (\Delta n + |\nabla n|^2 n) &= 0,
\end{align*}

with the following initial and boundary conditions:

\begin{align*}
(\rho, u, n)|_{t=0} &= (\rho_0, u_0, n_0), & x &\in \Omega, \\
u(x, t) = u_0(x) = 0, & n(x, t) = n_0(x), & x &\in \partial \Omega,
\end{align*}

where $u$ is the velocity field, $n$ the macroscopic average of the nematic liquid crystal orientation field, $\rho_0 \geq 0$, $|n_0| = 1$, and pressure $p = a \rho^\gamma$ with $\gamma > 1$, where $\gamma$ is the adiabatic constant (in the physically relevant case of a monoatomic gas, $\gamma = \frac{5}{3}$). This system is modeled after the theory of Oseen [1933] and Frank [1958]; see the articles [Ericksen 1962; Forster et al. 1971; Leslie 1966; 1968] or the books [Ericksen and Kinderlehrer 1987; Gennes and Prost 1993; Pasechnik et al. 2009; Stephen 1970; Xie 1988].

The system (1.1)–(1.3) is much more complicated than the compressible Navier–Stokes equations, because equation (1.3), like the situation with heat flow into a sphere, makes the strongly coupling term $\text{div} (\nabla n \otimes \nabla n) - \nabla \frac{|\nabla n|^2}{2}$ have a weak convergence. So far, the existence of weak solutions to the system remains open, though there are celebrated contributions by Lions [1998]; see also [Feireisl 2004;...]

This work was supported partly by NSFC grant 11071043, 11131005, and 11071069.

MSC2010: 76N10, 35Q35, 35Q30.

Keywords: strong solutions, compressible liquid crystals, local existence.
Feireisl et al. 2001]. Liu and Qing [2011] proved the global existence of finite energy weak solutions to the case where the free energy is replaced by the Ginzburg–Landau approximation energy,

$$\min_{n \in H^1(\Omega; \mathbb{R}^3)} \int_{\Omega} \frac{1}{2} |\nabla n|^2 + \frac{1}{4\sigma^2} (|n|^2 - 1)^2 \, dx.$$  

In the incompressible case, F. H. Lin and C. Liu, among others [Lin 1989; Lin and Liu 1995; Lin and Liu 2001; Lin and Liu 2000; Lin and Liu 1996; Calderer and Liu 2000], systematically studied the incompressible liquid crystal dynamics system based on the Ericksen–Leslie model (that is, the Ginzburg–Landau approximation case with $\rho$ being a constant in system (1.1) makes the velocity field divergence free) and proved the global existence of weak solutions, classical solutions, and partial regularity. Liu and Zhang [2009] also studied the existence of weak solutions to the incompressible liquid crystal system with the Ginzburg–Landau approximation and $\rho$ nonconstant.

It is well known that there exist no global solutions to the system (1.1)–(1.3) even in the incompressible case. Surprisingly, we can prove the local existence of a strong solution to the compressible liquid crystal system with initial density $\rho_0 \geq 0$. We gained enlightenment from the corresponding results of the compressible Navier–Stokes equations. There is a huge literature on the compressible Navier–Stokes equations, under the crucial assumption that the initial density $\rho_0$ is bounded below away from zero. The existence results were obtained by Nash, Itaya, Tani, Matsumura, and Nishida, among others. For general nonnegative initial density, Cho, Kim, and Choe [Choe and Kim 2003; Cho et al. 2004; Cho and Kim 2006] obtained the existence of a local strong solution to a compressible Navier–Stokes equation.

We first have the energy law

$$\frac{dE}{dt} = \int_{\Omega} \mu |\nabla u|^2 + \lambda |\nabla \gamma + |\nabla n|^2 \rho \frac{\gamma - 1}{\gamma - 1} \rho \rho \frac{\gamma - 1}{\gamma - 1} \rho \rho \frac{\gamma - 1}{\gamma - 1} \rho$$

with

$$E(t) = \int_{\Omega} \left( \frac{1}{2} \rho u^2 + \frac{\lambda}{2} |\nabla n|^2 + \frac{a}{\gamma - 1} \rho \right).$$

From the definition of velocity,

$$\frac{dx(X, t)}{dt} = u(x(X, t), t),$$

$$x(X, 0) = X. \tag{1.6}$$

The continuity equation can be rewritten as

$$\frac{d\rho(x(X, t), t)}{dt} + \rho \text{div } u = 0,$$
that is,
\[ \rho(x, t) = \rho_0 \exp \left( - \int_0^t \text{div} u \right). \]

We need the following regularity for \( \rho_0, n_0, \) and \( u_0 \):
\[ \rho_0 \in W^{1,6}(\Omega), \quad u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \quad n_0 \in H^3(\Omega). \]

We also need some compatibility condition on the initial data: for some \( g \in L^2, \)
\[ \mu \Delta u_0 - \lambda \text{div}(\nabla n_0 \otimes \nabla n_0 - \frac{1}{2} |\nabla n_0|^2 I) - a \nabla \rho_0^\gamma = \rho_0^\frac{7}{2} g. \]

The following is our main result.

**Theorem 1.1.** Assume \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \) and \((\rho_0, n_0, u_0)\) satisfies regularity condition (1.9) and compatibility condition (1.10). Then there exist a small time \( T^* > 0 \) and a unique strong solution \((\rho, n, u)\) of the compressible liquid crystal system (1.1)–(1.3) in \((0, T^*) \times \Omega\), satisfying initial and boundary conditions (1.4) and (1.5), such that
\[
\begin{align*}
\rho &\in C([0, T^*); W^{1,6}), & \rho_t &\in C([0, T^*); L^6), \\
u &\in C([0, T^*); H^1_0 \cap H^2) \cap L^2(0, T^*; W^{2,6}), & u_t &\in L^2(0, T^*; H^1_0), \\
n &\in C([0, T^*); H^2) \cap L^2(0, T^*; W^{2,6}), & n_t &\in C([0, T^*); H^1_0), \\
\sqrt{\rho} u_t &\in C([0, T^*); L^2). 
\end{align*}
\]

## 2. Approximation solutions

We now consider the linearized equations as follows: for fixed smooth functions \( v, d : \Omega \times [0, T] \to \mathbb{R}^3 \) with
\[
\frac{dx(X, t)}{dt} = v(x(X, t), t)
\]
and \( x(X, 0) = X, \) and \( v(x, 0) = u_0(x), \) \( d(x, 0) = n_0(x) \),
\[
\begin{align*}
\rho_t + \text{div}(\rho v) &\equiv 0, \\
(\rho u)_t + \text{div}(\rho v \otimes v) + a \nabla \rho^\gamma &\equiv \mu \Delta u - \lambda \text{div}(\nabla n \otimes \nabla n - \frac{1}{2} |\nabla n|^2 I), \\
n_t - \gamma \Delta n &\equiv \lambda |\nabla d|^2 d - v \cdot \nabla d,
\end{align*}
\]
with initial and boundary conditions
\[
\begin{align*}
(\rho, u, n)|_{t=0} &= (\rho_0 + \delta, u_0, n_0), & x \in \Omega, \\
u(x, t) &= u_0(x) = 0, & n(x, t) = n_0(x), & x \in \partial \Omega.
\end{align*}
\]
Here \( \delta > 0 \) is a constant, and \( \rho_0 \geq 0, |n_0| = 1. \)
We use the following notations: Suppose Banach spaces
\[ \mathcal{A} = L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,6}(\Omega)) \cap W^{1,1}_2((0, T) \times \Omega), \]
\[ \mathcal{B} = L^\infty(0, T; W^{2,6}(\Omega)) \cap W^{1,1}_2((0, T) \times \Omega) \cap W^{2,1}_2((0, T) \times \Omega) \]
with norm respectively
\[ \|v\|_\mathcal{A} = \|v\|_{L^\infty(0, T; H^2(\Omega))} + \|v\|_{L^2(0, T; W^{2,6}(\Omega))} + \|v_t\|_{L^2(0, T; H^1(\Omega))}, \]
\[ \|d\|_\mathcal{A} = \|d_t\|_{L^2(0, T; H^2(\Omega))} + \|d_t\|_{L^\infty(0, T; H^1(\Omega))} + \|d\|_{L^\infty(0, T; W^{2,6}(\Omega))}. \]

**Lemma 2.1.** For given \( v \) with \( \|v\|_\mathcal{A} \leq A \), the unique solution \( \rho \) of (2.1) satisfies
\[ \|\rho\|_{L^\infty(0, T; W^{1,6}(\Omega))} \leq cc_0(1 + T^{\frac{1}{2}}A) \exp(cT^{\frac{1}{2}}A), \]
\[ \|\rho_t\|_{L^\infty(0, T; L^6(\Omega))} \leq cc_0 A \exp(cT^{\frac{1}{2}}A). \]

In particular,
\[ \|P\|_{L^\infty(0, T; W^{1,6}(\Omega))} \leq cc_0(1 + T^{\frac{1}{2}}A) \exp(cT^{\frac{1}{2}}A), \]
\[ \|P_t\|_{L^\infty(0, T; L^6(\Omega))} \leq cc_0 A \exp(cT^{\frac{1}{2}}A), \]
where \( c \) is an absolute constant, perhaps dependent on \( \Omega, \lambda, \mu, \gamma, \) etc., and \( c_0 \) is a constant dependent on initial and boundary data.

**Proof.** Since
\[ \nabla \rho = \nabla \rho_0 \exp\left(- \int_0^t \text{div} v\right) - \rho_0 \int_0^t \nabla \text{div} v \exp\left(- \int_0^t \text{div} v\right), \]
\[ \rho_t = -\rho_0 \text{div} v \exp\left(- \int_0^t \text{div} v\right), \]
we have, from the Minkowski inequality,
\[ \|\nabla \rho\|_{L^6(\Omega)} \leq c \|\rho_0\|_{W^{1,6}(\Omega)} \left(1 + \left\| \int_0^t \nabla^2 v \right\|_{L^6(\Omega)} \right) \exp\left(\int_0^T \|\text{div} v\|_{L^\infty(\Omega)}\right) \]
\[ \leq c \|\rho_0\|_{W^{1,6}(\Omega)} \left(1 + \int_0^T \|\nabla^2 v\|_{L^6(\Omega)} \right) \exp\left(\int_0^T \|\text{div} v\|_{L^\infty(\Omega)}\right) \]
\[ \leq c \|\rho_0\|_{W^{1,6}(\Omega)} (1 + T^{\frac{1}{2}}\|v\|_X) \exp(cT^{\frac{1}{2}}\|v\|_X) \]
\[ \leq cc_0 (1 + T^{\frac{1}{2}}A) \exp(cT^{\frac{1}{2}}A), \]
\[ \|\rho_t\|_{L^6(\Omega)} \leq cc_0 \exp(cT^{\frac{1}{2}}A) \|v\|_{H^2(\Omega)} \leq cc_0 A \exp(cT^{\frac{1}{2}}A), \]
where \( X = L^2(0, T; W^{2,6}(\Omega)) \).
Lemma 2.2. Suppose \(\|v\|_{\delta} \leq A, \|d\|_{\delta} \leq B\). Then (2.3) with initial condition \(n(x, 0) = n_0(x)\) has a unique solution \(n\) and a constant \(K_1\), depending only on \(n_0\) and \(u_0\), such that, for \(T = T(A, B)\) small enough,

\[
\|n\|_{\bar{\delta}} = \|n_t\|_{L^2(0,T;H^2(\Omega))} + \|n_t\|_{L^\infty(0,T;H^1(\Omega))} + \|n\|_{L^\infty(0,T;W^{2,6}(\Omega))} \leq K_1.
\]

Proof. The existence of a solution to (2.3) is standard. We just give the estimates as follows. Differentiating (2.3) with respect to time \(t\),

\[
n_{tt} - \nu \Delta n_t = v(\|\nabla d\|_2^2 d + |\nabla d|^2 d_t) + (v_t \cdot \nabla) d - (v \cdot \nabla) d_t.
\]

Multiplying by \(\Delta n_t\), integrating over \(\Omega\), and using the Cauchy inequality, we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla n_t|^2 + v \int_{\Omega} |\Delta n_t|^2 = - \int_{\Omega} v(\|\nabla d\|_2^2 d + |\nabla d|^2 d_t) \cdot \Delta n_t + (v_t \cdot \nabla) d \cdot \Delta n_t - (v \cdot \nabla) d_t \cdot \Delta n_t
\]

\[
\leq \int_{\Omega} 2v|\nabla d||\nabla n_t||d||\Delta n_t| + v|\nabla d|^2|d_t||\Delta n_t|
\]

\[
+ \int_{\Omega} |\nabla v_t||\nabla d||\nabla n_t| + |v_t||\nabla^2 d||\nabla n_t| + |v||\nabla d_t||\Delta n_t|
\]

\[
= \sum_{i=1}^5 I_i.
\]

We have the following estimates for \(I_i\):

\[
I_1 = \int_{\Omega} 2v|\nabla d||\nabla n_t||d||\Delta n_t| \leq c \int_{\Omega} |\nabla d|^2|\nabla n_t|^2 + \frac{v}{6}\|\Delta n_t\|^2_{L^2(\Omega)},
\]

\[
I_2 = \int_{\Omega} v|\nabla d|^2|d_t||\Delta n_t| \leq c \int_{\Omega} |\nabla d|^4|d_t|^2 + \frac{v}{6}\|\Delta n_t\|^2_{L^2},
\]

\[
I_3 = \int_{\Omega} |\nabla v_t||\nabla d||\nabla n_t| \leq A^{-2}B^{-2} \int_{\Omega} |\nabla v_t|^2|\nabla d|^2 + A^2B^2 \int_{\Omega} |\nabla n_t|^2,
\]

\[
I_4 = \int_{\Omega} |v_t||\nabla d||\nabla n_t| \leq A^{-2}B^{-2} \int_{\Omega} |v_t|^2|\nabla d|^2 + A^2B^2 \int_{\Omega} |\nabla n_t|^2
\]

\[
\leq cA^{-2}B^{-2}\|\nabla v_t\|^2_{L^2(\Omega)}\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^6} + A^2B^2 \int_{\Omega} |\nabla n_t|^2,
\]

\[
I_5 = \int_{\Omega} |v||\nabla d_t||\Delta n_t| \leq \frac{3}{v} \int_{\Omega} |v|^2|\nabla d_t|^2 + \frac{v}{6}\|\Delta n_t\|^2_{L^2(\Omega)}.
\]

Substituting all the estimates into (2.11), we get

\[
\frac{d}{dt} \int_{\Omega} |\nabla n_t|^2 + v \int_{\Omega} |\Delta n_t|^2 \leq c \int_{\Omega} |\nabla d|^2|\nabla n_t||d||\Delta n_t| + c \int_{\Omega} |\nabla d|^4|d_t|^2
\]

\[
+ cA^{-2}B^{-2} \int_{\Omega} |\nabla v_t|^2|\nabla d|^2 + cA^2B^2 \int_{\Omega} |\nabla n_t|^2
\]

\[
+ c \int_{\Omega} |v|^2|\nabla d_t|^2 + cA^{-2}B^{-2}\|\nabla v_t\|^2_{L^2(\Omega)}\|\nabla^2 d\|_{L^6} + \|\nabla^2 d\|_{L^6}.
\]
that is,
\[
\int_\Omega |\nabla n_t|^2 + v \int_0^T \int_\Omega |\Delta n_t|^2 \\
\leq cB^6T + cA^2B^2T + c + cA^2B^2 \int_0^T \int_\Omega |\nabla n_t|^2 + c(n_0, u_0),
\]
where
\[
c(n_0, u_0) = c \int_\Omega |\Delta \nabla n_0|^2 + |\nabla n_0|^2 |\nabla^2 n_0|^2 + |\nabla n_0|^6 + c \int_\Omega |\nabla u_0|^2 |\nabla n_0|^2 + |u_0|^2 |\nabla^2 n_0|^2.
\]
Using Gronwall’s inequality, we obtain
\[
\int_\Omega |\nabla n_t|^2 \leq (cB^6T + cA^2B^2T + c_0) \exp(cA^2B^2T)
\]
and
\[
\int_\Omega |\nabla n_t|^2 + v \int_0^T \int_\Omega |\Delta n_t|^2 \leq c(B^6T + A^2B^2T + c_0)(1 + \exp(cA^2B^2T)).
\]
Taking \( T = T(A, B) \) small, we get
\[
\int_\Omega |\nabla n_t|^2 + v \int_0^T \int_\Omega |\Delta n_t|^2 \leq c.
\]
The elliptic estimates can be deduced from (2.3):
\[
\|n\|_{W^{2,6}(\Omega)} \leq \|n_t\|_{L^6} + \|v \cdot \nabla d\|_{L^6} + \|\nabla d\|^2\|d\|_{L^6} + \|n_0\|_{W^{2,6}} \\
\leq \|v \cdot \nabla d\|_{L^6} + \|\nabla d\|^2\|d\|_{L^6} + c_0.
\]
We estimate each item:
\[
\|v \cdot \nabla d\|_{L^6}
\leq \left( \int_\Omega |v|^6|\nabla d|^6 \right)^{\frac{1}{6}} \leq \left( \int_\Omega |v - u_0|^6|\nabla d|^6 \right)^{\frac{1}{6}} + \|u_0\|_{L^\infty} \left( \int_\Omega |\nabla d|^6 \right)^{\frac{1}{6}} \\
\leq cB \left( \int_\Omega |\nabla v - \nabla u_0|^2 \right)^{\frac{1}{2}} + c\|u_0\|_{L^\infty} \left( \int_\Omega |\nabla d - \nabla n_0|^6 \right)^{\frac{1}{6}} + c\|u_0\|_{L^\infty} \|\nabla n_0\|_{L^\infty} \\
\leq cB \left( \int_\Omega \int_0^T |\nabla v_t|^2 \right)^{\frac{1}{2}} + c_0 B^\frac{3}{2} \left( \int_\Omega \int_0^T |\nabla d_t|^2 \right)^{\frac{1}{6}} + c_0 \\
\leq cBT^\frac{1}{2} \|\nabla v_t\|_{L^2(Q_T)} + c_0 T^\frac{1}{3} B + c_0 \leq cABT^\frac{1}{2} + c_0 BT^\frac{1}{4} + c_0
and
\[
\|\nabla d\|^2 dL^6 = \left( \int_\Omega |\nabla d|^6 \right)^{\frac{1}{6}} \leq \left( \int_\Omega |\nabla d|^1 |d - n_0|^6 \right)^{\frac{1}{6}} + c_0 \left( \int_\Omega |\nabla d|^1 \right)^{\frac{1}{6}}
\]
\[
\leq cB^2 \left( \int_\Omega |d - n_0|^6 \right)^{\frac{1}{6}} + c_0 \left( \int_\Omega |\nabla d - \nabla n_0|^1 \right)^{\frac{1}{6}} + c_0
\]
\[
\leq cB^2 \left( \int_\Omega |\nabla d - \nabla n_0|^2 \right)^{\frac{1}{2}} + c_0 B \left( \int_\Omega |\nabla d - \nabla n_0|^6 \right)^{\frac{1}{6}} + c_0
\]
\[
\leq cAB^2 T^{\frac{1}{2}} + c_0 B^2 T^{\frac{1}{2}} + c_0.
\]
Taking \( T = T(A, B) \) small enough, we obtain the desired \( \|n\|_{W^{2,6}} \leq c_0 \). \qed

For (2.2) we have following Lemma.

**Lemma 2.3.** Under the conditions of Lemma 2.2, suppose \( n \) satisfies (2.3) and \( \rho \) (2.1). Then there exists a unique solution \( u \) satisfying (2.2), and there is a constant \( K_2 \), depending only on \( n_0 \) and \( u_0 \), such that, for \( T = T(A, B) \) small enough,

\[
(2.12) \quad \|u\|_{\mathcal{A}} \equiv \|u\|_{L^\infty(0,T;H^2(\Omega))} + \|u\|_{L^2(0,T;W^{2,6}(\Omega))} + \|u_t\|_{L^2(0,T;H^1(\Omega))} \leq K_2.
\]

**Proof.** Since
\[
\rho \geq \delta \exp \left( -\int_0^T |\nabla v|_{L^\infty((0,T) \times \Omega)} \right) > 0,
\]
the standard theory of parabolic equations implies the existence of the solution to (2.2). Differentiating (2.2) with respect to time \( t \), we get

\[
(2.13) \quad \rho u_{tt} - \mu \Delta u_t
\]
\[
= -\lambda \text{div}((\nabla d \otimes \nabla d_t) - \frac{1}{2} |\nabla d_t|^2 I) - \nabla (\rho v \cdot \nabla) v_t - (\rho v \cdot \nabla) v_t - (\rho v \cdot \nabla) v - (\rho v_t \cdot \nabla) v - \rho_t u_t.
\]

Multiplying by \( u_t \), integrating by parts, and using the continuity of (2.1), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 + \mu \int_\Omega |\nabla u_t|^2
\]
\[
= \lambda \int_\Omega ((\nabla d \otimes \nabla d_t) - \frac{1}{2} |\nabla d_t|^2 I) \cdot \nabla u_t
\]
\[
- \int_\Omega \nabla p_t \cdot u_t - (\rho v \cdot \nabla) v_t \cdot u_t - (\rho v \cdot \nabla) v \cdot u_t - \int_\Omega (\rho v_t \cdot \nabla) v \cdot u_t + \rho_t |u_t|^2
\]
\[
\leq 3\lambda \int_\Omega |\nabla d||\nabla d_t||\nabla u_t| + \int_\Omega p_t \text{div}(u_t) + \rho |v||\nabla v||u_t|
\]
\[
+ \int_\Omega \rho |v||\nabla v|^2|u_t| + \rho |v|^2|\nabla^2 v||u_t| + \rho |v||\nabla v||\nabla u_t|
\]
\[
= \sum_{i=1}^8 I_i.
\]
For each $I_i$ we have

$$I_1 = 3\lambda \int_\Omega |\nabla d| |\nabla d_t| |\nabla u_t| \leq c \int_\Omega |\nabla d|^2 |\nabla d_t|^2 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2 \leq c B^4 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2,$$

$$I_2 = \int_\Omega p_i \text{div}(u_i) \leq c \int_\Omega |p_i|^2 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2 \leq c_0 \exp \left( \int_0^T 2 \|\nabla v\|_{L^\infty(\Omega)} \right) \int_\Omega |\nabla v|^2 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2 \leq c_0 A^2 \exp(c A T^{\frac{1}{2}}) + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2,$$

$$I_3 = \int_\Omega |\rho| |v||\nabla v_t||u_t| \leq A^4 \int_\Omega |\rho| |u_t|^2 + c_0 A^{-2} \exp(c A T^{\frac{1}{2}}) \int_\Omega |\nabla v_t|^2,$$

$$I_4 = \int_\Omega |\rho| |v||\nabla v|^2 |u_t| \leq A^6 \int_\Omega |\rho| |u_t|^2 + c_0 \exp(c A T^{\frac{1}{2}}),$$

$$I_5 = \int_\Omega |\rho| |v|^2 |\nabla^2 v||u_t| \leq A^6 \int_\Omega |\rho| |u_t|^2 + c_0 \exp(c A T^{\frac{1}{2}}),$$

$$I_6 = \int_\Omega \rho |v||\nabla v||\nabla u_t| \leq c \int_\Omega \rho^2 |v|^2 |\nabla v|^2 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2 \leq c_0 A^4 \exp(c A T^{\frac{1}{2}}) + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2,$$

$$I_7 = \int_\Omega \rho |v_t| |\nabla v||u_t| \leq A^4 \int_\Omega \rho |u_t|^2 + A^{-4} \int_\Omega \rho |v_t|^2 |\nabla v|^2 \leq A^2 \int_\Omega \rho |u_t|^2 + c_0 A^{-2} \exp(c A T^{\frac{1}{2}}) \int_\Omega |v_t|^2,$$

$$I_8 = 2 \int_\Omega \rho |v||\nabla u_t||u_t| \leq c \int_\Omega \rho |u_t|^2 (\rho |v|^2) + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2 \leq c_0 A^2 \exp(c A T^{\frac{1}{2}}) \int_\Omega \rho |u_t|^2 + \frac{\mu}{12} \int_\Omega |\nabla u_t|^2.$$  

From the above estimates, we get

$$\int_\Omega \rho |u_t|^2 + \int_0^T \int_\Omega |\nabla u_t|^2 \leq c B^4 T + c_0 A^4 T \exp(c A T^{\frac{1}{2}}) + c_0 + c_0 A^4 \exp(c A T^{\frac{1}{2}}) \int_0^T \int_\Omega \rho |u_t|^2,$$

which implies that

$$\int_\Omega \rho |u_t|^2 + \int_0^T \int_\Omega |\nabla u_t|^2 \leq (c B^4 T + c_0 A^4 T \exp(c A T^{\frac{1}{2}})) c_0 A^4 T \exp(c A T^{\frac{1}{2}}).$$
Taking $T = T(A, B)$ small enough, we deduce

$$
\int_{\Omega} \rho |u_t|^2 + \int_0^T \int_{\Omega} |\nabla u_t|^2 \leq C(c_0).
$$

Finally, we estimate

$$\|u\|_{L^\infty(0,T;H^2(\Omega))} \quad \text{and} \quad \|u\|_{L^2(0,T;W^{2,6}(\Omega))}.$$

From (2.2), we get

$$\|u\|_{H^2(\Omega)} \leq c(\|\nabla p\|_{L^2(\Omega)} + \|\rho u_t\|_{L^2(\Omega)} + \|\nabla^2 n\nabla n\|_{L^2(\Omega)} + c(\|\rho v \cdot \nabla v\|_{L^2(\Omega)} + c_0)).$$

Now we have

$$\|\nabla p\|_{L^2(\Omega)} \leq c_0 \exp(cAT^{1/2}) + c_0 AT^{1/2} \exp(cAT^{1/2}),$$

$$\|\rho u_t\|_{L^2(\Omega)} \leq c_0 \exp(cAT^{1/2}) \|\sqrt{\rho} u_t\|_{L^2(\Omega)},$$

$$\|\nabla^2 n\nabla n\|_{L^2(\Omega)} \leq \|\nabla^2 n\|_{L^6(\Omega)} \|\nabla n\|_{L^2(\Omega)}^{1/2} \|\nabla n\|_{L^6(\Omega)}^{1/2} \leq K_1^2,$$

and

$$\|\rho v \cdot \nabla v\|_{L^2(\Omega)}^2 \leq \|\rho\|_{L^\infty(\Omega)}^2 \int_{\Omega} |v|^2 |\nabla v|^2$$

$$\leq c_0 \exp(cAT^{1/2}) \left( \int_{\Omega} |v - u_0|^2 |\nabla v|^2 + \|u_0\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla v - \nabla u_0|^2 + c_0 \right)$$

$$\leq c_0 \exp(cAT^{1/2}) \left( \int_{\Omega} \int_0^T \left| v_t \right|^2 |\nabla v|^2 + c_0 \int_{\Omega} \int_0^T \left| \nabla v_t \right|^2 + c_0 \right)$$

$$\leq c_0 \exp(cAT^{1/2}) (A^4 T + c_0 A^2 T + c_0).$$

Similarly, we have

$$\|\nabla p\|_{L^6(\Omega)} \leq c_0 \exp(cAT^{1/2}) + c_0 AT^{1/2} \exp(cAT^{1/2}),$$

$$\|\rho u_t\|_{L^2(0,T;L^6(\Omega))} \leq c_0 \exp(cAT^{1/2}) \|\nabla u_t\|_{L^2(0,T;L^2(\Omega))}$$

$$\leq c_0 \exp(cAT^{1/2}) C(c_0),$$

$$\|\nabla^2 n\nabla n\|_{L^2(0,T;L^6(\Omega))} \leq \|\nabla^2 n\|_{L^2(0,T;L^6(\Omega))} \|\nabla n\|_{L^\infty(\Omega)} \leq K_1^2.$$
and
\[ \| \rho v \cdot \nabla v \|_{L^2(0,T;L^6(\Omega))}^2 \]
\[ \leq \| \rho \|_{L^\infty(\Omega)}^2 \int_0^T \left( \int_\Omega |v|^6 |\nabla v|^6 \right) \frac{1}{3} \]
\[ \leq c_0 \exp(cAT^{\frac{1}{2}}) \int_0^T \| v \|_{L^\infty(\Omega)}^2 \| \nabla v \|_{L^\infty(\Omega)}^4 \times \left( \int_\Omega |\nabla v - \nabla u_0|^2 + 1 \right)^{\frac{1}{3}} \]
\[ \leq c_0 \exp(cAT^{\frac{1}{2}}) A^2 \int_0^T \| v \|_{L^\infty(\Omega)}^2 \times \left( \int_\Omega \left| \int_0^t \nabla v_t \right|^2 + 1 \right)^{\frac{1}{3}} \]
\[ \leq c_0 \exp(cAT^{\frac{1}{2}}) \left( T \int_0^T \int_\Omega |\nabla v_t|^2 + 1 \right)^{\frac{1}{3}} \times \left( \int_0^T \| v \|_{W^{2,6}(\Omega)}^2 \right)^{\frac{1}{3}} \]
\[ \leq c_0 \exp(cAT^{\frac{1}{2}}) (T A^2 + 1)^{\frac{1}{3}} A^\frac{4}{3} T^{\frac{1}{3}}. \]

Thus
\[ \int_\Omega \rho |u_t|^2 dx + \mu \int_0^T \int_\Omega |\nabla u_t|^2 dx dt + \| u \|_{L^\infty(0,T;H^2(\Omega))} + \| u \|_{L^2(0,T;W^{2,6}(\Omega))} \leq C(c_0). \]

This concludes the proof. \(\square\)

If \((n^\delta, u^\delta)\) denotes a unique solution of (2.2) and (2.3) with
\[ \rho(x, 0) = \rho_0 + \delta \]
and initial and boundary conditions, then taking \(\delta \to 0\), we obtain a unique solution \((n, u)\) of the linearized system (2.1)–(2.3) with \(\rho(x, 0) = \rho_0\) and initial and boundary conditions such that \(\| n \|_{\mathcal{B}} \leq K_1, \| u \|_{\mathcal{A}} \leq K_2\). So we can define a map
\[ \mathcal{T} : \mathcal{W} \to \mathcal{W}, \quad (d, v) \mapsto (n, u), \]
where Banach space
\[ \mathcal{W} = (\mathcal{A} \otimes \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \otimes \mathcal{B} \]
with
\[ \mathcal{C} = \{ (n, u) : \|(n, u)\|_{\mathcal{C}} = \| n \|_{L^2(0,T;H^2(\Omega))} + \| u \|_{L^2(0,T;H^1(\Omega))} < \infty \}. \]

The following lemma tells us that the map \(\mathcal{T}\) is contracted in the sense of weaker norm for \((d, v) \in \mathcal{W}\).

**Lemma 2.4.** There is a constant \(0 < \theta < 1\) such that for any \((d^i, v^i) \in \mathcal{W}, i = 1, 2,\)
\[ \| \mathcal{T}(d^1, v^1) - \mathcal{T}(d^2, v^2) \|_{\mathcal{C}} \leq \theta \|(d^1 - d^2, v^1 - v^2)\|_{\mathcal{C}} \]
for some small \(T > 0\).
Proof. Suppose \( \rho_i, n^i, \) and \( u^i \) are the solutions to (2.1)–(2.3) corresponding to given \((d^i, v^i) \in W\). Define \( \rho = \rho_2 - \rho_1, d = d^2 - d^1, v = v^2 - v^1, n = n^2 - n^1, u = u^2 - u^1, \) and 
\[
\rho_i = \rho_0 \exp \left( -\int_0^t \text{div} v^i \right),
\]
i = 1, 2. Then
\[
\begin{align*}
\rho_t + \text{div}(\rho v^2) & = -\text{div}(\rho_1 v), \\
n_t - v \Delta n & = v|\nabla d^2|^2 d^2 - v|\nabla d^1|^2 d^1 - v^2 \nabla d^2 + v^1 \nabla d^1, \\
\rho_2 u_t - \mu \Delta u & = (\rho_1 - \rho_2) u^1_t + \rho_1 v^1 \nabla v^1 - \rho_2 v^2 \nabla v^2 + \nabla p_1 \\
& - \nabla p_2 - \lambda \nabla \cdot (\nabla n^2 \otimes \nabla n^2 - \frac{1}{2} |\nabla n^2|^2 I) \\
& + \lambda \nabla \cdot (\nabla n^1 \otimes \nabla n^1 - \frac{1}{2} |\nabla n^1|^2 I).
\end{align*}
\]

Multiplying (2.16) by \( n \) and integrating over \( \Omega \), we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |n|^2 \, dx + v \int_{\Omega} |\nabla n|^2 \, dx \\
& \leq \int_{\Omega} |\nabla d^2|^2 d^2 \cdot n - |\nabla d^1|^2 d^1 \cdot n - v \nabla d^2 \cdot n - v^1 \nabla d \cdot n \\
& \leq \eta \int_{\Omega} (|\nabla d|^2 + |\nabla v|^2) + c(\eta, A, B) \int_{\Omega} |n|^2,
\end{align*}
\]
where \( c(\eta, A, B)(s) \) satisfies
\[
\int_0^T c(\eta, A, B)(s) \, ds \leq K_3
\]
for small \( T = T(A, B, \eta) \), where \( K_3 \) is a constant dependent on initial and boundary data \( c_0 \).

Differentiating (2.16) with respect to \( x_i \), multiplying by \( \nabla n \), and integrating over \( \Omega \), we deduce
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla n|^2 \, dx + \frac{v}{2} \int_{\Omega} |\nabla^2 n|^2 \, dx \\
& \leq \eta \int_{\Omega} (|\nabla v|^2 + |\nabla d|^2 + |\nabla^2 d|^2) + c(\eta, A, B) \int_{\Omega} |\nabla n|^2,
\end{align*}
\]
where \( c(\eta, A, B) \) satisfies (2.19), and we have used the following identities and estimates:
\[
\begin{align*}
\nabla d^2 \nabla^2 d^2 d^2 - \nabla d^1 \nabla^2 d^1 d^1 & = \nabla d \nabla^2 d^2 d^1 + \nabla d^1 \nabla^2 d^1 d^1 + \nabla d^1 \nabla^2 d^1 d, \\
|\nabla d^2|^2 \nabla^2 d^2 - |\nabla d^1|^2 \nabla^2 d^1 & = |\nabla d|^2 \nabla d + (|\nabla d|^2 - |\nabla d^1|^2) \nabla d^1, \\
\end{align*}
\]
\[ \int_{\Omega} |\nabla n|^2 |\nabla^2 d|^2 \leq \left( \int_{\Omega} |\nabla^2 d|^6 \right)^{\frac{1}{3}} \left( \int_{\Omega} |\nabla n|^2 \right)^{\frac{2}{3}} \leq cB^2 \|\nabla n\|_{L^2(\Omega)} \|\nabla^2 n\|_{L^2(\Omega)} \leq \frac{v}{2} \int_{\Omega} |\nabla^2 n|^2 + cB^4 \int_{\Omega} |\nabla n|^2. \]

Multiplying (2.15) by \( \rho \) and using the Minkowski inequality, we have

\[ \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\rho|^2 = \int_{\Omega} -\frac{1}{2} |\rho|^2 \text{div} \ v^2 - \int_{\Omega} \rho (\nabla \rho_1 v + \rho_1 \text{div} \ v) \]
\[ \leq c \int_{\Omega} |\rho|^2 |\nabla v|^2 + c \|\rho\|_{L^2(\Omega)} \|\nabla \rho_1 \|_{L^2(\Omega)} \|v\|_{L^6(\Omega)} \]
\[ + c \|\rho\|_{L^2(\Omega)} \|\rho_1 \|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \]
\[ \leq c \|v^2\|_{W^{2,6}(\Omega)} \|\rho\|_{L^2(\Omega)} + \eta \|\nabla v\|_{L^2(\Omega)} \]
\[ + c_0 \eta^{-1} \exp(cAT^{\frac{1}{2}}) \left( 1 + \left\| \int_0^t \nabla^2 v^1 \right\|^2_{L^2(\Omega)} \right) \|\rho\|_{L^2(\Omega)} \]
\[ \leq \eta \|\nabla v\|_{L^2(\Omega)} + c_0 \eta^{-1} \exp(cAT^{\frac{1}{2}}) \left( 1 + T \|\nabla^2 v^1\|_{L^2(0,T;L^6(\Omega))} \right) \|\rho\|_{L^2(\Omega)} \]
\[ \leq c_0 \eta^{-1} \exp(cAT^{\frac{1}{2}}) \left( 1 + TA^2 + \|v^2\|_{W^{2,6}(\Omega)} \right) \|\rho\|_{L^2(\Omega)} + \eta \|\nabla v\|_{L^2(\Omega)}, \]

that is,

(2.21) \[ \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\rho|^2 \leq \eta \|\nabla v\|_{L^2(\Omega)}^2 + c(\eta, A, T) \|\rho\|_{L^2(\Omega)}^2, \]

where \( c(\eta, A, T) \) satisfies (2.19).

Multiplying (2.17) by \( u \) and integrating over \( \Omega \), we deduce

(2.22) \[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_2 |u|^2 \, dx + \mu \int_{\Omega} |\nabla u|^2 \, dx \]
\[ = \int_{\Omega} -\rho_2 v^2 u \nabla u + (\rho_1 - \rho_2) u_1^2 u + \rho_1 v^1 \nabla v^1 \cdot u - \rho_2 v^2 \nabla^2 v \cdot u + (p_2 - p_1) \text{div} \ u \]
\[ + \lambda (\nabla n^2 \otimes \nabla n^2 - \frac{1}{2} |\nabla n|^2 I) \nabla u - \lambda (\nabla n^1 \otimes \nabla n^1 - \frac{1}{2} |\nabla n| I) \nabla u \]
\[ = \int_{\Omega} -\rho_2 v^2 u \nabla u + (\rho_1 - \rho_2) (u_1^2 + v^1 \nabla v^1) \cdot u \]
\[ - \rho_2 (v \nabla v^2 + v^1 \nabla v) \cdot u + (p_1 - p_2) \text{div} \ u \]
\[ + \lambda (\nabla n^2 \otimes \nabla n^2 - \frac{1}{2} |\nabla n|^2 I) \nabla u - \lambda (\nabla n^1 \otimes \nabla n^1 - \frac{1}{2} |\nabla n|^2 I) \nabla u \]
\[ \leq \eta \int_{\Omega} |\nabla v|^2 + \frac{2\mu}{3} \int_{\Omega} |\nabla u|^2 + c(\eta, A, B) \int_{\Omega} \rho_2 |u|^2 + |\rho|^2 + |\nabla n|^2, \]
where \( c(\eta, A, B) \) satisfying (2.19). Here we have used the key estimates
\[
\int \Omega \rho_2 |v \nabla v^2 + v^1 \nabla v| |u| \leq \|\nabla v^2\|_{L^6(\Omega)} \|\rho_2 u\|_{L^2(\Omega)} \|v\|_{L^6(\Omega)}
\]
\[
+ \|\nabla v\|_{L^2(\Omega)} \|\rho_2 u\|_{L^2(\Omega)} \|v^1\|_{L^\infty(\Omega)}
\]
\[
\leq c_0 \exp(c AT^{1/2}) \|\sqrt{\rho_2} u\|_{L^2(\Omega)} \|\nabla v^2\|_{H^1(\Omega)} \|\nabla v\|_{L^2(\Omega)}
\]
\[
+ c_0 \exp(c AT^{1/2}) \|\sqrt{\rho_2} u\|_{L^2(\Omega)} \|v^1\|_{H^1(\Omega)} \|\nabla v\|_{L^2(\Omega)}
\]
\[
\leq \eta \|\nabla v\|_{L^2(\Omega)}^2 + c\eta^{-1} A^2 \exp(c AT^{1/2}) \|\sqrt{\rho_2} u\|_{L^2(\Omega)^2},
\]
\[
\int \Omega |\nabla n| |\nabla u| |\nabla n| \leq \eta \int \Omega |\nabla u|^2 + c\eta^{-1} |\nabla n|^2 \int \Omega |\nabla n|^2
\]
\[
\leq \frac{\mu}{3} \int \Omega |\nabla u|^2 + cB^2 \int \Omega |\nabla n|^2
\]
and
\[
\int \Omega (\rho_1 - \rho_2)(u_1^1 + v^1 \nabla v^1) \cdot u \leq \|\rho\|_{L^2(\Omega)}^3 \|u_1^1 + v^1 \nabla v^1\|_{L^6(\Omega)} \|u\|_{L^6(\Omega)}
\]
\[
\leq c \|\rho\|_{L^2(\Omega)} \|u_1^1 + v^1 \nabla v^1\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}
\]
\[
\leq \frac{\mu}{3} \|\nabla u\|_{L^2(\Omega)}^2 + c(A, T(t)) \|\rho\|_{L^2(\Omega)}^2,
\]
where \( c(\eta, A, T(t)) \) satisfies (2.19).

Summing inequalities (2.18) and (2.20)–(2.22), we obtain
\[
\frac{d}{dt} \int \Omega |\rho|^2 + |n|^2 + |\nabla n|^2 + \rho_2 |u|^2 + \int \Omega |\nabla n|^2 + |\nabla^2 n|^2 + |\nabla u|^2
\]
\[
\leq c\eta \int \Omega |\nabla v|^2 + |\nabla d|^2 + |\nabla^2 d|^2 + c(\eta, A, B, T) \int \Omega |\rho|^2 + |n|^2 + |\nabla n|^2 + \rho_2 |u|^2,
\]
which implies, by (2.19) and taking \( T = T(\eta, A, B) \) small enough,
\[
\int \Omega |\rho|^2 + |n|^2 + |\nabla n|^2 + \rho_2 |u|^2
\]
\[
\leq \eta \exp \left( \int_0^T c(\eta, A, B)(s) ds \right) \int_0^T \int \Omega |\rho|^2 + |n|^2 + |\nabla n|^2 + \rho_2 |u|^2
\]
\[
\leq c\eta \int_0^T \int \Omega |\rho|^2 + |n|^2 + |\nabla n|^2 + \rho_2 |u|^2.
\]

Thus, taking \( \eta \) small, we obtain
\[
(2.23) \quad \|\rho\|_{L^\infty(0,T;L^2(\Omega))} + \|n\|_{L^\infty(0,T;H^1(\Omega))} + \|\sqrt{\rho_2} u\|_{L^\infty(0,T;L^2(\Omega))} \leq c
\]
and
\[ \int_0^T \int_\Omega |\nabla n|^2 + |\nabla^2 n|^2 + |\nabla u|^2 \leq \theta \int_0^T \int_\Omega |\nabla d|^2 + |\nabla^2 d|^2 + |\nabla v|^2 \]
with \(0 < \theta < 1\). Since \(n\) and \(u\) are zero on boundary, we finish the proof. \[\square\]

3. Proof of Theorem 1.1

Proof. By the contractibility of \(T\), we can easily obtain a unique solution \((n, u)\) of (1.3) and (1.2), and \(\rho\) is from \(u\) by formula (1.8), that is, \(\rho\) is a unique solution of (1.1). Lemmas 2.1–2.3 and the lower semicontinuity of norms imply that the solutions \((\rho, n, u)\) satisfy the same estimates. Multiplying (1.3) by \(n\), we get
\[ |n|^2 + (u \cdot \nabla)|n|^2 = \nu|\Delta n|^2 + (|n|^2 - 1)|\nabla n|^2, \]
that is,
\[ (|n|^2 - 1)_t + (u \cdot \nabla)(|n|^2 - 1) = \nu\Delta (|n|^2 - 1) + (|n|^2 - 1)|\nabla n|^2. \]
Define \(D = (|n|^2 - 1)\exp(\|\nabla n\|^2_{L^\infty(\Omega_T)} t)\), where \(\Omega_T = \Omega \times [0, T]\). Then
\[ D_t + (u \cdot \nabla)D = \nu\Delta D + (|\nabla n|^2 - \|\nabla n\|^2_{L^\infty(\Omega_T)})D \]
with \(D|_{\partial\Omega} = 0\). So from the maximum principle of parabolic equations, we deduce
\[ D \equiv 0 \quad \text{in} \quad ((0, T) \times \Omega). \]
Thus we complete the proof of the theorem. \[\square\]

References


Received May 16, 2011. Revised November 12, 2011.

YU-MING CHU  
SCHOOL OF MATHEMATICS AND COMPUTATION SCIENCES  
HUNAN CITY UNIVERSITY  
YIYANG, 413000  
CHINA  
chuyuming@hutc.zj.cn

XIAN-GAO LIU  
INSTITUTE OF MATHEMATICS  
FUDAN UNIVERSITY  
SHANGHAI, 200433  
CHINA  
xgliu@fudan.edu.cn

XIAO LIU  
INSTITUTE OF MATHEMATICS  
FUDAN UNIVERSITY  
SHANGHAI, 200433  
CHINA  
shaw0820@gmail.com
Energy and volume of vector fields on spherical domains 1
   FABIANO G. B. BRITO, ANDRÉ O. GOMES and GIOVANNI S. NUNES
Maps on 3-manifolds given by surgery 9
   BOLDIZSÁR KALMÁR and ANDRÁS I. STIPSICZ
Strong solutions to the compressible liquid crystal system 37
   YU-MING CHU, XIAN-GAO LIU and XIAO LIU
Presentations for the higher-dimensional Thompson groups $nV$ 53
   JOHANNA HENNIG and FRANCESCO MATUCCI
Resonant solutions and turning points in an elliptic problem with oscillatory boundary conditions 75
   ALFONSO CASTRO and ROSA Pardo
Relative measure homology and continuous bounded cohomology of topological pairs 91
   ROBERTO FRIGERIO and CRISTINA PAGLIANTINI
Normal enveloping algebras 131
   ALEXANDRE N. GRISHKOV, MARINA RASSKAZOVA and SALVATORE SICILIANO
Bounded and unbounded capillary surfaces in a cusp domain 143
   YASUNORI AOKI and DAVID SIEGEL
On orthogonal polynomials with respect to certain discrete Sobolev inner product 167
   FRANCISCO MARCELLÁN, RAMADAN ZEJNULLAHU, BUIJAR FEIZULLAHU and EDMUNDO HUERTAS
Green versus Lempert functions: A minimal example 189
   PASCAL THOMAS
Differential Harnack inequalities for nonlinear heat equations with potentials under the Ricci flow 199
   JIA-YONG WU
On overtwisted, right-veering open books 219
   PAOLO LISCA
Weakly Krull domains and the composite numerical semigroup ring $D + E[\Gamma^*]$ 227
   JUNG WOOK LIM
Arithmeticity of complex hyperbolic triangle groups 243
   MATTHEW STOVER