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Measure homology was introduced by Thurston in his notes about the geometry and topology of 3-manifolds, where it was exploited in the computation of the simplicial volume of hyperbolic manifolds. Zastrow and Hansen independently proved that there exists a canonical isomorphism between measure homology and singular homology (on the category of CW-complexes), and it was then shown by Löh that, in the absolute case, such isomorphism is in fact an isometry with respect to the L^1 -seminorm on singular homology and the total variation seminorm on measure homology. Löh's result plays a fundamental rôle in the use of measure homology as a tool for computing the simplicial volume of Riemannian manifolds.

This paper deals with an extension of Löh's result to the relative case. We prove that relative singular homology and relative measure homology are isometrically isomorphic for a wide class of topological pairs. Our results can be applied for instance in computing the simplicial volume of Riemannian manifolds with boundary.

Our arguments are based on new results about continuous (bounded) cohomology of topological pairs, which are probably of independent interest.

1. Introduction

Measure homology was introduced in [Thurston 1979], where it was exploited in the proof that the simplicial volume of a closed hyperbolic n -manifold is equal to its Riemannian volume divided by a constant only depending on n (this result is attributed in [Thurston 1979] to Gromov). In order to rely on measure homology, it is necessary to know that this theory “coincides” with the usual real singular homology, at least for a large class of spaces. The proof that measure homology and real singular homology of CW-pairs are isomorphic has appeared in [Hansen 1998; Zastrow 1998]. However, in order to exploit measure homology as a tool for computing the simplicial volume, one has to show that these homology theories are not only isomorphic, but also *isometric* (with respect to the seminorms introduced below). In the absolute case, this result is achieved in [Löh 2006]. Our paper is

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devoted to extending Löh's result to the context of relative homology of topological pairs. As mentioned in [Fujiwara and Manning 2011, Appendix A] and [Löh 2007, Remark 4.22], such an extension seems to raise difficulties that suggest that Löh's argument should not admit a straightforward translation into the relative context. For a detailed account about the notion of measure homology and its applications see, e.g., the introductions of [Zastrow 1998; Berlanga 2008].

In order to achieve our main results, we develop some aspects of the theory of continuous bounded cohomology of topological pairs. More precisely, we compare such theory with the usual bounded cohomology of pairs of groups and spaces. Park [2003] provided the algebraic foundations to the theory of relative bounded cohomology, extending Ivanov's [1985] homological algebra approach to the relative case. However, Park endows the bounded cohomology of a pair of spaces with a seminorm which is *a priori* different from the seminorm considered in this paper. In fact, the most common definition of simplicial volume is based on a specific L^1 -seminorm on singular homology, whose dual is just the L^∞ -seminorm on bounded cohomology defined in [Gromov 1982, Section 4.1]. This seminorm does not coincide *a priori* with Park's seminorm, so our results cannot be deduced from Park's arguments. More precisely, it is shown in [Park 2003, Theorem 4.6] that Gromov's and Park's norms are bi-Lipschitz equivalent (see Theorem 6.1 below). In [Park 2003, page 206] it is stated that it remains unknown if this equivalence is actually an isometry. In Section 6 we answer this question in the negative, providing examples showing that Park's and Gromov's seminorms indeed do not coincide in general.

1A. Relative singular homology of pairs. Let X be a topological space and $W \subseteq X$ a (possibly empty) subspace of X . For $n \in \mathbb{N}$ we denote by $C_n(X)$ the module of singular n -chains with real coefficients, i.e., the \mathbb{R} -module freely generated by the set $S_n(X)$ of singular n -simplices with values in X . The natural inclusion of W in X induces an inclusion of $C_n(W)$ into $C_n(X)$, and we denote by $C_n(X, W)$ the quotient space $C_n(X)/C_n(W)$. The usual differential of the complex $C_*(X)$ defines a differential $d_*: C_*(X, W) \rightarrow C_{*-1}(X, W)$. The homology of the resulting complex is the usual relative singular homology of the topological pair (X, W) , and will be denoted by $H_*(X, W)$.

The real vector space $C_n(X, W)$ can be endowed with a natural L^1 -norm, as follows. If $\alpha \in C_n(X, W)$, then

$$\|\alpha\|_1 = \inf \left\{ \sum_{\sigma \in S_n(X)} |a_\sigma|, \text{ where } \alpha = \left[\sum_{\sigma \in S_n(X)} a_\sigma \sigma \right] \text{ in } C_n(X)/C_n(W) \right\}.$$

Such a norm descends to a seminorm on $H_n(X, W)$, which is defined as follows: if $[\alpha] \in H_n(X, W)$, then

$$\|[\alpha]\|_1 = \inf\{\|\beta\|_1 \mid \beta \in C_n(X, W), d_n\beta = 0, [\beta] = [\alpha]\}$$

(this seminorm can be null on nonzero elements of $H_n(X, W)$). Of course, we recover the absolute homology modules of X just by setting $H_n(X) = H_n(X, \emptyset)$.

1B. Relative measure homology of pairs. We now recall the definition of relative measure homology of the pair (X, W) . We endow $S_n(X)$ with the compact-open topology and denote by $\mathbb{B}_n(X)$ the σ -algebra of Borel subsets of $S_n(X)$. If μ is a signed measure on $\mathbb{B}_n(X)$ (in this case we say for short that μ is a Borel measure on $S_n(X)$), the *total variation of μ* is defined by the formula

$$\|\mu\|_m = \sup_{A \in \mathbb{B}_n(X)} \mu(A) - \inf_{B \in \mathbb{B}_n(X)} \mu(B) \in [0, +\infty]$$

(the subscript m stands for *measure*). For every $n \geq 0$, the measure chain module $\mathcal{C}_n(X)$ is the real vector space of the Borel measures on $S_n(X)$ having finite total variation and admitting a compact determination set. The graded module $\mathcal{C}_*(X)$ can be given the structure of a complex via the boundary operator

$$\begin{aligned} \partial_n : \mathcal{C}_n(X) &\rightarrow \mathcal{C}_{n-1}(X), \\ \mu &\mapsto \sum_{j=0}^n (-1)^j \mu^j, \end{aligned}$$

where μ^j is the push-forward of μ under the map that takes a simplex $\sigma \in S_n(X)$ into the composition of σ with the usual inclusion of the standard $(n - 1)$ -simplex onto the j -th face of σ .

Let now W be a (possibly empty) subspace of X . It is proved in [Zastrow 1998, Proposition 1.10] that the σ -algebra $\mathbb{B}_n(W)$ of Borel subsets of $S_n(W)$ coincides with the set $\{A \cap S_n(W) \mid A \in \mathbb{B}_n(X)\}$. For every $\mu \in \mathcal{C}_n(W)$, the assignment

$$\nu(A) = \mu(A \cap S_n(W)), \quad A \in \mathbb{B}_n(X),$$

defines a Borel measure on $S_n(X)$, which is called the *extension* of μ . If μ has compact determination set and finite total variation then the same is true for ν , so that we have a natural inclusion $\mathcal{C}_n(W) \hookrightarrow \mathcal{C}_n(X)$ (see [Zastrow 1998, Proposition 1.10 and Lemma 1.11] for full details). The image of $\mathcal{C}_n(W)$ in $\mathcal{C}_n(X)$ will be simply denoted by $\mathcal{C}_n(W)$, and coincides with the set of the elements of $\mathcal{C}_n(X)$ which admit a compact determination set contained in $S_n(W)$. We denote by $\mathcal{C}_n(X, W)$ the quotient $\mathcal{C}_n(X)/\mathcal{C}_n(W)$.

It is readily seen that $\partial_n(\mathcal{C}_n(W)) \subseteq \mathcal{C}_{n-1}(W)$, so ∂_n induces a boundary operator $\mathcal{C}_n(X, W) \rightarrow \mathcal{C}_{n-1}(X, W)$, which will still be denoted by ∂_n . The homology of the complex $(\mathcal{C}_*(X, W), \partial_*)$ is the *relative measure homology of the pair (X, W)* , and it is denoted by $\mathcal{H}_*(X, W)$.

Just as in the case of singular homology, we may endow $\mathcal{H}_n(X, W)$ with a seminorm as follows. For every $\alpha \in \mathcal{C}_n(X, W)$ we set

$$\|\alpha\|_m = \inf \{ \|\mu\|_m, \text{ where } \mu \in \mathcal{C}_n(X), [\mu] = \alpha \text{ in } \mathcal{C}_n(X, W) = \mathcal{C}_n(X)/\mathcal{C}_n(W) \}.$$

Then, for every $[\alpha] \in \mathcal{H}_n(X, W)$ we set

$$\|[\alpha]\|_{\text{mh}} = \inf\{\|\beta\|_{\text{m}} \mid \beta \in \mathcal{C}_n(X, W), \partial_n \beta = 0, [\beta] = [\alpha]\}$$

(the subscript mh stands for *measure homology*). The absolute measure homology module $\mathcal{H}_n(X)$ can be defined just by setting $\mathcal{H}_n(X) = \mathcal{H}_n(X, \emptyset)$.

1C. Relative singular homology versus relative measure homology. For every $\sigma \in S_n(X)$ let us denote by δ_σ the atomic measure supported by the singleton $\{\sigma\} \subseteq S_n(X)$. The chain map

$$\begin{aligned} \iota_* : C_*(X, W) &\rightarrow \mathcal{C}_*(X, W), \\ \sum_{i=0}^k a_i \sigma_i &\mapsto \sum_{i=0}^k a_i \delta_{\sigma_i} \end{aligned}$$

induces a map

$$H_n(\iota_*) : H_n(X, W) \rightarrow \mathcal{H}_n(X, W), \quad n \in \mathbb{N},$$

which is obviously norm-nonincreasing for every $n \in \mathbb{N}$.

Theorem 1.1 [Zastrow 1998; Hansen 1998]. *Let (X, W) be a CW-pair. For every $n \in \mathbb{N}$, the map*

$$H_n(\iota_*) : H_n(X, W) \rightarrow \mathcal{H}_n(X, W)$$

is an isomorphism.

Zastrow's and Hansen's proofs of [Theorem 1.1](#) are based on the fact that relative measure homology satisfies the Eilenberg–Steenrod axioms for homology (on suitable categories of topological pairs). Therefore, their approach avoids the explicit construction of the inverse maps $H_n(\iota_*)^{-1}$, $n \in \mathbb{N}$, and does not give much information about the behavior of such inverse maps with respect to the seminorms introduced above. In the case when $W = \emptyset$, the fact that $H_n(\iota_*)$ is indeed an isometry was proved by Löh:

Theorem 1.2 [Löh 2006]. *If X is any connected CW-complex, then for every $n \in \mathbb{N}$ the map*

$$H_n(\iota_*) : H_n(X) \rightarrow \mathcal{H}_n(X)$$

is an isometric isomorphism.

Löh's proof of [Theorem 1.2](#) exploits deep results about the *bounded cohomology* of groups and topological spaces. In [Section 3](#) and [Section 4](#) we develop a suitable relative version of such results, which we use on page [125](#) to prove this:

Theorem 1.3. *Let (X, W) be a CW-pair, and let us suppose that the following conditions hold:*

- (1) X (whence W) is countable, and both X and W are connected;

- (2) the map $\pi_j(W) \rightarrow \pi_j(X)$ induced by the inclusion $W \hookrightarrow X$ is injective for $j = 1$, and it is an isomorphism for $j \geq 2$.

Then, for every $n \in \mathbb{N}$ the isomorphism

$$H_n(\iota_*) : H_n(X, W) \rightarrow \mathcal{H}_n(X, W)$$

is isometric.

In fact, we will deduce [Theorem 1.3](#) from [Theorem 1.7](#) below concerning the relationships between continuous (bounded) cohomology and singular (bounded) cohomology of topological pairs.

Definition 1.4. A CW-pair (X, W) is *good* if it satisfies conditions (1) and (2) in the statement of [Theorem 1.3](#).

We conjecture that [Theorem 1.3](#) holds even without the hypothesis that the pair (X, W) is good, so a brief comment about the places where this assumption comes into play is in order. The fact that W is connected and π_1 -injective in X allows us to exploit results regarding the bounded cohomology of a pair (G, A) , where G is a group and A is a subgroup of G . In order to deal with the case when W is *not* assumed to be π_1 -injective, one could probably build on results regarding the bounded cohomology of a pair (G, A) , where A, G are groups and $\varphi : A \rightarrow G$ is a homomorphism of A into G . This case is treated in [\[Park 2003\]](#) by means of a mapping cone construction. However, the mapping cone introduced there does not admit a norm inducing Gromov's seminorm in bounded cohomology, so Park's approach seems to be of no help to our purposes. Perhaps it is easier to drop from the hypotheses of [Theorem 1.3](#) the requirement that W be connected (provided that we still assume that every component of W is π_1 -injective in X). Several arguments in our proofs make use of cone constructions which are based on the choice of a basepoint in the universal coverings \tilde{X}, \tilde{W} of X, W . When W is connected (and π_1 -injective in X), the space \tilde{W} is realized as a connected subset of \tilde{X} , and this allows us to define compatible cone constructions on \tilde{X} and \tilde{W} . It is not clear how to replace these constructions when W is disconnected: one could probably build on the theory of homology and cohomology of a group with respect to any system of subgroups, as described for instance in [\[Bieri and Eckmann 1978\]](#) (see also [\[Mineyev and Yaman 2007\]](#)), but several difficulties arise which we have not been able to overcome. Finally, the assumption that $\pi_i(W)$ is isomorphic to $\pi_i(X)$ for every $i \geq 2$ plays a fundamental rôle in our proof of [Proposition 4.7](#) below. One could get rid of this assumption by using a result stated without proof in [\[Park 2003, Lemma 4.2\]](#), but at the moment we are not able to provide a proof for Park's statement (see [Remark 4.9](#) for a brief discussion of this issue).

1D. Locally convex pairs. We are able to prove that measure homology is isometric to singular homology also for a large family of pairs of metric spaces, namely for those pairs which support a *relative straightening* for simplices.

The *straightening procedure* for simplices was introduced in [Thurston 1979], and establishes an isometric isomorphism between the usual singular homology of a space and the homology of the complex of *straight chains*. Such a procedure was originally defined on hyperbolic manifolds, and has then been extended to the context of nonpositively curved Riemannian manifolds. In Section 2 we give the precise definition of *locally convex pair of metric spaces*. Then, following some ideas described in [Löh and Sauer 2009], for every locally convex pair (X, W) we define a straightening procedure which induces a chain map between relative measure chains and relative singular chains. It turns out that such a straightening induces a well-defined norm-nonincreasing map $\mathcal{H}_n(X, W) \rightarrow H_n(X, W)$. This map provides the desired norm-nonincreasing inverse of $H_n(\iota_*)$, so that we can prove (in Section 2D) the following:

Theorem 1.5. *Let (X, W) be a locally convex pair of metric spaces. Then the map*

$$H_n(\iota_*) : H_n(X, W) \rightarrow \mathcal{H}_n(X, W)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

The class of locally convex pairs is indeed quite large, including for example all the pairs $(M, \partial M)$, where M is a nonpositively curved complete Riemannian manifold with geodesic boundary ∂M .

Remark 1.6. Suppose that (X, W) is a locally convex pair, and let K be a connected component of W . An easy application of a metric version of Cartan–Hadamard theorem (see [Bridson and Haefliger 1999, II.4.1]) shows that $\pi_1(K)$ injects into $\pi_1(X)$, and $\pi_i(K) = \pi_i(X) = 0$ for every $i \geq 2$. In particular, if (X, W) is also a countable CW-pair and W is connected, then (X, W) is good, and the conclusion of Theorem 1.5 also descends from Theorem 1.3. Note however that the request that W be connected could be quite restrictive in several applications of our results. For example, it is well-known that the natural compactification of a complete finite-volume hyperbolic manifold with geodesic boundary and/or cusps is a manifold with boundary N admitting a locally CAT(0) (whence locally convex) metric that turns the pair $(N, \partial N)$ into a locally convex pair (see [Bridson and Haefliger 1999, pages 362–366], for example). We have discussed in [Frigerio and Pagliantini 2010] some properties of the simplicial volume of such manifolds, and in that context several interesting examples have in fact disconnected boundary. In [Pagliantini 2012] it is shown how to apply Theorem 1.5 for getting shorter proofs of the main results of [Frigerio and Pagliantini 2010].

1E. (Continuous) relative bounded cohomology. As mentioned above, our proof of [Theorem 1.3](#) involves the study of the relative bounded cohomology of topological pairs. Introduced in [[Gromov 1982](#)], the relative bounded cohomology of pairs (of groups or spaces) seems to be less clearly understood than absolute bounded cohomology. Here below we define the *continuous* (bounded) cohomology of topological pairs, and we put on (continuous) bounded cohomology Gromov's L^∞ -seminorm which is “dual” (in a sense to be specified below) to the seminorm on (measure) homology described above. Then, in [Section 4](#) we compare the continuous bounded cohomology of a good CW-pair to its usual singular bounded cohomology (see [Theorem 1.7](#) below). In [Section 5](#) we show how this result implies [Theorem 1.3](#).

Let us now state more precisely our results. For every $n \in \mathbb{N}$ we denote by $C^n(X)$ and $C^n(X, W)$ the algebraic duals of $C_n(X)$ and $C_n(X, W)$ (that is, the respective modules of singular n -cochains with real coefficients). We will often identify $C^n(X, W)$ with a submodule of $C^n(X)$ via the canonical isomorphism

$$C^n(X, W) \cong \{f \in C^n(X) \mid f|_{C_n(W)} = 0\}.$$

If $\delta^* : C^*(X, W) \rightarrow C^{*+1}(X, W)$ is the usual differential, the homology of the complex $(C^*(X, W), \delta^*)$ is the relative singular cohomology of the pair (X, W) , and it is denoted by $H^*(X, W)$.

We regard $S_n(X)$ as a subset of $C_n(X)$, so that for every cochain $\varphi \in C^n(X, W) \subseteq C^n(X)$ it makes sense to consider the restriction $\varphi|_{S_n(X)}$. In particular, we say that φ is *continuous* if $\varphi|_{S_n(X)}$ is (recall that $S_n(X)$ is endowed with the compact-open topology). If we set

$$C_c^*(X, W) = \{\varphi \in C^*(X, W) \mid \varphi \text{ is continuous}\},$$

then it is readily seen that $\delta^n(C_c^n(X, W)) \subseteq C_c^{n+1}(X, W)$, so $C_c^*(X, W)$ is a subcomplex of $C^*(X, W)$, whose homology is denoted by $H_c^*(X, W)$.

We now come to the definition of (continuous) bounded cohomology. We endow $C^n(X, W)$ with the L^∞ -norm defined by

$$\|f\|_\infty = \sup_{\sigma \in S_n(X)} |f(\sigma)| \in [0, \infty], \quad f \in C^n(X, W),$$

and introduce the following submodules of $C^*(X, W)$:

$$C_b^*(X, W) = \{f \in C^*(X, W) \mid \|f\|_\infty < \infty\},$$

$$C_{cb}^*(X, W) = C_b^*(X, W) \cap C_c^*(X, W).$$

The coboundary map δ^n is bounded, so $C_b^*(X, W)$ (resp. $C_{cb}^*(X, W)$) is a subcomplex of $C^*(X, W)$ (resp. of $C_c^*(X, W)$). Its homology is denoted by $H_b^*(X, W)$ (resp. $H_{cb}^*(X, W)$), and it is called the *bounded cohomology* (resp. *continuous*

bounded cohomology) of (X, W) . The L^∞ -norm on $C^*(X, W)$ descends (after suitable restrictions) to a seminorm on each of the modules $H^*(X, W)$, $H_c^*(X, W)$, $H_b^*(X, W)$, $H_{cb}^*(X, W)$. These seminorms will still be denoted by $\|\cdot\|_\infty$. The inclusion maps

$$\rho_b^* : C_{cb}^*(X, W) \hookrightarrow C_b^*(X, W), \quad \rho^* : C_c^*(X, W) \hookrightarrow C^*(X, W)$$

induce maps

$$H^*(\rho_b^*) : H_{cb}^*(X, W) \rightarrow H_b^*(X, W), \quad H^*(\rho^*) : H_c^*(X, W) \rightarrow H^*(X, W),$$

that are a priori neither injective nor surjective.

We are now ready to state our main result about (continuous) bounded cohomology of pairs, which is proved in [Section 4E](#):

Theorem 1.7. *Let (X, W) be a good CW-pair. Then the map*

$$H^n(\rho_b^*) : H_{cb}^n(X, W) \rightarrow H_b^n(X, W)$$

admits a right inverse which is an isometric embedding (in particular, $H^n(\rho_b^)$ is surjective) for every $n \in \mathbb{N}$.*

In the absolute case, when $W = \emptyset$, [Theorem 1.7](#) is proved in [[Frigerio 2011](#), [Theorem 1.2](#)]. In order to prove [Theorem 1.7](#) we suitably develop the theory of relative bounded cohomology of pairs of groups. In particular, our [Theorem 4.1](#) implies the following result, which is maybe of independent interest (see [Section 3](#) for the definition of $H_b^*(G, A)$, where G is a group and A is a subgroup of G):

Theorem 1.8. *Let (X, W) be a countable CW-pair. Also suppose that X, W are connected, and that the map $\pi_1(W) \rightarrow \pi_1(X)$ induced by the inclusion $W \hookrightarrow X$ is injective. Then for every $n \in \mathbb{N}$ there exists a norm-nonincreasing isomorphism*

$$H_b^n(\pi_1(X), \pi_1(W)) \rightarrow H_b^n(X, W).$$

If in addition the pair (X, W) is good, then this isomorphism is isometric.

In [Section 4F](#) we show how [Theorem 1.7](#) and [[Frigerio 2011](#), [Theorem 1.1](#)] can be exploited to prove the following:

Theorem 1.9. *Let (X, W) be a locally finite good CW-pair. Then the map*

$$H^n(\rho^*) : H_c^n(X, W) \rightarrow H^n(X, W)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

2. The case of locally convex pairs

The following definitions can be found for instance in [Bridson and Haefliger 1999]. Let (X, d) be a metric space (when d is fixed, we denote (X, d) simply by X). A *geodesic segment* in X is an isometric embedding of a bounded closed interval into X . The metric d (or the metric space $X = (X, d)$) is *geodesic* if every two points in X are joined by a geodesic segment (in particular, X is path-connected and locally path connected). Moreover, d (or $X = (X, d)$) is *globally convex* if it is geodesic and if any two geodesic segments $c_1 : [0, a] \rightarrow X$, $c_2 : [0, a] \rightarrow X$ such that $c_1(0) = c_2(0)$ satisfy the condition $d(c_1(ta), c_2(ta)) \leq td(c_1(a), c_2(a))$ for every $t \in [0, 1]$ (and in this case, X is contractible, see Lemma 2.1 below). We say that d (or $X = (X, d)$) is *locally convex* if every point in X has a neighborhood in which the restriction of d is convex (in particular, it is geodesic). A subspace $Y \subseteq X$ is *convex* if every geodesic segment (in X) joining any two points of Y is entirely contained in Y (in particular, if X is geodesic, then Y is path-connected).

Suppose that X is geodesic, complete and locally convex. Then it is locally contractible, hence it admits a universal covering $p : \tilde{X} \rightarrow X$. We endow \tilde{X} with the length metric induced by p , that is, the unique length metric \tilde{d} such that $p : (\tilde{X}, \tilde{d}) \rightarrow (X, d)$ is a local isometry (see [Bridson and Haefliger 1999, Proposition I.3.25]). Since (X, d) is complete and geodesic, the same is true for (\tilde{X}, \tilde{d}) . Moreover, the Cartan–Hadamard theorem for metric spaces (see [loc. cit., II.4.1]) implies that the space (\tilde{X}, \tilde{d}) is globally convex.

Let W be any subset of X . We say that (X, W) is a *locally convex pair of metric spaces* (or simply a *locally convex pair*) if the following conditions hold:

- (1) X is geodesic, complete and locally convex;
- (2) W is closed in X and locally path-connected;
- (3) every path-connected component of $p^{-1}(W) \subseteq \tilde{X}$ is convex in \tilde{X} .

Throughout the whole section we denote by (X, W) a locally convex pair of metric spaces, we fix a universal covering $p : \tilde{X} \rightarrow X$ (where \tilde{X} is endowed with the induced metric), and we denote by \tilde{W} the subset $p^{-1}(W) \subseteq \tilde{X}$ (on the contrary, in Section 4 we will denote by \tilde{W} a fixed connected component of $p^{-1}(W)$).

2A. Straight simplices. In order to properly define straight simplices we first need the following result, which is an immediate consequence of the Cartan–Hadamard theorem for metric spaces:

Lemma 2.1 [Bridson and Haefliger 1999, II.4.1]. *For every pair of points $p, q \in \tilde{X}$ there exists a unique geodesic segment in \tilde{X} joining p to q . Moreover, if $\alpha_{p,q} : [0, 1] \rightarrow \tilde{X}$ is a constant-speed parametrization of such a segment, then $\alpha_{p,q}$ continuously depends (with respect to the compact-open topology) on p and q . In particular, \tilde{X} is contractible.*

For $i \in \mathbb{N}$ we denote by e_i the point $(0, 0, \dots, 1, \dots, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}$ where the unique nonzero coefficient is at the i -th entry (entries are indexed by \mathbb{N} , so $(1, 0, \dots) = e_0$). We denote by Δ_p the standard p -simplex, that is, the convex hull of e_0, \dots, e_p , and we observe that with these notations we have $\Delta_p \subseteq \Delta_{p+1}$.

Let $k \in \mathbb{N}$, and let x_0, \dots, x_k be points in \tilde{X} . We recall here the well-known definition of *straight* simplex $[x_0, \dots, x_k] \in S_k(\tilde{X})$ with vertices x_0, \dots, x_k : if $k = 0$, then $[x_0]$ is the 0-simplex with image x_0 ; if straight simplices have been defined for every $h \leq k$, then $[x_0, \dots, x_{k+1}] : \Delta_{k+1} \rightarrow \tilde{X}$ is determined by the following condition: for every $z \in \Delta_k \subseteq \Delta_{k+1}$, the restriction of $[x_0, \dots, x_{k+1}]$ to the segment with endpoints z, e_{k+1} is a constant speed parametrization of the geodesic joining $[x_0, \dots, x_k](z)$ to x_{k+1} (the fact that $[x_0, \dots, x_{k+1}]$ is well-defined and continuous is an immediate consequence of [Lemma 2.1](#)).

2B. Nets. Let $\Gamma \cong \pi_1(X)$ be the group of covering automorphisms of $p : \tilde{X} \rightarrow X$, and observe that, since p is a local isometry, every element of Γ is an isometry of \tilde{X} .

Definition 2.2. A *net* in \tilde{X} is given by a subset $\tilde{\Lambda} \subseteq \tilde{X}$ and a locally finite collection of Borel sets $\{\tilde{B}_x\}_{x \in \tilde{\Lambda}}$ such that the following conditions hold:

- (1) $\tilde{X} = \bigcup_{x \in \tilde{\Lambda}} \tilde{B}_x$ and $\tilde{B}_x \cap \tilde{B}_y = \emptyset$ for every $x, y \in \tilde{\Lambda}$ with $x \neq y$.
- (2) $\gamma(\tilde{\Lambda}) = \tilde{\Lambda}$ for every $\gamma \in \Gamma$ and $\gamma(\tilde{B}_x) = \tilde{B}_{\gamma(x)}$ for every $x \in \tilde{\Lambda}$, $\gamma \in \Gamma$.
- (3) If \tilde{K} is a path-connected component of \tilde{W} , then $\tilde{K} \subseteq \bigcup_{x \in \tilde{\Lambda} \cap \tilde{K}} \tilde{B}_x$.

Lemma 2.3. *There exists a net.*

Proof. For every $q \in X$ let us denote by U_q an evenly covered open neighborhood of q in X (with respect to the universal covering $\tilde{X} \rightarrow X$). Since W is closed and locally path-connected, we may also suppose that $W \cap U_q$ is path-connected. Being metrizable, X is paracompact, so the open covering $\{U_q\}_{q \in X}$ admits a locally finite open refinement $\{V_i\}_{i \in I}$. Now fix a total ordering \leq on I in such a way that $i \leq j$ whenever $V_i \cap W \neq \emptyset$ and $V_j \cap W = \emptyset$, and let us set

$$B_i = V_i \setminus \left(\bigcup_{j < i} V_j \right).$$

By construction, the family $\{B_i\}_{i \in I}$ is locally finite in X . Moreover, every B_i is the intersection of an open set and a closed set, so it is a Borel subset of X . Therefore, up to replacing I with the subset $\{i \in I \mid B_i \neq \emptyset\}$, the family $\{B_i\}_{i \in I}$ provides a locally finite cover of X by nonempty Borel sets. For every $i \in I$ let us choose $x_i \in B_i$ in such a way that $x_i \in W$ whenever $B_i \cap W \neq \emptyset$, and let us set $\Lambda = \bigcup_{i \in I} \{x_i\}$. We also set $B_{x_i} = B_i$ for every $i \in I$.

We now define $\tilde{\Lambda} = p^{-1}(\Lambda)$. For every $i \in I$ we choose an element $\tilde{x}_i \in p^{-1}(x_i)$, and we take $q_i \in X$ in such a way that $B_{x_i} \subseteq U_{q_i}$. Being evenly covered, U_{q_i} lifts to

the disjoint union $p^{-1}(U_{q_i}) = \bigcup_{\gamma \in \Gamma} \gamma(\tilde{U}_{q_i})$, where \tilde{U}_{q_i} is the connected component of $p^{-1}(U_{q_i})$ containing \tilde{x}_i .

We are now ready to define \tilde{B}_x , where x is any element of $\tilde{\Lambda}$. In fact, every $x \in \tilde{\Lambda}$ uniquely determines an index $i \in I$ and an element $\gamma \in \Gamma$ such that $x = \gamma(\tilde{x}_i)$, and we can set $\tilde{B}_x = \gamma(\tilde{U}_{q_i} \cap p^{-1}(B_{x_i}))$. Of course \tilde{B}_x is a Borel subset of \tilde{X} .

It is now easy to check that the pair $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ provides a net: the local finiteness of the family $\{\tilde{B}_x, x \in \tilde{\Lambda}\}$ readily descends from the fact p is a covering and $\{B_x, x \in \Lambda\}$ is locally finite in X , and conditions (1) and (2) of [Definition 2.2](#) are an obvious consequence of our choices. We now show that condition (3) also holds. We fix $x \in \tilde{\Lambda}$ such that $\tilde{W} \cap \tilde{B}_x \neq \emptyset$. By construction we have $x \in \tilde{W}$, and there exist $\gamma \in \Gamma$ and $i \in I$ such that $\tilde{B}_x \subseteq \gamma(\tilde{U}_{q_i})$. Our assumption that $U_q \cap W$ is path-connected implies that $\gamma(\tilde{U}_{q_i}) \cap \tilde{W}$ is also path-connected, so the set $\tilde{B}_x \cap \tilde{W}$ is entirely contained in the path-connected component of \tilde{W} containing x , whence the conclusion. \square

2C. Straightening. We are now ready to define our straightening operator. Let $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ be a net. We denote by $S_n^{\tilde{\Lambda}}(\tilde{X}) \subseteq S_n(\tilde{X})$ the set of straight n -simplices in \tilde{X} with vertices in $\tilde{\Lambda}$. Then we let $\tilde{\text{str}}_n : C_n(\tilde{X}) \rightarrow C_n(\tilde{X})$ be the unique linear map such that for $\tilde{\sigma} \in S_n(\tilde{X})$

$$\tilde{\text{str}}_n(\tilde{\sigma}) = [x_0, \dots, x_n] \in S_n^{\tilde{\Lambda}}(\tilde{X}),$$

where $x_i \in \tilde{\Lambda}$ is such that $\tilde{\sigma}(e_i) \in \tilde{B}_{x_i}$ for $i = 0, \dots, n$.

Proposition 2.4. *The map $\tilde{\text{str}}_* : C_*(\tilde{X}) \rightarrow C_*(\tilde{X})$ satisfies the following properties:*

- (1) $d_{n+1} \circ \tilde{\text{str}}_{n+1} = \tilde{\text{str}}_n \circ d_{n+1}$ for every $n \in \mathbb{N}$.
- (2) $\tilde{\text{str}}_n(\gamma \circ \tilde{\sigma}) = \gamma \circ \tilde{\text{str}}_n(\tilde{\sigma})$ for every $n \in \mathbb{N}$, $\gamma \in \Gamma$, $\tilde{\sigma} \in S_n(\tilde{X})$.
- (3) $\tilde{\text{str}}_*(C_*(\tilde{W})) \subseteq C_*(\tilde{W})$.
- (4) *The induced chain map $C_*(\tilde{X}, \tilde{W}) \rightarrow C_*(\tilde{X}, \tilde{W})$, which we will still denote by $\tilde{\text{str}}_*$, is Γ -equivariantly homotopic to the identity.*

Proof. If $x_0, \dots, x_n \in \tilde{X}$, then it is easily seen that for every $i \leq n$ the i -th face of $[x_0, \dots, x_n]$ is given by $[x_0, \dots, \hat{x}_i, \dots, x_n]$; moreover since isometries preserve geodesics we have $\gamma \circ [x_0, \dots, x_n] = [\gamma(x_0), \dots, \gamma(x_n)]$ for every $\gamma \in \text{Isom}(\tilde{X})$. Together with property (2) in the definition of net, these facts readily imply points (1) and (2) of the proposition.

If $\tilde{\sigma} \in S_n(\tilde{W})$, then all the vertices of $\tilde{\sigma}$ lie in the same connected component \tilde{K} of \tilde{W} . By property (3) in the definition of net, the vertices of $\tilde{\text{str}}_n(\tilde{\sigma})$ still lie in \tilde{K} . Since (X, W) is a locally convex pair, the subset \tilde{K} is convex in \tilde{X} , so $\tilde{\text{str}}_n(\tilde{\sigma})$ belongs to $S_n(\tilde{W})$, whence (3).

Finally, for $\tilde{\sigma} \in S_n(\tilde{X})$, let $F_{\tilde{\sigma}} : \Delta_n \times [0, 1] \rightarrow \tilde{X}$ be defined by $F_{\tilde{\sigma}}(x, t) = \beta_x(t)$, where $\beta_x : [0, 1] \rightarrow \tilde{X}$ is the constant-speed parametrization of the geodesic

segment joining $\tilde{\sigma}(x)$ with $\tilde{\text{str}}(\tilde{\sigma})(x)$. We set $T_n(\tilde{\sigma}) = (F_{\tilde{\sigma}})_*(c)$, where c is the standard chain triangulating the prism $\Delta_n \times [0, 1]$ by $(n + 1)$ -simplices. The fact that $d_{n+1}T_n + T_{n-1}d_n = \text{Id} - \tilde{\text{str}}_n$ is now easily checked, while the Γ -equivariance of T_* is a consequence of property (2) of nets together with the fact that geodesics are preserved by isometries. As above, the fact that $T_n(C_n(\tilde{W})) \subseteq C_{n+1}(\tilde{W})$ is a consequence of the convexity of the components of \tilde{W} . \square

Let $\Lambda = p(\tilde{\Lambda})$, and let $S_*^\Lambda(X)$ be the subset of $S_*(X)$ given by those singular simplices which are obtained by composing a simplex in $S_*^\Lambda(\tilde{X})$ with the covering projection p . As a consequence of [Proposition 2.4](#) we get the following:

Proposition 2.5. *The map $\tilde{\text{str}}_*$ induces a chain map $\text{str}_* : C_*(X, W) \rightarrow C_*(X, W)$ which is homotopic to the identity.*

Remark 2.6. The maps $\tilde{\text{str}}_*$, str_* obviously depend on the net chosen for their construction. Such a dependence is however somewhat inessential in our arguments below. Henceforth we understand that a net $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ is fixed, and we denote by $\tilde{\text{str}}_*$, str_* the corresponding straightening operators.

We are now ready to construct a chain map $\theta_* : \mathcal{C}_*(X, W) \rightarrow C_*(X, W)$ whose induced map in homology will provide the desired norm-nonincreasing inverse of $H_*(\iota_*)$.

Fix a simplex $\sigma \in S_n^\Lambda(X)$. It is readily seen that the set $\text{str}_n^{-1}(\sigma)$ is a Borel subset of $S_n(X)$. Therefore, for every measure $\mu \in \mathcal{C}_n(X)$ it makes sense to set

$$c_\sigma(\mu) = \mu(\text{str}_n^{-1}(\sigma)) \in \mathbb{R}.$$

Lemma 2.7. *For every measure $\mu \in \mathcal{C}_n(X)$, the set*

$$\{\sigma \in S_n^\Lambda(X) \mid c_\sigma(\mu) \neq 0\}$$

is finite.

Proof. Since μ admits a compact determination set, it is sufficient to show that the family $\{\text{str}_n^{-1}(\sigma), \sigma \in S_n^\Lambda(X)\}$ is locally finite in $S_n(X)$. So, let us take $\sigma_0 \in S_n(X)$, and let $\tilde{\sigma}_0 \in S_n(\tilde{X})$ be a lift of σ_0 to \tilde{X} . For every $j = 0, \dots, n$, let Z_i be an open neighborhood of $\tilde{\sigma}_0(e_i)$ that intersects only a finite number of \tilde{B}_{x_i} 's, and let $\tilde{\Omega} \subseteq S_n(\tilde{X})$ be the set of n -simplices whose i -th vertex belongs to Z_i for every $i = 0, \dots, n$. Then $\tilde{\Omega}$ is an open neighborhood of $\tilde{\sigma}_0$ in $S_n(\tilde{X})$.

Let $p_n : S_n(\tilde{X}) \rightarrow S_n(X)$ be the map taking every $\tilde{\sigma} \in S_n(\tilde{X})$ into $p \circ \tilde{\sigma}$. It is proved in [\[Frigerio 2011, Lemma A.4\]](#) (see also [\[Löh 2006\]](#)) that p_n is a covering, whence an open map, so $\Omega = p_n(\tilde{\Omega})$ is an open neighborhood of σ_0 in $S_n(X)$. Moreover, by construction the set $\text{str}_n(\Omega) = \text{str}_n(p_n(\tilde{\Omega})) = p_n(\tilde{\text{str}}_n(\tilde{\Omega}))$ is finite, whence the conclusion. \square

By [Lemma 2.7](#) we can define the map

$$\theta_n : \mathcal{C}_n(X) \rightarrow C_n(X), \quad \theta_n(\mu) = \sum_{\sigma \in S_n^\Delta(X)} c_\sigma(\mu)\sigma.$$

Lemma 2.8. (1) $\theta_n \circ \partial_{n+1} = d_{n+1} \circ \theta_{n+1}$ for every $n \in \mathbb{N}$.

(2) $\theta_n(\mathcal{C}_n(W)) \subseteq C_n(W)$ for every $n \in \mathbb{N}$.

(3) $\|\theta_n(\mu)\|_1 \leq \|\mu\|_m$ for every $\mu \in \mathcal{C}_n(X)$, $n \in \mathbb{N}$.

Proof. Point (1) is a direct consequence of the fact that str_* is a chain map.

Since $\text{str}_n(C_n(W)) \subseteq C_n(W)$, if $\sigma \in S_n^\Delta(X) \setminus S_n(W)$, then $\text{str}_n^{-1}(\sigma) \cap S_n(W) = \emptyset$. Therefore, if $\mu \in \mathcal{C}_n(W) \subseteq \mathcal{C}_n(X)$, then $c_\sigma(\mu) = \mu(\text{str}_n^{-1}(\sigma)) = 0$, whence point (2).

Point (3) is a consequence of the fact that, if $\{Z_j\}_{j \in J}$ is a finite collection of pairwise disjoint Borel subsets of $S_n(X)$, then $\sum_{j \in J} |\mu(Z_j)| \leq \|\mu\|_m$. \square

2D. Concluding the proof of [Theorem 1.5](#). As a consequence of [Lemma 2.8](#), the map $\theta_* : \mathcal{C}_*(X) \rightarrow C_*(X)$ induces norm-nonincreasing maps

$$\bar{\theta}_* : \mathcal{C}_*(X, W) \rightarrow C_*(X, W), \quad H_*(\bar{\theta}_*) : \mathcal{H}_*(X, W) \rightarrow H_*(X, W).$$

Since we have already seen that $H_*(\iota_*) : H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is a norm-nonincreasing isomorphism, in order to prove that $H_*(\iota_*)$ is an isometry it is sufficient to show that $H_n(\bar{\theta}_*) \circ H_n(\iota_*)$ is the identity of $H_n(X, W)$ for every $n \in \mathbb{N}$. However, we have from the very definitions that $\bar{\theta}_n \circ \iota_n = \text{str}_n$ for every $n \in \mathbb{N}$, so the conclusion follows from [Proposition 2.5](#).

3. Relative bounded cohomology of groups

Let us recall some basic definitions and results about the bounded cohomology of groups. For full details we refer the reader to [[Gromov 1982](#); [Ivanov 1985](#); [Monod 2001](#)]. Henceforth, we denote by G a fixed group, which has to be thought as endowed with the discrete topology.

Definition 3.1 [[Ivanov 1985](#); [Monod 2001](#)]. A *Banach G -module* is a Banach space V with a (left) action of G such that $\|g \cdot v\| \leq \|v\|$ for every $g \in G$ and every $v \in V$. A G -morphism of Banach G -modules is a bounded G -equivariant linear operator.

From now on we refer to a Banach G -module simply as a G -module.

3A. Relative injectivity. A bounded linear map $\iota : A \rightarrow B$ of Banach spaces is *strongly injective* if there is a bounded linear map $\sigma : B \rightarrow A$ with $\|\sigma\| \leq 1$ and $\sigma \circ \iota = \text{Id}_A$ (in particular, ι is injective). We emphasize that, even when A and B are G -modules, the map σ is *not* required to be G -equivariant.

Definition 3.2. A G -module E is *relatively injective* if for every strongly injective G -morphism $\iota : A \rightarrow B$ of Banach G -modules and every G -morphism $\alpha : A \rightarrow E$ there is a G -morphism $\beta : B \rightarrow E$ satisfying $\beta \circ \iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

$$\begin{array}{ccccc}
 0 & \longrightarrow & A & \begin{array}{c} \xleftarrow{\sigma} \\ \xrightarrow{\iota} \end{array} & B \\
 & & \downarrow \alpha & \searrow \beta & \\
 & & E & &
 \end{array}$$

3B. Resolutions. A G -complex (or simply a *complex*) is a sequence of G -modules E^i and G -maps $\delta^i : E^i \rightarrow E^{i+1}$ such that $\delta^{i+1} \circ \delta^i = 0$ for every i , where i runs over $\mathbb{N} \cup \{-1\}$:

$$0 \rightarrow E^{-1} \xrightarrow{\delta^{-1}} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

Such a sequence will often be denoted by (E^*, δ^*) .

A G -chain map (or simply a *chain map*) between G -complexes (E^*, δ_E^*) and (F^*, δ_F^*) is a sequence of G -maps $\{\alpha^i : E^i \rightarrow F^i \mid i \geq -1\}$ such that $\delta_F^i \circ \alpha^i = \alpha^{i+1} \circ \delta_E^i$ for every $i \geq -1$. If α^* , β^* are chain maps between (E^*, δ_E^*) and (F^*, δ_F^*) which coincide in degree -1 , a G -homotopy between α^* and β^* is a sequence of G -maps $\{T^i : E^i \rightarrow F^{i-1} \mid i \geq 0\}$ such that $\delta_F^{i-1} \circ T^i + T^{i+1} \circ \delta_E^i = \alpha^i - \beta^i$ for every $i \geq 0$, and $T^0 \circ \delta_E^{-1} = 0$. We recall that, according to our definition of G -maps, both chain maps between G -complexes and G -homotopies between such chain maps have to be bounded in every degree.

A complex is *exact* if δ^{-1} is injective and $\ker \delta^{i+1} = \text{Im } \delta^i$ for every $i \geq -1$. A G -resolution (or simply a *resolution*) of a G -module E is an exact G -complex (E^*, δ^*) with $E^{-1} = E$. A resolution (E^*, δ^*) is *relatively injective* if E^n is relatively injective for every $n \geq 0$.

A *contracting homotopy* for a resolution (E^*, δ^*) is a sequence of linear maps $k^i : E^i \rightarrow E^{i-1}$ such that $\|k^i\| \leq 1$ for every $i \in \mathbb{N}$, $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k^0 \circ \delta^{-1} = \text{Id}_E$.

$$0 \longrightarrow E^{-1} \begin{array}{c} \xleftarrow{k^0} \\ \xrightarrow{\delta^{-1}} \end{array} E^0 \begin{array}{c} \xleftarrow{k^1} \\ \xrightarrow{\delta^0} \end{array} E^1 \begin{array}{c} \xleftarrow{k^2} \\ \xrightarrow{\delta^1} \end{array} \dots \begin{array}{c} \xleftarrow{k^n} \\ \xrightarrow{\delta^{n-1}} \end{array} E^n \begin{array}{c} \xleftarrow{k^{n+1}} \\ \xrightarrow{\delta^n} \end{array} \dots$$

Note however that it is not required that k^i be G -equivariant. A resolution is *strong* if it admits a contracting homotopy.

The following result can be proved by means of standard homological algebra arguments (see [Ivanov 1985] and [Monod 2001, Lemmas 7.2.4 and 7.2.6]).

Proposition 3.3. Let $\alpha : E \rightarrow F$ be a G -map between G -modules, let (E^*, δ_E^*) be a strong resolution of E , and suppose (F^*, δ_F^*) is a G -complex such that $F^{-1} = F$

and F^i is relatively injective for every $i \geq 0$. Then α extends to a chain map α^* , and any two extensions of α to chain maps are G -homotopic.

3C. Absolute bounded cohomology of groups. If E is a G -module, we denote by $E^G \subseteq E$ the submodule of G -invariant elements in E .

Let (E^*, δ^*) be a relatively injective strong resolution of the trivial G -module \mathbb{R} (such a resolution exists, see [Section 3D](#)). Since coboundary maps are G -maps, they restrict to the G -invariant submodules of the E^i 's. Thus $((E^*)^G, \delta^*|_G)$ is a subcomplex of (E^*, δ^*) . A standard application of [Proposition 3.3](#) now shows that the isomorphism type of the homology of $((E^*)^G, \delta^*|_G)$ does not depend on the chosen resolution (while the seminorm induced on such homology module by the norms on the E^i 's could depend on it). What is more, there exists a canonical isomorphism between the homology of any two such resolutions, which is induced by any extension of the identity of \mathbb{R} . For every $n \geq 0$, we now define the n -dimensional *bounded cohomology* module $H_b^n(G)$ of G as follows: if $n \geq 1$, then $H_b^n(G)$ is the n -th homology module of the complex $((E^*)^G, \delta^*|_G)$, while if $n = 0$ then $H_b^n(G) = \ker \delta^0 \cong \mathbb{R}$.

3D. The standard resolution. For every $n \in \mathbb{N}$, let $B^n(G)$ be the space of bounded real maps on G^{n+1} . We endow $B^n(G)$ with the supremum norm and with the diagonal action of G defined by $(g \cdot f)(g_0, \dots, g_n) = f(g^{-1}g_0, \dots, g^{-1}g_n)$, thus defining on $B^n(G)$ a structure of G -module. For $n \geq 0$ we define $\delta^n : B^n(G) \rightarrow B^{n+1}(G)$ by setting:

$$\delta^n(f)(g_0, g_1, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g}_i, \dots, g_{n+1}).$$

Moreover, we let $B^{-1}(G) = \mathbb{R}$ be the trivial G -module, and we define $\delta^{-1} : \mathbb{R} \rightarrow B^0(G)$ by setting $\delta^{-1}(t)(g) = t$ for every $g \in G$. The complex $(B^*(G), \delta^*)$ admits the following contracting homotopy:

$$(1) \quad v^n : B^n(G) \rightarrow B^{n-1}(G), \quad v^n(f)(g_0, \dots, g_{n-1}) = f(e, g_0, \dots, g_{n-1})$$

(for $n = 0$ we understand that $v^0(f) = f(e) \in \mathbb{R} = B^{-1}(G)$ for every $f \in B^0(G)$). Therefore, the complex $(B^*(G), \delta^*)$ provides a strong resolution of the trivial G -module \mathbb{R} , and we will see in [Proposition 3.5](#) below that such a resolution is also relatively injective. In fact, the complex $(B^*(G), \delta^*)$ is usually known as the *standard resolution of the trivial G -module* \mathbb{R} .

Remark 3.4. We briefly compare our notion of standard resolution with Ivanov's and Monod's ones. In [\[Ivanov 1985\]](#), for every $n \in \mathbb{N}$ the set $B^n(G)$ is denoted by $B(G^{n+1})$, and is turned into a Banach G -module by the action $g \cdot f(g_0, \dots, g_n) =$

$f(g_0, \dots, g_n \cdot g)$. Moreover, the sequence of modules $B(G^n)$, $n \in \mathbb{N}$, is equipped with a structure of G -complex

$$0 \rightarrow \mathbb{R} \xrightarrow{d_{-1}} B(G) \xrightarrow{d_0} B(G^2) \xrightarrow{d_1} \dots \xrightarrow{d_n} B(G^{n+2}) \xrightarrow{d_{n+1}} \dots,$$

where $d_{-1}(t)(g) = t$ and

$$\begin{aligned} d_n(f)(g_0, \dots, g_{n+1}) \\ = (-1)^{n+1} f(g_1, \dots, g_{n+1}) + \sum_{i=0}^n (-1)^{n-i} f(g_0, \dots, g_i g_{i+1}, \dots, g_{n+1}) \end{aligned}$$

for every $n \geq 0$ (here we are using Ivanov's notation also for the differential). Now, it is readily seen that Ivanov's resolution is isomorphic to our standard resolution via the isometric G -chain isomorphism $\varphi^* : B^*(G) \rightarrow B(G^{*+1})$ defined by

$$\varphi^n(f)(g_0, \dots, g_n) = f(g_n^{-1}, g_n^{-1} g_{n-1}^{-1}, \dots, g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1} g_0^{-1});$$

its inverse is given by

$$(\varphi^n)^{-1}(f)(g_0, \dots, g_n) = f(g_n^{-1} g_{n-1}, g_{n-1}^{-1} g_{n-2}, \dots, g_1^{-1} g_0, g_0^{-1}).$$

We observe that our contracting homotopy (1) is conjugated by φ^* into Ivanov's contracting homotopy [1985] for the complex $(B(G^*), d_*)$.

Our notation is much closer to Monod's one. In fact, in [Monod 2001] the more general case of a topological group G is addressed, and the n -th module of the standard G -resolution of \mathbb{R} is inductively defined by setting

$$C_b^0(G, \mathbb{R}) = C_b(G, \mathbb{R}), \quad C_b^n(G, \mathbb{R}) = C_b(G, C_b^{n-1}(G, \mathbb{R})),$$

where $C_b(G, E)$ denotes the space of *continuous* bounded maps from G to the Banach space E . However, as observed in [Monod 2001, Remarks 6.1.2 and 6.1.3], the case when G is an abstract group may be recovered from the general case just by equipping G with the discrete topology. In that case, our notion of standard resolution coincides with Monod's. (See also [Monod 2001, Remark 7.4.9].)

Proposition 3.5 [Ivanov 1985; Monod 2001]. *The standard resolution of \mathbb{R} as a G -module is relatively injective and strong.*

Proof. We have already shown that the standard resolution is strong. The fact that it is also relatively injective is proved in [Monod 2001, Proposition 4.4.1] (see also Remark 7.4.9 of the same reference). Alternatively, since our standard resolution is isometrically isomorphic to Ivanov's one (see Remark 3.4), the relative injectivity of the standard resolution may be deduced from [Ivanov 1985, Lemma 3.2.2]. \square

The seminorm induced on $H_b^*(G)$ by the standard resolution is called the *canonical seminorm*. It is shown in [Ivanov 1985] that the canonical seminorm coincides

with the infimum of all the seminorms induced on $H_b^*(G)$ by any relatively injective strong resolution of the trivial G -module \mathbb{R} (see also [Proposition 3.10](#) below).

3E. Relative bounded cohomology of groups. Let A be a subgroup of G . Henceforth, whenever E is a G -module we understand that E is endowed also with the natural structure of A -module induced by the inclusion of A in G .

Definition 3.6 [[Park 2003](#), Definitions 3.1 and 3.5]. Let (U^*, δ_U^*) be a relatively injective strong G -resolution of the trivial G -module \mathbb{R} and (V^*, δ_V^*) be a relatively injective strong A -resolution of the trivial A -module \mathbb{R} . By [Proposition 3.3](#), the identity of \mathbb{R} may be extended to an A -chain map $\lambda^* : U^* \rightarrow V^*$. The pair of resolutions (U^*, δ_U^*) , (V^*, δ_V^*) , together with the chain map λ^* , provides a *pair of resolutions for* $(G, A; \mathbb{R})$. We say that such a pair is

- (1) *allowable*, if the chain map λ^* commutes with the contracting homotopies of (U^*, δ_U^*) and (V^*, δ_V^*) ;
- (2) *proper*, if the map λ^n restricts to a surjective map $\widehat{\lambda}^n : (U^n)^G \rightarrow (V^n)^A$ for every $n \in \mathbb{N}$.

We denote by $\ker(U^n \rightarrow V^n)$ the kernel of λ^n . It is readily seen that the module $\ker(U^n \rightarrow V^n)^G \subseteq (U^n)^G$ coincides with the kernel of $\widehat{\lambda}^n$.

If the pair of resolutions (U^*, δ_U^*) , (V^*, δ_V^*) is proper, there exists an exact sequence

$$0 \longrightarrow \ker(U^n \rightarrow V^n)^G \longrightarrow (U^n)^G \xrightarrow{\widehat{\lambda}^n} (V^n)^A \longrightarrow 0,$$

which induces the long exact sequence

$$\cdots \longrightarrow H_b^{n-1}(A) \longrightarrow H^n(\ker(U^* \rightarrow V^*)^G) \longrightarrow H_b^n(G) \longrightarrow H_b^n(A) \longrightarrow \cdots$$

As observed in [[Park 2003](#)], if the pair (U^*, δ_U^*) , (V^*, δ_V^*) is also allowable, then the isomorphism type of $H^n(\ker(U^* \rightarrow V^*)^G)$ does not depend on the chosen proper allowable pair of resolutions (see also [Proposition 3.10](#) below). Such a module is called the *n -th bounded cohomology group of the pair (G, A)* , and it is denoted by $H_b^n(G, A)$.

3F. The standard pair of resolutions. The following result is proved in [[Park 2003](#), Propositions 3.1 and 3.18], and shows that, just as in the absolute case, there exists a canonical proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Strictly speaking, Park's notion of standard pair of resolutions is different from ours, since it is based on Ivanov's definition of standard resolution. However, the isomorphism described in [Remark 3.4](#) translates Park's results into the following:

Proposition 3.7. *The standard resolutions $B^*(G)$ and $B^*(A)$ of the trivial G - and A -module \mathbb{R} , together with the obvious restriction map $B^*(G) \rightarrow B^*(A)$, provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$.*

The seminorm induced on $H_b^*(G, A; \mathbb{R})$ by this resolution is called the *canonical seminorm*. In order to save some words, from now on we fix the following notation:

$$B^n(G, A) = \ker(B^n(G) \rightarrow B^n(A)).$$

3G. Morphisms of pairs of resolutions. Let (U^*, δ_U^*) , (V^*, δ_V^*) and (E^*, δ_E^*) , (F^*, δ_F^*) be pairs of resolutions for $(G, A; \mathbb{R})$. A *morphism* between such pairs is a pair of chain maps (α_G^*, α_A^*) such that:

- (1) $\alpha_G^* : U^* \rightarrow E^*$ (resp. $\alpha_A^* : V^* \rightarrow F^*$) is a G -chain map (resp. an A -chain map) extending the identity of $\mathbb{R} = U^{-1} = E^{-1}$ (resp. the identity of $\mathbb{R} = V^{-1} = F^{-1}$);
- (2) for every $n \in \mathbb{N}$, the following diagram commutes

$$\begin{array}{ccc} U^n & \longrightarrow & V^n \\ \downarrow \alpha_G^n & & \downarrow \alpha_A^n \\ E^n & \longrightarrow & F^n, \end{array}$$

where the horizontal rows represent the A -morphisms involved in the definition of a pair of resolutions.

By condition (2), if (α_G^*, α_A^*) is a morphism of pairs of resolutions, then α_G^* restricts to a chain map

$$\alpha_{G,A}^* : \ker(U^* \rightarrow V^*) \rightarrow \ker(E^* \rightarrow F^*),$$

which induces in turn a map

$$H^*(\alpha_{G,A}^*) : H^*(\ker(U^* \rightarrow V^*)^G) \rightarrow H^*(\ker(E^* \rightarrow F^*)^G).$$

Proposition 3.8. *If the pairs of resolutions*

$$(U^*, \delta_U^*), (V^*, \delta_V^*) \quad \text{and} \quad (E^*, \delta_E^*), (F^*, \delta_F^*)$$

are proper, the map $H^(\alpha_{G,A}^*)$ is an isomorphism.*

Proof. Our hypothesis ensures that we have the commutative diagram

$$\begin{array}{ccccccc} \dots & H^{n-1}((V^*)^A) & \longrightarrow & H^n(\ker(U^* \rightarrow V^*)^G) & \longrightarrow & H^n((U^*)^G) & \longrightarrow & H^n((V^*)^A) & \dots \\ & \downarrow H^{n-1}(\alpha_A^*) & & \downarrow H^n(\alpha_{G,A}^*) & & \downarrow H^n(\alpha_G^*) & & \downarrow H^n(\alpha_A^*) & \\ \dots & H^{n-1}((F^*)^A) & \longrightarrow & H^n(\ker(E^* \rightarrow F^*)^G) & \longrightarrow & H^n((E^*)^G) & \longrightarrow & H^n((F^*)^A) & \dots \end{array}$$

The discussion carried out in [Section 3C](#) implies that the vertical arrows corresponding to $H^*(\alpha_G^*)$ and $H^*(\alpha_A^*)$ are isomorphisms, so the conclusion follows from the Five Lemma. \square

Remark 3.9. At the moment we are not able to prove either that every two proper allowable pairs of resolutions for $(G, A; \mathbb{R})$ are related by a morphism of pairs of resolutions, or that any two such morphisms induce the same map in cohomology. In fact, whenever two proper allowable pairs of resolutions are given, using [Proposition 3.3](#) one can easily construct the needed chain maps α_G^* and α_A^* . However, some troubles arise in proving that such chain maps can be chosen so to fulfill condition (2) in the above definition of morphism of pairs of resolutions. Despite these difficulties, the results proved in [Propositions 3.8](#) and [3.10](#) are sufficient to our purposes.

Also observe that in the statement of [Proposition 3.8](#) we do not require the involved pairs of resolutions to be allowable. However, allowability plays a fundamental rôle in constructing a morphism of pairs of resolutions between any generic proper allowable pair of resolutions and the standard pair of resolutions (see [Proposition 3.10](#) below), and in getting explicit bounds on the norm of such a morphism.

The following result shows that, just as in the absolute case, the bounded cohomology of (G, A) is computed by any proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Moreover, the canonical seminorm coincides with the infimum of all the seminorms induced on $H_b^*(G, A)$ by any such pair of resolutions.

Proposition 3.10. *Let $(U^*, \delta_U^*), (V^*, \delta_V^*)$ be a proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Then there exists a morphism (α_G^*, α_A^*) between this pair of resolutions and the canonical pair of resolutions introduced in [Section 3F](#). Moreover, one may choose α_G^*, α_A^* in such a way that the induced map*

$$H^*(\alpha_{G,A}^*) : H^*(\ker(U^* \rightarrow V^*)^G) \rightarrow H^*(B^*(G, A)^G) \cong H_b^*(G, A)$$

is a norm-nonincreasing isomorphism.

Proof. Let k_G^* and k_A^* be the contracting homotopies of (U^*, δ_U^*) and (V^*, δ_V^*) , respectively. Define α_G^n and α_A^n by induction as follows:

$$(2) \quad \begin{aligned} \alpha_G^n(f)(g_0, \dots, g_n) &= \alpha_G^{n-1}(g_0(k_G^n g_0^{-1}(f)))(g_1, \dots, g_n) \in \mathbb{R}, \\ \alpha_A^n(f)(g_0, \dots, g_n) &= \alpha_A^{n-1}(g_0(k_A^n g_0^{-1}(f)))(g_1, \dots, g_n) \in \mathbb{R}. \end{aligned}$$

That α_G^* is indeed a G -chain map and α_A^* is an A -chain map is showed in the proof of [\[Monod 2001, Theorem 7.3.1\]](#). (Alternatively, one may easily check that the maps α_G^* and α_A^* are related to the maps given in [\[Ivanov 1985, Theorem 3.6\]](#)

via the isomorphism described in [Remark 3.4.](#)) Moreover, it is clear from the definitions that α_G^* and α_A^* are norm-nonincreasing in every degree.

Since the chain map $U^* \rightarrow V^*$ commutes with the contracting homotopies of (U^*, δ_U^*) and (V^*, δ_V^*) , the following diagram commutes:

$$\begin{array}{ccc} U^n & \longrightarrow & V^n \\ \downarrow \alpha_G^n & & \downarrow \alpha_A^n \\ B^n(G) & \longrightarrow & B^n(A). \end{array}$$

This implies that (α_G^*, α_A^*) is a morphism of pairs of resolutions. Now the conclusion follows from [Proposition 3.8.](#) \square

4. Relative (continuous) bounded cohomology of spaces

Throughout the whole section we denote by (X, W) a countable CW-pair. We also make the following:

Standing assumption: Both X and W are connected, and the inclusion of W in X induces an injective map on fundamental groups.

Being locally contractible, the space X admits a universal covering $p: \tilde{X} \rightarrow X$. We denote by \tilde{W} a fixed connected component of $p^{-1}(W) \subseteq \tilde{X}$. We also choose a basepoint $b_0 \in \tilde{W}$. This choice determines a canonical isomorphism between $\pi_1(X, p(b_0))$ and the group G of the covering automorphisms of \tilde{X} . We denote by $A \subseteq G$ the subgroup corresponding to $i_*(\pi_1(W, p(b_0)))$ under this isomorphism, where $i: W \rightarrow X$ is the inclusion. Observe that A coincides with the group of automorphisms of \tilde{X} that leave \tilde{W} invariant. In particular, for every $n \in \mathbb{N}$ the module $C_b^n(\tilde{X})$ (resp. $C_b^n(\tilde{W})$) admits a natural structure of G -module (resp. A -module). Moreover, the covering projection $p: \tilde{X} \rightarrow X$ defines a pull-back map $p^*: C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})$ which induces in turn an isometric isomorphism $C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})^G$. As a consequence, we get the natural identification

$$H_b^*(X, W) \cong H_b^*(C_b^*(\tilde{X}, \tilde{W})^G).$$

The straightening procedure described in [Section 2](#) shows that, when (X, W) is a locally convex pair of metric spaces, in order to compute the relative singular homology of (X, W) one may replace the singular complex $C_*(X, W)$ with the subcomplex of straight chains. As a consequence, it is easily seen that in order to compute the cohomology (resp. the bounded cohomology) of (X, W) one may replace the complex $C^*(\tilde{X}, \tilde{W})^G$ (resp. $C_b^*(\tilde{X}, \tilde{W})^G$) with the subcomplex of those invariant cochains whose value on each simplex only depends on the vertices of the simplex (recall that straight simplices in \tilde{X} only depend on their vertices). Following [\[Gromov 1982\]](#), we say that any such cochain is *straight*.

Observe that the definition of straight cochain makes sense even when it is not possible to properly define a straightening on singular chains. Let us briefly describe some known results about straight cochains in the absolute case (when $W = \emptyset$). If \tilde{X} is contractible, a classical result ensures that both straight cochains and singular cochains compute the cohomology of G , so the cohomology of straight cochains is isomorphic to the singular cohomology of X . An important result in [Gromov 1982, Section 2.3] shows that the same is true for bounded cohomology, even without the assumption that \tilde{X} is contractible. More precisely, both bounded straight cochains and bounded singular cochains compute the bounded cohomology of G , and they both induce the canonical seminorm on $H_b^*(G)$, so the cohomology of bounded straight cochains is isometrically isomorphic to the bounded cohomology of X . Moreover by [Monod 2001, Theorem 7.4.5], the bounded cohomology of G (whence of X) is computed also by *continuous* bounded straight cochains. Monod’s result plays a fundamental rôle in Löh’s description of the isometric isomorphism between measure homology and singular homology in the absolute case.

In this section we show that, in the case when $W \neq \emptyset$, continuous bounded straight cochains compute the bounded cohomology of the pair (G, A) , thus extending Monod’s result to the relative case (see Theorem 4.1).

Moreover, in the case when the pair (X, W) is good we prove that also $H_b^*(X, W)$ is isometrically isomorphic to $H_b^*(G, A)$, thus obtaining that the bounded cohomology of (X, W) is computed by continuous bounded straight cochains. Finally, in Section 4E we show that this result easily implies our Theorem 1.7.

4A. Bounded cochains versus continuous bounded straight cochains. We next give the precise definition of the complex of continuous bounded straight cochains. For every $n \in \mathbb{N}$ we consider the following Banach spaces:

$$\begin{aligned} C_{cbs}^n(\tilde{X}) &= \{f : \tilde{X}^{n+1} \rightarrow \mathbb{R}, f \text{ continuous and bounded}\}, \\ C_{cbs}^n(\tilde{W}) &= \{f : \tilde{W}^{n+1} \rightarrow \mathbb{R}, f \text{ continuous and bounded}\}, \end{aligned}$$

both endowed with the supremum norm. The diagonal G -action such that $g \cdot f(x_0, \dots, x_n) = f(g^{-1}x_0, \dots, g^{-1}x_n)$ for every $g \in G$ endows $C_{cbs}^n(\tilde{X})$ with a structure of G -module. The obvious coboundary maps $\delta^n : C_{cbs}^n(\tilde{X}) \rightarrow C_{cbs}^{n+1}(\tilde{X})$ given by

$$\delta^n(f)(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_{n+1})$$

define on $C_{cbs}^*(\tilde{X})$ a structure of G -complex. In the very same way one endows $C_{cbs}^*(\tilde{W})$ with a structure of A -complex. For every $n \in \mathbb{N}$, the inclusion $\tilde{W}^{n+1} \hookrightarrow \tilde{X}^{n+1}$ induces an obvious restriction $C_{cbs}^n(\tilde{X}) \rightarrow C_{cbs}^n(\tilde{W})$, whose kernel will be

denoted by $C_{cbs}^n(\tilde{X}, \tilde{W})$. Finally, for every $n \in \mathbb{N}$ we set

$$(3) \quad H_{cbs}^n(X, W) = H^n(C_{cbs}^*(\tilde{X}, \tilde{W})^G).$$

We will prove in Propositions 4.3 and 4.7 that both $C_b^*(\tilde{X})$, $C_b^*(\tilde{W})$ and $C_{cbs}^*(\tilde{X})$, $C_{cbs}^*(\tilde{W})$ provide proper pairs of resolutions for $(G, A; \mathbb{R})$. The pair of norm-nonincreasing chain maps

$$(4) \quad \begin{aligned} \eta_G^* : C_{cbs}^*(\tilde{X}) &\rightarrow C_b^*(\tilde{X}), & \eta_G^n(f)(\sigma) &= f(\sigma(e_0), \dots, \sigma(e_n)), \\ \eta_A^* : C_{cbs}^*(\tilde{W}) &\rightarrow C_b^*(\tilde{W}), & \eta_A^n(f)(\sigma) &= f(\sigma(e_0), \dots, \sigma(e_n)) \end{aligned}$$

allows us to identify $C_{cbs}^*(\tilde{X})$ with the subcomplex of $C_b^*(\tilde{X})$ of continuous bounded straight cochains on \tilde{X} , and likewise with \tilde{W} in place of \tilde{X} . Moreover, it is readily seen that the pair (η_G^*, η_A^*) is a morphism of resolutions. Therefore, Proposition 3.8 implies that the induced map in cohomology

$$H^*(\eta_{G,A}^*) : H_{cbs}^*(X, W) = H^*(C_{cbs}^*(\tilde{X}, \tilde{W})^G) \rightarrow H^*(C_b^*(\tilde{X}, \tilde{W})^G) = H_b^*(X, W)$$

is an isomorphism. Moreover, the explicit description of $\eta_{G,A}^*$ shows that $H^*(\eta_{G,A}^*)$ is norm-nonincreasing.

Under the assumption that the pair (X, W) is good, the isomorphism $H^*(\eta_{G,A}^*)$ is in fact an isometry. This fact is proved in the following subsections, and will play a fundamental rôle in our proof of Theorem 1.7.

We now describe briefly the content of the following subsections. In Section 4B we define a morphism of resolutions (β_G^*, β_A^*) between the standard pair of resolutions and continuous bounded straight cochains via an *ad hoc* construction, and we show that this morphism induces an isometric isomorphism in cohomology. Then, under the assumption that (X, W) is good, we prove in Proposition 4.7 that bounded cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$, so we may exploit Proposition 3.10 to construct a morphism of pairs of resolutions (α_G^*, α_A^*) between bounded cochains and the standard pair of resolutions for $(G, A; \mathbb{R})$. This morphism induces a norm-nonincreasing isomorphism in cohomology, so in order to prove that the isomorphism $H^*(\eta_{G,A}^*)$ is isometric we will be left to show that the composition $\beta_{G,A}^* \circ \alpha_{G,A}^*$ induces the inverse of $H^*(\eta_{G,A}^*)$ in cohomology; in other words, that the following diagram commutes:

$$\begin{array}{ccc} & H_b^*(G, A) & \\ H^*(\beta_{G,A}^*) \swarrow & & \nwarrow H^*(\alpha_{G,A}^*) \\ H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W). \end{array}$$

We can summarize the results just described in the following theorem, whose proof is carried out in Subsections 4B, 4C, 4D.

Theorem 4.1. *For every $n \in \mathbb{N}$ the map*

$$H^n(\beta_{G,A}^*) : H_b^n(G, A) \rightarrow H_{cbs}^n(X, W)$$

is an isometric isomorphism, and the map

$$H^n(\eta_{G,A}^*) : H_{cbs}^n(X, W) \rightarrow H_b^n(X, W)$$

is a norm-nonincreasing isomorphism. In particular, the composition

$$H^n(\eta_{G,A}^*) \circ H^n(\beta_{G,A}^*)$$

is a norm-nonincreasing isomorphism between $H_b^n(G, A)$ and $H_b^n(X, W)$. If, in addition, (X, W) is good, then $H^n(\eta_{G,A}^)$ is an isometry, and $H_b^n(G, A)$ and $H_b^n(X, W)$ are isometrically isomorphic.*

In fact, one may notice that the proof that $H^n(\beta_{G,A}^*)$ is an isometric isomorphism still works without the assumption that X and W are countable.

4B. Mapping standard resolutions into continuous bounded straight cochains.

We begin with a generalization of [Frigerio 2011, Lemma 5.1]:

Lemma 4.2. *There exists a continuous map $\chi : \tilde{X} \rightarrow [0, 1]$ with the following properties:*

- (1) *For every $x \in \tilde{X}$ there exists a neighborhood U_x of $x \in \tilde{X}$ such that the set $\{g \in G \mid \text{supp}(\chi) \cap g(U_x) \neq \emptyset\}$ is finite.*
- (2) *For every $x \in \tilde{X}$, we have $\sum_{g \in G} \chi(g \cdot x) = 1$. (Note that the sum on the left-hand side is finite by (1).)*
- (3) *For every $w \in \tilde{W}$ and every $g \in G \setminus A$, we have $\chi(g \cdot w) = 0$, whence $\sum_{g \in A} \chi(g \cdot w) = 1$.*
- (4) *We have $\chi(b_0) = 1$, so $\chi(g \cdot b_0) = 0$ for every $g \neq 1$.*

Proof. Recall that $p : \tilde{X} \rightarrow X$ is the universal covering of X . Using that W is a subcomplex of X , one can easily construct an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that every U_i is contractible (whence evenly covered with respect to $p : \tilde{X} \rightarrow X$) and $U_i \cap W$ is path-connected for every $i \in I$ (for example, if $\epsilon > 0$ is small enough and $x \in X$, the contractible ϵ -neighborhood $N_\epsilon(x)$ of x constructed in [Hatcher 2002, page 522] intersects any subcomplex of X in a contractible, whence path-connected, subset). Now choose $i_0 \in I$ such that $p(b_0) \in U_{i_0}$, and set $J = \{i \in I \mid U_i \cap W \neq \emptyset\}$ (so $i_0 \in J$).

For every U_i we choose an open subset $H_i \subseteq \tilde{X}$ in such a way that the following conditions hold:

- (a) $p|_{H_i} : H_i \rightarrow U_i$ is a homeomorphism.
- (b) $p^{-1}(U_i) = \bigcup_{g \in G} g(H_i)$ and $g(H_i) \cap g'(H_i) = \emptyset$ for every $g \neq g'$.

- (c) $b_0 \in H_{i_0}$.
- (d) $H_i \cap \tilde{W} \neq \emptyset$ for every $i \in J$.

We now set $U'_i = U_i \setminus \{p(b_0)\}$ for every $i \neq i_0$, $U'_{i_0} = U_{i_0}$, and $\mathcal{U}' = \{U'_i\}_{i \in I}$. Let also $H'_i = H_i \cap p^{-1}(U'_i)$. Since $U_i \cap W$ is path-connected, condition (d) easily implies that

$$H_i \cap p^{-1}(W) = H_i \cap \tilde{W} \quad \text{for every } i \in I,$$

whence

$$(5) \quad H'_i \cap p^{-1}(W) = H'_i \cap \tilde{W} \quad \text{for every } i \in I.$$

Since every CW-complex is paracompact (see [Miyazaki 1952; Bourgin 1952], for instance), we may now take a partition of unity $\{\varphi_i\}_{i \in I}$ adapted to \mathcal{U}' , and let $\psi_i : \tilde{X} \rightarrow \mathbb{R}$ be the map which coincides with $\varphi_i \circ p$ on H'_i and is null outside H'_i . We finally set

$$\chi = \sum_{i \in I} \psi_i.$$

The fact that χ satisfies properties (1) and (2) of the statement is proved in [Frigerio 2011, Lemma 5.1]. Moreover, for every $w \in \tilde{W}$ and $g \in G \setminus A$ we have $g \cdot w \in p^{-1}(W) \setminus \tilde{W}$, so Equation (5) implies that $g \cdot w$ does not belong to any H'_i . This implies point (3). Finally, since $p(b_0) \notin U'_i$ for every $i \neq i_0$, we have necessarily $\varphi_i(p(b_0)) = 0$ for every $i \neq i_0$, and $\varphi_{i_0}(p(b_0)) = 1$. By (c) this implies that $\psi_{i_0}(b_0) = 1$, whence $\chi(b_0) = 1$, as desired. \square

Proposition 4.3. *The pair $(C_{cbs}^*(\tilde{X}), \delta^*)$, $(C_{cbs}^*(\tilde{W}), \delta^*)$ provides a proper allowable pair of resolutions for $(G, A; \mathbb{R})$.*

Proof. The fact that $(C_{cbs}^*(\tilde{X}), \delta^*)$ (resp. $(C_{cbs}^*(\tilde{W}), \delta^*)$) provides a relatively injective resolution of \mathbb{R} as a trivial G -module (resp. A -module) is proved in [Monod 2001, Theorem 7.4.5]. (To apply that result our CW-complexes X and W should be locally compact, whence locally finite; but these conditions are used in Monod's proof only to ensure the existence of a suitable Bruhat function on \tilde{X} and on \tilde{W} ; in our case of interest the fact that G and A are discrete allows us to explicitly describe such a map; see Lemma 4.2.)

It is readily seen that these resolutions admit the contracting homotopies

$$(6) \quad \begin{aligned} t_G^n(f)(x_1, \dots, x_n) &= f(b_0, x_1, \dots, x_n), \quad f \in C_{cbs}^n(\tilde{X}), \quad (x_1, \dots, x_n) \in \tilde{X}^n, \\ t_A^n(f)(w_1, \dots, w_n) &= f(b_0, w_1, \dots, w_n), \quad f \in C_{cbs}^n(\tilde{W}), \quad (w_1, \dots, w_n) \in \tilde{W}^n. \end{aligned}$$

This clearly implies that the A -chain map $\gamma^* : C_{cbs}^*(\tilde{X}) \rightarrow C_{cbs}^*(\tilde{W})$ induced by the inclusion $\tilde{W} \hookrightarrow \tilde{X}$ commutes with the contracting homotopies.

In order to conclude we have to show that γ^* restricts to a surjective map

$$\widehat{\gamma}^* : C_{cbs}^*(\widetilde{X})^G \rightarrow C_{cbs}^*(\widetilde{W})^A.$$

Let $f : \widetilde{W}^{n+1} \rightarrow \mathbb{R}$ be an A -invariant bounded continuous map. The inclusion $\widetilde{W}^{n+1} \hookrightarrow \widetilde{X}^{n+1}$ induces a homeomorphism ψ between \widetilde{W}^{n+1}/A and a closed subset K of \widetilde{X}^{n+1}/G (recall that W is a CW-subcomplex of X , so it is closed in X). Therefore, f defines a bounded continuous map \bar{f} on K , and by Tietze's theorem we may extend \bar{f} to a bounded continuous map $\bar{g} : \widetilde{X}^{n+1}/G \rightarrow \mathbb{R}$. If g is obtained by precomposing \bar{g} with the projection $\widetilde{X}^{n+1} \rightarrow \widetilde{X}^{n+1}/G$, then $g \in C_{cbs}^n(\widetilde{X})^G$, and $\widehat{\gamma}^n(g) = f$. We have thus shown that $\widehat{\gamma}^*$ is surjective, and this concludes the proof. \square

We are now ready to describe a morphism of pairs of resolutions (β_G^*, β_A^*) between the standard pair of resolutions for $(G, A; \mathbb{R})$ and the complexes of straight cochains. Let

$$\beta_G^n : B^n(G) \rightarrow C_{cbs}^n(\widetilde{X}), \quad \beta_A^n : B^n(A) \rightarrow C_{cbs}^n(\widetilde{W})$$

be defined as follows:

$$\begin{aligned} \beta_G^n(f)(x_0, \dots, x_n) &= \sum_{(g_0, \dots, g_n) \in G^{n+1}} \chi(g_0^{-1}x_0) \cdots \chi(g_n^{-1}x_n) \cdot f(g_0, \dots, g_n), \\ \beta_A^n(f)(w_0, \dots, w_n) &= \sum_{(g_0, \dots, g_n) \in A^{n+1}} \chi(g_0^{-1}w_0) \cdots \chi(g_n^{-1}w_n) \cdot f(g_0, \dots, g_n). \end{aligned}$$

Lemma 4.4. *For every $f \in B^n(G)$, $(g_0, \dots, g_n) \in G^{n+1}$ we have*

$$\beta_G^n(f)(g_0b_0, \dots, g_nb_0) = f(g_0, \dots, g_n).$$

Proof. By Lemma 4.2(4), for every $(\gamma_0, \dots, \gamma_n) \in G^{n+1}$ we have

$$\begin{aligned} \chi(\gamma_0^{-1}g_0b_0) \cdots \chi(\gamma_n^{-1}g_nb_0) \cdot f(\gamma_0, \dots, \gamma_n) \\ = \begin{cases} f(g_0, \dots, g_n) & \text{if } \gamma_i = g_i \text{ for every } i, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and this readily implies the conclusion. \square

Proposition 4.5. *The pair (β_G^*, β_A^*) provides a well-defined morphism of pairs of resolutions. For every $n \in \mathbb{N}$ the induced map*

$$H^n(\beta_{G,A}^*) : H_b^n(G, A) \rightarrow H_{cbs}^n(X, W)$$

is an isometric isomorphism.

Proof. We begin by showing that β_G^* is a G -map. So, take $f \in B^n(G)$, $g \in G$, and $(x_0, \dots, x_n) \in \tilde{X}^{n+1}$. By definition we have

$$\begin{aligned}\beta_G^n(g \cdot f)(x_0, \dots, x_n) &= \sum_{(g_0, \dots, g_n) \in G^{n+1}} \chi(g_0^{-1}x_0) \cdots \chi(g_n^{-1}x_n) \cdot f(g^{-1}g_0, \dots, g^{-1}g_n), \\ (g \cdot \beta_G^n(f))(x_0, \dots, x_n) &= \sum_{(g_0, \dots, g_n) \in G^{n+1}} \chi(g_0^{-1}g^{-1}x_0) \cdots \chi(g_n^{-1}g^{-1}x_n) \cdot f(g_0, \dots, g_n),\end{aligned}$$

and an easy change of variables implies that β_G^n is a G -map. A similar argument shows that β_A^n is an A -map. We now check that β_G^* is a chain map. By [Lemma 4.2\(2\)](#), for every $x_i \in \tilde{X}$ we have $\sum_{g \in G} \chi(g^{-1}x_i) = 1$, so if $(g_0, \dots, g_{n+1}) \in G^{n+2}$ and $(x_0, \dots, x_{n+1}) \in \tilde{X}^{n+2}$ are fixed, then

$$\begin{aligned}\chi(g_0^{-1}x_0) \cdots \chi(\widehat{g_i^{-1}x_i}) \cdots \chi(g_{n+1}^{-1}x_{n+1}) \\ = \sum_{g \in G} \chi(g_0^{-1}x_0) \cdots \chi(g^{-1}x_i) \cdots \chi(g_{n+1}^{-1}x_{n+1})\end{aligned}$$

and $\beta_G^n(f)(x_0, \dots, \widehat{x_i}, \dots, x_{n+1})$ is equal to

$$\sum_{(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}) \in G^{n+1}} \chi(g_0^{-1}x_0) \cdots \chi(\widehat{g_i^{-1}x_i}) \cdots \chi(g_{n+1}^{-1}x_{n+1}) \cdot f(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}),$$

which in turn equals

$$\sum_{(g_0, \dots, g_i, \dots, g_{n+1}) \in G^{n+2}} \chi(g_0^{-1}x_0) \cdots \chi(g_i^{-1}x_i) \cdots \chi(g_{n+1}^{-1}x_{n+1}) \cdot f(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}).$$

From this equality it is easy to deduce that $\delta^n(\beta_G^n(f)) = \beta_G^{n+1}(\delta^n(f))$, and this proves that β_G^* is a chain map. Since χ has been chosen in such a way that [Lemma 4.2\(3\)](#) holds, the same argument may be exploited to show that β_A^* is also a chain map.

Using again [Lemma 4.2\(3\)](#), it is easily checked that the restriction $\beta_G^n(f)|_{\tilde{W}^{n+1}}$ coincides with the map $\beta_A^n(f|_{A^{n+1}})$ for every $f \in B^n(G)$. As a consequence, the pair (β_G^*, β_A^*) is a morphism of pairs of resolutions, and [Proposition 3.8](#) implies that $H^*(\beta_{G,A}^*)$ is an isomorphism. Moreover, $H^n(\beta_{G,A}^*)$ is obviously norm-non-increasing for every $n \in \mathbb{N}$.

Recall now that [Proposition 3.10](#) provides a morphism of pairs of resolutions

$$\zeta_G^* : C_{cbs}^*(\tilde{X}) \rightarrow B^*(G), \quad \zeta_A^* : C_{cbs}^*(\tilde{W}) \rightarrow B^*(A),$$

which induces a norm-nonincreasing isomorphism

$$H^*(\zeta_{G,A}^*) : H_{cbs}^*(X, W) \rightarrow H_b^*(G, A).$$

In order to conclude it is sufficient to show that for every $n \in \mathbb{N}$ the composition $\zeta_G^n \circ \beta_G^n$ is the identity of $B^n(G)$.

The proof of [Proposition 3.10](#) implies that the map ζ_G^n can be described by the following inductive formula:

$$\zeta_G^n(f)(g_0, \dots, g_n) = \zeta_G^{n-1}(g_0(t_G^n(g_0^{-1}(f))))(g_1, \dots, g_n),$$

where t_G^* is the contracting homotopy for the resolution $C_{cbS}^*(\tilde{X})$ described in [Equation \(6\)](#). As a consequence, an easy induction shows that $\zeta_G^n(f)(g_0, \dots, g_n) = f(g_0 b_0, \dots, g_n b_0)$ for every $f \in C_{cbS}^n(\tilde{X})$, $(g_0, \dots, g_n) \in G^{n+1}$. By [Lemma 4.4](#), this implies that $\zeta_G^n \circ \beta_G^n$ is the identity of $B^n(G)$, whence the conclusion. \square

4C. Ivanov's contracting homotopy. In order to show that, under the hypothesis that (X, W) is good, bounded cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$, we first recall Ivanov's construction of a contracting homotopy for the resolution $C_b^*(\tilde{X})$.

It is shown in [\[Ivanov 1985\]](#) that one can construct an infinite tower of bundles

$$(7) \quad \dots \xrightarrow{p_m} X_m \xrightarrow{p_{m-1}} X_{m-1} \xrightarrow{p_{m-2}} \dots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1,$$

where $X_1 = \tilde{X}$, $\pi_i(X_m) = 0$ for every $i \leq m$, $\pi_i(X_m) = \pi_i(X)$ for every $i > m$ and each map $p_m : X_{m+1} \rightarrow X_m$ is a principal H_m -bundle for some topological connected abelian group H_m , which has the homotopy type of a $K(\pi_{m+1}(X), m)$. Moreover, the induced chain maps $p_m^* : C_b^*(X_m) \rightarrow C_b^*(X_{m+1})$ admit left inverse chain maps $A_m^* : C_b^*(X_{m+1}) \rightarrow C_b^*(X_m)$ obtained by averaging cochains over the preimages in X_{m+1} of simplices in X_m , in such a way that the A_m 's are norm-nonincreasing.

Denote by $W_m \subseteq X_m$ the preimage $p_{m-1}^{-1}(p_{m-2}^{-1}(\dots(p_1^{-1}(\tilde{W})))) \subseteq X_m$ (so W_{m+1} is a principal H_m -bundle over W_m for every $m \geq 1$). We denote simply by

$$p_m : W_{m+1} \rightarrow W_m$$

the restriction of p_m to W_{m+1} . It follows from Ivanov's construction that each A_m^* induces a norm-nonincreasing chain map $C_b^*(W_{m+1}) \rightarrow C_b^*(W_m)$, which will still be denoted by A_m^* .

Lemma 4.6. *Suppose that (X, W) is good. Then $\pi_i(W_m) = 0$ for every $i \leq m$.*

Proof. Of course, it is sufficient to prove that $\pi_i(W_m) \cong \pi_i(X_m)$ for every $i \in \mathbb{N}$, $m \in \mathbb{N}$. Let us prove this last statement by induction on m . Since the inclusion map $W \hookrightarrow X$ is π_1 -injective we have $\pi_1(W_1) = \pi_1(X_1) = 0$. Therefore, since coverings induce isomorphisms on homotopy groups of order at least two, the case $m = 1$ follows from the fact that the pair (X, W) is good. The inductive step follows from an easy application of the Five Lemma to the following commutative diagram,

which descends in turn from the naturality of the homotopy exact sequences for the bundles $X_{m+1} \rightarrow X_m$, $W_{m+1} \rightarrow W_m$:

$$\begin{array}{ccccccccc}
 \pi_{i+1}(W_m) & \longrightarrow & \pi_i(H_m) & \longrightarrow & \pi_i(W_{m+1}) & \longrightarrow & \pi_i(W_m) & \longrightarrow & \pi_{i-1}(H_m) \\
 \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\
 \pi_{i+1}(X_m) & \longrightarrow & \pi_i(H_m) & \longrightarrow & \pi_i(X_{m+1}) & \longrightarrow & \pi_i(X_m) & \longrightarrow & \pi_{i-1}(H_m). \quad \square
 \end{array}$$

Now suppose that (X, W) is good. We choose basepoints $w_m \in W_m$ in such a way that $p_m(w_{m+1}) = w_m$ for every $m \geq 1$, and $w_1 \in W_1 = \tilde{W}$ coincides with the basepoint b_0 fixed above. Since X_m is m -connected, for every $n \leq m$ it is possible to construct a map $L_n^m : S_n(X_m) \rightarrow S_{n+1}(X_m)$ that associates to every $\sigma \in S_n(X_m)$ a cone of σ over w_m (see [Ivanov 1985]). We stress that, since W_m is also m -connected, if $\sigma \in S_n(W_m) \subseteq S_n(X_m)$, then $L_n^m(\sigma)$ can be chosen to belong to $S_{n+1}(W_m)$. The maps L_n^m , $n \leq m$, induce a (partial) homotopy between the identity and the null map of $C_*(X_m)$, which in turn induces a (partial) contracting homotopy $\{k_m^n\}_{n \leq m}$ for the (partial) complex $\{C_b^n(X_m)\}_{n \leq m}$. Since $L_n^m(S_n(W_m)) \subseteq S_{n+1}(W_m)$, this contracting homotopy induces a (partial) contracting homotopy for $\{C_b^n(W_m)\}_{n \leq m}$, which we still denote by k_m^n . Moreover, it is possible to choose these contracting homotopies in a compatible way, in the sense that the equality $A_m^{n-1} \circ k_{m+1}^n \circ p_m^n = k_m^n$ holds for every $n \leq m$ (see again [Ivanov 1985]). Thanks to this compatibility condition, one can finally define the contracting homotopy

$$k_G^* : C_b^*(\tilde{X}) \rightarrow C_b^{*-1}(\tilde{X}),$$

via the formula

$$k_G^n = A_1^{n-1} \circ \dots \circ A_{m-1}^{n-1} \circ k_m^n \circ p_{m-1}^n \circ \dots \circ p_2^n \circ p_1^n \quad \text{for any } m \geq n.$$

The very same formula defines a contracting homotopy for $C_b^*(\tilde{W})$. By construction, the restriction map $C_b^*(\tilde{X}) \rightarrow C_b^*(\tilde{W})$ commutes with these contracting homotopies, and it obviously restricts to a surjective map $C_b^*(\tilde{X})^G \rightarrow C_b^*(\tilde{W})^A$. Since $C_b^n(\tilde{X})$, $C_b^n(\tilde{W})$ are relatively injective for every $n \geq 0$ (see [Ivanov 1985]), we have finally proved the following:

Proposition 4.7. *The pair $(C_b^*(\tilde{X}), \delta^*)$, $(C_b^*(\tilde{W}), \delta^*)$ provides a proper pair of resolutions for $(G, A; \mathbb{R})$. If in addition (X, W) is good, then this pair of resolutions is also allowable.*

Corollary 4.8. *For every $n \in \mathbb{N}$, the map*

$$H^n(\eta_{G,A}^*) : H_{cbs}^n(X, W) \rightarrow H_b^n(X, W)$$

is a norm-nonincreasing isomorphism.

Proof. By [Proposition 4.7](#), bounded cochains provide a proper pair of resolutions for $(G, A; \mathbb{R})$, so [Proposition 3.8](#) implies that $H^n(\eta_{G,A}^*)$ is an isomorphism. That it is norm-nonincreasing is a direct consequence of its explicit description. \square

Remark 4.9. The fact that the pair of resolutions $(C_b^*(\tilde{X}), \delta^*), (C_b^*(\tilde{W}), \delta^*)$ is allowable is stated in [\[Park 2003, Lemma 4.2\]](#) under the only assumption that (X, W) is a pair of connected CW-pairs. However, at the moment we are not able to prove such a statement without the assumption that (X, W) is good. For example, let us suppose that X is simply connected and W is a point (so that $\pi_n(W)$ injects into $\pi_n(X)$ for every $n \in \mathbb{N}$, and $X_1 = \tilde{X} = X, W_1 = \tilde{W} = W$). Then for every $n \in \mathbb{N}$ there exists only one simplex in $S_n(W)$, namely the constant n -simplex σ_n^W . Therefore, the only possible contracting homotopy for W is given by the map which sends the cochain $\varphi \in C_b^n(W)$ to the cochain $k_A^n(\varphi)$ such that $k_A^n(\varphi)(\sigma_{n-1}^W) = \varphi(\sigma_n^W)$. On the other hand, it is not difficult to show that $\pi_i(W_m) = \pi_{i+1}(X)$ for every $i < m$, and $\pi_i(W_m) = 0$ for every $i \geq m$. Therefore, if $\pi_{i+1}(X) \neq 0$, then $\pi_i(W_m) \neq 0$ for every $m > i$. This readily implies that for $m > i$ one cannot construct *cone-like* operators $L_j^m : C_j(X_m) \rightarrow C_{j+1}(X_m), j \leq i$, such that $d_{j+1}L_j^m + L_{j-1}^m d_j = \text{Id}$ and $L_j^m(C_j(W_m)) \subseteq C_{j+1}(W_m)$ for every $j \leq i$, so it is not clear how to show that the pair of resolutions $C_b^*(\tilde{X}), C_b^*(\tilde{W})$ is allowable. This difficulty already arises for the pair (S^2, q) , where q is any point of the 2-dimensional sphere S^2 .

Some troubles arise also in the case when the inclusion induces surjective (but not bijective) maps between the homotopy groups of W and of X . For instance, if X is the Euclidean 3-space and $W = S^2$, then $X_m = X$ for every $m \in \mathbb{N}$, so $W_m = W$ for every $m \in \mathbb{N}$, and, if i is sufficiently high, the partial complex $\{C_j(X, W)\}_{j \leq i}$ does not support a relative cone-like operator. Also observe that, if $\{W'_m, m \in \mathbb{N}\}$ is the tower of bundles constructed starting from W just as X_m is constructed starting from X , then the only map $W'_m \rightarrow W_m = S^2 \subseteq \mathbb{R}^3 = X_m$ which commutes with the projections of W'_m and X_m onto $W_1 = S^2$ and $X_1 = \mathbb{R}^3$ is the projection $W'_m \rightarrow W_1 = S^2$. As a consequence, also in this case it is not clear why the pair of resolutions $C_b^*(\tilde{X}), C_b^*(\tilde{W})$ should be allowable.

4D. Proof of Theorem 4.1. We now come back to the proof of [Theorem 4.1](#). By [Proposition 4.5](#) and [Corollary 4.8](#), we are only left to show that, under the assumption that (X, W) is good, the isomorphism

$$H^n(\eta_G^*) : H_{cbs}^n(X, W) \rightarrow H_b(X, W)$$

is isometric for every $n \in \mathbb{N}$.

So, suppose that (X, W) is good. By [Proposition 4.7](#) bounded cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Therefore, [Proposition 3.10](#) provides a morphism of pairs of resolutions

$$\alpha_G^* : C_b^*(\tilde{X}) \rightarrow B^*(G), \quad \alpha_A^* : C_b^*(\tilde{W}) \rightarrow B^*(A),$$

such that the induced map $H^*(\alpha_{G,A}^*)$ is a norm-nonincreasing isomorphism.

We already know that all the maps in the diagram

$$\begin{array}{ccc}
 & H_b^*(G, A) & \\
 H^*(\beta_{G,A}^*) \swarrow & & \nwarrow H^*(\alpha_{G,A}^*) \\
 H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W).
 \end{array}$$

are norm-nonincreasing isomorphisms, so in order to conclude it is sufficient to show that the diagram commutes. This fact is obviously implied by the following result, which concludes the proof of [Theorem 4.1](#).

Proposition 4.10. *Suppose that (X, W) is good. Then, for every $n \in \mathbb{N}$ the composition*

$$\alpha_{G,A}^n \circ \eta_{G,A}^n \circ \beta_{G,A}^n : B^n(G, A) \rightarrow B^n(G, A)$$

is equal to the identity of $B^n(G, A)$.

Proof. Since the composition $\alpha_{G,A}^n \circ \eta_{G,A}^n \circ \beta_{G,A}^n$ coincides with the restriction of $\alpha_G^n \circ \eta_G^n \circ \beta_G^n$ to $B^n(G, A) \subseteq B^n(G)$, it is sufficient to show that $\alpha_G^n \circ \eta_G^n \circ \beta_G^n$ is the identity of $B^n(G)$.

Before going into the needed computations, let us stress that the definition of α_G^* involves the contracting homotopy for the resolution $C_b^*(\tilde{X})$ described in [Section 4C](#). Being based on a non-explicit averaging procedure, this contracting homotopy cannot be described by an explicit formula, and the same is true for the chain map α_G^* . However, the explicit description of the composition $\alpha_G^* \circ \eta_G^*$ is sufficient to our purposes.

In fact, we already know from [Lemma 4.4](#) that

$$\beta_G^n(f)(g_0 b_0, \dots, g_n b_0) = f(g_0, \dots, g_n)$$

for every $f \in B^n(G)$, $(g_0, \dots, g_n) \in G^{n+1}$. Therefore, in order to conclude it is sufficient to prove that

$$(8) \quad \alpha_G^n(\eta_G^n(f))(g_0, \dots, g_n) = f(g_0 b_0, \dots, g_n b_0)$$

for every $f \in C_{cbs}^n(\tilde{X})$. So, let t_G^* and k_G^* be the contracting homotopies for continuous bounded straight cochains and for bounded cochains, respectively; see [\(6\)](#) and [\(7\)](#). We first show that for every $n \in \mathbb{N}$ we have

$$(9) \quad k_G^n \circ \eta_G^n = \eta_G^{n-1} \circ t_G^n.$$

Fix $f \in C_{cbs}^n(\tilde{X})$ and $\sigma \in S_{n-1}(\tilde{X})$, and let us compute $k_G^n(\eta_G^n(f))(\sigma)$. With notation as in [Section 4C](#), we choose $m \geq n$ and set

$$f_m = p_{m-1}^n(\dots p_1^n(\eta_G^n(f))) \in C_b^n(X_m).$$

Then, if σ_m is any lift of σ in X_m , we have $k_m^n(f_m)(\sigma_m) = f_m(\sigma'_m)$, where $\sigma'_m \in S_n(X_m)$ has vertices $w_m, \sigma_m(e_0), \dots, \sigma_m(e_{n-1})$. It readily follows that

$$k_m^n(f_m)(\sigma_m) = f(b_0, \sigma(e_0), \dots, \sigma(e_{n-1})).$$

We have thus shown that the cochain $k_m^n(f_m)$ is constant on all the lifts of σ in X_m . By definition, the value of $k_G^n(\eta_G^n(f))(\sigma)$ is obtained by suitably averaging the values taken by $k_m^n(f_m)$ on such lifts, so we finally get

$$k_G^n(\eta_G^n(f))(\sigma) = f(b_0, \sigma(e_0), \dots, \sigma(e_{n-1})),$$

whence (9).

Recall now that the map α_G^* is explicitly described (in terms of the contracting homotopy k_G^*) in [Proposition 3.10](#); see (2). Therefore, (2) and (9) readily imply that the composition $\alpha_G^n \circ \eta_G^n$ can be described by the following inductive formula:

$$\alpha_G^n(\eta_G^n(f))(g_0, \dots, g_n) = \alpha_G^{n-1}(g_0(\eta_G^{n-1}(t_G^n(g_0^{-1}(f)))))(g_1, \dots, g_n).$$

An easy induction now implies (8), whence the conclusion. \square

4E. Proof of Theorem 1.7. We next describe how [Theorem 1.7](#) can be deduced from [Theorem 4.1](#). For every $n \in \mathbb{N}$ the module $C_{cb}^n(\tilde{X})$ (resp. $C_{cb}^n(\tilde{W})$) admits a natural structure of G -module (resp. A -module). Moreover, it is proved in [[Frigerio 2011](#), Lemma 6.1] that the isometric isomorphism $C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})^G$ induced by the covering projection $p: \tilde{X} \rightarrow X$ restricts to an isometric isomorphism $C_{cb}^*(X, W) \rightarrow C_{cb}^*(\tilde{X}, \tilde{W})^G$, which induces in turn a natural identification

$$(10) \quad H_{cb}^*(X, W) \cong H^*(C_{cb}^*(\tilde{X}, \tilde{W})^G).$$

The G -chain map $v_G^*: C_{cbs}^*(\tilde{X}) \rightarrow C_{cb}^*(\tilde{X})$ defined by

$$v_G^n(f)(\sigma) = f(\sigma(e_0), \dots, \sigma(e_n)) \quad \text{for every } n \in \mathbb{N}, f \in C_{cbs}^n(\tilde{X}), \sigma \in S_n(\tilde{X}),$$

obviously restricts to a chain map $v_{G,A}^*: C_{cbs}^*(\tilde{X}, \tilde{W})^G \rightarrow C_{cb}^*(\tilde{X}, \tilde{W})^G$. Under the identifications described in (3) and (10), this chain map induces the norm-nonincreasing map

$$H^*(v_{G,A}^*) : H_{cbs}^*(X, W) \rightarrow H_{cb}^*(X, W)$$

(we cannot realize $H^*(v_{G,A}^*)$ as the map induced by a morphism of pairs of resolutions just because we are not able to prove that the pair $C_{cb}^*(\tilde{X}), C_{cb}^*(\tilde{W})$ provides a pair of resolutions for $(G, A; \mathbb{R})$; see [Remark 4.11](#) below).

It readily follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc}
 H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W) \\
 & \searrow^{H^*(v_{G,A}^*)} & \nearrow^{H^*(\rho_b^*)} \\
 & & H_{cb}^*(X, W)
 \end{array}$$

where $H^*(\rho_b^*) : H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ is the map described in the Introduction.

Now suppose that (X, W) is good. Then [Theorem 4.1](#) implies that the map $H^*(\eta_{G,A}^*)$ is an isometric isomorphism, so the map $H^*(v_{G,A}^*) \circ H^*(\eta_{G,A}^*)^{-1}$ provides a right inverse to $H^*(\rho_b^*)$. Since $H^*(v_{G,A}^*)$ is norm-nonincreasing, this map is an isometric embedding, and this concludes the proof of [Theorem 1.7](#).

Remark 4.11. Suppose that (X, W) is good. If we were able to prove that the complexes $C_{cb}^*(\tilde{X})$, $C_{cb}^*(\tilde{W})$ provide a proper pair of resolutions for $(G, A; \mathbb{R})$, then we could prove that $H^*(\rho_b^*) : H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ is an isometric isomorphism for every good pair (X, W) . However, it is not clear why Ivanov's contracting homotopies should take continuous cochains into continuous cochains, thus restricting to contracting homotopies for $C_{cb}^*(\tilde{X})$, $C_{cb}^*(\tilde{W})$.

4F. (Unbounded) continuous cohomology of pairs. We conclude the section by proving [Theorem 1.9](#), which asserts that, when (X, W) is a locally finite good CW-pair, the map

$$H^*(\rho^*) : H_c^*(X, W) \rightarrow H^*(X, W)$$

is an isometric isomorphism.

We first observe that, since W is closed in X , the subspace $S_n(W)$ is closed in $S_n(X)$ for every $n \in \mathbb{N}$. Moreover, since X is locally finite, it is metrizable, and this implies that $S_n(X)$ is also metrizable. Therefore, by Tietze's theorem, every continuous cochain on W extends to a continuous cochain on X ; i.e., the restriction map $C_c^*(X) \rightarrow C_c^*(W)$ is surjective. As a consequence, both rows of the following commutative diagram are exact:

$$\begin{array}{ccccccccc}
 H_c^{n+1}(X) & \longrightarrow & H_c^{n+1}(W) & \longrightarrow & H_c^n(X, W) & \longrightarrow & H_c^n(X) & \longrightarrow & H_c^n(W) \\
 \downarrow & & \downarrow & & \downarrow^{H^n(\rho^*)} & & \downarrow & & \downarrow \\
 H^{n+1}(X) & \longrightarrow & H^{n+1}(W) & \longrightarrow & H^n(X, W) & \longrightarrow & H^n(X) & \longrightarrow & H^n(W).
 \end{array}$$

We know from [[Frigerio 2011](#), Theorem 1.1] that, in the absolute case, the vertical arrows are isomorphisms, and the Five Lemma implies now that $H^n(\rho^*)$ is an isomorphism. We are left to show that it is also an isometry.

The inclusions $C_b^*(X, W) \hookrightarrow C^*(X, W)$, $C_{cb}^*(X, W) \hookrightarrow C_c^*(X, W)$ induce the comparison maps $c^* : H_b^*(X, W) \rightarrow H^*(X, W)$, $c_c^* : H_{cb}^*(X, W) \rightarrow H_c^*(X, W)$ and

it follows from the very definitions that for every $\varphi \in H^n(X, W)$, $\varphi_c \in H_c^n(X, W)$ the following equalities hold:

$$\begin{aligned}\|\varphi\|_\infty &= \inf\{\|\psi\|_\infty \mid \psi \in H_b^n(X, W), c^n(\psi) = \varphi\}, \\ \|\varphi_c\|_\infty &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^n(X, W), c_c^n(\psi_c) = \varphi_c\},\end{aligned}$$

where we understand that $\inf \emptyset = +\infty$. Moreover, since $H^*(\rho^*) \circ c_c^* = c^* \circ H^*(\rho_b^*)$, for every $\varphi_c \in H_c^*(X, W)$ we have

$$\begin{aligned}\|H^*(\rho^*)(\varphi_c)\|_\infty &= \inf\{\|\psi\|_\infty \mid \psi \in H_b^*(X, W), c^*(\psi) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), c^*(H^*(\rho_b^*)(\psi_c)) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), H^*(\rho^*)(c_c^*(\psi_c)) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), c_c^*(\psi_c) = \varphi_c\} = \|\varphi_c\|_\infty,\end{aligned}$$

where the second equality is due to [Theorem 1.7](#) (recall that locally finite CW-pairs are countable). The proof of [Theorem 1.9](#) is now complete.

5. The duality principle

This section is mainly devoted to the proof of [Theorem 1.3](#). As already mentioned in the Introduction, once a suitable duality pairing between measure homology and continuous bounded cohomology is established, [Theorem 1.3](#) can be easily deduced from [Theorem 1.7](#).

5A. Duality between singular homology and bounded cohomology. Let us begin by recalling the well-known duality between bounded cohomology and singular homology. Let (X, W) be any pair of topological spaces. By definition, $C^n(X, W)$ is the algebraic dual of $C_n(X, W)$, and it is readily seen that the L^∞ -norm on $C^n(X, W)$ is dual to the L^1 -norm on $C_n(X, W)$. As a consequence, $C_b^n(X, W)$ coincides with the topological dual of $C_n(X, W)$. This does *not* imply that $H_b^n(X, W)$ is the topological dual of $H_n(X, W)$, because taking duals of normed chain complexes does not commute in general with homology (see [\[Löh 2008\]](#) for a detailed discussion of this issue). However, if we denote by

$$\langle \cdot, \cdot \rangle : H_b^n(X, W) \times H_n(X, W) \rightarrow \mathbb{R}$$

the *Kronecker product* induced by the pairing $C_b^n(X, W) \times C_n(X, W) \rightarrow \mathbb{R}$, then an application of Hahn–Banach theorem (for details, see [\[Löh 2007, Theorem 3.8\]](#), for instance) gives the following:

Proposition 5.1. *For every $\alpha \in H_n(X, W)$ we have*

$$\|\alpha\|_1 = \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(X, W), \langle \varphi, \alpha \rangle = 1 \right\},$$

where we understand that $\sup \emptyset = 0$.

5B. Duality between measure homology and continuous bounded cohomology.

The topological dual of $\mathcal{C}_*(X, W)$ does not admit an easy description, so in order to compute seminorms in $\mathcal{H}_*(X, W)$ via duality more work is needed. We first observe that, if μ is any measure on $S_n(X)$ with compact determination set and f is any continuous function on $S_n(X)$, it makes sense to integrate f with respect to μ . Therefore, for every $n \in \mathbb{N}$ the bilinear pairing

$$\langle \cdot, \cdot \rangle : C_{cb}^n(X, W) \times \mathcal{C}_n(X, W) \rightarrow \mathbb{R}, \quad \langle f, \mu \rangle = \int_{S_n(X)} f(\sigma) d\mu(\sigma)$$

is well-defined. It readily follows from the definitions that $|\langle f, \mu \rangle| \leq \|f\|_\infty \cdot \|\mu\|_m$ for every $f \in C_{cb}^n(X, W)$, $\mu \in \mathcal{C}_n(X, W)$, so $C_{cb}^*(X, W)$ lies in the topological dual of $\mathcal{C}_*(X, W)$. Moreover, for every $i \in \mathbb{N}$, $f \in C_{cb}^i(X, W)$ and $\mu \in \mathcal{C}_{i+1}(X, W)$ we have $\langle \delta f, \mu \rangle = \langle f, \partial \mu \rangle$, so this pairing defines a Kronecker product

$$\langle \cdot, \cdot \rangle : H_{cb}^n(X, W) \times \mathcal{H}_n(X, W) \rightarrow \mathbb{R}$$

such that

$$(11) \quad |\langle \varphi_c, \alpha \rangle| \leq \|\varphi_c\|_\infty \cdot \|\alpha\|_{mh} \quad \text{for every } \varphi_c \in H_{cb}^n(X, W), \alpha \in \mathcal{H}_n(X, W).$$

The following proposition is an immediate consequence of inequality (11), and provides a sort of weak duality theorem for continuous bounded cohomology and measure homology. The term “weak” refers to the fact that while [Proposition 5.1](#) allows to compute seminorms in homology in terms of seminorms in bounded cohomology, here only an inequality is established. However, this turns out to be sufficient to our purposes. Moreover, once [Theorem 1.3](#) is proved, one could easily prove that (in the case of good CW-pairs) the inequality of [Proposition 5.2](#) is in fact an equality, thus recovering a “full” duality between continuous bounded cohomology and measure homology.

Proposition 5.2. *For every $\alpha \in \mathcal{H}_n(X, W)$ we have*

$$\|\alpha\|_{mh} \geq \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle \varphi_c, \alpha \rangle = 1 \right\},$$

where we understand that $\sup \emptyset = 0$.

To conclude the proof of [Theorem 1.3](#), we need one more result, which follows readily from the definitions and ensures that the Kronecker products introduced above are compatible with each other:

Proposition 5.3. *For every $\varphi_c \in H_{cb}^n(X, W)$, $\alpha \in H_n(X, W)$ we have*

$$\langle H^n(\rho_b^*)(\varphi_c), \alpha \rangle = \langle \varphi_c, H_n(\iota_*)(\alpha) \rangle.$$

Proof of Theorem 1.3. Suppose that (X, W) is a good CW-pair. We already know that the map $H_*(\iota_*) : H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is a norm-nonincreasing isomorphism, so we are left to show that $\|H_*(\iota_*)(\alpha)\|_{\text{mh}} \geq \|\alpha\|_1$ for every $\alpha \in H_*(X, W)$.

However, for every $\alpha \in H_n(X, W)$ we have

$$\begin{aligned} \|H_n(\iota_*)(\alpha)\|_{\text{mh}} &\geq \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle \varphi_c, H_n(\iota_*)(\alpha) \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle H^n(\rho_b^*)(\varphi_c), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(X, W), \langle \varphi, \alpha \rangle = 1 \right\} \\ &= \|\alpha\|_1, \end{aligned}$$

where the inequality is due to Proposition 5.2, the first equality to Proposition 5.3, the second equality to Theorem 1.7, and the last equality to Proposition 5.1. \square

Remark 5.4. Let (X, W) be any CW-pair. The arguments described in this section show that if $H^*(\rho_b^*) : H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ admits a norm-nonincreasing right inverse, then the map $H_*(\iota_*) : H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is an isometric isomorphism.

6. A comparison with Park’s seminorms

Park [2003] describes an algebraic foundation of relative bounded cohomology of pairs, both in the case of a pair of groups (G, A) equipped with a homomorphism $A \rightarrow G$ and in the case of a pair of path-connected topological spaces (X, W) equipped with a continuous map $W \rightarrow X$. However, recall from the Introduction that the seminorms considered by Park are quite different from the ones considered in this paper, which go back to [Gromov 1982]. In this section we investigate the relationships between our seminorms and the seminorms introduced in [Park 2003], proving in particular that there exist examples for which they are *not* isometric to each other.

6A. Park’s mapping cone for homology. Let (X, W) be a countable CW-pair, where both X and W are connected, and let us suppose that the inclusion $i : W \hookrightarrow X$ induces an injective map on the fundamental groups (several considerations here below also hold without this last assumption, but this is not relevant to our purposes). We also denote by $i_* : C_*(W) \rightarrow C_*(X)$ the map induced by the inclusion i . The homology mapping cone complex of (X, W) is the complex

$$(C_*(W \rightarrow X), \bar{d}_*) = (C_*(X) \oplus C_{*-1}(W), \bar{d}_*),$$

where

$$\begin{aligned} \bar{d}_n : C_n(X) \oplus C_{n-1}(W) &\rightarrow C_{n-1}(X) \oplus C_{n-2}(W) \\ (u_n, v_{n-1}) &\mapsto (d_n u_n + i_{n-1}(v_{n-1}), -d_{n-1} v_{n-1}), \end{aligned}$$

and d_* denotes the usual differential both of $C_*(X)$ and of $C_*(W)$. The homology of the mapping cone $(C_*(W \rightarrow X), \bar{d}_*)$ is denoted by $H_*(W \rightarrow X)$. For every $\omega \in [0, \infty)$ one can endow $C_*(W \rightarrow X)$ with the L^1 -norm

$$\|(u, v)\|_1(\omega) = \|u\|_1 + (1 + \omega)\|v\|_1,$$

which induces in turn a seminorm (still denoted by $\|\cdot\|_1(\omega)$) on $H_*(W \rightarrow X)$ (in fact, in [Park 2004] the case $\omega = \infty$ is also considered, but this is not relevant to our purposes).

As observed in [Park 2004], the chain map

$$(12) \quad \beta_* : C_*(W \rightarrow X) \rightarrow C_*(X, W) = C_*(X)/C_*(W), \quad \beta_*(u, v) = [u]$$

induces an isomorphism

$$H_*(\beta_*) : H_*(W \rightarrow X) \rightarrow H_*(X, W).$$

The explicit description of β_* implies that

$$\|H_*(\beta_*)(\alpha)\|_1 \leq \|\alpha\|_1(0) \leq \|\alpha\|_1(\omega)$$

for every $\alpha \in H_*(W \rightarrow X)$, $\omega \in [0, \infty)$.

6B. Park's mapping cone for bounded cohomology. We define the mapping cone for bounded cohomology as the (topological) dual of the mapping cone for homology. More precisely, we fix $\omega \in [0, \infty)$, and endow $C_*(W \rightarrow X)$ with the norm $\|\cdot\|_1(\omega)$. It is readily seen that the topological dual of $C_n(W \rightarrow X) = C_n(X) \oplus C_{n-1}(W)$ is isometrically isomorphic to the space

$$C_b^n(W \rightarrow X) = C_b^n(X) \oplus C_b^{n-1}(W)$$

endowed with the L^∞ -norm $\|\cdot\|_\infty(\omega)$ defined by

$$\|(f, g)\|_\infty(\omega) = \max\{\|f\|_\infty, (1 + \omega)^{-1}\|g\|_\infty\}.$$

In other words, the pairing

$$C_b^*(W \rightarrow X) \times C_*(W \rightarrow X) \rightarrow \mathbb{R}, \quad ((f, f'), (a, a')) \mapsto f(a) - f'(a')$$

realizes $C_b^*(W \rightarrow X)$ as the topological dual of $C_*(W \rightarrow X)$, and an easy computation shows that the norm $\|\cdot\|_\infty(\omega)$ just introduced on $C_b^*(W \rightarrow X)$ coincides with the operator norm (with respect to the norm $\|\cdot\|_1(\omega)$ fixed on $C_*(W \rightarrow X)$). Therefore, if $i^* : C_b^*(X) \rightarrow C_b^*(W)$ is the cochain map induced by the inclusion, then the

cohomology mapping cone complex of (X, W) is the complex $(C_b^*(W \rightarrow X), \bar{\delta}^*)$, where $\bar{\delta}^*$ is defined as the dual map of \bar{d}_* , and admits therefore the following explicit description (see [Park 2003] for the details):

$$\begin{aligned} \bar{\delta}^n : C_b^n(X) \oplus C_b^{n-1}(W) &\rightarrow C_b^{n+1}(X) \oplus C_b^n(W) \\ (f_n, g_{n-1}) &\mapsto (\delta^n f_n, -i^n(f_n) - \delta^{n-1} g_{n-1}) \end{aligned}$$

(here δ^* denotes the usual differential both of $C_b^*(X)$ and of $C_b^*(W)$). The cohomology of the complex $(C_b^*(W \rightarrow X), \bar{\delta}^*)$ is denoted by $H_b^*(W \rightarrow X)$. Just as in the case of homology, the L^∞ -norm $\|\cdot\|_\infty(\omega)$ on $C_b^n(W \rightarrow X)$ descends to a seminorm (still denoted by $\|\cdot\|_\infty(\omega)$) on $H_b^*(W \rightarrow X)$.

The chain map

$$\beta^* : C_b^*(X, W) \rightarrow C_b^*(W \rightarrow X), \quad \beta^*(f) = (f, 0)$$

is the dual of the chain map β_* introduced in Equation (12) above, and induces an isomorphism

$$H^*(\beta^*) : H_b^*(X, W) \rightarrow H_b^*(W \rightarrow X)$$

such that

$$\|H^*(\beta^*)(\varphi)\|_\infty(\omega) \leq \|H^*(\beta^*)(\varphi)\|_\infty(0) \leq \|\varphi\|_\infty$$

for every $\varphi \in H_b^*(X, W)$, $\omega \in [0, \infty)$. More precisely:

Theorem 6.1 [Park 2003, Theorem 4.6]. *For every $n \in \mathbb{N}$, the isomorphism $H^n(\beta^*)$ is such that*

$$\frac{1}{n+2} \|\varphi\|_\infty \leq \|H^n(\beta^*)(\varphi)\|_\infty(0) \leq \|\varphi\|_\infty \quad \text{for every } \varphi \in H_b^n(X, W).$$

It is asked in [Park 2003] whether $H^*(\beta^*)$ is actually an isometry or not. We show in Proposition 6.4 below that there exist examples for which $H^*(\beta^*)$ is *not* an isometry.

6C. Mapping cones and duality. In the previous subsection we have seen that, for every $\omega \geq 0$, the normed space $(C_b^*(W \rightarrow X), \|\cdot\|_\infty(\omega))$ coincides with the topological dual of the normed space $(C_*(W \rightarrow X), \|\cdot\|_1(\omega))$. We may therefore apply the duality result proved in [Löh 2007, Theorem 3.14], and obtain the following:

Proposition 6.2. *If the map*

$$H^*(\beta^*) : (H_b^*(X, W), \|\cdot\|_\infty) \rightarrow (H_b^*(W \rightarrow X), \|\cdot\|_\infty(\omega))$$

is an isometric isomorphism, then

$$\|H_*(\beta_*)(\alpha)\|_1 = \|\alpha\|_1(\omega)$$

for every $\alpha \in H_(X, W)$.*

6D. An explicit example. Let M be a compact, connected, oriented manifold with connected boundary, and suppose that the inclusion $i : \partial M \rightarrow M$ induces an injective homomorphism $i_* : \pi_1(\partial M) \rightarrow \pi_1(M)$.

We denote by $[M, \partial M]$ the (real) fundamental class in $H_n(M, \partial M)$ and we set

$$[\partial M \rightarrow M] = H_n(\beta_*)^{-1}([M, \partial M]) \in H_n(\partial M \rightarrow M).$$

The L^1 -seminorm $\|[M, \partial M]\|_1$ of the real fundamental class of M is usually known as the *simplicial volume* of M , and it is denoted simply by $\|M\|$. Similarly, the L^1 -seminorm of the real fundamental class $[\partial M] \in H_{n-1}(\partial M)$ is the simplicial volume of ∂M , and it is denoted by $\|\partial M\|$.

Lemma 6.3. *We have*

$$\|[\partial M \rightarrow M]\|_1(\omega) \geq \|M\| + (1 + \omega)\|\partial M\|.$$

Proof. It is shown in [Park 2004] that, if $\alpha \in C_i(M)$ is such that $d_i\alpha \in C_{i-1}(\partial M)$ (so that α defines an element $[\alpha] \in H_i(M, \partial M)$), then

$$H_i(\beta_*)^{-1}([\alpha]) = [(\alpha, -d_i\alpha)].$$

Therefore, if $\alpha \in C_n(M)$ is a representative of the fundamental class $[M, \partial M] \in H_n(M, \partial M)$, then $(\alpha, -d_n\alpha)$ is a representative of $[\partial M \rightarrow M] \in H_n(\partial M \rightarrow M)$. If (α', γ) is any other representative of such a class, then by definition of mapping cone there exist $x \in C_{n+1}(M)$ and $y \in C_n(\partial M)$ such that:

$$\alpha - \alpha' = d_{n+1}x + i_n(y) \quad \text{and} \quad \gamma + d_n\alpha = -d_n y.$$

These equalities readily imply that $[\alpha'] = [\alpha]$ in $H_n(M, \partial M)$ and $[\gamma] = [-d_n\alpha]$ in $H_{n-1}(\partial M)$. As a consequence, since $d_n\alpha$ is a representative of the fundamental class of ∂M , we have $\|\alpha'\|_1 \geq \|[\alpha']\|_1 = \|M\|$ and $\|\gamma\|_1 \geq \|[\gamma]\|_1 = \|\partial M\|$, whence

$$\|(\alpha', \gamma)\|_1(\omega) \geq \|M\| + (1 + \omega)\|\partial M\|.$$

The conclusion follows from the fact that (α', γ) is an arbitrary representative of $[\partial M \rightarrow M]$. \square

Proposition 6.4. *Let M be a compact connected oriented hyperbolic n -manifold with connected geodesic boundary. Then, for every $\omega \in [0, \infty)$ the isomorphism*

$$H^n(\beta^*) : (H_b^n(M, \partial M), \|\cdot\|_\infty) \rightarrow (H_b^n(\partial M \rightarrow M), \|\cdot\|_\infty(\omega))$$

is not isometric.

Proof. It is well-known that the inclusion $\partial M \hookrightarrow M$ induces an injective map on fundamental groups. Moreover, since ∂M is a closed oriented hyperbolic $(n-1)$ -manifold, we also have $\|\partial M\| > 0$. By Proposition 6.2, if $H^n(\beta^*)$ were an isometry

we would have $\|[\partial M \rightarrow M]\|_1(\omega) = \|[M, \partial M]\|_1 = \|M\|$, and this contradicts Lemma 6.3. \square

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