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ON OVERTWISTED, RIGHT-VEERING OPEN BOOKS

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ON OVERTWISTED, RIGHT-VEERING OPEN BOOKS

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We exhibit infinitely many overtwisted, right-veering, non-destabilizable open books, thus providing infinitely many counterexamples to a conjecture of Honda, Kazez and Matić. The page of all our open books is a four-holed sphere and the underlying 3-manifolds are lens spaces.

1. Introduction

The purpose of this note is to construct infinitely many counterexamples to a conjecture of Honda, Kazez and Matić from [Honda et al. 2009]. For the basic notions of contact topology not recalled below we refer the reader to [Etnyre 2003; Geiges 2008].

Let S be a compact, oriented surface with boundary and $\mathrm{Map}(S, \partial S)$ the group of orientation-preserving diffeomorphisms of S that restrict to ∂S as the identity, up to isotopies fixing ∂S pointwise. An *open book* (also known as an *abstract open book*) is a pair (S, Φ) where S is a surface as above and $\Phi \in \mathrm{Map}(S, \partial S)$. Giroux [2002] introduced a fundamental operation of *stabilization* $(S, \Phi) \to (S', \Phi')$ on open books, and proved the existence of a 1-1 correspondence between the set of open books modulo stabilization and the set of contact 3-manifolds modulo isomorphism (see, for example, [Etnyre 2006] for details). Honda, Kazez and Matić [Honda et al. 2007] showed that a contact 3-manifold is tight if and only if it corresponds to an equivalence class of open books (S, Φ) all of whose monodromies Φ are *right-veering* (in the sense of [Honda et al. 2007, Section 2]). In [Goodman 2005; Honda et al. 2007] it is also showed that every open book can be made right-veering after a sequence of stabilizations. Honda, Kazez and Matić [Honda et al. 2009] proved that when S is a holed torus, the contact structure corresponding to (S, Φ) is tight if and only if Φ is right-veering, and conjectured that a non-destabilizable right-veering open book corresponds to a tight contact 3-manifold. The Honda–Kazez–Matić conjecture was recently disproved by Lekili [2011], who produced a counterexample (S, Φ) with S equal to a four-holed sphere and whose underlying 3-manifold is the Poincaré homology sphere.

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Keywords: contact surgery, destabilizable diffeomorphisms, Giroux's correspondence, open books, overtwisted contact structures, right-veering diffeomorphisms.

We shall now describe our examples. Denote by $\delta_\gamma \in \mathrm{Map}(S, \partial S)$ the class of a positive Dehn twist along a simple closed curve $\gamma \subset S$.

Theorem 1.1. *Let S be an oriented four-holed sphere, and a, b, c, d, e the simple closed curves on S shown in Figure 1.*

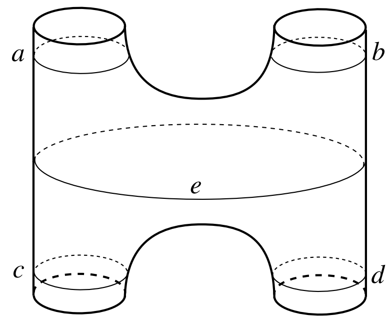

Figure 1. The four-holed sphere S.

Let $h, k \geq 1$ be integers. Define $\Phi_{h,k} := \delta_a^h \delta_b \delta_c \delta_d \delta_e^{-k-1} \in \mathrm{Map}(S, \partial S)$. Then

- *the underlying 3-manifold $Y_{(S,\Phi_{h,k})}$ is the lens space*
$$L((h+1)(2k-1)+2,\,(h+1)k+1);$$
- *the associated contact structure $\xi_{(S,\Phi_{h,k})}$ is overtwisted;*
- *$\Phi_{h,k}$ is right-veering;*
- *$(S, \Phi_{h,k})$ is not destabilizable.*

Warning: in the above statement we adopt the convention that the lens space $L(p, q)$ is the oriented 3-manifold obtained by performing a rational surgery along an unknot in S^3 with coefficient $-p/q$.

We prove Theorem 1.1 in Section 2. The proof can be outlined as follows. In Proposition 2.1 we use elementary arguments to determine a contact surgery presentation for the contact 3-manifold $(Y_{(S,\Phi_{h,k})}, \xi_{(S,\Phi_{h,k})})$, and in Corollary 2.2 we apply Proposition 2.1 and a few Kirby calculus moves to identify the underlying 3-manifold $Y_{(S,\Phi_{h,k})}$. In Proposition 2.3 we appeal to calculations from [Lekili 2011] to deduce that the contact Ozsváth–Szabó invariant of $\xi_{(S,\Phi_{h,k})}$ vanishes, and we conclude from the fact that $Y_{(S,\Phi_{h,k})}$ is a lens space that $\xi_{(S,\Phi_{h,k})}$ must be overtwisted. That $\Phi_{h,k}$ is right-veering in Lemma 2.4 follows directly from [Arıkan and Durusoy 2012, Theorem 4.3], but it can also be deduced by imitating the proof of [Lekili 2011, Theorem 1.2], that is, by applying [Honda et al. 2007, Corollary 3.4]. Finally, we use results from [Arıkan 2008; Lekili 2011] to conclude that $(S, \Phi_{h,k})$ is not destabilizable.

2. Proof of Theorem 1.1

Recall that every contact structure has a *contact surgery presentation*. We refer the reader to [Ding and Geiges 2004] for this fact and the basic properties of contact surgeries, and to [Lisca and Stipsicz 2004] for the use of the "front notation" in contact surgery presentations, in particular for the meaning of Figure 2 below.

Proposition 2.1. *For $h, k \geq 1$, the contact structure $\xi_{(S,\Phi_{h,k})}$ has the contact surgery presentation given by Figure 2.*

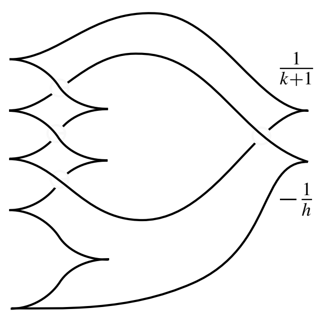

Figure 2. Contact surgery presentation for $\xi_{(S,\Phi_{h,k})}$, $h, k \geq 1$.

Proof. Figure 3 (a) represents an open book (A, f), where A is an annulus and f is a positive Dehn twist along the core of A. The associated contact 3-manifold is the standard contact 3-sphere (S^3, ξ_{st}), the annulus A can be viewed as the page of an open book decomposition of S^3, and the curve κ in the picture can be made Legendrian via an isotopy of the contact structure, in such a way that the contact framing on κ coincides with the framing induced on it by the page (see [Etnyre 2006, Figure 11]). The knot κ is the unique Legendrian unknot in (S^3, ξ_{st}) having Thurston–Bennequin invariant $\mathrm{tb}(\kappa) = -1$ and rotation number $\mathrm{rot}(\kappa) = 0$. A suitable choice of orientation for κ uniquely specifies its *negative* oriented Legendrian stabilization κ_-, which satisfies $\mathrm{tb}(\kappa_-) = -2$ and $\mathrm{rot}(\kappa_-) = -1$. As shown in [Etnyre 2006], κ_- can be realized as sitting on the page of a Giroux stabilization (A', f') of (A, f). This is illustrated in Figure 3 (b), assuming the orientation on κ was taken to be "counterclockwise" in Figure 3 (a). Finally, Figure 3 (c) shows an open book (S, f'') obtained by Giroux stabilizing (A', f') and containing both κ_- and $(\kappa_-)_-$ in S (κ_- was also given the "counterclockwise" orientation in Figure 3 (b)). Clearly (S, f'') still corresponds to (S^3, ξ_{st}), and it is well-known that κ_-, $(\kappa_-)_-$ are the two Legendrian knots illustrated in Figure 2 (when oriented "clockwise" in that picture). By definition, $\Phi_{h,k}$ is obtained by precomposing f'' with $k + 1$ negative Dehn twists along parallel copies of κ_- and h positive Dehn twists along parallel copies of $(\kappa_-)_-$. Moreover, if $m \neq 0$ is an integer, $\frac{1}{m}$-contact surgery along any Legendrian knot λ is equivalent to $\frac{m}{|m|}$-contact surgeries along $|m|$ Legendrian push-offs of λ [Ding and Geiges 2004]. Since page and contact framings coincide, and by [Etnyre 2006, Theorem 5.7] positive (negative, respectively) Dehn twists correspond to -1-contact surgeries ($+1$-contact surgeries, respectively), it is easy to check that the resulting contact structure is given by the contact surgery presentation of Figure 2. □

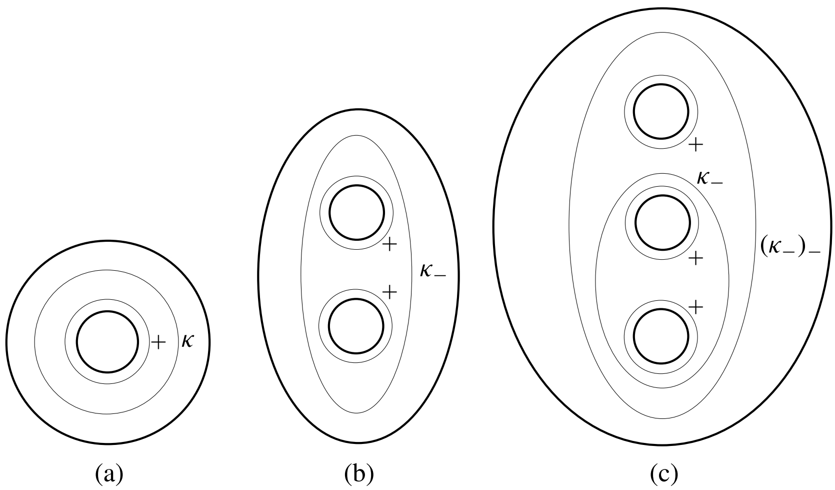

Figure 3. Determination of the contact surgery presentation.

Corollary 2.2. *For $h, k \geq 1$, the oriented 3-manifold underlying the open book $(S, \Phi_{h,k})$ is the lens space $L((h+1)(2k-1)+2, (h+1)k+1)$.*

Proof. Using the fact that the two Legendrian unknots illustrated in Figure 2 have Thurston–Bennequin invariants -2 and -3, it is easy to check that the topological surgery underlying Figure 2 is given by the first (upper left) picture of Figure 4. Two $+1$-blowups and two inverse slam-dunks give the second picture, while the

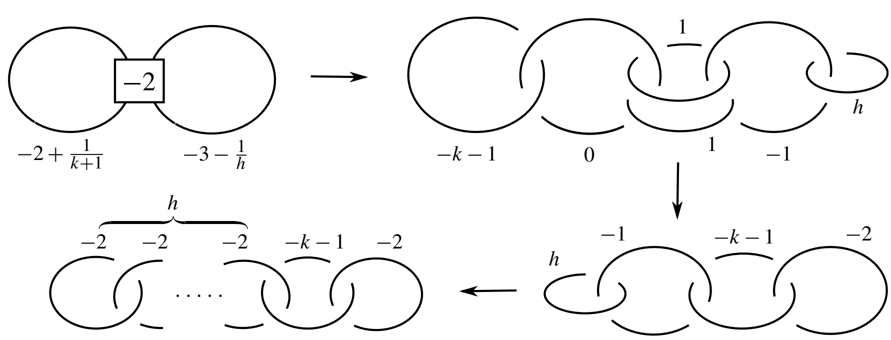

Figure 4. Determination of the underlying 3-manifold.

third picture is obtained from the second one by sliding the -1-framed knot over the 0-framed knot and then applying two $+1$-blow-downs. The last picture is obtained simply converting the h-framed unknot in the third picture into the string of -2-framed unknots via a sequence of -1-blowups and a final $+1$-blowdown. The last picture shows that the underlying 3-manifold $Y_{(S,\Phi_{h,k})}$ is obtained by performing a rational surgery on an unknot in S^3 with coefficient $-p/q$, where

$$\frac{p}{q} = 2 - \cfrac{1}{k+1-\cfrac{1}{2-\cfrac{1}{\ddots - \frac{1}{2}}}} = \frac{(h+1)(2k-1)+2}{(h+1)k+1}.$$

Therefore, according to our conventions $Y_{(S,\Phi_{h,k})}$ can be identified with the lens space $L((h+1)(2k-1)+2, (h+1)k+1)$. □

Proposition 2.3. *For $h, k \geq 1$, the contact structure $\xi_{(S,\Phi_{h,k})}$ is overtwisted.*

Proof. By [Giroux 2000; Honda 2000] a contact structure on a lens space is either overtwisted or Stein fillable. Moreover, Stein fillable contact structures have nonzero contact Ozsváth–Szabó invariant [Ozsváth and Szabó 2005]. Finally, [Lekili 2011, Theorem 1.3] immediately implies that the contact invariant of $(S, \Phi_{h,k})$ vanishes, therefore $\xi_{(S,\Phi_{h,k})}$ must be overtwisted. □

Lemma 2.4. *For $h, k \geq 1$, the diffeomorphism class*

$$\Phi_{h,k} = \delta_a^h \delta_b \delta_c \delta_d \delta_e^{-k-1} \in \mathrm{Map}(S, \partial S)$$

is right-veering.

Proof. The lemma follows immediately from the statement of Theorem 4.3 in [Arıkan and Durusoy 2012]. Alternatively, one can imitate the proof of Theorem 1.2 of [Lekili 2011]. Indeed, applying Corollary 3.4 from [Honda et al. 2007] to the monodromy $\Phi_1 = \delta_e^{-k-1}$ and a properly embedded arc $\gamma_{cd} \subset S$ disjoint from the curve e and connecting the components ∂_c and ∂_d of ∂S parallel to the curves c and d shows that $\Phi_2 = \delta_d \delta_e^{-k-1}$ is right-veering with respect to ∂_d. Another application of the corollary to Φ_2 and γ_{cd} shows that $\Phi_3 = \delta_c \delta_d \delta_e^{-k-1}$ is right-veering with respect to ∂_c. Moreover, since δ_c is right-veering with respect to ∂_c and the composition of right-veering diffeomorphisms is still right-veering [Honda et al. 2007], Φ_3 is right-veering with respect to ∂_d as well. Appying the corollary in the same way to Φ_3 and an arc connecting the components of ∂S parallel to the curves a and b yields the statement of the lemma. □

Proof of Theorem 1.1. Corollary 2.2, Proposition 2.3 and Lemma 2.4 establish the first three portions of the statement. Thus we only need to show that $(S, \Phi_{h,k})$ is not destabilizable for every $h, k \geq 1$. If $(S, \Phi_{h,k})$ were destabilizable, it would be a stabilization of an open book (S', Φ'), where S' is a three-holed sphere and $\Phi' = \tau_1^{a_1}\tau_2^{a_2}\tau_3^{a_3}$, where $a_i \in \mathbb{Z}$ and τ_i is a positive Dehn twist along a simple closed curve parallel to the i-th boundary components of S', $i = 1, 2, 3$. By [Arıkan 2008, Theorem 1.2], $\xi_{(S,\Phi_{h,k})}$ is tight if and only if $a_i \geq 0$, $i = 1, 2, 3$. Therefore, by Proposition 2.3 at least one of these exponents must be strictly negative. But the proof of Theorem 1.2 of [Lekili 2011] shows that when one of the a_i's is negative, any stabilization of (S', Φ') to an open book with page a four-holed sphere is not right-veering. This would contradict Lemma 2.4, therefore we conclude that $(S, \Phi_{h,k})$ cannot be destabilizable. □

Note added in proof: after the submission of the present paper the author was informed of unpublished work of A. Wand containing, in particular, a different proof of Proposition 2.3.

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