WEAKLY KRULL DOMAINS AND THE Composite Numerical Semigroup Ring $D + E[\Gamma^*]$

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Let $D \subseteq E$ be an extension of integral domains, $\Gamma$ a numerical semigroup with $\Gamma \subseteq \mathbb{N}_0$, $\Gamma^* = \Gamma \setminus \{0\}$ and $R = D + E[\Gamma^*]$. In this paper, we completely characterize when $R$ is a weakly Krull domain, an AWFD or a GWFD. We also prove that $R$ is never a WFD.

Introduction

We first review some preliminaries. Let $D$ be an integral domain with quotient field $\text{qf}(D)$ and let $F(D)$ denote the set of nonzero fractional ideals of $D$. Recall that the $v$-operation on $D$ is a star-operation on $F(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$, where $I^{-1} = \{x \in \text{qf}(D) \mid xI \subseteq D\}$. The $t$-operation on $D$ is a star-operation defined by $I \mapsto I_t := \bigcup \{J_v \mid J \subseteq I \text{ with } J \in F(D) \text{ finitely generated}\}$. An $I \in F(D)$ is said to be a $v$-ideal if $I_v = I$, and a $t$-ideal if $I_t = I$. A $v$-ideal $I$ is said to be of finite type if $I = J_v$ for some finitely generated fractional ideal $J$ of $D$. A $t$-ideal $M$ of $D$ is called a maximal $t$-ideal if $M$ is maximal among proper integral $t$-ideals of $D$. It is well known that maximal $t$-ideals are prime ideals. Let $t$-$\text{Max}(D)$ be the set of maximal $t$-ideals of $D$. Then $t$-$\text{Max}(D) \neq \emptyset$ if $D$ is not a field. An $I \in F(D)$ is said to be $t$-invertible if $(II^{-1})_t = D$; equivalently, $II^{-1} \not\subseteq M$ for each $M \in t$-$\text{Max}(D)$.

Let $T(D)$ be the abelian group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $I \ast J = (IJ)_t$, and let $\text{Inv}(D)$ and $\text{Prin}(D)$ be the subgroups of $T(D)$ consisting respectively of invertible fractional ideals of $D$ and nonzero principal fractional ideals of $D$. Then it is clear that $\text{Prin}(D) \subseteq \text{Inv}(D) \subseteq T(D)$. The $t$-class group of $D$ is an abelian group $\text{Cl}(D) = T(D)/\text{Prin}(D)$ and the Picard group $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$ is a subgroup of $\text{Cl}(D)$. The local $t$-class group $G(D)$ of $D$ is defined by $G(D) = \text{Cl}(D)/\text{Pic}(D)$.

Let $X^1(D)$ stand for the set of height-one prime ideals of $D$. We say that $D$ is a weakly Krull domain if $D = \bigcap_{P \in X^1(D)} D_P$ and this intersection has finite character, i.e., each nonzero element $d \in D$ is a unit in $D_P$ for all but a finite number of $P$’s in $X^1(D)$; $D$ is a weakly factorial domain (WFD) if every nonzero nonunit element of $D$ is a product of primary elements; $D$ is an almost weakly factorial domain.

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(AWFD) if for each nonzero nonunit element \(d \in D\), there exists a positive integer \(n = n(d)\) such that \(d^n\) is a product of primary elements; and \(D\) is a generalized weakly factorial domain (GWFD) if each nonzero prime ideal of \(D\) contains a primary element. (Recall that a nonzero nonunit \(d \in D\) is called a primary element of \(D\) if \((d)\) is a primary ideal of \(D\).) It is well known that

\[
\text{WFD} \Rightarrow \text{AWFD} \Rightarrow \text{GWFD} \Rightarrow \text{weakly Krull domain}
\]

and a weakly Krull domain has \(t\)-dimension one. (The \(t\)-dimension of \(D\), abbreviated \(t \dim(D)\), is the supremum of lengths of chains of prime \(t\)-ideals of \(D\). Hence \(t \dim(D) = 1\) if and only if each maximal \(t\)-ideal of \(D\) has height-one.) Also, it was shown in [Anderson and Zafrullah 1990, Theorem] that a weakly Krull domain \(D\) is a WFD if and only if \(\text{Cl}(D) = 0\), and in [Anderson et al. 1992, Theorem 3.4] that a weakly Krull domain \(D\) is an AWFD if and only if \(\text{Cl}(D)\) is torsion. We note that \(t \dim(D[\Gamma]) = t \dim(D[X])\) for any numerical semigroup \(\Gamma\) [Chang et al. 2012, Theorem 1.5].

Let \(\mathbb{N}_0\) (resp., \(\mathbb{Z}\)) be the set of nonnegative integers (resp., integers). A semigroup \(\Gamma\) is called a numerical semigroup if \(\Gamma\) is a subset of \(\mathbb{N}_0\) containing \(0\) and generates \(\mathbb{Z}\) as a group. It is known that if \(\Gamma\) is a numerical semigroup, then \(\Gamma\) is finitely generated and \(\mathbb{N}_0 \setminus \Gamma\) is a finite set. Hence there exists the largest nonnegative integer which is not contained in \(\Gamma\). This number is called the Frobenius number of \(\Gamma\) and is denoted by \(F(\Gamma)\).

Throughout this article, \(D \subseteq E\) denotes an extension of integral domains, \(q f(D)\) (resp., \(q f(E)\)) is the quotient field of \(D\) (resp., \(E\)), \(\overline{D}\) means the integral closure of \(D\), \(X\) is an indeterminate over \(E\), \(\Gamma\) is a numerical semigroup with \(\Gamma \subseteq \mathbb{N}_0\) and \(D[\Gamma]\) is the numerical semigroup ring of \(\Gamma\) over \(D\). Note that each element \(f \in D[\Gamma]\) is uniquely expressible in the form

\[
f = a_1X^{a_1} + \cdots + a_kX^{a_k},\]

where \(a_i \in D\) and \(\alpha_i \in \Gamma\) with \(\alpha_1 < \cdots < \alpha_k\). Let \(\Gamma^* = \Gamma \setminus \{0\}\), \(R = D + E[\Gamma^*]\), \(T = D + XE[X]\) and \(T_n = D + X^nE[X]\) for integers \(n \geq 2\), i.e., \(R = \{f \in E[\Gamma] \mid f(0) \in D\}\), \(T = \{f \in E[X] \mid f(0) \in D\}\) and \(T_n = R\) when \(\Gamma = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}\). Then \(D[\Gamma] \subseteq R \subseteq E[\Gamma]\) and \(T_{F(\Gamma)+1} \subseteq R \subseteq T \subseteq E[X]\). For an \(f \in q f(D)[\Gamma]\), \(c(f)\) means the fractional ideal of \(D\) generated by the coefficients of \(f\). If \(I\) is an ideal of \(D[\Gamma]\), then \(c(I)\) denotes the ideal of \(D\) generated by the coefficients of all the polynomials in \(I\).

In multiplicative ideal theory, the \(D + E[\Gamma^*]\) construction has been extensively studied by several authors for its interest in constructing examples with prescribed properties. As a special kind of pullbacks, this has become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in this construction.

Anderson et al. [2003a; 2006] (see also [Anderson and Chang 2007]) studied when the domains \(D[X^2, X^3]\), \(D + XE[X]\) and \(D + X^2E[X]\) are weakly Krull
domains, WFDs, AWFDs or GWFDs. In fact, they showed that $D[X^2, X^3]$ is a weakly Krull domain if and only if $D$ is a weakly Krull UMT-domain [Anderson et al. 2003a, Proposition 2.7]; if char$(D) \neq 0$, then $D[X^2, X^3]$ is an AWFD if and only if $D[X^2, X^3]$ is a GWFD [Anderson and Chang 2007, Corollary 2.11]; $D + XE[X]$ is a weakly Krull domain if and only if $D + X^2E[X]$ is a weakly Krull domain [Anderson et al. 2006, Theorem 4.3]; and $D + XE[X]$ is an AWFD if and only if $D + XE[X]$ is a GWFD [Anderson and Chang 2007, Corollary 2.10]. The main purpose of this paper is to determine how certain properties of $D$, $E$ and $\Gamma$ influence those of $R$, and vice versa. This also extends the results for the domains $D[X^2, X^3]$, $D + XE[X]$ and $D + X^2E[X]$ to any composite numerical semigroup ring $D + E[\Gamma^*]$.

In Section 1, we investigate weakly Krull domains, AWFDs and GWFDs in the context of numerical semigroup rings $D[\Gamma]$ which coincide with the domains $R = D + E[\Gamma^*]$ when $D = E$. We prove that $D[\Gamma]$ is a weakly Krull domain if and only if $D$ is a weakly Krull UMT-domain, and that if char$(D) \neq 0$, then $D[\Gamma]$ is an AWFD if and only if $D[\Gamma]$ is a GWFD, if and only if $D$ is an almost weakly factorial quasi-AGCD-domain, if and only if $D$ is a generalized weakly factorial quasi-AGCD-domain.

In Section 2, we study when the domain $R = D + E[\Gamma^*]$ is a weakly Krull domain, an AWFD or a GWFD, where $D \subseteq E$. We show that $R$ is a weakly Krull domain if and only if $T = D + XE[X]$ is a weakly Krull domain, and that if char$(E) \neq 0$, then $R$ is an AWFD if and only if $R$ is a GWFD, if and only if $T$ is an AWFD, if and only if $R$ is a GWFD. We also prove that $R$ is never a WFD.

1. Weakly Krull domains as numerical semigroup rings

In this section, we characterize when the numerical semigroup ring $D[\Gamma]$ is a weakly Krull domain, an AWFD or a GWFD.

The first two lemmas are well known for the general semigroup rings, but we include their proofs for the convenience of the reader.

**Lemma 1.1** [El Baghdadi et al. 2002, Lemma 2.3]. Let $D$ be an integral domain and $\Gamma$ be a numerical semigroup. The following statements hold for an $I \in \mathbf{F}(D)$:

1. $(ID[\Gamma])^{-1} = I^{-1}D[\Gamma]$.
2. $(ID[\Gamma])_v = I_vD[\Gamma]$.
3. $(ID[\Gamma])_t = I_tD[\Gamma]$.

**Proof.** (1) Since $(ID[\Gamma])(I^{-1}D[\Gamma]) \subseteq D[\Gamma]$, $I^{-1}D[\Gamma] \subseteq (ID[\Gamma])^{-1}$. Conversely, let $f \in (ID[\Gamma])^{-1}$. Then $fID[\Gamma] \subseteq D[\Gamma]$ and hence $c(f)I \subseteq D$. Hence $c(f) \subseteq I^{-1}$, and therefore $f \in c(f)D[\Gamma] \subseteq I^{-1}D[\Gamma]$. Thus the equality holds.

(2) By (1), $(ID[\Gamma])_v = ((ID[\Gamma])^{-1})^{-1} = (I^{-1}D[\Gamma])^{-1} = I_vD[\Gamma]$. 


(3) Let $f_1, \ldots, f_n$ be nonzero elements of $ID[\Gamma]$. Then we have

\[
((f_1, \ldots, f_n)D[\Gamma])_v \subseteq ((c(f_1), \ldots, c(f_n))D[\Gamma])_v
\]

\[
= (c(f_1), \ldots, c(f_n))_v D[\Gamma]
\]

\[
\subseteq I_tD[\Gamma]
\]

by (2), i.e., $(ID[\Gamma])_t \subseteq I_tD[\Gamma]$. For the reverse inclusion, let $J$ be a nonzero finitely generated subideal of $I$. Then \(J_vD[\Gamma] = (JD[\Gamma])_v \subseteq (ID[\Gamma])_t\) by (2). Hence \(I_tD[\Gamma] \subseteq (ID[\Gamma])_t\). Thus we have the desired equality. \(\Box\)

**Lemma 1.2** [Anderson and Chang 2005, Corollary 2.3]. Let $D$ be an integral domain, $\Gamma$ be a numerical semigroup and let $Q$ be a maximal $t$-ideal of $D[\Gamma]$ such that $Q \cap D \neq (0)$. Then $Q = (Q \cap D)D[\Gamma]$. In particular, $Q \cap D$ is a maximal $t$-ideal of $D$.

**Proof.** The containment $(Q \cap D)D[\Gamma] \subseteq Q$ is obvious. For the converse, it suffices to show that $c(Q) \subseteq Q$. Suppose to the contrary that $c(Q) \nsubseteq Q$. Then

\[
Q \nsubseteq c(Q)D[\Gamma].
\]

Since $Q$ is a maximal $t$-ideal of $D[\Gamma]$, $(c(Q)D[\Gamma])_t = D[\Gamma]$. Therefore $c(Q)_v = D$ by Lemma 1.1(3), and hence $c(f)_v = D$ for some $f \in Q$. Let $0 \neq d \in Q \cap D$ and choose $0 \neq g \in (d, f)^{-1}$. Then $gd \in D[\Gamma]$ and hence $g \in qf(D)[\Gamma]$. Also, we have $fg \in D[\Gamma]$. Hence it follows from [Gilmer 1992, Theorem 28.1] that

\[
c(g) \subseteq c(g)_v = (c(f)^{m+1}c(g))_v = (c(f^m)c(fg))_v = (fg)_v \subseteq D,
\]

where $m$ is the degree of $g$. So $g \in c(g)D[\Gamma] \subseteq D[\Gamma]$, which implies that $(d, f)^{-1} = D[\Gamma]$. This contradicts the fact that $Q$ is a maximal $t$-ideal of $D[\Gamma]$. Therefore $c(Q) \subseteq Q$, and thus $Q \subseteq (Q \cap D)D[\Gamma]$. The second assertion is an immediate consequence of Lemma 1.1(3). \(\Box\)

An integral domain $B$ is said to be a $UMT$-domain if every upper to zero (a nonzero prime ideal of $B[X]$ which contracts to zero in $B$) $Q$ of $B[X]$ is a maximal $t$-ideal (equivalently, is $t$-invertible). Now, we give the numerical semigoup ring version of [Anderson et al. 1993, Proposition 4.11].

**Theorem 1.3.** Let $D$ be an integral domain and $\Gamma$ be a numerical semigroup with $\Gamma \nsubseteq \mathbb{N}_0$. Then the following assertions are equivalent.

(1) $D[\Gamma]$ is a weakly Krull domain.

(2) $D[X]$ is a weakly Krull domain.

(3) $D$ is a weakly Krull UMT-domain.
Proof. (1) $\Rightarrow$ (3) Assume $D[\Gamma]$ is a weakly Krull domain. Then $t$-dim($D[\Gamma]$) = 1 [Anderson et al. 1992, Lemma 2.1]. Let $P$ be a prime $t$-ideal of $D$. Then $PD[\Gamma]$ is a prime $t$-ideal of $D[\Gamma]$ by Lemma 1.1(3); so $\text{ht}(P) = \text{ht}_{D[\Gamma]}(PD[\Gamma]) = 1$; so $t$-dim($D$) = 1. Since $t$-dim($D[\Gamma]$) = 1, we have $t$-dim($D[X]$) = 1 by [Chang et al. 2012, Theorem 1.5]. Therefore every upper to zero in $D[X]$ is a maximal $t$-ideal, and thus $D$ is a UMT-domain. Note that

$$D = \bigcap_{P \in X^1(D)} D_P$$

by [Kang 1989, Proposition 2.9]. To show that this intersection has finite character, let $d \in D \setminus \{0\}$. Since $D[\Gamma]$ is a weakly Krull domain, $d$ belongs to only finitely many height-one prime ideals of $D[\Gamma]$, and hence there exists only a finite number of height-one prime ideals of $D$ containing $d$. Thus $D$ is a weakly Krull domain.

(3) $\Rightarrow$ (1) Assume that $D$ is a weakly Krull UMT-domain and let $Q$ be a maximal $t$-ideal of $D[\Gamma]$ with $Q \cap D \neq \{0\}$. By Lemma 1.2, $Q = (Q \cap D)D[\Gamma]$ and $Q \cap D$ is a maximal $t$-ideal of $D$. Since $t$-dim($D$) = 1 [Anderson et al. 1992, Lemma 2.1], $\text{ht}(Q \cap D) = 1$; so $\text{ht}_{D[\Gamma]}Q \leq 2$ (cf. [Kaplansky 1970, Theorem 37]). If $\text{ht}_{D[\Gamma]}Q = 2$, then there exists a nonzero prime ideal $P \subsetneq Q$ which contracts to zero in $D$. Note that $P = M \cap D[\Gamma]$ for some prime ideal $M$ of $D[X]$ [Chang et al. 2012, Proposition 1.1]. Since $M \cap D = \{0\}$ and $D$ is a UMT-domain, $M$ is a maximal $t$-ideal of $D[X]$. Hence $P$ is a maximal $t$-ideal of $D[\Gamma]$ [Chang et al. 2012, Theorem 1.4]. This contradicts the choice of $P$. Thus $t$-dim($D[\Gamma]$) = 1. By [Kang 1989, Proposition 2.9], we have $D[\Gamma] = \bigcap_{Q \in X^1(D[\Gamma])} D[\Gamma]Q$. We claim that this intersection has finite character. Let $f \in D[\Gamma] \setminus \{0\}$ and set

$$\mathcal{F} = \{Q \in X^1(D[\Gamma]) \mid f \in Q\},$$

$$\mathcal{F}_1 = \{Q \in \mathcal{F} \mid Q \cap D \in X^1(D)\},$$

and

$$\mathcal{F}_2 = \{Q \in \mathcal{F} \mid Q \cap D = \{0\}\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. If $\mathcal{F}_1$ is an infinite set, then $c(f)$ belongs to infinitely many height-one prime ideals of $D$ by Lemma 1.2. This is absurd, because $D$ is a weakly Krull domain. Hence $\mathcal{F}_1$ is a finite set. Note that $qf(D)[\Gamma]$ is a one-dimensional Noetherian domain; so $qf(D)[\Gamma]$ is a weakly Krull domain. Hence $\mathcal{F}_2$ is also a finite set. Therefore $\mathcal{F}$ is a finite set. Thus $D[\Gamma]$ is a weakly Krull domain.

(2) $\Leftrightarrow$ (3) See [Anderson et al. 1993, Proposition 4.11].

Recall that if $D \subseteq E$ is an extension of integral domains, then $E$ is said to be a root extension of $D$ if for each $z \in E$, there is a positive integer $n = n(z)$ such that $z^n \in D$. A domain $B$ is called an almost Prüfer $v$-multiplication domain (APvMD) (resp., almost GCD-domain (AGCD-domain)) if for each $0 \neq a, b \in B$, there exists a positive integer $n = n(a, b)$ such that $(a^n, b^n)_v$ is $t$-invertible (resp., principal).
It is known that $B$ is a weakly Krull P$v$MD if and only if $B[X]$ is weakly Krull and $B$ is integrally closed [Anderson et al. 1993, Corollary 4.13]. We weaken the hypothesis and obtain the following result.

**Corollary 1.4.** Let $D$ be an integral domain and $\Gamma$ be a numerical semigroup.

1. $D$ is a weakly Krull AP$v$MD if and only if $D[\Gamma]$ is a weakly Krull domain and $D \subseteq \overline{D}$ is a root extension.

2. $D$ is an almost weakly factorial AGCD-domain if and only if $D[\Gamma]$ is a weakly Krull domain, $\text{Cl}(D)$ is torsion and $D \subseteq \overline{D}$ is a root extension.

**Proof.** (1) By [Li 2012, Theorem 3.8], a domain $B$ is an AP$v$MD if and only if $B$ is a UMT-domain and $B \subseteq B$ is a root extension. Thus the result follows from Theorem 1.3.

(2) By [Li 2012, Theorem 3.1], a domain $B$ is an AGCD-domain if and only if $B$ is an AP$v$MD and $\text{Cl}(B)$ is torsion. Also, by [Anderson et al. 1992, Theorem 3.4], $B$ is an AWFD if and only if $B$ is a weakly Krull domain and $\text{Cl}(B)$ is torsion. Thus the result is an immediate consequence of Theorem 1.3 and (1). □

Let $S$ be a saturated multiplicative subset of a domain $B$ and let $N(S) = \{0 \neq b \in B \mid (b, s)_v = B \text{ for all } s \in S\}$ be the $m$-complement of $S$. We say that $S$ is an almost splitting set if for each $0 \neq b \in B$, there exists a positive integer $n = n(b)$ such that $b^n = st$ for some $s \in S$ and $t \in N(S)$. Following [Anderson and Chang 2007], $B$ is called a quasi-AGCD-domain if $B \setminus \{0\}$ is an almost splitting set in $B[X]$. It was shown that if $B$ is integrally closed, then the notion of quasi-AGCD-domains coincides with that of AGCD-domains [Chang 2005, Proposition 2.6]. The next corollary characterizes when the numerical semigroup ring $D[\Gamma]$ is an AWFD or a GWFD.

**Corollary 1.5.** Let $D$ be an integral domain with $\text{char}(D) \neq 0$ and $\Gamma$ be a numerical semigroup with $\Gamma \subseteq \mathbb{N}_0$. Then the following conditions are equivalent.

1. $D[\Gamma]$ is an AWFD.
2. $D[\Gamma]$ is a GWFD.
3. $D[X]$ is an AWFD.
4. $D[X]$ is a GWFD.
5. $D$ is an almost weakly factorial quasi-AGCD-domain.
6. $D$ is a generalized weakly factorial quasi-AGCD-domain.
7. $D$ is a weakly Krull quasi-AGCD-domain.

**Proof.** Let $\text{char}(D) = p$.

(1) $\Rightarrow$ (2) This is well known.
(1) $\iff$ (3) By [Anderson et al. 1992, Theorem 3.4], an integral domain $B$ is an AWFD if and only if $B$ is a weakly Krull domain and $\text{Cl}(B)$ is torsion, and by Theorem 1.3, $D[\Gamma]$ is a weakly Krull domain if and only if $D[X]$ is a weakly Krull domain. By [Chang et al. 2012, Lemma 2.7], $\text{Pic}(qf(D)[\Gamma])$ is torsion if and only if $\text{char}(D) \neq 0$. Since $\text{Cl}(D[\Gamma]) = \text{Cl}(D[X]) \oplus \text{Pic}(qf(D)[\Gamma])$ [Anderson and Chang 2004, Theorem 5], $\text{Cl}(D[\Gamma])$ is torsion if and only if $\text{Cl}(D[X])$ is torsion and $\text{char}(D) \neq 0$. Thus this equivalence follows from these facts.

(4) $\Rightarrow$ (2) By [Anderson et al. 2003b, Theorem 2.2], a domain $B$ is a GWFD if and only if $t\text{-dim}(B) = 1$ and for each $P \in X^1(B)$, $P = \sqrt{bB}$ for some $b \in B$. Assume that $D[X]$ is a GWFD and let $P \in X^1(D[\Gamma])$. Since $t\text{-dim}(D[\Gamma]) = t\text{-dim}(D[X]) = 1$ [Chang et al. 2012, Theorem 1.5], it suffices to show that $P = \sqrt{fD[\Gamma]}$ for some $f \in D[\Gamma]$. If $P \cap D \neq (0)$, then $P = (P \cap D)[\Gamma]$ by Lemma 1.2. Since $D[X]$ is a GWFD, $(P \cap D)[X] = \sqrt{aD[X]}$ for some $a \in P \cap D$. It is easy to see that $P = \sqrt{fD[\Gamma]}$. Next, suppose that $P \cap D = (0)$. Then there exists a prime $t$-ideal of $D[X]$ such that $P = Q \cap D[\Gamma]$ [Chang et al. 2012, Theorem 1.5]. Since $D[X]$ is a GWFD, $Q = \sqrt{fD[X]}$ for some $f \in D[X]$. Also, since $\text{char}(D) = p > 0$, there exists a positive integer $n$ such that $f^{p^n} \in D[\Gamma]$. An easy calculation shows that $P = \sqrt{f^{p^n}D[\Gamma]}$. Thus $D[\Gamma]$ is a GWFD.

(2) $\Rightarrow$ (4) This direction is an easy modification of the proof of (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (5) See [Anderson and Chang 2007, Corollary 2.9].

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) These implications are obvious.

(7) $\Rightarrow$ (1) Assume that $D$ is a weakly Krull quasi-AGCD-domain. Then $D$ is a UMT-domain and $\text{Cl}(D[X])$ is torsion [Anderson and Chang 2007, Theorem 2.4]. Hence $D[\Gamma]$ is a weakly Krull domain by Theorem 1.3. Also, it follows from [Anderson and Chang 2004, Theorem 5; Chang et al. 2012, Lemma 2.7] that $\text{Cl}(D[\Gamma])$ is torsion. Thus $D[\Gamma]$ is an AWFD [Anderson et al. 1992, Theorem 3.4].

We end this section by noting that $D[\Gamma]$ is never a WFD. We also show that $D[\Gamma]$ need not be an AWFD if $\text{char}(D) = 0$.

**Remark 1.6.** (1) Let $B$ be an integral domain with quotient field $K$. In [Gilmer and Martin 1990, Theorem 7], Gilmer and Martin showed that if $B$ is a seminormal domain and $B + X^n B[X] \subseteq B[\Gamma]$, then $\text{Pic}(B[\Gamma]) = \text{Pic}(B) \oplus (W_n/L)$, where $L \subseteq W_n$ are the subgroups of the group $U(B[X]/X^n B[X])$ of units of $B[X]/X^n B[X]$ defined by $W_n = \{1 + Xf + X^n B[X] \mid f \in B[X]\}$ and $L = \{1 + Xf + X^n B[X] \mid 1 + Xf \in B[\Gamma]\}$. Note that $\text{Cl}(B[\Gamma]) = \text{Cl}(B[X]) \oplus \text{Pic}(K[\Gamma])$ [Anderson and Chang 2004, Theorem 5] and that $B$ is a WFD if and only if $B$ is a weakly Krull domain and $\text{Cl}(B) = 0$ [Anderson and Zafrullah 1990, Theorem]. If $D[\Gamma]$ is a WFD, then $\text{Cl}(D[\Gamma]) = 0$, and hence $\text{Pic}(qf(D)[\Gamma]) = 0$. Therefore $W_n = L$;
so \(1 + X + X^n qf(D)[X] \in L\), which implies that \(1 \in \Gamma\). Thus, if \(\Gamma\) is a proper numerical semigroup, then \(D[\Gamma]\) is never a WFD.

(2) If \(D[\Gamma]\) is an AWFD, then \(\text{Cl}(D[\Gamma])\) is torsion [Anderson et al. 1992, Theorem 3.4]; so \(\text{Pic}(qf(D)[\Gamma])\) is torsion [Anderson and Chang 2004, Theorem 5]. Hence \(\text{char}(D) \neq 0\) [Chang et al. 2012, Lemma 2.7]. This shows that the condition that \(\text{char}(D) \neq 0\) is essential in Corollary 1.5.

(3) It is known that a generalized unique factorization domain (GUFD) is a weakly factorial GCD-domain [Anderson et al. 1995, Theorem 7], and hence integrally closed. (See [Anderson et al. 1995] for the definition and some characterizations of a GUFD.) Thus, if \(\Gamma\) is a numerical semigroup with \(\Gamma \not\subseteq \mathbb{N}_0\), then \(D[\Gamma]\) is not a GUFD by (1). In fact, \(D[\Gamma]\) is not integrally closed; so \(D[\Gamma]\) is never a GUFD.

2. Weakly Krull domains and the ring \(D + E[\Gamma^*]\) when \(D \not\subseteq E\)

For a domain \(A\), \(\text{Spec}(A)\) stands for the set of prime ideals of \(A\). Assume that \(D \not\subseteq E\) is an extension of integral domains, \(\Gamma\) is a numerical semigroup with \(\Gamma \not\subseteq \mathbb{N}_0\) and let \(R = D + E[\Gamma^*]\), \(T = D + XE[X]\), \(T_n = D + X^n E[X]\) and \(\Delta_n = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}\) for integers \(n \geq 2\). Note that \(D[\Gamma] \not\subseteq R \not\subseteq T\) and \(T_n \not\subseteq T\). In this section, we characterize when the domains \(R\) and \(T_n\) are weakly Krull domains, AWFDs or GWFDs. To do this, we need two lemmas.

**Lemma 2.1.** Let \(R = D + E[\Gamma^*]\) and \(T = D + XE[X]\). If \(Q\) is a prime ideal of \(R\), then there exists a unique prime ideal of \(T\) lying over \(Q\). Thus the natural map \(\phi: \text{Spec}(T) \to \text{Spec}(R),\) given by \(P \mapsto P \cap R\), is an order-preserving bijection. In particular, \(\text{ht}_T(XE[X]) = \text{ht}_R(E[\Gamma^*])\).

**Proof.** Let \(Q\) be a prime ideal of \(R\). Since \(T\) is an integral extension of \(R\), there exists a prime ideal \(P\) of \(T\) such that \(Q = P \cap R\) [Kaplansky 1970, Theorem 44]. Note that \(E[\Gamma^*] \subseteq Q\) if and only if \(XE[X] \subseteq P\). If \(E[\Gamma^*] \subseteq Q\), then \(P\) is the unique prime ideal of \(T\) lying over \(Q\) because \(R/XE[X] \cong D \cong R/E[\Gamma^*]\). If \(E[\Gamma^*] \not\subseteq Q\), then \(X^{F(\Gamma)+1} f \not\in Q\) for some \(f \in E[X]\); so

\[g = \frac{X^{F(\Gamma)+1} f g}{X^{F(\Gamma)+1} f} \in R_Q\]

for any \(g \in T\). Hence \(T_Q R_Q \cap T = R_Q\). Thus \(Q R_Q \cap T\) is the unique prime ideal of \(T\) lying over \(Q\).

Let \(n\) be an integer \(\geq 2\). Then it is clear that if \(\Gamma = \Delta_n\), then \(R = T_n\). Hence Lemma 2.1 also shows that \(\text{ht}_T(XE[X]) = \text{ht}_{T_n}(X^n E[X])\).

**Remark 2.2.** Let \(\Gamma = \{\alpha_1, \ldots, \alpha_n\} \cup \Delta_{F(\Gamma)+1}\) with \(1 < \alpha_1 < \cdots < \alpha_n < F(\Gamma) + 1\) and \(R = D + E[\Gamma^*]\).
(1) Let \( g \in (R : E[\Gamma^*]) \). Then \( gE[\Gamma^*] \subseteq R \); hence for each \( \alpha \in \Gamma^* \), \( g x^\alpha = a_\alpha + f_\alpha \) for some \( a_\alpha \in D \) and \( f_\alpha \in E[\Gamma^*] \). Therefore \( g x^{\alpha + F(\Gamma)} = (a_\alpha + f_\alpha) x^{F(\Gamma)} \in R \), which means that \( a_\alpha = 0 \). Hence \( g x^\alpha = f_\alpha \in E[\Gamma^*] \), and so \( g \in \bigcap_{\alpha \in \Gamma^*} \{ f \mid f \in E[\Gamma^*] \} \). The reverse containment is obvious. Thus we have

\[
(R : E[\Gamma^*]) = \bigcap_{\alpha \in \Gamma^*} \left\{ \frac{1}{x^\alpha} f \mid f \in E[\Gamma^*] \right\}.
\]

(2) It is clear that \( E[\Gamma] \subseteq (R : E[\Gamma^*]) \) because \( X^{F(\Gamma)} \in (R : E[\Gamma^*]) \setminus E[\Gamma] \). Let \( g \in (R : E[\Gamma^*]) \). Then \( X^{F(\Gamma)+1} g \in R \); so we can write

\[
X^{F(\Gamma)+1} g = \sum_{i=0}^{n} g_i X^{\alpha_i} + X^{F(\Gamma)+1} h
\]

for some \( g_i \in E \) and \( h \in E[X] \). (For the sake of convenience, set \( \alpha_0 = 0 \).) Fix a \( k \in \{1, \ldots, n\} \). Then we have \( X^{2F(\Gamma)+1} = \sum_{i=0}^{k-1} g_i X^{F(\Gamma)+1} - \sum_{i=k+1}^{n} g_i X^{\alpha_i} - \sum_{i=k+1}^{n} g_i X^{\alpha_i - 1} + h \in R \); so \( g_k = 0 \) for all \( k = 1, \ldots, n \). Also, we have

\[
X^{F(\Gamma)+2} = g_0 X + X^{F(\Gamma)+2} \in R; \text{ so } g_0 = 0. \text{ Therefore } X^{F(\Gamma)+1} g = X^{F(\Gamma)+1} + h,
\]

and hence \( g \in E[X] \). Thus \( E[\Gamma] \subseteq (R : E[\Gamma^*]) \subseteq E[X] \). In particular, if \( \Gamma = \Delta F(\Gamma)+1 \), then \( E[X] \subseteq (R : E[\Gamma^*]) \); so \( (R : E[\Gamma^*]) = E[X] \).

(3) Lemma 4.2 of [Anderson et al. 2006] cannot be extended to any proper numerical semigroup, i.e., it may happen that \( (R : E[\Gamma^*]) \not\subseteq E[X] \) for some \( \Gamma \not\subseteq \mathbb{N}_0 \). For instance, if \( \Gamma = \{2\} \cup \Delta_4 \), then \( X \in E[X] \setminus (R : E[\Gamma^*]) \).

**Lemma 2.3.** The following statements hold for \( R = D + E[\Gamma^*] \).

1. \( E[\Gamma^*] \) is a prime \( t \)-ideal of \( R \).
2. \( E[\Gamma^*] \) is a maximal \( t \)-ideal of \( R \) if and only if \( qf(D) \cap E = D \).

**Proof.** (1) Let \( \Gamma = \{\alpha_1, \ldots, \alpha_k\} \cup \Delta F(\Gamma)+1 \) such that \( 0 < \alpha_1 < \cdots < \alpha_k < F(\Gamma)+1 \). Since \( R/E[\Gamma^*] \cong D \), \( E[\Gamma^*] \) is a prime ideal of \( R \). It suffices to show that \( E[\Gamma^*] \) is a \( v \)-ideal of \( R \), because each \( v \)-ideal is a \( t \)-ideal.

**Case 1.** \( \{\alpha_1, \ldots, \alpha_k\} \) is empty. In this case, \( (R : E[\Gamma^*]) = E[X] \) by Remark 2.2(2); so we need to show that \( (R : E[X]) = E[\Gamma^*] \). It is clear that \( E[\Gamma^*] \subseteq (R : E[X]) \). For the converse, let \( f \in (R : E[X]) \). Then \( f \in E[X] \subseteq R \). Since \( 1 \in E[X] \), \( f \in E \). Also, since \( X \in E[X] \), \( f(0) = 0 \); so \( f \in E[\Gamma^*] \).

**Case 2.** \( \{\alpha_1, \ldots, \alpha_k\} \) is nonempty. Deny the conclusion, and then there exists a polynomial \( g = g_0 + \sum_{i=1}^{k} g_i X^{\alpha_i} + \sum_{i=F(\Gamma)+1}^{l} g_i X^i \in (E[\Gamma^*])_v \setminus E[\Gamma^*] \). Hence \( g(\Gamma : E[\Gamma^*]) \not\subseteq R \). Let \( f \in (R : E[\Gamma^*]) \). Then \( f \in E[X] \) by Remark 2.2(2); so we can write \( f = \sum_{i=0}^{m} f_i X^i \). Note that

\[
f g = f_0 g_0 + \sum_{i=1}^{\alpha_1} f_i X^i + (f_0 g_{\alpha_1} + f_{\alpha_1} g_0) X^{\alpha_1} + X^{\alpha_1+1} h_1
\]
for some \( h_1 \in E[X] \). Since \( fg \in R \) and \( g_0 \neq 0 \), \( f_1 = \cdots = f_{\alpha_1 - 1} = 0 \); so \( f = f_0 + \sum_{i=\alpha_1}^{m} f_i X^i \). Note that \( 2\alpha_1 \in \Gamma^* \); so \( 2\alpha_1 \geq F(\Gamma) + 1 \) or \( 2\alpha_1 = \alpha_p \) for some \( p \in \{2, \ldots, k\} \). If \( 2\alpha_1 \geq F(\Gamma) + 1 \), then we have

\[
fg = f_0 g_0 + (f_0 g_{\alpha_1} + f_{\alpha_1} g_0) X^{\alpha_1} + g_0 \sum_{i=\alpha_1+1}^{\alpha_2-1} f_i X^i + (f_0 g_{\alpha_2} + f_{\alpha_2} g_0) X^{\alpha_2} + X^{\alpha_2+1} h_2
\]

for some \( h_2 \in E[X] \). Again, since \( fg \in R \), \( f_{\alpha_1 + 1} = \cdots = f_{\alpha_2 - 1} = 0 \). By repeating this process, we have \( f_i = 0 \) for all \( i \in \mathbb{N}_0 \setminus \Gamma \), and hence \( f \in R \). Therefore \( (R : E[\Gamma^*]) = R \). However, this is impossible because \( X^{F(\Gamma)} \in (R : E[\Gamma^*]) \setminus R \).

If \( 2\alpha_1 = \alpha_p \) for some \( p \in \{2, \ldots, k\} \), a simple modification of the proof of the previous case leads to the same conclusion because \( 2\alpha_l \geq F(\Gamma) + 1 \) for some \( l \leq k \).

In either case, \( E[\Gamma^*] \) is a \( v \)-ideal, and thus \( E[\Gamma^*] \) is a \( t \)-ideal of \( R \).

(2) This appears in [Lim 2012, Lemma 1.2].

Now, we are ready to give a necessary and sufficient condition for the domain \( R \) to be a weakly Krull domain.

**Theorem 2.4.** Let \( R = D + E[\Gamma^*] \), \( T = D + X E[X] \), \( T_n = D + X^n E[X] \) and \( \Delta_n = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\} \) for integers \( n \geq 2 \). Then the following statements are equivalent.

1. \( R \) is a weakly Krull domain.
2. \( T \) is a weakly Krull domain.
3. \( T_n \) is a weakly Krull domain.
4. \( X^n E[X] \) is a height-one maximal \( t \)-ideal of \( T_n \) and \( E[\Delta_n] \) is a weakly Krull domain.
5. \( E_{D\setminus\{0\}} \) is a field, \( qf(D) \cap E = D \) and \( E[X] \) is a weakly Krull domain.

**Proof.** (2) \( \Rightarrow \) (1) Let \( T \) be a weakly Krull domain. Let \( \Gamma = \{\alpha_1, \ldots, \alpha_k\} \cup \Delta_F(\Gamma)_{+1} \) be such that \( 0 < \alpha_1 < \cdots < \alpha_k < F(\Gamma) + 1 \). Then \( T = \bigcap_{P \in X^1(T)} T_P \) and this intersection has finite character. Note that \( X E[X] \) is a height-one prime ideal of \( T \) [Anderson et al. 2006, Theorem 3.4]; so \( E[\Gamma^*] \) is a height-one prime ideal of \( R \) by Lemma 2.1. We claim that \( R = \bigcap_{P \cap R \in X^1(R)} R_{P \cap R} \), where \( P \) ranges over all height-one prime ideals of \( T \). Suppose to the contrary that there exists an element \( f \) in \( \bigcap_{P \cap R \in X^1(R)} R_{P \cap R} \setminus R \). Note that \( f \in T \), and hence we can write \( f = \sum_{i=0}^{m} f_i X^i \). Then there exists a polynomial \( g \in R \setminus E[\Gamma^*] \) such that \( fg \in R \). Since \( g(0) \neq 0 \), the same argument as in the proof of Lemma 2.3(1) shows that \( f \in R \), which contradicts the choice of \( f \). Thus the equality holds. Since \( T = \bigcap_{P \in X^1(T)} T_P \) has finite character, it is clear that the intersection \( R = \bigcap_{P \cap R \in X^1(R)} R_{P \cap R} \) also has finite character. Thus \( R \) is a weakly Krull domain.

(2) \( \Rightarrow \) (3) This implication was already shown in the proof of (2) \( \Rightarrow \) (1).
(3) \(\Rightarrow (4)\) Assume that \(T_n\) is a weakly Krull domain. Then \(t\)-dim\((T_n)\) = 1 [Anderson et al. 1992, Lemma 2.1]; so \(X^n E[X]\) is a maximal \(t\)-ideal of \(T_n\) by Lemma 2.3(1).

Let \(S = \{X^m \mid m \in \Delta_n\}\). Then \(E[\Delta_n]_S = E[X, X^{-1}] = (T_n)_S\) is a weakly Krull domain [Anderson et al. 1993, Proposition 4.7]. Note that \(XE[X]\) is a height-one prime ideal of \(E[X]\); so \(X^n E[X]\) is a height-one prime ideal of \(E[\Delta_n]\) [Chang et al. 2012, Proposition 1.1]; so \(E[\Delta_n]_X^E[X]\) is a one-dimensional quasi-local domain. Hence \(E[\Delta_n]_X^E[X]\) is a weakly Krull domain. We claim that \(E[\Delta_n] = E[\Delta_n]_S \cap E[\Delta_n]_X^E[X]\). Let \(f = f_0 + \sum_{i=0}^{k_1} f_i X^i\) and \(h = h_0 + \sum_{i=0}^{k_2} h_i X^i\) be nonzero elements of \(E[\Delta_n]\) with \(h(0) \neq 0\) and let \(g = \sum_{i=0}^{k_3} g_i X^i \in E[X] \setminus \{0\}\) with \(g(0) \neq 0\) satisfying \(\frac{g}{X^m} = \frac{f}{E[X]} \in E[\Delta_n]_S \cap E[\Delta_n]_X^E[X]\) for some nonnegative integer \(m\). Then \(X^m f = gh\); so \(m = 0\). By comparing coefficients of \(f, gh\), it is easy to see that \(g_i = 0\) for all \(i = 1, \ldots, n - 1\). Hence \(\frac{g}{X^m} \in E[\Delta_n]\). The reverse inclusion is clear. Thus \(E[\Delta_n]\) is a weakly Krull domain.

(4) \(\Rightarrow (5)\) By [Zafarullah 2003, Lemma 2.6], \(ht_t(XE[X]) = \dim(E_{D\setminus\{0\}}[X])\). By (4), \(ht_t(X^n E[X]) = 1\); so the comment before Remark 2.2 establishes that

\[\dim(E_{D\setminus\{0\}}[X]) = 1.\]

Thus \(E_{D\setminus\{0\}}\) is a field. Also, since \(X^n E[X]\) is a maximal \(t\)-ideal of \(T_n\), \(qf(D) \cap E = D\) by Lemma 2.3(2). Finally, it follows directly from Theorem 1.3 that \(E[X]\) is a weakly Krull domain.

(5) \(\Rightarrow (2)\) [Anderson et al. 2006, Theorem 3.4].

(1) \(\Rightarrow (2)\) In the proof of (2) \(\Leftrightarrow (4)\), the integer \(n \geq 2\) was arbitrary; so it suffices to show that \(XF^{(\Gamma)+1} E[X]\) is a height-one maximal \(t\)-ideal of \(TF^{(\Gamma)+1}\) and \(E[\Delta_F^{(\Gamma)+1}]\) is a weakly Krull domain. Assume that \(R\) is a weakly Krull domain. Since \(t\)-dim\((R)\) = 1 [Anderson et al. 1992, Lemma 2.1], \(E[\Gamma^*]\) is a height-one maximal \(t\)-ideal of \(R\) by Lemma 2.3(1); so \(XF^{(\Gamma)+1} E[X]\) is a height-one maximal \(t\)-ideal of \(T_{\Delta_F^{(\Gamma)+1}}\) by Lemma 2.1 and the remark before Remark 2.2. Let \(S_1 = \{X^\alpha \mid \alpha \in \Delta_F^{(\Gamma)+1}\}\) and \(S_2 = \{X^\alpha \mid \alpha \in \Gamma\}\). Then \(E[\Delta_F^{(\Gamma)+1}]_{S_1} = R_{S_2}\) is a weakly Krull domain [Anderson et al. 1993, Proposition 4.7]. Also, \(E[\Delta_F^{(\Gamma)+1}]_{X_F^{(\Gamma)+1} E[X]}\) is a weakly Krull domain because it is one-dimensional quasi-local. Note that \(E[\Delta_F^{(\Gamma)+1}]_{X_F^{(\Gamma)+1} E[X]} = E[\Delta_F^{(\Gamma)+1}]_{S_1} \cap E[\Delta_F^{(\Gamma)+1}]_{X_F^{(\Gamma)+1} E[X]}\) as in the proof of (3) \(\Rightarrow (4)\). Thus \(E[\Delta_F^{(\Gamma)+1}]\) is a weakly Krull domain.

\[\square\]

**Corollary 2.5.** Let \(R = D + E[\Gamma^*]\), \(T = D + XE[X]\), \(T_n = D + X^n E[X]\) and \(\Delta_n = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}\) for integers \(n \geq 2\). If \(\text{char}(E) \neq 0\), then the following statements are equivalent.

(1) \(R\) is an AWFD.

(2) \(R\) is a GWFD.

(3) \(T\) is an AWFD.
(4) \( T \) is a GWFD.

(5) \( T_n \) is an AWFD.

(6) \( T_n \) is a GWFD.

(7) \( X^n E[X] \) is a maximal \( t \)-ideal of \( T_n \), \( E[\Delta_n] \) is an AWFD and for each \( 0 \neq e \in E \), there exist an integer \( m = m(e) \geq 1 \) and a unit \( u \) of \( E \) such that \( ue^m \in D \).

(8) \( X^n E[X] \) is a maximal \( t \)-ideal of \( T_n \), \( E[\Delta_n] \) is a GWFD and for each \( 0 \neq e \in E \), there exist an integer \( m = m(e) \geq 1 \) and a unit \( u \) of \( E \) such that \( ue^m \in D \).}

(9) \( qf(D) \cap E = D, E[X] \) is an AWFD and for each \( 0 \neq e \in E \), there exist an integer \( m = m(e) \geq 1 \) and a unit \( u \) of \( E \) such that \( ue^m \in D \).

(10) \( qf(D) \cap E = D, E[X] \) is a GWFD and for each \( 0 \neq e \in E \), there exist an integer \( m = m(e) \geq 1 \) and a unit \( u \) of \( E \) such that \( ue^m \in D \).

Proof. (1) \( \Rightarrow \) (2) and (5) \( \Rightarrow \) (6) Their definitions lead to these implications.

(3) \( \Leftrightarrow \) (9) [Anderson et al. 2006, Theorem 3.5].

(4) \( \Leftrightarrow \) (10) [Anderson and Chang 2007, Corollary 2.10].

(7) \( \Leftrightarrow \) (8) and (9) \( \Leftrightarrow \) (10) See Corollary 1.5.

(7) \( \Rightarrow \) (9) This equivalence follows from Corollary 1.5 and Lemma 2.3(2).

(3) \( \Rightarrow \) (1) Assume that \( T \) is an AWFD. Then \( T \) is a weakly Krull domain [Anderson et al. 1992, Theorem 3.4]. Hence \( E[X] \) is a weakly Krull domain by Theorem 2.4. Let \( S = \{ X^m \mid m \in \mathbb{N}_0 \} \). Since \( X \) is a prime element of \( E[X] \), \( Cl(E[X]) = Cl(T_S) \) is torsion [Anderson et al. 1993, Corollary 4.9]; so \( E[X] \) is an AWFD [Anderson et al. 1992, Theorem 3.4]. Let \( f \in R \setminus \{0\} \). Then there exists an integer \( m \geq 1 \) such that \( f^m = X^l f_1 \cdots f_r \) for some nonnegative positive integer \( l \) and primary elements \( f_1, \ldots, f_r \) of \( E[X] \) with nonzero constant terms. Also, since \( \text{char}(E) \neq 0 \), there exists an integer \( k \geq F(\Gamma) + 1 \) such that \( f_i^k \in E[\Gamma] \) for all \( i = 1, \ldots, r \); so \( f^mk = X^l f_1^k \cdots f_r^k \in E[\Gamma] \). Fix an \( i \in \{1, \ldots, r\} \), and we claim that \( \sqrt{f_i^k E[\Gamma]} \) is a prime ideal of \( E[\Gamma] \) [Anderson et al. 2003b, Lemma 2.1]. Note that \( \sqrt{f_i^k E[X]} = \sqrt{f_i^k E[X]} = X E[X] \). If \( \sqrt{f_i^k E[X]} = X E[X] \), then an easy calculation using a similar method as in the proof of (2) \( \Rightarrow \) (1) in Theorem 2.4 shows that \( \sqrt{f_i^k E[\Gamma]} = E[\Gamma^*] \) is a prime ideal. Assume that \( \sqrt{f_i^k E[X]} \neq X E[X] \). Since \( f_i(0) \neq 0 \), \( f_i^k E[X, X^{-1}] \) is a primary ideal of \( E[X, X^{-1}] \); so \( f_i^k E[X, X^{-1}] \cap E[\Gamma] \) is primary in \( E[\Gamma] \). It is easy to see that \( \sqrt{f_i^k E[X, X^{-1}]} \cap E[\Gamma] = \sqrt{f_i^k E[\Gamma]} \). Hence \( \sqrt{f_i^k E[\Gamma]} \) is a prime ideal. Therefore we may assume that \( f_1, \ldots, f_r \) are primary elements of \( E[\Gamma] \) with nonzero constant terms and write \( f^m = X^l f_1 \cdots f_r \) as above. Note that for each \( i = 1, \ldots, r \), there exist a unit \( u_i \) of \( E \) and an integer \( a_i \geq F(\Gamma) + 1 \) such that
$u_i f_i(0)^{a_i} \in D$ as in the proof of (3) $\iff$ (9); so $u_i f_i^{a_i} \in R$. Let
\[ a = a_1 \cdots a_r, \quad \hat{a}_i = \frac{a}{a_i}, \quad \text{and} \quad u = \hat{u}_1^{\hat{a}_1} \cdots \hat{u}_r^{\hat{a}_r}. \]

Then $uf^{am} = X^{al}(u_1 f_1^{a_1})^{\hat{a}_1} \cdots (u_r f_r^{a_r})^{\hat{a}_r}$ and $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]} = \sqrt{f_i E[\Gamma]}$ for each $i = 1, \ldots, r$. Since $t\dim(E[\Gamma]) = 1$, $(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]$ is a primary ideal of $E[\Gamma]$ [Anderson et al. 2003b, Lemma 2.1] for each $1 \leq i \leq r$.

**Claim.** For each $1 \leq i \leq r$, $(u_i f_i^{a_i})^{\hat{a}_i} R$ is a primary ideal of $R$.

**Proof.** Note that $(u_i f_i^{a_i})^{\hat{a}_i} R \subset R$ and fix an $i \in \{1, \ldots, r\}$. We also note that $t\dim(R) = 1$ because $R$ is a weakly Krull domain by Theorem 2.4. Hence, by [Anderson et al. 2003b, Lemma 2.1], it suffices to show that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R}$ is a prime ideal of $R$. If $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i}} E[\Gamma] = E[\Gamma^*]$, then it is easy to see that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R} = E[\Gamma^*]$ is a prime ideal of $R$. Assume that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]} \neq E[\Gamma^*]$. Then $(u_i f_i(0)^{a_i})^{\hat{a}_i} R \neq 0$. Now, we show that $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R = (u_i f_i^{a_i})^{\hat{a}_i} R$. Let $h \in (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R$. Note that we have
\[ (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R \subseteq (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap E[\Gamma] \]
\[ = (u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma] \]
by adapting the proof of (2) $\Rightarrow$ (1) in Theorem 2.4. So, we can write $h = (u_i f_i^{a_i})^{\hat{a}_i} g$ for some $g \in E[\Gamma]$. Then
\[ g(0) = \frac{(u_i f_i(0)^{a_i})^{\hat{a}_i}}{h(0)} \in qf(D) \cap E = D \]
by Theorem 2.4; so $g \in R$. Therefore $h \in (u_i f_i^{a_i})^{\hat{a}_i} R$, and hence
\[ (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R \subseteq (u_i f_i^{a_i})^{\hat{a}_i} R. \]

The reverse inclusion is clear, and hence $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R = (u_i f_i^{a_i})^{\hat{a}_i} R$. Since $(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]$ is a primary ideal of $E[\Gamma]$, $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}]$ is a primary ideal of $E[X, X^{-1}]$. Therefore $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R} = \sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R}$ is a prime ideal of $R$, and thus $(u_i f_i^{a_i})^{\hat{a}_i} R$ is a primary ideal of $R$. The claim is proved. \(\square\)

If $l = 0$, then $uf(0)^{am} = (u_1 f_1(0)^{a_1})^{\hat{a}_1} \cdots (u_r f_r(0)^{a_r})^{\hat{a}_r} \in D$; so $u$ is a unit of $D$ because $u$ is a unit of $E$. If $l \geq 1$, then $f^{am} = u^{-1} X^{al}(u_1 f_1^{a_1})^{\hat{a}_1} \cdots (u_r f_r^{a_r})^{\hat{a}_r}$. Since $u^{-1} X^{al} E[\Gamma]$ is a primary ideal of $E[\Gamma]$, $u^{-1} X^{al} R$ is a primary ideal of $R$ by imitating the previous proof. Hence $f^{am}$ is a product of primary elements of $R$, and thus $R$ is an AWFD.

(2) $\Rightarrow$ (8) Assume that $R$ is a GWFD and fix an integer $n \geq 2$. Then $R$ is a weakly Krull domain [Anderson et al. 2003b, Corollary 2.3]; so $X^n E[X]$ is a height-one maximal $t$-ideal of $T_n$ by Theorem 2.4.
Next, we claim that $E[\Delta_n]$ is a GWFD. Let $S_1 = \{X^m \mid m \in \Delta_n\}$ and $S_2 = \{X^m \mid m \in \Gamma\}$. Then $E[\Delta_n]_{S_1} = E[X, X^{-1}] = R_{S_2}$ is a GWFD. Let $Q$ be a nonzero prime ideal of $E[\Delta_n]$. If $Q \cap S_1 \neq \emptyset$, then $Q$ contains a primary element $X^n$ of $E[\Delta_n]$. If $Q \cap S_1 = \emptyset$, then $QE[\Delta_n]_{S_1}$ is a prime ideal of $E[\Delta_n]_{S_1}$; so $QE[\Delta_n]_{S_1}$ contains a primary element $f \in E[X, X^{-1}]$. Note that $X$ is a unit of $E[X, X^{-1}]$ and $f^k \in E[\Delta_n]$ for some integer $k \geq 1$ because $\text{char}(E) \neq 0$; so we may assume that $f \in E[\Delta_n]$ with $f(0) \neq 0$. Then

$$fE[\Delta_n] \subseteq fE[\Delta_n]_{S_1} \cap E[\Delta_n] \subseteq QE[\Delta_n]_{S_1} \cap E[\Delta_n] = Q;$$

so $Q$ contains a primary element $f$. Hence $E[\Delta_n]$ is a GWFD.

In order to check the final condition, let $e \in E \setminus \{0\}$. If $e$ is a unit of $E$, then we have nothing to prove. So, we assume that $e$ is not a unit of $E$ and let $h = e + X \in E[X]$. Since $c(h)_v = E, hE[X] = hqf(E)[X] \cap E[X]$ [Anderson and Chang 2007, Lemma 2.1(1)]; so $hE[X]$ is a height-one prime ideal. Let $P = hE[X] \cap R$. Since $e$ is not a unit of $E$, $X^{\text{char}(E)+1} \not\in P$; so $X^\alpha \not\in P$ for all $\alpha \in \Gamma$. Therefore $hE[X, X^{-1}] = PR_{S_2} \subseteq R_{S_2}$, and hence $ht_R(P) = 1$. Since $R$ is a GWFD, $P = \sqrt{gR}$ for some primary element $g \in R$ [Anderson et al. 2003b, Theorem 2.2]. Suppose to the contrary that $g(0) = 0$. Since $E_{D\setminus\{0\}}$ is a field by Theorem 2.4, $\frac{1}{e} = \frac{e'}{d}$ for some $0 \neq d \in D$ and $e' \in E$; so $e'h = d + e'X \in T$. Since $\text{char}(E) \neq 0$, $(e'h)^k \in hE[X] \cap R = P$ for some integer $k \geq 1$. Hence $(e'h)^k \in gR$ for some integer $l \geq 1$. However, this is impossible because $e \neq 0$. Therefore $g(0) \neq 0$. It is clear that $gR_{S_2}$ is a primary ideal of $R_{S_2}, gR_{S_2} \cap E[X] = gE[X], PR_{S_2} = \sqrt{gR} \subseteq R_{S_2}$ and $PR_{S_2} \cap E[X] = hE[X]$. Hence $gE[X]$ is a $hE[X]$-primary ideal. Therefore $g = uh^m$ for some $u \in qf(E)$ and some integer $m \geq 1$; so $ue^m = g(0) \in D$. Thus $u$ is a unit of $E$.

(3) $\Rightarrow$ (5) and (6) $\Rightarrow$ (8) These implications can be obtained by applying $\Gamma = \Delta_n$ to the proofs of (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (8), respectively. 

We are closing this paper by showing that $R = D + E[\Gamma^*]$ is never a WFD and the assumption “$\text{char}(E) = 0$” is essential in Corollary 2.5.

Remark 2.6. Assume that $R = D + E[\Gamma^*]$ is a WFD or an AWFD. Let $h = 1 + X \in E[X], P = hE[X] \cap R$ and let $M$ be a maximal $t$-ideal of $R$. If $M = E[\Gamma^*]$, then $PR_M = R_M$ because $1 + (-1)^{\text{char}(E)}X^{(\text{char}(E)+1)} \in P \setminus E[\Gamma^*]$. Assume that $M \neq E[\Gamma^*]$. Since $c(h)_v = E, hqf(E)[X] \cap hE[X] = hE[X]$ [Anderson and Chang 2007, Lemma 2.1(1)]. Let $S = \{X^m \mid m \in \Gamma\}$. Then $PE[X, X^{-1}] = hE[X, X^{-1}]$; so $PR_M = hR_M$ is principal. Hence $P$ is $t$-locally principal, and thus $P$ is $t$-invertible [Anderson et al. 1992, Lemma 2.2].

(1) If $R$ is a WFD, then $P = gR$ for some $g \in R$ with $g(0) \neq 0$ [Anderson and Zafrullah 1990, Theorem]. Note that $hE[X, X^{-1}] = gE[X, X^{-1}]$; so $g = uh$ for some unit $u$ of $E$. Hence $uh \in R$, which is impossible. Thus $R$ is not a WFD.
Assume that \( R \) is an AWFD. Then \( P^m = gR \) for some integer \( m \geq 1 \) and \( g \in R \) with \( g(0) \neq 0 \) [Anderson et al. 1992, Theorem 3.4]. We note that

\[
h^m E[X, X^{-1}] = g E[X, X^{-1}];
\]

so \( uh^m = g \) for some unit \( u \) of \( E \). Hence \( uh^m \in R \). However, this can not happen if \( \text{char}(E) = 0 \). Thus \( R \) is never an AWFD whenever \( \text{char}(E) = 0 \).

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**References**


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