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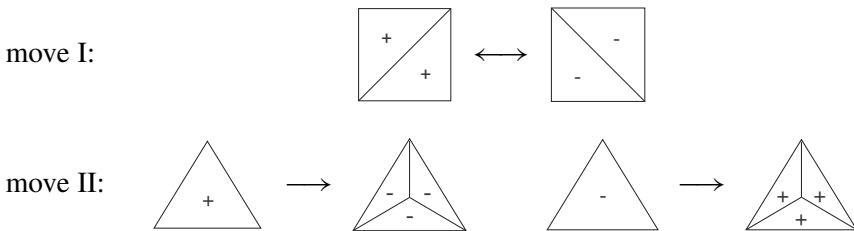
# EXTENDING TRIANGULATIONS OF THE 2-SPHERE TO THE 3-DISK PRESERVING A 4-COLORING

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We prove that any triangulation of a 2-dimensional sphere with a proper strict 4-coloring on its vertices can be seen as the boundary of a triangulation of a 3-dimensional disk with the same vertices in such a way that the 4-coloring remains proper.

## 1. Introduction

We are interested in triangulations of the sphere whose vertices can be assigned four colors in a way that is *proper* (no adjacent vertices receive the same color) and *strict* (all colors are used). Elichou, Gravier and Payan [2002] proved that all such triangulations can be obtained from the tetrahedron by sequences of the following moves:



Here the signs are defined by the 4-coloring in the following manner. A proper vertex 4-coloring of a triangulation induces a 3-coloring on the edges if we regard the four colors as the elements of the field of order 4; each edge is assigned the sum (or difference) of the colors of its end points. By fixing an order on the three edge colors we get a signing on the triangles: “+” if the colors on the boundary are ordered counterclockwise and “-” if the colors on the boundary are ordered clockwise. Thus move I does not change the coloring on the vertices, and move II extends the coloring on the vertices in a unique way to the central vertex.

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Disregarding signs and the unidirectional arrows, these are the same moves of [Pachner 1991], which can be seen as the gluing of tetrahedra onto the triangulation. This observation led the author to wonder whether or not any triangulation of a sphere with a proper strict 4-coloring can be seen as the boundary of a properly vertex-colored triangulation of the 3-dimensional disk with the same vertices, even though the extended triangulation has many extra edges.

The answer to this question in the affirmative is the main result of this paper, and the result stated at the start of this introduction follows from this as a corollary.

## 2. Triangulations, $\Delta$ -structures and other algebraic topological prerequisites

We clarify some topological definitions that we use in this paper. Eliahou et al. [2002] defined a triangulation of a closed surface  $S$  as a finite graph, loop-free but possibly with multiple edges, embedded in the surface  $S$  and subdividing it into triangular faces. However, this definition, by allowing parallel edges, differs from the usual meaning given by many topologists, which defines a triangulation of  $S$  as an ordered pair  $(K, h)$ , where  $K$  is a simplicial complex and  $h : |K| \rightarrow S$  is a homeomorphism from the geometric realization of  $K$  to the space  $S$  (for example, see [Rotman 1988]). Instead, we use as the definition of a triangulation on a space  $X$  what is called in [Hatcher 2002] a  $\Delta$ -complex structure on  $X$ .

Let

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

be the *standard  $n$ -simplex*. A  $k$ -dimensional face of  $\Delta^n$  (with  $k < n$ ) is a subset of  $\Delta^n$  where  $n - k$  coordinates  $t_{i_1}, \dots, t_{i_{n-k}}$  are equal to zero. The union of all the faces of  $\Delta^n$  is the *boundary* of  $\Delta^n$ , denoted by  $\partial\Delta^n$ . The *open  $n$ -simplex*  $\mathring{\Delta}^n$  is  $\Delta^n - \partial\Delta^n$ , the interior of  $\Delta^n$ :

$$\mathring{\Delta}^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1 \text{ and } t_i > 0 \text{ for all } i\}.$$

A  $\Delta$ -complex structure on a space  $X$  (which we also call a triangulation on  $X$ ) is a (finite) collection of maps  $\sigma_\alpha : \Delta^n \rightarrow X$ , with  $n$  depending on the index  $\alpha$ , such that:

- (i) The restriction  $\sigma_\alpha|_{\mathring{\Delta}^n}$  is injective, and each point in  $X$  is in the image of exactly one such restriction  $\sigma_\alpha|_{\mathring{\Delta}^n}$ .
- (ii) Each restriction  $\sigma_\alpha$  to a face (of dimension  $k < n$ ) of  $\Delta^n$  is one of the maps  $\sigma_\beta : \Delta^k \rightarrow X$ . Here we identify the ( $k$ -dimensional) face of  $\Delta^n$  with  $\Delta^k$  by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- (iii) A set  $A \subset X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\Delta^n$  for each  $\sigma_\alpha$ .

We call a space  $X$  provided with a  $\Delta$ -complex structure  $T$  a *triangulated space*.

We call the images of the 0-simplex  $\sigma_\alpha : \Delta^0 \rightarrow X$  the *vertices* in  $X$ , the images of the 1-simplex  $\sigma_\alpha : \Delta^1 \rightarrow X$  the *edges* in  $X$ , the images of the 2-simplex the *triangles* in  $X$ , and the images of the 3-simplex the *tetrahedra* in  $X$ . The *graph* of a triangulation is formed by its sets of vertices and edges, and a (proper) coloring on a triangulation is a (proper) coloring on its graph.

The *star* of a vertex  $v$  in  $X$  is formed by the images of the maps  $\sigma_\alpha : \Delta^n \rightarrow X$  that contain  $v$ :

$$\text{st}(v) = \bigcup_{v \in \sigma_\alpha(\Delta^n)} \sigma_\alpha(\Delta^n).$$

The *deletion* of a vertex  $v$  in  $X$  is formed by the images of the maps  $\sigma_\alpha : \Delta^n \rightarrow X$  that do not contain  $v$ :

$$\text{dl}(v) = \bigcup_{v \notin \sigma_\alpha(\Delta^n)} \sigma_\alpha(\Delta^n).$$

A  $\Delta$ -complex structure on a space  $X$  induces a  $\Delta$ -complex structure on its cone  $CX = X \times [0, 1]/X \times \{0\}$  in a natural way. For each map  $\sigma_\alpha : \Delta^k \rightarrow X$ , we have two maps:  $\bar{\sigma}_\alpha : \Delta^k \rightarrow CX$  given by  $\bar{\sigma}_\alpha(t_0, \dots, t_k) = (\sigma_\alpha(t_0, \dots, t_k), 1)$ , and  $\hat{\sigma}_\alpha : \Delta^{k+1} \rightarrow CX$  given by

$$\hat{\sigma}_\alpha(t_0, \dots, t_{k+1}) = \left( \sigma_\alpha \left( \frac{t_0}{1-t_{k+1}}, \dots, \frac{t_k}{1-t_{k+1}} \right), 1-t_{k+1} \right)$$

(for  $t_{k+1} \neq 1$ ) and  $\hat{\sigma}_\alpha(0, \dots, 0, 1) = *$ , where  $*$  is the point  $X \times \{0\}$  in  $CX$ . Finally we complete the triangulation with the map  $\sigma_* : \Delta^0 \rightarrow CX$  given by  $\sigma_*(\Delta^0) = *$ . We call this construction the *cone of the triangulation* of  $X$ .

For the proof of the main result we need to introduce the following topological concept. Consider a 3-disk  $D$  with a  $\Delta$ -complex structure and an edge  $e$  on the boundary of the 3-disk that is in the boundary of a triangle  $t$  such that its interior  $\mathring{t}$  is contained in the interior of  $D$ . Then the *inner star* of the edge  $e$ ,

$$\mathring{\text{st}}(e) = \bigcup_{e \subseteq \sigma_\alpha(\Delta^n)} \sigma_\alpha(\mathring{\Delta}^n),$$

is topologically a half open ball,  $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1, z \geq 0\}$ , which is split into two pieces  $\Omega_1$  and  $\Omega_2$  by the triangle  $t$  (or  $\mathring{t} \cup \mathring{e}$ ). This means that  $\Omega_1$  and  $\Omega_2$  are the two connected components of  $\mathring{\text{st}}(e) \setminus (\mathring{t} \cup \mathring{e})$ . Now consider  $\Omega_1 \cup \Omega_2$  as a subspace of  $D \setminus (\mathring{t} \cup \mathring{e})$  and a subspace of  $(\Omega_1 \cup \mathring{t} \cup \mathring{e}) \coprod (\Omega_2 \cup \mathring{t} \cup \mathring{e})$  (where each component  $(\Omega_i \cup \mathring{t} \cup \mathring{e})$  is given the subspace topology induced from  $D$ ). By gluing<sup>1</sup>  $D \setminus (\mathring{t} \cup \mathring{e})$  and  $(\Omega_1 \cup \mathring{t} \cup \mathring{e}) \coprod (\Omega_2 \cup \mathring{t} \cup \mathring{e})$  via  $\Omega_1 \cup \Omega_2$ , we get a new space

<sup>1</sup>If a space  $A$  is a subspace of two spaces  $X$  and  $Y$ , the *gluing* of  $X$  and  $Y$  via  $A$  is the space obtained by taking the quotient of  $X \coprod Y$  that identifies the two copies of  $A$  (as a subspace of  $X$  and as a subspace of  $Y$ ).

that we say to be obtained from  $D$  by opening a *fissure* through the edge  $e$  along the triangle  $t$ .

### 3. The main result and a corollary

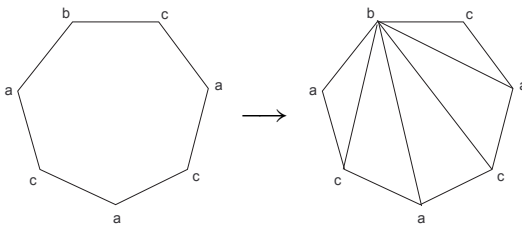
Our main result can be stated as follows:

**Theorem 1.** *If  $\psi$  is a proper strict 4-coloring of a triangulation  $T$  of the 2-sphere  $\mathbb{S}^2$ , then there exists a triangulation  $T'$  of the 3-disk  $\mathbb{D}^3$  such that  $T$  is the triangulation induced by  $T'$  on the boundary of the disk, the vertices of  $T'$  are in  $T$  and  $\psi$  is still a proper 4-coloring of  $T'$ .*

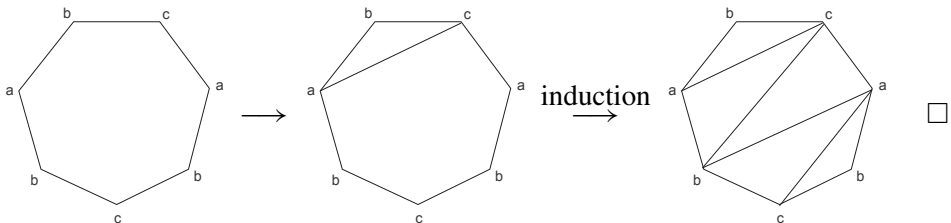
To prove this theorem, we make use of the following lemma, which can be seen as a version of the theorem for one dimension lower:

**Lemma 2.** *If  $\psi$  is a proper strict 3-coloring of a triangulation  $T$  of the circle  $\mathbb{S}^1$ , then there exists a triangulation  $T'$  of the disk  $\mathbb{D}^2$  such that  $T$  is the triangulation induced by  $T'$  on the boundary of the disk, the vertices of  $T'$  are in  $T$  and  $\psi$  is still a proper 3-coloring of  $T'$ .*

*Proof.* In this case  $T$  is just a cycle graph, and thus if  $\psi$  is a proper strict 3-coloring, then there are three consecutive vertices  $v_1, v_2$  and  $v_3$  with distinct colors  $a, b$  and  $c$ . Suppose that the middle vertex  $v_2$  is colored by  $b$ . If  $v_2$  is the only vertex in  $T$  colored by  $b$ , then we can add edges linking  $v_2$  with all vertices of  $T$ , and therefore we get a triangulation  $T'$  of the disk  $\mathbb{D}^2$  with the desired properties.



If  $v_2$  is not the only vertex in  $T$  colored by  $b$ , then we can add an edge linking  $v_1$  with  $v_3$ , and then complete, by induction on the number of vertices of  $T$ , the triangulation on the disk whose boundary is the cycle  $v_1, v_3, \dots, v_n$ .



*Proof of Theorem 1.* The topological procedures used in this proof are:

- (1) taking the triangulated cone of a triangulated 2-disk,
- (2) attaching a triangulated cone of a triangulated 2-disk without inner vertices to a previously triangulated 3-disk,
- (3) “grafting” a triangulated 3-disk into another triangulated 3-disk using a fissure (see the end of [Section 2](#)),
- (4) gluing two previously triangulated 3-disks along a shared triangle on their surface, and
- (5) gluing two triangulated cones of 2-disks along two adjacent shared triangles on their surface.

The details of the procedures follow.

First we consider the case where the triangulation  $T$  has no parallel edges (edges between the same pair of vertices), so the star and the deletion (see [Section 2](#)) of any vertex of  $T$  are simplicial disks.

We start by proving that in the triangulation  $T$  there exists a vertex  $v$  that is adjacent to a cycle colored by three colors. We take a triangle colored by (say)  $a$ ,  $b$  and  $c$  and consider the region formed by the triangles colored by the same colors. Since  $\psi$  is a strict 4-coloring, this region has a nonempty boundary, and any vertex of the boundary satisfies the required condition, because its link necessarily has vertices colored by  $d$  and by two colors amongst  $a$ ,  $b$  and  $c$  that differ from the color of the vertex itself.

Now, if  $v$  is the only vertex in  $T$  colored by its color  $\psi(v)$ , then we take the cone of the deletion of  $v$  to get a triangulation  $T'$  of the disk  $\mathbb{D}^3$  with the desired properties. This is procedure (1).

If, on the contrary,  $v$  is not the only vertex in  $T$  colored by  $\psi(v)$ , we remove  $v$ , use [Lemma 2](#) to triangulate the region  $R$  bounded by the link of  $v$  to get a triangulation  $T''$  of  $\mathbb{S}^2$  that is still strict 4-colored and with one less vertex than  $T$ , use induction to get a triangulation of the disk  $\mathbb{D}^3$  bounded by this, and attach its cone to the region  $R$  to obtain the desired triangulation  $T'$  of  $\mathbb{D}^3$ . This is procedure (2).

In the case where the triangulation does have parallel edges, we take two parallel edges  $e$  and  $e'$  linking two vertices  $v_i$  and  $v_j$  colored by two colors (say 1 and 2). The two parallel edges form a cycle  $C$  that splits the sphere into two (triangulated) 2-disks  $D_1$  and  $D_2$ .

Suppose that we have a strict 4-coloring for at least one of the disks (say  $D_1$  has vertices colored by all four colors). Then, gluing the edges  $e$  and  $e'$ , we get a (triangulated) sphere  $S_1$  with fewer vertices<sup>2</sup> than the original sphere. To the other disk we attach along the cycle  $C$  its cone (formed by two triangles  $t_e$  and  $t_{e'}$ ) in

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<sup>2</sup>Since the cycle  $C$  is formed by only two edges, each disk must contain inner vertices.

order to get another (triangulated) sphere  $S_2$  with fewer vertices<sup>3</sup> than the original sphere. The disk  $D_2$  is colored by at least three colors (say 1, 2 and 3), so we color the inserted vertex on  $S_2$  with the color 4 in order to guarantee we have a strict 4-coloring on  $S_2$ . Therefore, by induction on the number of vertices,  $S_1$  is the boundary of a triangulated 3-disk  $\mathcal{D}'_1$  without inner vertices and preserving the properness of the 4-coloring, and  $S_2$  is the boundary of a triangulated 3-disk  $\mathcal{D}'_2$  with the same properties.

Now we perform the following surgery: take the edge  $e^*$  on  $S_1$  resulting from the gluing of the parallel edges  $e$  and  $e'$ , and search in  $\mathcal{D}'_1$  for a triangle  $t$  containing that edge and a vertex colored with the color 4 (Such a triangle exists because the edge belongs to some tetrahedron.)

If  $t$  is not on the surface  $S_1$ , then it is adjacent to two tetrahedra  $\tau_1$  and  $\tau_2$  in  $\mathcal{D}'_1$ . We replace  $t$  by two copies of it,  $t_1$  (adjacent to  $\tau_1$ ) and  $t_2$  (adjacent to  $\tau_2$ ), sharing the same vertices and the same edges except the edge  $e^*$ , which is replaced by the old parallel edges  $e$  and  $e'$ . In other words, we are opening a fissure in  $\mathcal{D}'_1$  through the edge  $e^*$  along the triangle  $t$ . Suppose, without loss of generality, that  $e$  is an edge of  $t_1$  and  $e'$  is an edge of  $t_2$ . Then we identify the triangles  $t_1$  and  $t_2$  with the triangles  $t_e$  and  $t_{e'}$  in  $S_2$  (which are in  $\mathcal{D}'_2$ ) that made the cone of the cycle  $C$  (formed by  $e$  and  $e'$ ). This “grafting” of the disk  $\mathcal{D}'_2$  in the disk  $\mathcal{D}'_1$  produces the desired triangulated 3-disk. This is procedure (3).

If  $t$  is on the surface  $S_1$ , then there exists a triangle  $t'$  on  $D_1$  with the same vertices. If the triangle  $t'$  is adjacent to the edge  $e$  (resp.  $e'$ ), then we identify the triangle  $t$  with the triangle on  $S_2$  that belongs to the cone of the cycle  $C$  and is adjacent to the edge  $e'$  (resp.  $e$ ). This gluing of the disk  $\mathcal{D}'_1$  with the disk  $\mathcal{D}'_2$  produces the desired triangulated 3-disk. This is procedure (4).

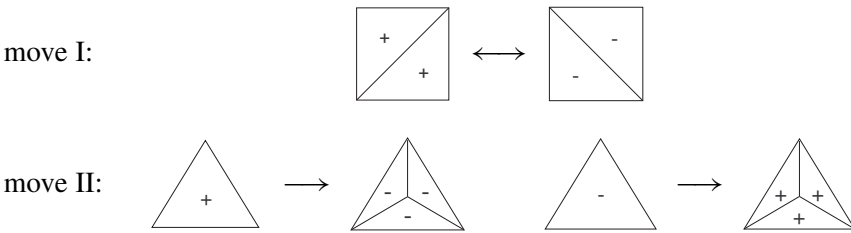
Finally, if neither of the disks  $D_1$  and  $D_2$  has a strict 4-coloring (we can suppose that  $D_1$  is colored by 1, 2 and 3 and  $D_2$  is colored by 1, 2 and 4), then we remove from  $D_1$  the edge  $e$  and the triangle  $t_1$  in  $D_1$  incident to it, and remove from  $D_2$  the edge  $e'$  and the triangle  $t_2$  in  $D_2$  incident to it. We then get two new 2-disks  $D'_1$  and  $D'_2$ . We take the triangulated cones of them both. Let  $v_1$  be the opposite vertex to the edge  $e$  in the triangle  $t_1$ ,  $v_2$  the opposite vertex to the edge  $e'$  in the triangle  $t_2$ ,  $v_1^*$  the cone vertex of the cone of  $D'_1$ , and  $v_2^*$  the cone vertex of the cone of  $D'_2$ . Then we get the desired triangulation of the 3-disk by gluing the two cones by identifying the triangle in the cone of  $D'_1$  having vertices  $v_1$ ,  $v_i$  and  $v_1^*$  with the triangle in the cone of  $D'_2$  having vertices  $v_2$ ,  $v_i$  and  $v_2^*$ , and identifying the triangle in the cone of  $D'_1$  having vertices  $v_1$ ,  $v_j$  and  $v_1^*$  with the triangle in the cone of  $D'_2$  having vertices  $v_2$ ,  $v_j$  and  $v_2^*$  (Recall that the two ends of the parallel edges  $e$  and  $e'$  were called  $v_i$  and  $v_j$ , and note that through the identifications, the triangles  $t_1$

<sup>3</sup>Since the disk  $D_1$  has vertices of all colors, at least two vertices must be deleted from the original triangulated sphere in order to get the disk  $D_2$ .

and  $t_2$  appear on the boundary of the 3-disk as faces of the cones of  $D'_2$  and  $D'_1$ , respectively.) This is procedure (5). □

As a corollary, we get an alternative proof of [Eliahou et al. 2002, Theorem 1.3]:

**Theorem 3.** *Suppose we are given a triangulation  $T$  of the sphere with signed faces. Then the signing comes from a proper strict 4-coloring of  $T$  if and only if  $T$  comes from the tetrahedron with the same sign on all its faces, by means of a sequence of signed diagonal flips (move I) and/or divisions of a triangle into three triangles (by adding a vertex  $v$  inside the triangle and edges joining  $v$  to the vertices of the triangle) with the opposite sign (move II).*



*Proof.* Sufficiency is easy, and we take the proof from [Eliahou et al. 2002]: move I does not change the coloring, and move II extends the coloring in a unique way.

For necessity, we take the triangulation of  $\mathbb{D}^3$  obtained in Theorem 1, choose one tetrahedron, and by adding the adjacent tetrahedra one by one, get a sequence of moves I and II (The signs of the faces are determined by the coloring, as was observed in the paragraph following Theorem 1.)

We have only to see that this sequence of attaching adjacent tetrahedra can be done while keeping the topology of a 3-disk in each step. We prove by induction on the number of tetrahedra that this can be done independently of the tetrahedron we choose to start with.

Recall that the triangulation produced in Theorem 1 is obtained by one of the following procedures: (1) taking the triangulated cone of a triangulated 2-disk, (2) attaching a triangulated cone of a triangulated 2-disk without inner vertices to a previously triangulated 3-disk, (3) “grafting” a triangulated 3-disk into another triangulated 3-disk, (4) gluing two previously triangulated 3-disks along a shared triangle on their surface, and (5) gluing two triangulated cones of 2-disks along two adjacent shared triangles on their surface.

In procedure (1) we have to prove that, given a triangulation of the 2-disk and a triangle in it, there exists a sequence of attaching adjacent triangles, starting from the given triangle and ending with the given triangulation, such that in each step we have a triangulation of a 2-disk. This can be easily proved by induction on the number of triangles of the triangulation. If there exists an edge that cuts the 2-disk



in two, then we get, by induction, a sequence of attaching triangles on the disk that contains the starting triangle, and we continue the sequence on the other disk starting with the triangle incident to the cutting edge. If there are no such cutting edges and the triangulation has more than one triangle (the case of a one-triangle triangulation is trivial), then we choose one triangle incident to a boundary edge different from the starting triangle, remove it, and use induction to get a sequence of attaching triangles from the starting triangle to this triangulation of the disk with the triangle removed, and complete the sequence by attaching this last triangle.

In procedure (2), we have by induction sequences for any starting tetrahedron both in the smaller 3-disk and in the attaching cone. If the starting tetrahedron is in the smaller 3-disk, then we take a sequence in the smaller 3-disk for that tetrahedron and complete it with a sequence in the attaching cone. If the starting tetrahedron is in the cone, then we take a sequence in the cone starting at that tetrahedron and a sequence in the smaller 3-disk starting at the tetrahedron adjacent to that tetrahedron, and proceed in the following way. Start with the starting tetrahedron, follow it with the sequence in the smaller 3-disk, and complete the sequence with the rest of the sequence in the cone.

In the “grafting” case (3), if the starting tetrahedron is in the 3-disk  $\mathcal{D}'_2$  that is grafted into the other 3-disk  $\mathcal{D}'_1$  (see the final part of the proof of [Theorem 1](#)) using an inner triangle, then we take, by induction, a sequence in  $\mathcal{D}'_1$  starting at that tetrahedron and complete the sequence in the other disk by starting at one of the tetrahedra incident to the triangle where the “grafting” is done. If the starting tetrahedron is in the other 3-disk ( $\mathcal{D}'_1$ ), then from a sequence in that disk starting at that tetrahedron, given by induction, we take the partial sequence from that starting tetrahedron to the first tetrahedron in the sequence that is incident to the triangle where the “grafting” is done, then follow with a sequence in the grafted disk  $\mathcal{D}'_2$  that starts at the tetrahedron adjacent to the latter tetrahedron, and finish with the rest of the sequence in  $\mathcal{D}'_1$ .

In the case (4) of two 3-disks attached by a single triangle, we start with a given tetrahedron, take the sequence given by induction in the disk that contains it, and complete the sequence in the other disk starting at the tetrahedron incident to the gluing triangle.

In the final case (5), we have the cones of two disks  $D'_1$  and  $D'_2$  glued by identifying a certain pair of adjacent triangles on each cone. Without loss of generality, suppose that the starting tetrahedron is in the cone of the disk  $D'_1$ . We know that for a cone of a triangulated 2-disk there is a sequence of attaching tetrahedra from any starting tetrahedron to the whole triangulation keeping the 3-disk topology in each step. So we can find such a sequence for the cone of the disk  $D'_1$  starting at the given tetrahedron, say  $T_1, T_2, \dots, T_k$ , and another such sequence for the cone of the disk  $D_2$  (that is, the disk  $D'_2$  with the triangle with vertices  $v_i, v_2$  and  $v_j$

attached to it; see the last part of the proof of [Theorem 1](#)), starting at the cone of this last triangle, say  $T'_1, T'_2, \dots, T'_l$ . Thus we obtain a sequence for the whole 3-disk by omitting  $T'_1$  and composing these two sequences as  $T_1, T_2, \dots, T_k, T'_2, \dots, T'_l$ .  $\square$

#### 4. Comments and a conjecture

We have seen the analogy between [Theorem 1](#) and [Lemma 2](#) and how the second result is used in the proof of the first. This leads us to conjecture the following:

**Conjecture 4.** If  $\psi$  is a proper strict  $(n + 2)$ -coloring of a triangulation  $T$  of the  $n$ -sphere  $\mathbb{S}^n$ , then there is a triangulation  $T'$  of the disk  $\mathbb{D}^{n+1}$  such that  $\partial T' = T$ , the vertices of  $T'$  are in  $T$ , and  $\psi$  is still a proper a  $(n + 2)$ -coloring of  $T'$ .

It is not clear if and how the proof of [Theorem 1](#) can be adapted to higher dimensions even in the weaker case of triangulations without parallel edges (the usual definition for triangulation). This is because, for triangulations of spheres of dimension greater than 4, the link of a vertex is not necessarily a sphere of lower dimension. However, if we consider only the case of piecewise linear triangulations (where the link of any simplex is a piecewise linear sphere [[Thurston 1997](#)]), the first part of the proof of [Theorem 1](#) seems to apply recursively to prove [Conjecture 4](#). Also, for piecewise linear triangulations, a weaker version of this conjecture (allowing inner vertices) follows from [[Hatcher and Wahl 2010](#), Lemma 3.1].

Another open problem is to what extent the proof of [Theorem 3](#) depends on the triangulations of the 3-disk obtained in the proof of [Theorem 1](#). In other words, given a triangulation of the 3-disk, not necessarily of the type used in the proof of [Theorem 1](#), we want to know if there is a sequence of attaching tetrahedra from an initial tetrahedron to the final triangulation such that the topology of the 3-disk is kept in each step.

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# PACIFIC JOURNAL OF MATHEMATICS

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Extending triangulations of the 2-sphere to the 3-disk preserving a 4-coloring	257
RUI PEDRO CARPENTIER	
Orthogonal quantum group invariants of links	267
LIN CHEN and QINGTAO CHEN	
Some properties of squeezing functions on bounded domains	319
FUSHENG DENG, QIAN GUAN and LIYOU ZHANG	
Representations of little $q$ -Schur algebras	343
JIE DU, QIANG FU and JIAN-PAN WANG	
Renormalized weighted volume and conformal fractional Laplacians	379
MARÍA DEL MAR GONZÁLEZ	
The $L_4$ norm of Littlewood polynomials derived from the Jacobi symbol	395
JONATHAN JEDWAB and KAI-UWE SCHMIDT	
On a conjecture of Kaneko and Ohno	419
ZHONG-HUA LI	
Categories of unitary representations of Banach–Lie supergroups and restriction functors	431
STÉPHANE MERIGON, KARL-HERMANN NEEB and HADI SALMASIAN	
Odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields	471
LI REN, QIANG MU and YONGZHENG ZHANG	
Interior derivative estimates for the Kähler–Ricci flow	491
MORGAN SHERMAN and BEN WEINKOVE	
Two-dimensional disjoint minimal graphs	503
LINFENG ZHOU	