

*Pacific
Journal of
Mathematics*

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Volume 257 No. 2

June 2012

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Let $X_0^*(k, n, s)$ denote the sum of all multiple zeta-star values of weight k , depth n and height s . Kaneko and Ohno conjectured that, for any positive integers m, n, s with $m, n \geq s$, the difference

$$(-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients. We give a proof of this conjecture.

1. Introduction

Given a sequence $\mathbf{k} = (k_1, \dots, k_n)$ of positive integers with $k_1 > 1$, the weight $\text{wt}(\mathbf{k})$, depth $\text{dep}(\mathbf{k})$ and height $\text{ht}(\mathbf{k})$ are defined by

$$\text{wt}(\mathbf{k}) = k_1 + \dots + k_n, \quad \text{dep}(\mathbf{k}) = n, \quad \text{ht}(\mathbf{k}) = \#\{i \mid k_i \geq 2\},$$

respectively. For such a sequence \mathbf{k} , there are two well-studied real numbers: the multiple zeta value $\zeta(\mathbf{k})$, defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_n) = \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \dots m_n^{k_n}},$$

and the multiple zeta-star value $\zeta^*(\mathbf{k})$, defined by

$$\zeta^*(\mathbf{k}) = \zeta^*(k_1, \dots, k_n) = \sum_{m_1 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}.$$

We call the values $\zeta(\mathbf{k})$ and $\zeta^*(\mathbf{k})$ with weight $\text{wt}(\mathbf{k})$, depth $\text{dep}(\mathbf{k})$ and height $\text{ht}(\mathbf{k})$.

The well-known Ohno–Zagier relation [2001] is a class of relations about the sums of multiple zeta values of fixed weight, depth and height. For integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$, we denote by $X_0(k, n, s)$ the sum of all multiple zeta values of weight k , depth n and height s . The Ohno–Zagier relation says that

This work was partially supported by the National Natural Science Foundation of China (grant no. 11001201), and the Japan Society for the Promotion of Science Postdoctoral Fellowship for Foreign Researchers.

MSC2010: 11M32, 33C20.

Keywords: multiple zeta-star values, generalized hypergeometric function.

$$X_0(k, n, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

More explicitly, Ohno and Zagier gave the generating function expression

$$\sum_{\substack{k \geq n+s \\ n \geq s \geq 1}} X_0(k, n, s) u^{k-n-s} v^{n-s} t^{s-1} = \frac{1}{uv-t} \left\{ 1 - \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (u^n + v^n - \alpha^n - \beta^n)\right) \right\},$$

where α and β are determined by $\alpha + \beta = u + v$ and $\alpha\beta = t$. In [Li 2010], we showed that the Ohno–Zagier relation can be deduced from the regularized double shuffle relation. In [Li 2008], we generalized the concept height to i -height, studied sums of multiple zeta values of fixed weight, depth and general height, and expressed a kind of generating function of these sums in terms of generalized hypergeometric functions.

Similarly, we denote by $X_0^*(k, n, s)$ the sum of all multiple zeta-star values of weight k , depth n and height s for integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$. The authors of [Aoki et al. 2008] considered a generating function $\Phi_0^*(u, v, t)$ of sums $X_0^*(k, n, s)$, where

$$\Phi_0^*(u, v, t) = \sum_{k \geq n+s, n \geq s \geq 1} X_0^*(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.$$

It was proved there that $\Phi_0^*(u, v, t)$ can be expressed by a special value of the generalized hypergeometric function ${}_3F_2$ as

$$(1) \quad \Phi_0^*(u, v, t) = \frac{1}{(1-v)(1-\beta)} {}_3F_2\left(\begin{matrix} 1-\beta, 1-\beta+u, 1 \\ 2-v, 2-\beta \end{matrix}; 1\right),$$

where α, β are determined by $\alpha + \beta = u + v$ and $\alpha\beta = uv - t^2$, and the generalized hypergeometric function ${}_3F_2$ is defined as (see [Bailey 1935])

$${}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{n! (\beta_1)_n (\beta_2)_n} z^n,$$

with the Pochhammer symbol $(a)_n$ given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1) \cdots (a+n-1) & \text{if } n > 0. \end{cases}$$

Similarly to [Li 2008], the authors of [Aoki et al. 2011] considered a kind of generating function of sums of multiple zeta-star values of fixed weight, depth and general height, and represented this generating function via generalized hypergeometric functions.

Since the generating function $\Phi_0^*(u, v, t)$ is represented by ${}_3F_2$ as in (1), it is expected that in general $X_0^*(k, n, s)$ can't be written as a polynomial of zeta values

with rational coefficients. While in [2010] Kaneko and Ohno considered some kind of duality of multiple zeta-star values, and proposed the following conjecture.

Conjecture [Kaneko and Ohno 2010]. For any positive integers m, n, s satisfying $m, n \geq s$, we have

$$(-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

Kaneko and Ohno showed that this is true for $s = 1$. Using the result of [Aoki et al. 2008] about the generating function $\Phi_0^*(u, v, 0)$, Yamazaki [2010] gave another proof of this case. Note that the Kaneko–Ohno theorem for their conjecture in the case $s = 1$ can be restated as

$$(2) \quad u \Phi_0^*(-u, v, 0) - v \Phi_0^*(-v, u, 0) = \frac{1}{u} - \frac{1}{v} + \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} \left((\Gamma(v)\Gamma(1-v))^2 - (\Gamma(u)\Gamma(1-u))^2 \right).$$

The purpose of this paper is to give a proof of the Kaneko–Ohno conjecture. In fact, similarly to (2), we give an expression of $u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t)$ by gamma functions in Theorem 2.2. Our proof is based on the expression of $\Phi_0^*(u, v, t)$ given in [Aoki et al. 2008], and hence is similar to the one of [Yamazaki 2010] for the special case $s = 1$.

In Section 2, we state our main result and give some corollaries. In Section 3, we prepare a result about generalized hypergeometric series ${}_3F_2$. In the last section, we give the proof of the main theorem.

2. Statement of the main result

Main theorem. As in Section 1, we denote by $X_0^*(k, n, s)$ the sum of all multiple zeta-star values of weight k , depth n and height s for integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$. Let $\Phi_0^*(u, v, t)$ be the generating function defined by

$$\Phi_0^*(u, v, t) = \sum_{k \geq n+s, n \geq s \geq 1} X_0^*(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.$$

For variables u, v, t , we define a and b by the conditions $a + b = -u + v$ and $ab = -uv - t^2$. Equivalently, we have

$$a, b = \frac{-u + v \pm \sqrt{(u+v)^2 + 4t^2}}{2}.$$

After that we define the function $A(u, v, a, b)$ by

$$(3) \quad A(u, v, a, b) = \frac{1}{2\pi} \left\{ \frac{\cos \pi u}{\sin \pi v} - \frac{\cos \pi v}{\sin \pi u} + \cos \pi(a-b)(\cot \pi u - \cot \pi v) \right\}.$$

Note that $A(u, v, a, b) = A(u, v, b, a)$, which shall play an important role in the proof of our main theorem. We can express $A(u, v, a, b)$ by gamma functions as in the following lemma.

Lemma 2.1. *We have*

$$(4) \quad A(u, v, a, b) = \frac{1}{\Gamma(u+a)\Gamma(1-u-a)} \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(a)\Gamma(1-a)} + \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(b)\Gamma(1-b)} \right),$$

$$(5) \quad A(u, v, a, b) = \frac{1}{\Gamma(u+b)\Gamma(1-u-b)} \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(b)\Gamma(1-b)} + \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(a)\Gamma(1-a)} \right).$$

Proof. Equation (5) follows from (4) and the fact $A(u, v, a, b) = A(u, v, b, a)$. Using the well-known reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

we find that the right-hand side of Equation (4) becomes

$$\frac{\sin \pi(u+a)}{\pi} \left(\frac{\sin \pi a}{\sin \pi v} + \frac{\sin \pi b}{\sin \pi u} \right),$$

which is equal to

$$\frac{1}{2\pi} \left(\frac{\cos \pi u - \cos \pi(v+a-b)}{\sin \pi v} + \frac{\cos \pi(u+a-b) - \cos \pi v}{\sin \pi u} \right).$$

Now it is easy to finish the proof. □

The main theorem of this paper is this:

Theorem 2.2. *We have*

$$(6) \quad u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t) \\ = \frac{u-v}{ab} + A(u, v, a, b) \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}.$$

Some remarks. By the definition of the generating function $\Phi_0^*(u, v, t)$, it is easy to see that

$$(7) \quad u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t) \\ = \sum_{\substack{m, n \geq s \\ s \geq 1}} (-1)^s \left((-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s) \right) \\ \times u^{m+1-s} v^{n+1-s} t^{2s-2} \\ + \sum_{n \geq s \geq 1} (-1)^{n+s} X_0^*(n+s, s, s) (u^{n+1-s} - v^{n+1-s}) t^{2s-2}.$$

Since we have the expansion

$$\Gamma(1 - x) = \exp\left(\gamma x + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} x^n\right),$$

where γ is Euler’s constant, we know that Theorem 2.2 indeed implies the Kaneko–Ohno conjecture.

Corollary 2.3. *For any positive integers m, n, s with $m, n \geq s$, the difference*

$$(-1)^m X_0^*(m + n + 1, n + 1, s) - (-1)^n X_0^*(m + n + 1, m + 1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients.

Taking into account to the second term of the right-hand side of (7), we have another corollary.

Corollary 2.4. *For any positive integers k, s with $k \geq 2s$, the sum $X_0^*(k, s, s)$ can be expressed as a polynomial of zeta values with rational coefficients.*

Note that this is an immediate consequence of the symmetric sum formula for multiple zeta-star values (see [Hoffman 1992, Theorem 2.1]).

Letting $t = 0$ in Theorem 2.2, we can derive (2). In fact, in this case, we can assume that $a = -u$ and $b = v$. For $A(u, v, a, b)$, we use the equivalent equation (4). Then using Theorem 2.2, we get (2).

3. A result about generalized hypergeometric series ${}_3F_2$

To prove the main theorem of this paper, we introduce the following result.

Proposition 3.1. *For $a, b, c \in \mathbb{C}$ with sufficient small real parts, we have*

$$\begin{aligned} (8) \quad & {}_3F_2\left(\begin{matrix} a, b, c \\ a+b, 1+c \end{matrix}; 1\right) \\ &= \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} (\psi(1+c-b) - \psi(a) - \psi(b) - \gamma) \\ &\quad - \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a)_n(1-b)_n}{nn!(1+c-b)_n}, \end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

To save space, from now on we will denote the special value

$${}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; 1\right) \quad \text{by} \quad {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right).$$

To prove the proposition, we need two transformation formulas. The first one,

from [Bailey 1935, Section 3.8, (1), p. 21], is

$$(9) \quad {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right) = \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)} {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \beta_2-\alpha_3 \\ \alpha_1+\alpha_2-\beta_1+1, \beta_2 \end{matrix}\right) \\ + \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\alpha_2-\beta_1)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} \\ \times {}_3F_2\left(\begin{matrix} \beta_1-\alpha_1, \beta_1-\alpha_2, \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 \\ \beta_1-\alpha_1-\alpha_2+1, \beta_1+\beta_2-\alpha_1-\alpha_2 \end{matrix}\right),$$

provided that $\operatorname{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\operatorname{Re}(\alpha_3 - \beta_1 + 1) > 0$. The second one, from [Bailey 1935, Examples 7, p. 98] is

$$(10) \quad {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right) \\ = \frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} {}_3F_2\left(\begin{matrix} \beta_1-\alpha_1, \beta_1-\alpha_2, \alpha_3 \\ \beta_1, \beta_1+\beta_2-\alpha_1-\alpha_2 \end{matrix}\right),$$

provided that $\operatorname{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\operatorname{Re}(\beta_2 - \alpha_3) > 0$.

Proof of Proposition 3.1. Taking a parameter ε such that $|\varepsilon|$ is sufficient small, we have

$${}_3F_2\left(\begin{matrix} a, b, c \\ a+b, 1+c \end{matrix}\right) = \lim_{\varepsilon \rightarrow 0} {}_3F_2\left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix}\right).$$

Now we consider the series

$${}_3F_2\left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix}\right).$$

Applying (9), we get

$$(11) \quad {}_3F_2\left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix}\right) \\ = \frac{\Gamma(a+b+\varepsilon)\Gamma(\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)} {}_3F_2\left(\begin{matrix} a, b, 1-\varepsilon \\ 1-\varepsilon, 1+c-\varepsilon \end{matrix}\right) \\ + \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1-\varepsilon)\Gamma(1+c)} {}_3F_2\left(\begin{matrix} a+\varepsilon, b+\varepsilon, 1 \\ 1+\varepsilon, 1+c \end{matrix}\right).$$

To the first ${}_3F_2$ -series in the right-hand side of (11), we apply the Gaussian summation formula (see [Bailey 1935, Section 1.3, (1)])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{n! (\beta)_n} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha_1-\alpha_2)}{\Gamma(\beta-\alpha_1)\Gamma(\beta-\alpha_2)}$$

for $\text{Re}(\beta - \alpha_1 - \alpha_2) > 0$, and apply (10) to the second ${}_3F_2$ -series in the right-hand side of (11), we obtain

$$\begin{aligned} & {}_3F_2\left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix}\right) \\ &= \frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)} \\ &\quad - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} {}_3F_2\left(\begin{matrix} \varepsilon, 1-b, a+\varepsilon \\ 1+\varepsilon, 1+c-b \end{matrix}\right). \end{aligned}$$

To the ${}_3F_2$ -series in the right-hand side of the above equation, we split it into two terms as $\sum_{n=0}^\infty a_n = a_0 + \sum_{n=1}^\infty a_n$. Then we see that

$${}_3F_2\left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix}\right)$$

is equal to

$$\begin{aligned} & \frac{1}{\varepsilon} \left(\frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)} \right. \\ & \quad \left. - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} \right) \\ & \quad - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} \sum_{n=1}^\infty \frac{(a+\varepsilon)_n(1-b)_n}{(n+\varepsilon)n!(1+c-b)_n}. \end{aligned}$$

Finally, let ε go to 0 to finish the proof. For the first two lines of the above expression, we use l'Hôpital's rule and the fact that $\psi(1) = -\gamma$. □

4. Proof of the main theorem

In this section, we prove Theorem 2.2.

Lemma 4.1. *Let α and β be determined by $\alpha + \beta = u + v$ and $\alpha\beta = uv - t^2$. We have*

$$\begin{aligned} \Phi_0^*(u, v, t) &= \frac{\Gamma(\beta-\alpha)\Gamma(1-\beta)\Gamma(v)\Gamma(1-v)}{\Gamma(1-\alpha)\Gamma(1+u-\alpha)\Gamma(1+\alpha-u)} \sum_{n=0}^\infty \frac{(\alpha)_n(1-\beta)_n}{n!(1+\alpha-\beta)_n} \frac{\alpha-u}{n+\alpha-u} \\ &\quad + \frac{\Gamma(\alpha-\beta)\Gamma(1-\alpha)\Gamma(v)\Gamma(1-v)}{\Gamma(1-\beta)\Gamma(1+u-\beta)\Gamma(1+\beta-u)} \sum_{n=0}^\infty \frac{(\beta)_n(1-\alpha)_n}{n!(1+\beta-\alpha)_n} \frac{\beta-u}{n+\beta-u}. \end{aligned}$$

Proof. A result of Aoki, Komбу, and Ohno [Aoki et al. 2008] about the generating function $\Phi_0^*(u, v, t)$ gives that

$$\begin{aligned} \Phi_0^*(u, v, t) &= \frac{\Gamma(\beta - \alpha)\Gamma(1 - v)}{\Gamma(1 - \alpha)\Gamma(1 + u - \alpha)} \int_0^1 s^{-\beta}(1 - s)^{v-1} {}_2F_1\left(\begin{matrix} \alpha, \alpha - u \\ 1 + \alpha - \beta \end{matrix}; s\right) ds \\ &\quad + \frac{\Gamma(\alpha - \beta)\Gamma(1 - v)}{\Gamma(1 - \beta)\Gamma(1 + u - \beta)} \int_0^1 s^{-\alpha}(1 - s)^{v-1} {}_2F_1\left(\begin{matrix} \beta, \beta - u \\ 1 + \beta - \alpha \end{matrix}; s\right) ds. \end{aligned}$$

Here ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; s\right)$ is the Gaussian hypergeometric function, given by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; s\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} s^n.$$

Hence, we have

$$\begin{aligned} \int_0^1 s^{-\beta}(1 - s)^{v-1} {}_2F_1\left(\begin{matrix} \alpha, \alpha - u \\ 1 + \alpha - \beta \end{matrix}; s\right) ds &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha - u)_n}{n!(1 + \alpha - \beta)_n} \int_0^1 s^{n-\beta}(1 - s)^{v-1} ds \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha - u)_n}{n!(1 + \alpha - \beta)_n} \frac{\Gamma(1 + n - \beta)\Gamma(v)}{\Gamma(1 + n + v - \beta)}. \end{aligned}$$

Now it is easy to finish the proof. □

Recall that we have defined a and b by $a + b = -u + v$ and $ab = -uv - t^2$. Using Lemma 4.1, we immediately get the following result.

Lemma 4.2. *We have*

$$u\Phi_0^*(-u, v, t) - v\Phi_0^*(-v, u, t) = F(u, v, a, b) + F(u, v, b, a),$$

where $F(u, v, a, b)$ is defined by

$$\begin{aligned} &\frac{\Gamma(b - a)}{\Gamma(1 - u - a)\Gamma(1 + u + a)} \\ &\times \left(\frac{u\Gamma(v)\Gamma(1 - v)\Gamma(1 - b)}{\Gamma(1 - a)} \sum_{n=0}^{\infty} \frac{(a)_n(1 - b)_n}{n!(1 + a - b)_n} \frac{u + a}{n + u + a} \right. \\ &\quad \left. - \frac{v\Gamma(u)\Gamma(1 - u)\Gamma(1 + a)}{\Gamma(1 + b)} \sum_{n=0}^{\infty} \frac{(1 + a)_n(-b)_n}{n!(1 + a - b)_n} \frac{u + a}{n + u + a} \right). \end{aligned}$$

Since we have

$$\sum_{n=0}^{\infty} \frac{(a)_n(1 - b)_n}{n!(1 + a - b)_n} \frac{u + a}{n + u + a} = \sum_{n=0}^{\infty} \frac{(a)_n(-b)_n}{n!(a - b)_n} \frac{(a - b)(u + a)(n - b)}{-b(n + a - b)(n + u + a)},$$

and

$$\frac{n - b}{(n + a - b)(n + u + a)} = \frac{-a}{u + b} \frac{1}{n + a - b} + \frac{v}{u + b} \frac{1}{n + u + a},$$

we get

$$\sum_{n=0}^{\infty} \frac{(a)_n(1-b)_n}{n!(1+a-b)_n} \frac{u+a}{n+u+a} = \frac{a(u+a)\Gamma(1+a-b)}{b(u+b)\Gamma(1+a)\Gamma(1-b)} - \frac{v(a-b)}{b(u+b)} {}_3F_2\left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix}\right).$$

In the above, we have used Gaussian summation formula for Gaussian hypergeometric function at unit argument. Similarly, we have

$$\sum_{n=0}^{\infty} \frac{(1+a)_n(-b)_n}{n!(1+a-b)_n} \frac{u+a}{n+u+a} = \frac{b(u+a)\Gamma(1+a-b)}{a(u+b)\Gamma(1+a)\Gamma(1-b)} + \frac{u(a-b)}{a(u+b)} {}_3F_2\left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix}\right).$$

Hence we get the following lemma.

Lemma 4.3. *We have*

$$F(u, v, a, b) = F_1(u, v, a, b) + F_2(u, v, a, b),$$

where

$$\begin{aligned} F_1(u, v, a, b) &= \frac{(u+a)\Gamma(b-a)\Gamma(1+a-b)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)} \left(\frac{u\Gamma(v)\Gamma(1-v)}{b\Gamma(a)\Gamma(1-a)} - \frac{v\Gamma(u)\Gamma(1-u)}{a\Gamma(b)\Gamma(1-b)} \right), \\ F_2(u, v, a, b) &= \frac{uv(a-b)\Gamma(b-a)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)} \\ &\quad \times \left(\frac{\Gamma(v)\Gamma(1-v)\Gamma(-b)}{\Gamma(1-a)} - \frac{\Gamma(u)\Gamma(1-u)\Gamma(a)}{\Gamma(1+b)} \right) {}_3F_2\left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix}\right). \end{aligned}$$

Now we compute $F_1(u, v, a, b) + F_1(u, v, b, a)$ and $F_2(u, v, a, b) + F_2(u, v, b, a)$.

Lemma 4.4. *The sum $F_1(u, v, a, b) + F_1(u, v, b, a)$ equals*

$$\frac{u-v}{ab} + \frac{(a-b)uv}{ab(u+a)(u+b)} \Gamma(b-a)\Gamma(1+a-b) A(u, v, a, b).$$

Proof. Using the reflection formula for gamma function, we see that

$$\begin{aligned} F_1(u, v, a, b) + F_1(u, v, b, a) &= \Gamma(b-a)\Gamma(1+a-b) \\ &\quad \times \left\{ \frac{\sin \pi(u+a)}{\pi(u+b)} \left(\frac{u \sin \pi a}{b \sin \pi v} - \frac{v \sin \pi b}{a \sin \pi u} \right) - \frac{\sin \pi(u+b)}{\pi(u+a)} \left(\frac{u \sin \pi b}{a \sin \pi v} - \frac{v \sin \pi a}{b \sin \pi u} \right) \right\}. \end{aligned}$$

The term in braces in this expression is

$$(12) \quad \frac{1}{2\pi(u+b)} \left(\frac{u(\cos \pi u - \cos \pi v \cos \pi(a-b) + \sin \pi v \sin \pi(a-b))}{b \sin \pi v} \right. \\ \left. - \frac{v(\cos \pi u \cos \pi(a-b) - \sin \pi u \sin \pi(a-b) - \cos \pi v)}{a \sin \pi u} \right) \\ - \frac{1}{2\pi(u+a)} \left(\frac{u(\cos \pi u - \cos \pi v \cos \pi(b-a) + \sin \pi v \sin \pi(b-a))}{a \sin \pi v} \right. \\ \left. - \frac{v(\cos \pi u \cos \pi(b-a) - \sin \pi u \sin \pi(b-a) - \cos \pi v)}{b \sin \pi u} \right).$$

Picking up the common factors, and noting the identities

$$\frac{1}{b(u+b)} - \frac{1}{a(u+a)} = \frac{v(a-b)}{ab(u+a)(u+b)}, \\ \frac{1}{a(u+b)} - \frac{1}{b(u+a)} = \frac{u(b-a)}{ab(u+a)(u+b)}, \\ \frac{u}{b(u+b)} + \frac{v}{a(u+b)} + \frac{u}{a(u+a)} + \frac{v}{b(u+a)} = \frac{2(v-u)}{ab},$$

we see that (12) becomes

$$\frac{uv(a-b)}{ab(u+a)(u+b)} A(u, v, a, b) + \frac{(v-u) \sin \pi(a-b)}{ab\pi},$$

which finishes the proof. □

Lemma 4.5. *The sum $F_2(u, v, a, b) + F_2(u, v, b, a)$ equals*

$$\frac{(b-a)uv}{ab(u+a)(u+b)} \Gamma(b-a) \Gamma(1+a-b) A(u, v, a, b) \\ + A(u, v, a, b) \frac{\Gamma(a) \Gamma(1-a) \Gamma(b) \Gamma(1-b) \Gamma(u+a) \Gamma(u+b)}{\Gamma(u) \Gamma(v)}.$$

Proof. Applying Proposition 3.1 to the ${}_3F_2$ -series in $F_2(u, v, a, b)$, we find that $F_2(u, v, a, b)$ becomes

$$\frac{(a-b) \Gamma(a-b) \Gamma(b-a) \Gamma(u+b)}{\Gamma(u) \Gamma(v) \Gamma(1-u-a)} \left(\frac{\Gamma(v) \Gamma(1-v)}{\Gamma(a) \Gamma(1-a)} - \frac{\Gamma(u) \Gamma(1-u)}{\Gamma(-b) \Gamma(1+b)} \right) \\ \times \left(\psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n (1+b)_n}{nn!(1+v)_n} \right),$$

which is just

$$A(u, v, a, b) \frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \times \left(\psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right).$$

Hence, using the equality $A(u, v, a, b) = A(u, v, b, a)$, we find that

$$\begin{aligned} &F_2(u, v, a, b) + F_2(u, v, b, a) \\ &= A(u, v, a, b) \frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \left(\frac{(1+a)_n(b)_n}{nn!(1+v)_n} - \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right) + \psi(b) - \psi(-b) + \psi(-a) - \psi(a) \right\}. \end{aligned}$$

It is easy to see that

$$\sum_{n=1}^{\infty} \left(\frac{(1+a)_n(b)_n}{nn!(1+v)_n} - \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right) = \frac{b-a}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!(1+v)_n},$$

which equals

$$\frac{b-a}{ab} \frac{\Gamma(1+u)\Gamma(1+v)}{\Gamma(1+u+a)\Gamma(1+u+b)} + \frac{1}{b} - \frac{1}{a}$$

by the Gaussian summation formula. Applying the formulas

$$\psi(-x) - \psi(x) - \frac{1}{x} = \pi \cot \pi x,$$

and

$$\pi \cot \pi a - \pi \cot \pi b = \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}{\Gamma(b-a)\Gamma(1+a-b)},$$

we finish the proof. □

Proof of Theorem 2.2. The theorem follows from Lemmas 4.2, 4.3, 4.4 and 4.5. □

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Received June 27, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

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Typeset in L^AT_EX

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PACIFIC JOURNAL OF MATHEMATICS

Volume 257 No. 2 June 2012

Extending triangulations of the 2-sphere to the 3-disk preserving a 4-coloring	257
RUI PEDRO CARPENTIER	
Orthogonal quantum group invariants of links	267
LIN CHEN and QINGTAO CHEN	
Some properties of squeezing functions on bounded domains	319
FUSHENG DENG, QIAN GUAN and LIYOU ZHANG	
Representations of little q -Schur algebras	343
JIE DU, QIANG FU and JIAN-PAN WANG	
Renormalized weighted volume and conformal fractional Laplacians	379
MARÍA DEL MAR GONZÁLEZ	
The L_4 norm of Littlewood polynomials derived from the Jacobi symbol	395
JONATHAN JEDWAB and KAI-UWE SCHMIDT	
On a conjecture of Kaneko and Ohno	419
ZHONG-HUA LI	
Categories of unitary representations of Banach–Lie supergroups and restriction functors	431
STÉPHANE MERIGON, KARL-HERMANN NEEB and HADI SALMASIAN	
Odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields	471
LI REN, QIANG MU and YONGZHENG ZHANG	
Interior derivative estimates for the Kähler–Ricci flow	491
MORGAN SHERMAN and BEN WEINKOVE	
Two-dimensional disjoint minimal graphs	503
LINFENG ZHOU	



0030-8730(201206)257:2;1-5