Pacific Journal of Mathematics

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Volume 257 No. 2 June 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 257, No. 2, 2012

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We give a maximum principle proof of interior derivative estimates for the Kähler–Ricci flow, assuming local uniform bounds on the metric.

1. Introduction

Let $(M, \hat{\omega})$ be a Kähler manifold of complex dimension n. Let $\omega = \omega(t)$ be a solution of the Kähler–Ricci flow on $M \times [0, T]$, for some $T > 0$:

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0, \tag{1-1}$$

with ω_0 a smooth initial Kähler metric.

Fix a point $p \in M$ and denote by $B_r \subset M$ the open ball centered at p of radius r for $0 < r < 1$ with respect to $\hat{\omega}$. We assume that r is sufficiently small so that $\overline{B_r}$ is contained in a single holomorphic coordinate chart. Our main result is as follows:

Theorem 1.1. *Let $N > 1$ satisfy*

$$\frac{1}{N}\hat{\omega} \leq \omega \leq N\hat{\omega}, \quad \text{on } \overline{B_r} \times [0, T]. \tag{1-2}$$

Then for each $m = 0, 1, 2, \ldots$ there exist constants C and C_m depending only on $\hat{\omega}$ and T such that on $B_{r/2} \times (0, T]$,

(i) $|\hat{\nabla}\omega|^2_{\omega} \leq C\dfrac{N^3}{r^2 t}$, *for $\hat{\nabla}$ the covariant derivative with respect to $\hat{\omega}$.*

(ii) $|\mathrm{Rm}|^2_{\omega} \leq C_0\dfrac{N^8}{r^4 t^2}$.

(iii) $|\nabla^m_{\mathbb{R}}\mathrm{Rm}|^2_{\omega} \leq C_m\left(\dfrac{N^4}{r^2 t}\right)^{m+2}$ *for $m = 1, 2, \ldots$, where $\nabla_{\mathbb{R}}$ is the real covariant derivative with respect to the metric ω.*

Research supported in part by NSF grants DMS-0848193 and DMS-1105373. In addition, Sherman was supported in part by a California State Faculty Support Grant (SFSG) and Weinkove by a Sloan Fellowship.
MSC2010: 53C44, 53C55.
Keywords: Ricci flow, Kähler, interior estimates.

Moreover, if we allow the constants C and C_m to depend also on ω_0 then the estimates (i), (ii) and (iii) hold with each factor of t on the right hand side replaced by 1.

We prove this result using the maximum principle. Note that by work of Shi [1989a; 1989b] it was already known that a bound on curvature as in (ii) implies (iii) (nevertheless, we include a proof here, for the sake of completeness). Theorem 1.1 implies the following:

Corollary 1.2. *Let $N > 1$ satisfy*

$$\frac{1}{N}\hat{\omega} \leq \omega \leq N\hat{\omega} \quad \text{on } \overline{B}_r \times [0, T]. \tag{1-3}$$

Then for each $m = 0, 1, 2, \ldots$ there exist constants C_m, α_m, β_m and γ_m depending only on m, $\hat{\omega}$ and T such that

$$|\hat{\nabla}^m_{\mathbb{R}}\omega|_{\hat{\omega}} \leq C_m \frac{N^{\alpha_m}}{r^{\beta_m} t^{\gamma_m}} \quad \text{on } B_{r/2} \times (0, T], \tag{1-4}$$

Moreover, if we allow the constants C_m, α_m and β_m to depend also on ω_0 then (1-4) *holds with $\gamma_m = 0$.*

Namely, a local uniform estimate for the metric along the Kähler–Ricci flow implies local derivative estimates to all orders. This fact in itself is not new. Indeed the local PDE theory of Evans [1982] and Krylov [1982] can be applied to the Kähler–Ricci flow equation (see, for example, [Chow et al. 2007] or the generalization in [Gill 2011]). The key point here is to establish this via Theorem 1.1 whose proof uses only elementary maximum principle arguments.

The form of the estimate (1-4) may be useful for applications and does not seem to be written down explicitly elsewhere in the literature. When considering the Kähler–Ricci flow on projective varieties, it is often the case that one obtains a uniform estimate for the metric ω away from a subvariety (see [Song and Tian 2009; Song and Weinkove 2011a; 2011b; 2011c; Tian and Zhang 2006; Tsuji 1988; Zhang 2009], for example). Theorem 1.1 can be used to replace global arguments. To illustrate, suppose that $\omega = \omega(t)$ solves the Kähler–Ricci flow on a compact Kähler manifold M and there exists an analytic hypersurface $D \subset M$ whose associated line bundle $[D]$ admits a holomorphic section s vanishing to order 1 along D. Assume that

$$\frac{1}{C}|s|^{\alpha}_{H}\hat{\omega} \leq \omega \leq \frac{C}{|s|^{\alpha}_{H}}\hat{\omega} \quad \text{on } (M \setminus D) \times [0, T] \tag{1-5}$$

for some positive constants C and α, where H is a Hermitian metric on $[D]$. An elementary argument shows that Theorem 1.1 implies the existence of C_m, α_m and

γ_m such that

$$|\hat{\nabla}^m_{\mathbb{R}}\omega|_{\hat{\omega}} \leq \frac{C_m}{t^{\gamma_m}|s|_H^{\alpha_m}}, \quad \text{on } (M \setminus D) \times (0, T] \tag{1-6}$$

for each $m = 1, 2, \ldots$. Moreover we can take $\gamma_m = 0$ if we allow C_m and α_m to depend on the initial metric ω_0. Estimates of the form of (1-6) are used, for example, in [Song and Weinkove 2011b; 2011c]. In particular, Corollary 1.2 gives an alternative proof of the results in Section 4 of [Song and Weinkove 2011b].

Finally we remark that since our result is completely local, we may and do assume that $M = \mathbb{C}^n$, $p = 0$ and $\hat{\omega}$ is the Euclidean metric. We will write g and $\hat{g}$ for the Kähler metrics associated to ω and $\hat{\omega}$. All magnitudes $|\cdot|$ are taken with respect to the metric g. We shall use the letter C (as well as C', C'', etc.) for a uniform constant (depending only on m, $\hat{\omega}$, and T) which may differ from line to line.

In Sections 2, 3 and 4 we prove parts (i), (ii) and (iii) of Theorem 1.1 respectively. In Section 5 we give a proof of Corollary 1.2.

2. Bound on the first derivative of the metric

In this section we prove the estimate on the first derivative of the metric g, and so establish Theorem 1.1(i). This gives a local parabolic version of Calabi's [1958] well-known "third-order" estimate for the complex Monge–Ampère equation (used by Yau [1978] in his solution of the Calabi conjecture). There exist now many generalizations of Calabi's estimate [Cherrier 1987; Tosatti 2010; Tosatti et al. 2008; Zhang and Zhang 2011]. A global parabolic Calabi estimate was applied to the case of the Kähler–Ricci flow in [Cao 1985]. Phong, Sesum and Sturm [Phong et al. 2007] later gave a neat and explicit computation in this which we will make use of here for our local estimate.

We wish to bound the quantity

$$S = |\hat{\nabla} g|^2 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \hat{\nabla}_i g_{k\bar{q}} \overline{\hat{\nabla}_j g_{l\bar{p}}} \tag{2-1}$$

where we write $\hat{\nabla}$ for the covariant derivative with respect to $\hat{g}$. Write $r_0 = r$ and let ψ be a nonnegative C^∞ cut-off function that is identically equal to 1 on $\overline{B_{r_1}}$ and vanishes outside B_r, where $r_0 > r_1 > r/2$. We may assume that

$$|\nabla\psi|^2,\ |\Delta\psi| \leq C\frac{N}{r^2}, \tag{2-2}$$

where $\Delta = \nabla^{\bar{j}}\nabla_{\bar{j}} = g^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$. Thus

$$(\partial_t - \Delta)(\psi^2 S) \leq \psi^2(\partial_t - \Delta)S + C\frac{N}{r^2}S + 2\left|\langle\nabla\psi^2,\ \nabla S\rangle\right|, \tag{2-3}$$

where we are writing $\langle \nabla F, \nabla G \rangle = g^{i\bar{j}} \partial_i F \partial_{\bar{j}} G$ for functions F, G. Following the notation in [Phong et al. 2007], we introduce the endomorphism $h^i{}_k = \hat{g}^{i\bar{j}} g_{\bar{j}k}$ and let X be the tensor with components $X^k_{il} = (\nabla_i h \cdot h^{-1})^k{}_l$, so that $S = |X|^2$. Note that X is the difference of the Christoffel symbols of g and $\hat{g}$.

An application of Young's inequality gives

$$2\left|\langle \nabla \psi^2,\ \nabla S \rangle\right| \le \psi^2 (|\nabla X|^2 + |\overline{\nabla} X|^2) + C \frac{N}{r^2} S. \tag{2-4}$$

We now use the evolution equation for S derived by Phong, Sesum and Sturm [ibid., (2.51)] which, in the case where $\hat{\omega}$ is Euclidean, has the simple form:

$$(\partial_t - \Delta) S \;=\; -\left(|\nabla X|^2 + |\overline{\nabla} X|^2\right). \tag{2-5}$$

Combining (2-3), (2-4), and (2-5) we find

$$(\partial_t - \Delta)(\psi^2 S) \le C \frac{N}{r^2} S. \tag{2-6}$$

We now need to use the evolution equation for $\operatorname{tr} h$ from [Cao 1985], which is a parabolic version of an estimate from [Aubin 1978; Yau 1978]. More precisely, we can apply equations (2.28) and (2.31) of [Phong et al. 2007] and use the fact that the fixed metric is Euclidean to obtain

$$(\partial_t - \Delta)(\operatorname{tr} h) = -\hat{g}^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \hat{\nabla}_i g_{\bar{l}p} \overline{\hat{\nabla}_j g_{\bar{k}q}}. \tag{2-7}$$

Hence

$$(\partial_t - \Delta)(\operatorname{tr} h) \;\le\; -\frac{S}{N}. \tag{2-8}$$

Let $f(t)$ denote either the function t or the constant 1. Then $0 \le f(t) \le \max(T, 1)$ and $f'(t) = 1$ or 0 so that we get, for any positive constant B,

$$(\partial_t - \Delta)(f(t)\psi^2 S + B \operatorname{tr} h) \;\le\; C \frac{N}{r^2} S - \frac{B}{N} S.$$

Let $B = (N^2/r^2)(C+1)$. Then, by the maximum principle, the maximum of $f(t)\psi^2 S + B \operatorname{tr} h$ on $\overline{B_r} \times [0, T]$ can only occur at $t = 0$ or on the boundary of $\overline{B_r}$, where $\psi = 0$. Since $\operatorname{tr} h \le nN$, we have

$$S \le C \frac{N^3}{f(t) r^2} \quad \text{on } \overline{B_{r_1}} \times (0, T]. \tag{2-9}$$

giving part (i) of Theorem 1.1.

3. Bound on curvature

We now prove part (ii) of Theorem 1.1. For global estimates of this type, see [Chau 2004; Phong et al. 2011]. We fix a smaller radius r_2 satisfying $r_1 > r_2 > r/2$. In this section we let ψ be a cut-off function, identically 1 on $\overline{B_{r_2}}$ and identically 0 outside B_{r_1}. As before we may assume $|\Delta\psi|, |\nabla\psi|^2 \le CN/r^2$ for some uniform constant C. Calculate

$$(\partial_t - \Delta) R_{\bar{j}i\bar{l}k} = - R_{\bar{j}i}{}^{p\bar{q}} R_{\bar{l}k\bar{q}p} + R_{\bar{l}i}{}^{p\bar{q}} R_{\bar{j}k\bar{q}p} - R_{\bar{j}p\bar{l}}{}^{\bar{q}} R^{p}{}_{i\bar{q}k} - R_{\bar{j}p} R^{p}{}_{i\bar{l}k} - R_{\bar{l}p} R_{\bar{j}i}{}^{p}{}_{k}, \tag{3-1}$$

and therefore (see [Hamilton 1982])

$$(\partial_t - \Delta)|\mathrm{Rm}|^2 \le -|\nabla \mathrm{Rm}|^2 - |\overline{\nabla}\mathrm{Rm}|^2 + C|\mathrm{Rm}|^3, \tag{3-2}$$

where we are writing $|\mathrm{Rm}|^2 = R_{\bar{j}i\bar{l}k} R^{i\bar{j}k\bar{l}}$ etc.

As before we set $f(t) = t, 1$. We introduce the function

$$\tilde{S} = fS + C_1 N \operatorname{tr} h \tag{3-3}$$

where C_1 is a large uniform constant. Note that by (2-9) we have $\tilde{S} \le C\frac{N^3}{r^2}$ at every $(x, t) \in \overline{B_{r_1}} \times [0, T]$. Furthermore $\tilde{S}$ satisfies

$$(\partial_t - \Delta)\tilde{S} \le -f(|\nabla X|^2 + |\overline{\nabla} X|^2) - C_2 S \tag{3-4}$$

where $C_2 = C_1 - f' \gg 1$ is uniform. Let $K = C_3 N^4/r^2$ where $C_3 \gg 1$ is a uniform constant. Note that we may assume $K/2 \le K - \tilde{S} \le K$. We will establish our bound for $|\mathrm{Rm}|$ by using a maximum principle argument for the function

$$F = f^2 \frac{\psi^2 |\mathrm{Rm}|^2}{K - \tilde{S}} + \tilde{B}\tilde{S}, \tag{3-5}$$

where $\tilde{B} = C_4/N^3$ with $C_4 \gg 1$ uniform. We begin by computing

$$\begin{aligned}(\partial_t - \Delta)\left(\psi^2 \frac{|\mathrm{Rm}|^2}{K - \tilde{S}}\right) = &- \Delta\psi^2 \frac{|\mathrm{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{(\partial_t - \Delta)|\mathrm{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{(\partial_t - \Delta)\tilde{S}}{(K - \tilde{S})^2}|\mathrm{Rm}|^2 \\ &- 2\psi^2 \frac{|\nabla\tilde{S}|^2 |\mathrm{Rm}|^2}{(K - \tilde{S})^3} - 4\,\mathrm{Re}\, \frac{\psi\langle\nabla\psi, \nabla|\mathrm{Rm}|^2\rangle}{K - \tilde{S}} \\ &- 4\,\mathrm{Re}\, \frac{\psi\langle\nabla\psi, \nabla\tilde{S}\rangle|\mathrm{Rm}|^2}{(K - \tilde{S})^2} - 2\,\mathrm{Re}\, \frac{\psi^2\langle\nabla|\mathrm{Rm}|^2, \nabla\tilde{S}\rangle}{(K - \tilde{S})^2}\end{aligned}$$

and thus

$$
\begin{aligned}
(3\text{-}6)\quad & (\partial_t - \Delta)\left(\psi^2 \frac{|\mathrm{Rm}|^2}{K-\tilde{S}}\right) \\
& \le \frac{1}{(K-\tilde{S})^2}\Big[|\Delta\psi^2|(K-\tilde{S})|\mathrm{Rm}|^2 \\
&\text{terms (2)–(4)} \qquad + \psi^2(K-\tilde{S})\left(C|\mathrm{Rm}|^3 - |\nabla \mathrm{Rm}|^2 - |\overline{\nabla}\mathrm{Rm}|^2\right) \\
&\text{terms (5)–(7)} \qquad + \psi^2\left(-f|\nabla X|^2 - f|\overline{\nabla}X|^2 - C_2 S\right)|\mathrm{Rm}|^2 \\
&\text{terms (8), (9)} \qquad - \frac{2}{K-\tilde{S}}\psi^2|\nabla\tilde{S}|^2|\mathrm{Rm}|^2 + 16|\nabla\psi|^2(K-\tilde{S})|\mathrm{Rm}|^2 \\
&\text{terms (10), (11)} \qquad + \tfrac{1}{2}\psi^2(K-\tilde{S})|\nabla\mathrm{Rm}|^2 + \tfrac{1}{2}\psi^2(K-\tilde{S})|\overline{\nabla}\mathrm{Rm}|^2 \\
&\text{terms (12), (13)} \qquad + \frac{1}{K-\tilde{S}}\psi^2|\nabla\tilde{S}|^2|\mathrm{Rm}|^2 + 4|\nabla\psi|^2(K-\tilde{S})|\mathrm{Rm}|^2 \\
&\text{term (14)} \qquad + \frac{4}{K-\tilde{S}}\psi^2|\nabla\tilde{S}|^2|\mathrm{Rm}|^2 \\
&\text{terms (15), (16)} \qquad + \tfrac{1}{2}\psi^2(K-\tilde{S})|\nabla\mathrm{Rm}|^2 + \tfrac{1}{2}\psi^2(K-\tilde{S})|\overline{\nabla}\mathrm{Rm}|^2\Big].
\end{aligned}
$$

We wish to bound (3-6) in terms of $|\mathrm{Rm}|^2$. Label the terms (1), (2), ..., (16), as shown. The bad terms are (1), (2), and (9)–(16), while the remaining terms are all good. One sees that

$$(1) + (9) + (13) \le C\frac{N}{Kr^2}|\mathrm{Rm}|^2,$$

while $[(10)+(11)+(15)+(16)] + [(3)+(4)] \le 0$ and $(12) + \frac{1}{2}(8) \le 0$. It remains only to bound the terms (2) and (14). For (2) we argue as follows: we may assume that at a maximum for the function F we have a lower bound of the form

$$f|\mathrm{Rm}| \ge CK, \quad C \gg 1, \tag{3-7}$$

for if not we can apply a maximum principle argument immediately: At any $(x,t) \in \overline{B_{r_1}} \times (0,T]$ we would have $F \le CK + C/r^2$, which implies that

$$f^2|\mathrm{Rm}|^2 \le C\frac{N^8}{r^4} \quad \text{on } \overline{B_{r_2}} \times (0,T].$$

Now since $\hat{\omega}$ is Euclidean we have

$$\left|\overline{\nabla}X\right|^2 = |\mathrm{Rm} - \widehat{\mathrm{Rm}}|^2 = |\mathrm{Rm}|^2. \tag{3-8}$$

Hence, using (3-7), we have $(2) + \frac{1}{2}(6) \le 0$. Finally, to control (14) we use

$$|\nabla\tilde{S}|^2 \le 4f^2S(|\nabla X|^2 + |\overline{\nabla}X|^2) + 2nC_1^2N^4S. \tag{3-9}$$

Here we have made use of a well-known estimate (computed in [Yau 1978]) that implies that $|\nabla \operatorname{tr} h|^2 \le nN^2 S$. Now we find (14) + $\frac{1}{2}$[(5) + (6) + (7)] ≤ 0 if in $K = C_3 N^4/r^2$ we choose $C_3 \gg C_1$. In total then we have

$$(\partial_t - \Delta)\left(\frac{\psi^2 |\mathrm{Rm}|^2}{K - \tilde{S}}\right) \le \frac{C}{N^3}|\mathrm{Rm}|^2. \tag{3-10}$$

Therefore

$$(\partial_t - \Delta)\left(\frac{\psi^2 f^2 |\mathrm{Rm}|^2}{K - \tilde{S}} + \tilde{B}\tilde{S}\right) \le -\frac{f}{N^3}|\mathrm{Rm}|^2, \tag{3-11}$$

if in $\tilde{B} = C_4/N^3$ we pick C_4 large enough. This implies that the maximum of F on $\overline{B_{r_1}} \times [0, T]$ can only occur at $t = 0$ or on the boundary of $\overline{B_r}$, where $\psi = 0$. Hence F is bounded above by C/r^2. Therefore at any (x, t) in $\overline{B_{r_2}} \times [0, T]$ we have $f^2|\mathrm{Rm}|^2 \le C' N^4/r^4$. Comparing with our comments following (3-7) we arrive at the estimate

$$|\mathrm{Rm}|^2 \le C\frac{N^8}{f(t)^2 r^4} \quad \text{on } \overline{B_{r_2}} \times (0, T]. \tag{3-12}$$

4. Higher-order estimates

We finish the proof of Theorem 1.1 by establishing bounds on the derivatives of curvature, following the basic idea of Shi [1989a; 1989b] (see also [Bando 1987; Chow and Knopf 2004; Chow et al. 2006]). Our setting here is slightly different from that of Shi, who assumes that curvature is uniformly bounded (independent of t) but that (1-2) does not necessarily hold. Although the result we need can be recovered from what is known in the literature, we include the short proof for the sake of completeness. Fix a sequence of radii $r = r_0 > r_1 > r_2 > \cdots > r/2$. For a fixed m we will denote by ψ a cutoff function that is zero outside $B_{r_{m+1}}$ and identically 1 on $\overline{B_{r_{m+2}}}$.

We now work in real coordinates, writing, in this section, ∇ for the real covariant derivative $\nabla_{\mathbb{R}}$. Write ∇^m for $\nabla\nabla\cdots\nabla$ (m times). The key evolution equation we need is due to Hamilton [1982]:

$$(\partial_t - \Delta)|\nabla^m \mathrm{Rm}|^2 = -|\nabla^{m+1}\mathrm{Rm}|^2 + \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm}, \tag{4-1}$$

where we are writing $S * T$ to denote a linear combination of the tensors S and T contracted with respect to the metric g. To clarify (4-1), we take Δ here to be the complex Laplacian, which, acting on functions, is half the usual Riemannian Laplace operator. When comparing to the formula in [Hamilton 1982] note that Hamilton's Ricci flow equation includes a factor of 2 that is not present in (1-1).

We will show inductively that

$$\left|\nabla^m \mathrm{Rm}\right|^2 \leq C\left(\frac{N^4}{f(t)r^2}\right)^{m+2} \quad \text{on } \overline{B_{r_{m+2}}} \times (0,T] \tag{4-2}$$

for every $m \geq 0$, the base case $m=0$ having already been established in Section 3. Assume (4-2) holds for every value $< m$. Let $A = N^4/r^2$. We will apply the maximum principle argument to the function

$$F = \psi^2 f^{m+2}|\nabla^m \mathrm{Rm}|^2 + Bf^{m+1}|\nabla^{m-1}\mathrm{Rm}|^2 \tag{4-3}$$

where $B = C_1 A$ with $C_1 \gg 1$ a large uniform constant. Let $(x_0, t_0) \in \overline{B_{r_{m+1}}} \times [0,T]$ be the point at which F achieves a maximum. We may assume that (x_0,t_0) lies in $B_{r_{m+1}} \times (0,T]$, otherwise, by the inductive hypothesis, we are finished. Suppose first that $f^{m+2}|\nabla^m \mathrm{Rm}|^2 \leq A^{m+2}$ at the point (x_0,t_0). Then at any $(x,t) \in \overline{B_{r_{m+2}}} \times [0,T]$ we have

$$f^{m+2}|\nabla^m \mathrm{Rm}|^2 \;\leq\; A^{m+2} + f^{m+1}B|\nabla^{m-1}\mathrm{Rm}|^2\big|_{(x_0,t_0)}, \tag{4-4}$$

and our claim follows by the inductive hypothesis. Otherwise we have

$$f^{m+2}|\nabla^m \mathrm{Rm}|^2 > A^{m+2} \text{ at } (x_0,t_0). \tag{4-5}$$

We note that by the inductive hypothesis we always have

$$|\nabla^i \mathrm{Rm}||\nabla^j \mathrm{Rm}| \;\leq\; C\left(\frac{A}{f}\right)^{\frac{i+j}{2}+2} \quad \text{when } i, j < m. \tag{4-6}$$

At (x_0,t_0),

$$\begin{aligned}
0 &\leq (\partial_t - \Delta)F \\
&\leq C\psi^2 f^{m+1}|\nabla^m \mathrm{Rm}|^2 + |\Delta\psi^2| f^{m+2}|\nabla^m \mathrm{Rm}|^2 - \psi^2 f^{m+2}|\nabla^{m+1}\mathrm{Rm}|^2 \\
&\quad + C\psi^2 f^{m+2}|\mathrm{Rm}||\nabla^m \mathrm{Rm}|^2 + C\psi^2 f^{m+2}(A/f)^{m/2+2}|\nabla^m \mathrm{Rm}| \\
&\quad + Cf^{m+2}\psi|\nabla\psi||\nabla^{m+1}\mathrm{Rm}||\nabla^m \mathrm{Rm}| \\
&\quad + CBf^m(A/f)^{m+1} - Bf^{m+1}|\nabla^m \mathrm{Rm}|^2 \\
&\quad + CBf^{m+1}|\mathrm{Rm}||\nabla^{m-1}\mathrm{Rm}|^2 + CBf^{m+1}(A/f)^{(m+3)/2}|\nabla^{m-1}\mathrm{Rm}| \\
&\leq Cf^{m+1}A|\nabla^m \mathrm{Rm}|^2 + Cf^{m/2}A^{m/2+2}|\nabla^m \mathrm{Rm}| \\
&\quad - C_1 A f^{m+1}|\nabla^m \mathrm{Rm}|^2 + CA^{m+3}f^{-1} \\
&\leq -f^{m+1}A|\nabla^m \mathrm{Rm}|^2 + C'A^{m+3}f^{-1},
\end{aligned} \tag{4-7}$$

where the final inequality follows from (4-5) and by taking the uniform constant C_1 in $B = C_1 A$ uniformly large enough. Hence $f^{m+2}|\nabla^m \mathrm{Rm}|^2 \leq C'A^{m+2}$ at (x_0,t_0)

and then, arguing in a similar way to (4-4) above, this completes the inductive step. Thus (4-2) is established.

5. Proof of Corollary 1.2

There are various ways to deduce Corollary 1.2 from Theorem 1.1. We could directly apply standard local parabolic theory (as discussed in [Chau 2004; Phong et al. 2011] for example), or the method in [Chow and Knopf 2004]. However, in our setting, we do not even need that $g(t)$ is a solution of a parabolic equation and instead we use an argument similar to one in [Song and Weinkove 2011b] which uses only standard linear elliptic theory and some embedding theorems.

Fix a time $t \in (0, T]$. Regarding $g_{i\bar{j}}$ as a set of n^2 functions, we consider the equations

$$\hat{\Delta} g_{i\bar{j}} = -\sum_k R_{k\bar{k}i\bar{j}} + \sum_{k,p,q} g^{q\bar{p}} \partial_k g_{i\bar{q}} \partial_{\bar{k}} g_{p\bar{j}} =: Q_{i\bar{j}}. \tag{5-1}$$

where $\hat{\Delta} = \sum_k \partial_k \partial_{\bar{k}}$. For each fixed i, j, we can regard (5-1) as Poisson's equation $\hat{\Delta} g_{i\bar{j}} = Q_{i\bar{j}}$.

For the purposes of this section we will say that a quantity Z is *uniformly bounded* if there exist constants C, α, β, γ depending only on $\hat{\omega}$ and T such that $Z \leq C N^{\alpha} r^{-\beta} t^{-\gamma}$. In the case when the constants may depend on ω_0, we insist that $\gamma = 0$.

Let $r = r_0 > r_1 > \cdots > r/2$ be as above. Fix $p > 2n$. From what we have proved, each $\|Q_{i\bar{j}}\|_{L^p(B_{r_2})}$ is uniformly bounded. Applying the standard elliptic estimates for the Poisson equation [Gilbarg and Trudinger 2001, Theorem 9.11] to (5-1) we see that the Sobolev norm $\|g_{i\bar{j}}\|_{L^p_2(B_{r_3})}$ is uniformly bounded. Morrey's embedding theorem [ibid., Theorem 7.17] gives that $\|g_{i\bar{j}}\|_{C^{1+\kappa}(B_{r_4})}$ is uniformly bounded for some $0 < \kappa < 1$.

The key observation we now need is that the mth derivative of $Q_{i\bar{j}}$ can be written as a finite sum $\sum_s A_s * B_s$ where each A_s or B_s is either a covariant derivative of Rm or a quantity involving derivatives of g up to order at most $m+1$. Hence if g is uniformly bounded in $C^{m+1+\kappa}$ then each $Q_{i\bar{j}}$ is uniformly bounded in $C^{m+\kappa}$, after possibly passing to a slightly smaller ball.

Applying this observation with $m = 0$ we see that each $\|Q_{i\bar{j}}\|_{C^{\kappa}(B_{r_4})}$ is uniformly bounded. The standard Schauder estimates for the Poisson equation [Gilbarg and Trudinger 2001, Theorem 4.8] give that $\|g_{i\bar{j}}\|_{C^{2+\kappa}(B_{r_5})}$ is uniformly bounded.

We can now apply a bootstrapping argument. Applying the observation with $m = 1$ we see that $Q_{i\bar{j}}$ is uniformly bounded in $C^{1+\kappa}$ on a slightly smaller ball and so on. This completes the proof of the corollary.

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Received July 18, 2011.

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Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 257 No. 2 June 2012

0030-8730(201206)257:2;1-5