

*Pacific  
Journal of  
Mathematics*

**INTERIOR DERIVATIVE ESTIMATES FOR THE  
KÄHLER-RICCI FLOW**

MORGAN SHERMAN AND BEN WEINKOVE

# INTERIOR DERIVATIVE ESTIMATES FOR THE KÄHLER–RICCI FLOW

MORGAN SHERMAN AND BEN WEINKOVE

**We give a maximum principle proof of interior derivative estimates for the Kähler–Ricci flow, assuming local uniform bounds on the metric.**

## 1. Introduction

Let  $(M, \hat{\omega})$  be a Kähler manifold of complex dimension  $n$ . Let  $\omega = \omega(t)$  be a solution of the Kähler–Ricci flow on  $M \times [0, T]$ , for some  $T > 0$ :

$$(1-1) \quad \frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0,$$

with  $\omega_0$  a smooth initial Kähler metric.

Fix a point  $p \in M$  and denote by  $B_r \subset M$  the open ball centered at  $p$  of radius  $r$  for  $0 < r < 1$  with respect to  $\hat{\omega}$ . We assume that  $r$  is sufficiently small so that  $\bar{B}_r$  is contained in a single holomorphic coordinate chart. Our main result is as follows:

**Theorem 1.1.** *Let  $N > 1$  satisfy*

$$(1-2) \quad \frac{1}{N} \hat{\omega} \leq \omega \leq N \hat{\omega}, \quad \text{on } \bar{B}_r \times [0, T].$$

*Then for each  $m = 0, 1, 2, \dots$  there exist constants  $C$  and  $C_m$  depending only on  $\hat{\omega}$  and  $T$  such that on  $B_{r/2} \times (0, T]$ ,*

- (i)  $|\hat{\nabla} \omega|_{\omega}^2 \leq C \frac{N^3}{r^2 t}$ , for  $\hat{\nabla}$  the covariant derivative with respect to  $\hat{\omega}$ .
- (ii)  $|\text{Rm}|_{\omega}^2 \leq C_0 \frac{N^8}{r^4 t^2}$ .
- (iii)  $|\nabla_{\mathbb{R}}^m \text{Rm}|_{\omega}^2 \leq C_m \left( \frac{N^4}{r^2 t} \right)^{m+2}$  for  $m = 1, 2, \dots$ , where  $\nabla_{\mathbb{R}}$  is the real covariant derivative with respect to the metric  $\omega$ .

---

Research supported in part by NSF grants DMS-0848193 and DMS-1105373. In addition, Sherman was supported in part by a California State Faculty Support Grant (SFSG) and Weinkove by a Sloan Fellowship.

MSC2010: 53C44, 53C55.

Keywords: Ricci flow, Kähler, interior estimates.

Moreover, if we allow the constants  $C$  and  $C_m$  to depend also on  $\omega_0$  then the estimates (i), (ii) and (iii) hold with each factor of  $t$  on the right hand side replaced by 1.

We prove this result using the maximum principle. Note that by work of Shi [1989a; 1989b] it was already known that a bound on curvature as in (ii) implies (iii) (nevertheless, we include a proof here, for the sake of completeness). [Theorem 1.1](#) implies the following:

**Corollary 1.2.** *Let  $N > 1$  satisfy*

$$(1-3) \quad \frac{1}{N} \hat{\omega} \leq \omega \leq N \hat{\omega} \quad \text{on } \bar{B}_r \times [0, T].$$

Then for each  $m = 0, 1, 2, \dots$  there exist constants  $C_m, \alpha_m, \beta_m$  and  $\gamma_m$  depending only on  $m, \hat{\omega}$  and  $T$  such that

$$(1-4) \quad |\hat{\nabla}_{\mathbb{R}}^m \omega|_{\hat{\omega}} \leq C_m \frac{N^{\alpha_m}}{r^{\beta_m} t^{\gamma_m}} \quad \text{on } B_{r/2} \times (0, T],$$

Moreover, if we allow the constants  $C_m, \alpha_m$  and  $\beta_m$  to depend also on  $\omega_0$  then (1-4) holds with  $\gamma_m = 0$ .

Namely, a local uniform estimate for the metric along the Kähler–Ricci flow implies local derivative estimates to all orders. This fact in itself is not new. Indeed the local PDE theory of Evans [1982] and Krylov [1982] can be applied to the Kähler–Ricci flow equation (see, for example, [Chow et al. 2007] or the generalization in [Gill 2011]). The key point here is to establish this via [Theorem 1.1](#) whose proof uses only elementary maximum principle arguments.

The form of the estimate (1-4) may be useful for applications and does not seem to be written down explicitly elsewhere in the literature. When considering the Kähler–Ricci flow on projective varieties, it is often the case that one obtains a uniform estimate for the metric  $\omega$  away from a subvariety (see [Song and Tian 2009; Song and Weinkove 2011a; 2011b; 2011c; Tian and Zhang 2006; Tsuji 1988; Zhang 2009], for example). [Theorem 1.1](#) can be used to replace global arguments. To illustrate, suppose that  $\omega = \omega(t)$  solves the Kähler–Ricci flow on a compact Kähler manifold  $M$  and there exists an analytic hypersurface  $D \subset M$  whose associated line bundle  $[D]$  admits a holomorphic section  $s$  vanishing to order 1 along  $D$ . Assume that

$$(1-5) \quad \frac{1}{C} |s|_H^\alpha \hat{\omega} \leq \omega \leq \frac{C}{|s|_H^\alpha} \hat{\omega} \quad \text{on } (M \setminus D) \times [0, T]$$

for some positive constants  $C$  and  $\alpha$ , where  $H$  is a Hermitian metric on  $[D]$ . An elementary argument shows that [Theorem 1.1](#) implies the existence of  $C_m, \alpha_m$  and

$\gamma_m$  such that

$$(1-6) \quad |\hat{\nabla}_{\mathbb{R}}^m \omega|_{\hat{\omega}} \leq \frac{C_m}{t^{\gamma_m} |S|_H^{\alpha_m}}, \quad \text{on } (M \setminus D) \times (0, T]$$

for each  $m = 1, 2, \dots$ . Moreover we can take  $\gamma_m = 0$  if we allow  $C_m$  and  $\alpha_m$  to depend on the initial metric  $\omega_0$ . Estimates of the form of (1-6) are used, for example, in [Song and Weinkove 2011b; 2011c]. In particular, Corollary 1.2 gives an alternative proof of the results in Section 4 of [Song and Weinkove 2011b].

Finally we remark that since our result is completely local, we may and do assume that  $M = \mathbb{C}^n$ ,  $p = 0$  and  $\hat{\omega}$  is the Euclidean metric. We will write  $g$  and  $\hat{g}$  for the Kähler metrics associated to  $\omega$  and  $\hat{\omega}$ . All magnitudes  $|\cdot|$  are taken with respect to the metric  $g$ . We shall use the letter  $C$  (as well as  $C'$ ,  $C''$ , etc.) for a uniform constant (depending only on  $m$ ,  $\hat{\omega}$ , and  $T$ ) which may differ from line to line.

In Sections 2, 3 and 4 we prove parts (i), (ii) and (iii) of Theorem 1.1 respectively. In Section 5 we give a proof of Corollary 1.2.

## 2. Bound on the first derivative of the metric

In this section we prove the estimate on the first derivative of the metric  $g$ , and so establish Theorem 1.1(i). This gives a local parabolic version of Calabi’s [1958] well-known “third-order” estimate for the complex Monge–Ampère equation (used by Yau [1978] in his solution of the Calabi conjecture). There exist now many generalizations of Calabi’s estimate [Cherrier 1987; Tosatti 2010; Tosatti et al. 2008; Zhang and Zhang 2011]. A global parabolic Calabi estimate was applied to the case of the Kähler–Ricci flow in [Cao 1985]. Phong, Sesum and Sturm [Phong et al. 2007] later gave a neat and explicit computation in this which we will make use of here for our local estimate.

We wish to bound the quantity

$$(2-1) \quad S = |\hat{\nabla} g|^2 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \hat{\nabla}_i g_{k\bar{q}} \overline{\hat{\nabla}_j g_{l\bar{p}}}$$

where we write  $\hat{\nabla}$  for the covariant derivative with respect to  $\hat{g}$ . Write  $r_0 = r$  and let  $\psi$  be a nonnegative  $C^\infty$  cut-off function that is identically equal to 1 on  $\overline{B_{r_1}}$  and vanishes outside  $B_r$ , where  $r_0 > r_1 > r/2$ . We may assume that

$$(2-2) \quad |\nabla \psi|^2, |\Delta \psi| \leq C \frac{N}{r^2},$$

where  $\Delta = \nabla^{\bar{j}} \nabla_{\bar{j}} = g^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$ . Thus

$$(2-3) \quad (\partial_t - \Delta)(\psi^2 S) \leq \psi^2 (\partial_t - \Delta)S + C \frac{N}{r^2} S + 2 |\langle \nabla \psi^2, \nabla S \rangle|,$$

where we are writing  $\langle \nabla F, \nabla G \rangle = g^{i\bar{j}} \partial_i F \partial_{\bar{j}} G$  for functions  $F, G$ . Following the notation in [Phong et al. 2007], we introduce the endomorphism  $h^i_k = \hat{g}^{i\bar{j}} g_{\bar{j}k}$  and let  $X$  be the tensor with components  $X^k_{i\bar{l}} = (\nabla_i h \cdot h^{-1})^k_{\bar{l}}$ , so that  $S = |X|^2$ . Note that  $X$  is the difference of the Christoffel symbols of  $g$  and  $\hat{g}$ .

An application of Young’s inequality gives

$$(2-4) \quad 2 |\langle \nabla \psi^2, \nabla S \rangle| \leq \psi^2 (|\nabla X|^2 + |\bar{\nabla} X|^2) + C \frac{N}{r^2} S.$$

We now use the evolution equation for  $S$  derived by Phong, Sesum and Sturm [ibid., (2.51)] which, in the case where  $\hat{\omega}$  is Euclidean, has the simple form:

$$(2-5) \quad (\partial_t - \Delta) S = - (|\nabla X|^2 + |\bar{\nabla} X|^2).$$

Combining (2-3), (2-4), and (2-5) we find

$$(2-6) \quad (\partial_t - \Delta)(\psi^2 S) \leq C \frac{N}{r^2} S.$$

We now need to use the evolution equation for  $\text{tr } h$  from [Cao 1985], which is a parabolic version of an estimate from [Aubin 1978; Yau 1978]. More precisely, we can apply equations (2.28) and (2.31) of [Phong et al. 2007] and use the fact that the fixed metric is Euclidean to obtain

$$(2-7) \quad (\partial_t - \Delta)(\text{tr } h) = -\hat{g}^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \hat{\nabla}_i g_{\bar{l}p} \overline{\hat{\nabla}_j g_{\bar{k}q}}.$$

Hence

$$(2-8) \quad (\partial_t - \Delta)(\text{tr } h) \leq -\frac{S}{N}.$$

Let  $f(t)$  denote either the function  $t$  or the constant 1. Then  $0 \leq f(t) \leq \max(T, 1)$  and  $f'(t) = 1$  or 0 so that we get, for any positive constant  $B$ ,

$$(\partial_t - \Delta)(f(t)\psi^2 S + B \text{tr } h) \leq C \frac{N}{r^2} S - \frac{B}{N} S.$$

Let  $B = (N^2/r^2)(C + 1)$ . Then, by the maximum principle, the maximum of  $f(t)\psi^2 S + B \text{tr } h$  on  $\bar{B}_r \times [0, T]$  can only occur at  $t = 0$  or on the boundary of  $\bar{B}_r$ , where  $\psi = 0$ . Since  $\text{tr } h \leq nN$ , we have

$$(2-9) \quad S \leq C \frac{N^3}{f(t)r^2} \text{ on } \bar{B}_{r_1} \times (0, T].$$

giving part (i) of Theorem 1.1.

### 3. Bound on curvature

We now prove part (ii) of [Theorem 1.1](#). For global estimates of this type, see [\[Chau 2004; Phong et al. 2011\]](#). We fix a smaller radius  $r_2$  satisfying  $r_1 > r_2 > r/2$ . In this section we let  $\psi$  be a cut-off function, identically 1 on  $\overline{B_{r_2}}$  and identically 0 outside  $B_{r_1}$ . As before we may assume  $|\Delta\psi|, |\nabla\psi|^2 \leq CN/r^2$  for some uniform constant  $C$ . Calculate

$$(3-1) \quad \begin{aligned} (\partial_t - \Delta)R_{\bar{j}i\bar{l}k} &= -R_{\bar{j}i}{}^{p\bar{q}}R_{\bar{l}k\bar{q}p} + R_{\bar{l}i}{}^{p\bar{q}}R_{\bar{j}k\bar{q}p} - R_{\bar{j}p}{}^{\bar{q}}R^p{}_{i\bar{q}k} \\ &\quad - R_{\bar{j}p}{}^p R^p{}_{i\bar{l}k} - R_{\bar{l}p}R_{\bar{j}i}{}^p{}_{k}, \end{aligned}$$

and therefore (see [\[Hamilton 1982\]](#))

$$(3-2) \quad (\partial_t - \Delta)|\text{Rm}|^2 \leq -|\nabla\text{Rm}|^2 - |\bar{\nabla}\text{Rm}|^2 + C|\text{Rm}|^3,$$

where we are writing  $|\text{Rm}|^2 = R_{\bar{j}i\bar{l}k}R^{i\bar{j}k\bar{l}}$  etc.

As before we set  $f(t) = t, 1$ . We introduce the function

$$(3-3) \quad \tilde{S} = fS + C_1 N \text{tr} h$$

where  $C_1$  is a large uniform constant. Note that by [\(2-9\)](#) we have  $\tilde{S} \leq C\frac{N^3}{r^2}$  at every  $(x, t) \in \overline{B_{r_1}} \times [0, T]$ . Furthermore  $\tilde{S}$  satisfies

$$(3-4) \quad (\partial_t - \Delta)\tilde{S} \leq -f(|\nabla X|^2 + |\bar{\nabla} X|^2) - C_2 S$$

where  $C_2 = C_1 - f' \gg 1$  is uniform. Let  $K = C_3 N^4 / r^2$  where  $C_3 \gg 1$  is a uniform constant. Note that we may assume  $K/2 \leq K - \tilde{S} \leq K$ . We will establish our bound for  $|\text{Rm}|$  by using a maximum principle argument for the function

$$(3-5) \quad F = f^2 \frac{\psi^2 |\text{Rm}|^2}{K - \tilde{S}} + \tilde{B} \tilde{S},$$

where  $\tilde{B} = C_4 / N^3$  with  $C_4 \gg 1$  uniform. We begin by computing

$$\begin{aligned} (\partial_t - \Delta) \left( \psi^2 \frac{|\text{Rm}|^2}{K - \tilde{S}} \right) &= -\Delta \psi^2 \frac{|\text{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{(\partial_t - \Delta)|\text{Rm}|^2}{K - \tilde{S}} + \psi^2 \frac{(\partial_t - \Delta)\tilde{S}}{(K - \tilde{S})^2} |\text{Rm}|^2 \\ &\quad - 2\psi^2 \frac{|\nabla\tilde{S}|^2 |\text{Rm}|^2}{(K - \tilde{S})^3} - 4 \text{Re} \frac{\psi \langle \nabla\psi, \nabla|\text{Rm}|^2 \rangle}{K - \tilde{S}} \\ &\quad - 4 \text{Re} \frac{\psi \langle \nabla\psi, \nabla\tilde{S} \rangle |\text{Rm}|^2}{(K - \tilde{S})^2} - 2 \text{Re} \frac{\psi^2 \langle \nabla|\text{Rm}|^2, \nabla\tilde{S} \rangle}{(K - \tilde{S})^2} \end{aligned}$$

and thus

$$\begin{aligned}
 (3-6) \quad (\partial_t - \Delta) \left( \psi^2 \frac{|\mathbf{Rm}|^2}{K - \tilde{S}} \right) &\leq \frac{1}{(K - \tilde{S})^2} \left[ |\Delta \psi^2| (K - \tilde{S}) |\mathbf{Rm}|^2 \right. \\
 \text{terms (2)–(4)} &\quad + \psi^2 (K - \tilde{S}) (C |\mathbf{Rm}|^3 - |\nabla \mathbf{Rm}|^2 - |\bar{\nabla} \mathbf{Rm}|^2) \\
 \text{terms (5)–(7)} &\quad + \psi^2 (-f |\nabla X|^2 - f |\bar{\nabla} X|^2 - C_2 S) |\mathbf{Rm}|^2 \\
 \text{terms (8), (9)} &\quad - \frac{2}{K - \tilde{S}} \psi^2 |\nabla \tilde{S}|^2 |\mathbf{Rm}|^2 + 16 |\nabla \psi|^2 (K - \tilde{S}) |\mathbf{Rm}|^2 \\
 \text{terms (10), (11)} &\quad + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla \mathbf{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\bar{\nabla} \mathbf{Rm}|^2 \\
 \text{terms (12), (13)} &\quad + \frac{1}{K - \tilde{S}} \psi^2 |\nabla \tilde{S}|^2 |\mathbf{Rm}|^2 + 4 |\nabla \psi|^2 (K - \tilde{S}) |\mathbf{Rm}|^2 \\
 \text{term (14)} &\quad + \frac{4}{K - \tilde{S}} \psi^2 |\nabla \tilde{S}|^2 |\mathbf{Rm}|^2 \\
 \text{terms (15), (16)} &\quad \left. + \frac{1}{2} \psi^2 (K - \tilde{S}) |\nabla \mathbf{Rm}|^2 + \frac{1}{2} \psi^2 (K - \tilde{S}) |\bar{\nabla} \mathbf{Rm}|^2 \right].
 \end{aligned}$$

We wish to bound (3-6) in terms of  $|\mathbf{Rm}|^2$ . Label the terms (1), (2),  $\dots$ , (16), as shown. The bad terms are (1), (2), and (9)–(16), while the remaining terms are all good. One sees that

$$(1) + (9) + (13) \leq C \frac{N}{Kr^2} |\mathbf{Rm}|^2,$$

while  $[(10) + (11) + (15) + (16)] + [(3) + (4)] \leq 0$  and  $(12) + \frac{1}{2}(8) \leq 0$ . It remains only to bound the terms (2) and (14). For (2) we argue as follows: we may assume that at a maximum for the function  $F$  we have a lower bound of the form

$$(3-7) \quad f |\mathbf{Rm}| \geq CK, \quad C \gg 1,$$

for if not we can apply a maximum principle argument immediately: At any  $(x, t) \in \bar{B}_{r_1} \times (0, T]$  we would have  $F \leq CK + C/r^2$ , which implies that

$$f^2 |\mathbf{Rm}|^2 \leq C \frac{N^8}{r^4} \quad \text{on } \bar{B}_{r_2} \times (0, T].$$

Now since  $\hat{\omega}$  is Euclidean we have

$$(3-8) \quad |\bar{\nabla} X|^2 = |\mathbf{Rm} - \widehat{\mathbf{Rm}}|^2 = |\mathbf{Rm}|^2.$$

Hence, using (3-7), we have  $(2) + \frac{1}{2}(6) \leq 0$ . Finally, to control (14) we use

$$(3-9) \quad |\nabla \tilde{S}|^2 \leq 4f^2 S (|\nabla X|^2 + |\bar{\nabla} X|^2) + 2nC_1^2 N^4 S.$$

Here we have made use of a well-known estimate (computed in [Yau 1978]) that implies that  $|\nabla \operatorname{tr} h|^2 \leq nN^2S$ . Now we find (14) +  $\frac{1}{2}[(5) + (6) + (7)] \leq 0$  if in  $K = C_3N^4/r^2$  we choose  $C_3 \gg C_1$ . In total then we have

$$(3-10) \quad (\partial_t - \Delta) \left( \frac{\psi^2 |\operatorname{Rm}|^2}{K - \tilde{S}} \right) \leq \frac{C}{N^3} |\operatorname{Rm}|^2.$$

Therefore

$$(3-11) \quad (\partial_t - \Delta) \left( \frac{\psi^2 f^2 |\operatorname{Rm}|^2}{K - \tilde{S}} + \tilde{B} \tilde{S} \right) \leq -\frac{f}{N^3} |\operatorname{Rm}|^2,$$

if in  $\tilde{B} = C_4/N^3$  we pick  $C_4$  large enough. This implies that the maximum of  $F$  on  $\overline{B_{r_1}} \times [0, T]$  can only occur at  $t = 0$  or on the boundary of  $\overline{B_r}$ , where  $\psi = 0$ . Hence  $F$  is bounded above by  $C/r^2$ . Therefore at any  $(x, t)$  in  $\overline{B_{r_2}} \times [0, T]$  we have  $f^2 |\operatorname{Rm}|^2 \leq C'N^4/r^4$ . Comparing with our comments following (3-7) we arrive at the estimate

$$(3-12) \quad |\operatorname{Rm}|^2 \leq C \frac{N^8}{f(t)^2 r^4} \quad \text{on } \overline{B_{r_2}} \times (0, T].$$

#### 4. Higher-order estimates

We finish the proof of [Theorem 1.1](#) by establishing bounds on the derivatives of curvature, following the basic idea of Shi [[1989a](#); [1989b](#)] (see also [[Bando 1987](#); [Chow and Knopf 2004](#); [Chow et al. 2006](#)]). Our setting here is slightly different from that of Shi, who assumes that curvature is uniformly bounded (independent of  $t$ ) but that (1-2) does not necessarily hold. Although the result we need can be recovered from what is known in the literature, we include the short proof for the sake of completeness. Fix a sequence of radii  $r = r_0 > r_1 > r_2 > \dots > r/2$ . For a fixed  $m$  we will denote by  $\psi$  a cutoff function that is zero outside  $B_{r_{m+1}}$  and identically 1 on  $\overline{B_{r_{m+2}}}$ .

We now work in real coordinates, writing, in this section,  $\nabla$  for the real covariant derivative  $\nabla_{\mathbb{R}}$ . Write  $\nabla^m$  for  $\nabla \nabla \dots \nabla$  ( $m$  times). The key evolution equation we need is due to Hamilton [[1982](#)]:

$$(4-1) \quad (\partial_t - \Delta) |\nabla^m \operatorname{Rm}|^2 = -|\nabla^{m+1} \operatorname{Rm}|^2 + \sum_{i+j=m} \nabla^i \operatorname{Rm} * \nabla^j \operatorname{Rm} * \nabla^m \operatorname{Rm},$$

where we are writing  $S * T$  to denote a linear combination of the tensors  $S$  and  $T$  contracted with respect to the metric  $g$ . To clarify (4-1), we take  $\Delta$  here to be the complex Laplacian, which, acting on functions, is half the usual Riemannian Laplace operator. When comparing to the formula in [[Hamilton 1982](#)] note that Hamilton’s Ricci flow equation includes a factor of 2 that is not present in (1-1).

We will show inductively that

$$(4-2) \quad |\nabla^m \text{Rm}|^2 \leq C \left( \frac{N^4}{f(t)r^2} \right)^{m+2} \quad \text{on } \overline{B_{r_{m+2}}} \times (0, T]$$

for every  $m \geq 0$ , the base case  $m = 0$  having already been established in [Section 3](#). Assume (4-2) holds for every value  $< m$ . Let  $A = N^4/r^2$ . We will apply the maximum principle argument to the function

$$(4-3) \quad F = \psi^2 f^{m+2} |\nabla^m \text{Rm}|^2 + B f^{m+1} |\nabla^{m-1} \text{Rm}|^2$$

where  $B = C_1 A$  with  $C_1 \gg 1$  a large uniform constant. Let  $(x_0, t_0) \in \overline{B_{r_{m+1}}} \times [0, T]$  be the point at which  $F$  achieves a maximum. We may assume that  $(x_0, t_0)$  lies in  $B_{r_{m+1}} \times (0, T]$ , otherwise, by the inductive hypothesis, we are finished. Suppose first that  $f^{m+2} |\nabla^m \text{Rm}|^2 \leq A^{m+2}$  at the point  $(x_0, t_0)$ . Then at any  $(x, t) \in \overline{B_{r_{m+2}}} \times [0, T]$  we have

$$(4-4) \quad f^{m+2} |\nabla^m \text{Rm}|^2 \leq A^{m+2} + f^{m+1} B |\nabla^{m-1} \text{Rm}|^2 \Big|_{(x_0, t_0)},$$

and our claim follows by the inductive hypothesis. Otherwise we have

$$(4-5) \quad f^{m+2} |\nabla^m \text{Rm}|^2 > A^{m+2} \text{ at } (x_0, t_0).$$

We note that by the inductive hypothesis we always have

$$(4-6) \quad |\nabla^i \text{Rm}| |\nabla^j \text{Rm}| \leq C \left( \frac{A}{f} \right)^{\frac{i+j}{2}+2} \quad \text{when } i, j < m.$$

At  $(x_0, t_0)$ ,

$$(4-7) \quad \begin{aligned} 0 &\leq (\partial_t - \Delta) F \\ &\leq C \psi^2 f^{m+1} |\nabla^m \text{Rm}|^2 + |\Delta \psi^2| f^{m+2} |\nabla^m \text{Rm}|^2 - \psi^2 f^{m+2} |\nabla^{m+1} \text{Rm}|^2 \\ &\quad + C \psi^2 f^{m+2} |\text{Rm}| |\nabla^m \text{Rm}|^2 + C \psi^2 f^{m+2} (A/f)^{m/2+2} |\nabla^m \text{Rm}| \\ &\quad + C f^{m+2} \psi |\nabla \psi| |\nabla^{m+1} \text{Rm}| |\nabla^m \text{Rm}| \\ &\quad + C B f^m (A/f)^{m+1} - B f^{m+1} |\nabla^m \text{Rm}|^2 \\ &\quad + C B f^{m+1} |\text{Rm}| |\nabla^{m-1} \text{Rm}|^2 + C B f^{m+1} (A/f)^{(m+3)/2} |\nabla^{m-1} \text{Rm}| \\ &\leq C f^{m+1} A |\nabla^m \text{Rm}|^2 + C f^{m/2} A^{m/2+2} |\nabla^m \text{Rm}| \\ &\quad - C_1 A f^{m+1} |\nabla^m \text{Rm}|^2 + C A^{m+3} f^{-1} \\ &\leq -f^{m+1} A |\nabla^m \text{Rm}|^2 + C' A^{m+3} f^{-1}, \end{aligned}$$

where the final inequality follows from (4-5) and by taking the uniform constant  $C_1$  in  $B = C_1 A$  uniformly large enough. Hence  $f^{m+2} |\nabla^m \text{Rm}|^2 \leq C' A^{m+2}$  at  $(x_0, t_0)$

and then, arguing in a similar way to (4-4) above, this completes the inductive step. Thus (4-2) is established.

### 5. Proof of Corollary 1.2

There are various ways to deduce Corollary 1.2 from Theorem 1.1. We could directly apply standard local parabolic theory (as discussed in [Chau 2004; Phong et al. 2011] for example), or the method in [Chow and Knopf 2004]. However, in our setting, we do not even need that  $g(t)$  is a solution of a parabolic equation and instead we use an argument similar to one in [Song and Weinkove 2011b] which uses only standard linear elliptic theory and some embedding theorems.

Fix a time  $t \in (0, T]$ . Regarding  $g_{i\bar{j}}$  as a set of  $n^2$  functions, we consider the equations

$$(5-1) \quad \hat{\Delta} g_{i\bar{j}} = - \sum_k R_{k\bar{k}i\bar{j}} + \sum_{k,p,q} g^{q\bar{p}} \partial_k g_{i\bar{q}} \partial_{\bar{k}} g_{p\bar{j}} =: Q_{i\bar{j}}.$$

where  $\hat{\Delta} = \sum_k \partial_k \partial_{\bar{k}}$ . For each fixed  $i, j$ , we can regard (5-1) as Poisson’s equation  $\hat{\Delta} g_{i\bar{j}} = Q_{i\bar{j}}$ .

For the purposes of this section we will say that a quantity  $Z$  is *uniformly bounded* if there exist constants  $C, \alpha, \beta, \gamma$  depending only on  $\hat{\omega}$  and  $T$  such that  $Z \leq CN^\alpha r^{-\beta} t^{-\gamma}$ . In the case when the constants may depend on  $\omega_0$ , we insist that  $\gamma = 0$ .

Let  $r = r_0 > r_1 > \dots > r/2$  be as above. Fix  $p > 2n$ . From what we have proved, each  $\|Q_{i\bar{j}}\|_{L^p(B_{r_2})}$  is uniformly bounded. Applying the standard elliptic estimates for the Poisson equation [Gilbarg and Trudinger 2001, Theorem 9.11] to (5-1) we see that the Sobolev norm  $\|g_{i\bar{j}}\|_{L^p_2(B_{r_3})}$  is uniformly bounded. Morrey’s embedding theorem [ibid., Theorem 7.17] gives that  $\|g_{i\bar{j}}\|_{C^{1+\kappa}(B_{r_4})}$  is uniformly bounded for some  $0 < \kappa < 1$ .

The key observation we now need is that the  $m$ th derivative of  $Q_{i\bar{j}}$  can be written as a finite sum  $\sum_s A_s * B_s$  where each  $A_s$  or  $B_s$  is either a covariant derivative of  $Rm$  or a quantity involving derivatives of  $g$  up to order at most  $m + 1$ . Hence if  $g$  is uniformly bounded in  $C^{m+1+\kappa}$  then each  $Q_{i\bar{j}}$  is uniformly bounded in  $C^{m+\kappa}$ , after possibly passing to a slightly smaller ball.

Applying this observation with  $m = 0$  we see that each  $\|Q_{i\bar{j}}\|_{C^\kappa(B_{r_4})}$  is uniformly bounded. The standard Schauder estimates for the Poisson equation [Gilbarg and Trudinger 2001, Theorem 4.8] give that  $\|g_{i\bar{j}}\|_{C^{2+\kappa}(B_{r_5})}$  is uniformly bounded.

We can now apply a bootstrapping argument. Applying the observation with  $m = 1$  we see that  $Q_{i\bar{j}}$  is uniformly bounded in  $C^{1+\kappa}$  on a slightly smaller ball and so on. This completes the proof of the corollary.

## References

- [Aubin 1978] T. Aubin, “Équations du type Monge–Ampère sur les variétés Kählériennes compactes”, *Bull. Sci. Math.* (2) **102**:1 (1978), 63–95. [MR 81d:53047](#) [Zbl 0374.53022](#)
- [Bando 1987] S. Bando, “Real analyticity of solutions of Hamilton’s equation”, *Math. Z.* **195**:1 (1987), 93–97. [MR 88i:53073](#) [Zbl 0606.58051](#)
- [Calabi 1958] E. Calabi, “Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens”, *Michigan Math. J.* **5**:2 (1958), 105–126. [MR 21 #5219](#) [Zbl 0113.30104](#)
- [Cao 1985] H.-D. Cao, “Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds”, *Invent. Math.* **81**:2 (1985), 359–372. [MR 87d:58051](#) [Zbl 0574.53042](#)
- [Chau 2004] A. Chau, “Convergence of the Kähler–Ricci flow on noncompact Kähler manifolds”, *J. Differential Geom.* **66**:2 (2004), 211–232. [MR 2005g:53118](#) [Zbl 1082.53070](#)
- [Cherrier 1987] P. Cherrier, “Équations de Monge–Ampère sur les variétés Hermitiennes compactes”, *Bull. Sci. Math.* (2) **111**:4 (1987), 343–385. [MR 89d:58131](#) [Zbl 0629.58028](#)
- [Chow and Knopf 2004] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs **110**, Amer. Math. Soc., Providence, RI, 2004. [MR 2005e:53101](#) [Zbl 1086.53085](#)
- [Chow et al. 2006] B. Chow, P. Lu, and L. Ni, *Hamilton’s Ricci flow*, Graduate Studies in Mathematics **77**, Amer. Math. Soc., Providence, RI, 2006. [MR 2008a:53068](#) [Zbl 1118.53001](#)
- [Chow et al. 2007] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, I: Geometric aspects*, Mathematical Surveys and Monographs **135**, Amer. Math. Soc., Providence, RI, 2007. [MR 2008f:53088](#) [Zbl 1157.53034](#)
- [Evans 1982] L. C. Evans, “Classical solutions of fully nonlinear, convex, second-order elliptic equations”, *Comm. Pure Appl. Math.* **35**:3 (1982), 333–363. [MR 83g:35038](#) [Zbl 0469.35022](#)
- [Gilbarg and Trudinger 2001] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 2001. [MR 2001k:35004](#) [Zbl 1042.35002](#)
- [Gill 2011] M. Gill, “Convergence of the parabolic complex Monge–Ampère equation on compact Hermitian manifolds”, *Comm. Anal. Geom.* **19**:2 (2011), 277–303. [MR 2835881](#) [Zbl 06031039](#)
- [Hamilton 1982] R. S. Hamilton, “Three-manifolds with positive Ricci curvature”, *J. Differential Geom.* **17**:2 (1982), 255–306. [MR 84a:53050](#) [Zbl 0504.53034](#)
- [Krylov 1982] N. V. Krylov, “Ограниченно неоднородные эллиптические и параболические уравнения”, *Izv. Akad. Nauk SSSR Ser. Mat.* **46**:3 (1982), 487–523. Translated as “Boundedly nonhomogeneous elliptic and parabolic equations” in *Math. USSR Izv.* **20**:3 (1983), 459–492. [MR 84a:35091](#) [Zbl 0529.35026](#)
- [Phong et al. 2007] D. H. Phong, N. Sesum, and J. Sturm, “Multiplier ideal sheaves and the Kähler–Ricci flow”, *Comm. Anal. Geom.* **15**:3 (2007), 613–632. [MR 2009a:32037](#) [Zbl 1143.53064](#)
- [Phong et al. 2011] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, “On the convergence of the modified Kähler–Ricci flow and solitons”, *Comment. Math. Helv.* **86**:1 (2011), 91–112. [MR 2012d:53218](#) [Zbl 1210.53066](#)
- [Shi 1989a] W.-X. Shi, “Deforming the metric on complete Riemannian manifolds”, *J. Differential Geom.* **30**:1 (1989), 223–301. [MR 90i:58202](#) [Zbl 0676.53044](#)
- [Shi 1989b] W.-X. Shi, “Ricci deformation of the metric on complete noncompact Riemannian manifolds”, *J. Differential Geom.* **30**:2 (1989), 303–394. [MR 90f:53080](#) [Zbl 0686.53037](#)
- [Song and Tian 2009] J. Song and G. Tian, “The Kähler–Ricci flow through singularities”, preprint, 2009. [arXiv 0909.4898](#)

- [Song and Weinkove 2011a] J. Song and B. Weinkove, “The Kähler–Ricci flow on Hirzebruch surfaces”, *J. Reine Angew. Math.* **659** (2011), 141–168. MR 2012g:53142 Zbl 05971442
- [Song and Weinkove 2011b] J. Song and B. Weinkove, “Contracting exceptional divisors by the Kähler–Ricci flow”, preprint, 2011. arXiv 1003.0718
- [Song and Weinkove 2011c] J. Song and B. Weinkove, “Contracting exceptional divisors by the Kähler–Ricci flow, II”, preprint, 2011. arXiv 1102.1759
- [Tian and Zhang 2006] G. Tian and Z. Zhang, “On the Kähler–Ricci flow on projective manifolds of general type”, *Chinese Ann. Math. Ser. B* **27**:2 (2006), 179–192. MR 2007c:32029 Zbl 1102.53047
- [Tosatti 2010] V. Tosatti, “Adiabatic limits of Ricci-flat Kähler metrics”, *J. Differential Geom.* **84**:2 (2010), 427–453. MR 2011m:32039 Zbl 1208.32024
- [Tosatti et al. 2008] V. Tosatti, B. Weinkove, and S.-T. Yau, “Taming symplectic forms and the Calabi–Yau equation”, *Proc. Lond. Math. Soc.* (3) **97**:2 (2008), 401–424. MR 2009h:32032 Zbl 1153.53054
- [Tsuji 1988] H. Tsuji, “Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type”, *Math. Ann.* **281**:1 (1988), 123–133. MR 89e:53075 Zbl 0631.53051
- [Yau 1978] S.-T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I”, *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411. MR 81d:53045 Zbl 0369.53059
- [Zhang 2009] Z. Zhang, “Scalar curvature bound for Kähler–Ricci flows over minimal manifolds of general type”, *Int. Math. Res. Not.* **2009**:20 (2009), 3901–3912. MR 2010j:32038 Zbl 1180.53068
- [Zhang and Zhang 2011] X. Zhang and X. Zhang, “Regularity estimates of solutions to complex Monge–Ampère equations on Hermitian manifolds”, *J. Funct. Anal.* **260**:7 (2011), 2004–2026. MR 2011m:32074 Zbl 1215.53038 arXiv 1007.2627

Received July 18, 2011.

MORGAN SHERMAN  
DEPARTMENT OF MATHEMATICS  
CALIFORNIA POLYTECHNIC STATE UNIVERSITY  
BUILDING 25  
SAN LUIS OBISPO, CA 93407  
UNITED STATES

[sherman1@calpoly.edu](mailto:sherman1@calpoly.edu)  
<http://www.calpoly.edu/~sherman1/>

BEN WEINKOVE  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF CALIFORNIA, SAN DIEGO  
9500 GILMAN DRIVE #0112  
LA JOLLA, CA 92093  
UNITED STATES

[weinkove@math.ucsd.edu](mailto:weinkove@math.ucsd.edu)  
<http://www.math.ucsd.edu/~bweinkov/>

# PACIFIC JOURNAL OF MATHEMATICS

<http://pacificmath.org>

Founded in 1951 by

E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

## EDITORS

V. S. Varadarajan (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[pacific@math.ucla.edu](mailto:pacific@math.ucla.edu)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Darren Long  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
[long@math.ucsb.edu](mailto:long@math.ucsb.edu)

Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

Robert Finn  
Department of Mathematics  
Stanford University  
Stanford, CA 94305-2125  
[finn@math.stanford.edu](mailto:finn@math.stanford.edu)

Jiang-Hua Lu  
Department of Mathematics  
The University of Hong Kong  
Pokfulam Rd., Hong Kong  
[jhlu@maths.hku.hk](mailto:jhlu@maths.hku.hk)

Jie Qing  
Department of Mathematics  
University of California  
Santa Cruz, CA 95064  
[qing@cats.ucsc.edu](mailto:qing@cats.ucsc.edu)

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Alexander Merkurjev  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[merkurev@math.ucla.edu](mailto:merkurev@math.ucla.edu)

Jonathan Rogawski  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[jonr@math.ucla.edu](mailto:jonr@math.ucla.edu)

## PRODUCTION

[pacific@math.berkeley.edu](mailto:pacific@math.berkeley.edu)

Silvio Levy, Scientific Editor

Mathew Cargo, Senior Production Editor

## SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI  
CALIFORNIA INST. OF TECHNOLOGY  
INST. DE MATEMÁTICA PURA E APLICADA  
KEIO UNIVERSITY  
MATH. SCIENCES RESEARCH INSTITUTE  
NEW MEXICO STATE UNIV.  
OREGON STATE UNIV.

STANFORD UNIVERSITY  
UNIV. OF BRITISH COLUMBIA  
UNIV. OF CALIFORNIA, BERKELEY  
UNIV. OF CALIFORNIA, DAVIS  
UNIV. OF CALIFORNIA, LOS ANGELES  
UNIV. OF CALIFORNIA, RIVERSIDE  
UNIV. OF CALIFORNIA, SAN DIEGO  
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ  
UNIV. OF MONTANA  
UNIV. OF OREGON  
UNIV. OF SOUTHERN CALIFORNIA  
UNIV. OF UTAH  
UNIV. OF WASHINGTON  
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

See inside back cover or [pacificmath.org](http://pacificmath.org) for submission instructions.

---

The subscription price for 2012 is US \$420/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

A NON-PROFIT CORPORATION

Typeset in L<sup>A</sup>T<sub>E</sub>X

Copyright ©2012 by Pacific Journal of Mathematics

# PACIFIC JOURNAL OF MATHEMATICS

Volume 257    No. 2    June 2012

---

|   |     |
|---|-----|
| Extending triangulations of the 2-sphere to the 3-disk preserving a 4-coloring                  | 257 |
| RUI PEDRO CARPENTIER  |     |
| Orthogonal quantum group invariants of links  | 267 |
| LIN CHEN and QINGTAO CHEN   |     |
| Some properties of squeezing functions on bounded domains                                       | 319 |
| FUSHENG DENG, QIAN GUAN and LIYOU ZHANG   |     |
| Representations of little $q$ -Schur algebras   | 343 |
| JIE DU, QIANG FU and JIAN-PAN WANG  |     |
| Renormalized weighted volume and conformal fractional Laplacians                                | 379 |
| MARÍA DEL MAR GONZÁLEZ  |     |
| The $L_4$ norm of Littlewood polynomials derived from the Jacobi symbol                         | 395 |
| JONATHAN JEDWAB and KAI-UWE SCHMIDT   |     |
| On a conjecture of Kaneko and Ohno  | 419 |
| ZHONG-HUA LI  |     |
| Categories of unitary representations of Banach–Lie supergroups and restriction functors        | 431 |
| STÉPHANE MERIGON, KARL-HERMANN NEEB and HADI SALMASIAN  |     |
| Odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields | 471 |
| LI REN, QIANG MU and YONGZHENG ZHANG  |     |
| Interior derivative estimates for the Kähler–Ricci flow   | 491 |
| MORGAN SHERMAN and BEN WEINKOVE   |     |
| Two-dimensional disjoint minimal graphs   | 503 |
| LINFENG ZHOU  |     |