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Under the assumption of Gauss curvature vanishing at infinity, we prove Meeks' conjecture: the number of disjointly supported minimal graphs in $\mathbb{R}^3$ is at most two.

1. Introduction

Let Ω be an open subset in $\mathbb{R}^2$ and denote its boundary by $\partial\Omega$. As we know, if a function $u(x)$ defined on Ω satisfies the equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0, \tag{1}$$

$G = \{(x, u(x)) : x \in \Omega\}$ is called a minimal graph in $\mathbb{R}^3$. We say the minimal graph G is supported on Ω if $u|_{\partial\Omega} = 0$ and $u \geq 0$.

Meeks [2005] has conjectured that the number of disjointly supported minimal graphs with zero boundary values over an open subset in $\mathbb{R}^2$ is at most 2. In fact, for arbitrary dimension, Meeks and Rosenberg [2005] proved if a set of disjointly supported minimal graphs have bounded gradient, then the number of the graphs must be finite. Later, Li and Wang [2001] gave an upper bound of the number of the graphs without any assumption on the growth rate of each graph. As a corollary, when minimal graphs are two dimensional in $\mathbb{R}^3$, they obtained the number is at most 24. At the same time, Spruck [2002] proved that there are at most two admissible sublinear growth solution pairs of Equation (1) defined over disjoint domains. Recently, by using angular density, Tkachev [2009] showed the number of two dimensional disjointly supported minimal graphs is less than or equals 3.

Observing the similarity between disjoint d-massive sets and disjointly supported minimal graphs, we can apply the method for proving the finiteness theorem of disjoint d-massive sets in $\mathbb{R}^2$ [Li and Wang 1999] to study disjoint minimal graphs. We obtain the following theorem:

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Theorem 1.1. *Suppose* $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$ *is a set of disjointly supported minimal graphs in* $\mathbb{R}^3$, *where each* Ω_i *is an open subset in* $\mathbb{R}^2$. *If the Gauss curvature* $K_i(x)$ *of each graph satisfies*

$$K_i(x) \to 0 \quad \text{as } |x| \to \infty,$$

then the number k is at most two.

By choosing a different region of integration, one obtains an improvement on a theorem of Spruck [2002]:

Corollary 1.2. *Suppose* $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$ *is a set of disjointly supported minimal graphs in* $\mathbb{R}^3$, *where each* Ω_i *is an open subset in* $\mathbb{R}^2$. *If each graph has sublinear growth*, *then k is at most two.*

2. Proof of Theorem 1.1

We denote the 3-dimensional ball of radius R centered at the origin of $\mathbb{R}^3$ by $B^3(R)$ and the 2-dimensional sphere of radius R by $S^2(R)$. The key is to estimate the sum of all curves' length $\ell(G_i \cap S^2(R))$ when R is sufficiently large.

Theorem 2.1. *Suppose* $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$ *is a set of disjointly supported minimal graphs in* $\mathbb{R}^3$, *where the Gauss curvature* $K_i(x)$ *of each* G_i *satisfies*

$$K_i(x) \to 0 \quad \text{as } |x| \to \infty.$$

For a sufficiently large radius R, *we have the bound*

$$\sum_{i=1}^k \ell(G_i \cap S^2(R)) \le \pi^2 R + o(1)R.$$

In the particular case when $k = 3$, *we have the better estimate*

$$\sum_{i=1}^3 \ell(G_i \cap S^2(R)) \le 2\sqrt{2}\pi R + o(1)R.$$

Before proving this, we introduce a lemma.

Lemma 2.2. *Let* $B^3_+(R)$ *be a* 3*-dimensional upper half-ball with radius* R *and let* $S^2_+(R)$ *be a* 2*-dimensional upper half-sphere. Suppose* $\pi_i : G_i \to \mathbb{R}^2$ *is the natural projection map. If* $\Sigma_1, \Sigma_2, \ldots, \Sigma_s$ *are planes in* $\mathbb{R}^3$ *such that the interiors of* $\pi_i(\Sigma_i \cap B^3_+(R))$ *are pairwise disjoint for sufficiently large* R, *we have*

$$\sum_{i=1}^s \ell(\Sigma_i \cap S^2_+(R)) \le \pi^2 R.$$

Moreover, *when* $s = 3$, *we have the better estimate*

$$\sum_{i=1}^3 \ell(\Sigma_i \cap S^2_+(R)) \le 2\sqrt{2}\pi R.$$

Proof. Suppose $D(R) = \{(x_1, x_1, 0) : x_1^2 + x_2^2 \le R^2\}$ is a disk in $\mathbb{R}^3$. Since each Σ_i is a plane, $\Sigma_i \cap D(R)$ is a chord; let θ_i be the corresponding central angle. Here we only need to consider the case that the union $\bigcup_{i=1}^{s}(\Sigma_i \cap D(R))$ is a polygon; otherwise, one can add more planes still satisfying the required conditions and such that the union of chords becomes a polygon.

If the center of the disk $D(R)$ is in the interior of the polygon or on one of the edges of the polygon, each central angle θ_i satisfies $0 < \theta_i \le \pi$. Since the interiors of the $\pi_i(\Sigma_i \cap B_+^3(R))$ are pairwise disjoint, a simple computation yields the bound

$$\ell(\Sigma_i \cap S_+^2(R)) \le \pi R \sin\frac{\theta_i}{2}.$$

on the length of the arc $\ell(\Sigma_i \cap S_+^2(R))$. The right-hand side achieves the maximum if and only if Σ_i is perpendicular to the disk $D(R)$. Thus

$$\begin{aligned}\sum_{i=1}^{s} \ell(\Sigma_i \cap S_+^2(R)) &\le \sum_{i=1}^{s} \pi R \sin\frac{\theta_i}{2} \le \pi R s \sin\Big(\frac{1}{s}\sum_{i}^{s}\frac{\theta_i}{2}\Big) \\ &\le \pi R s \sin\frac{\pi}{s} \le \pi^2 R.\end{aligned} \tag{2}$$

In the second inequality, we have used the concave property of the sine function on the interval $[0, \pi]$.

For the special case when $s = 3$, one gets from (2)

$$\sum_{i=1}^{3} \ell(\Sigma_i \cap S_+^2(R)) \le 3\pi R \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}\pi R. \tag{3}$$

If the center of the disk $D(R)$ is outside the polygon, there exists an i_0 such that $\theta_{i_0} > \pi$. For simplicity, let us assume $i_0 = s$. A similar computation leads to

$$\ell(\Sigma_i \cap S_+^2(R)) \le \begin{cases} \pi R \sin\dfrac{\theta_i}{2} & \text{for } 1 \le i \le s-1, \\ R\theta_s & \text{for } i = s. \end{cases}$$

In the first case, equality holds if and only if Σ_i is perpendicular to the disk, and in the second, if and only if Σ_s is in the same plane of the disk $D(R)$. Hence

$$\begin{aligned}\sum_{i=1}^{s} \ell(\Sigma_i \cap S_+^2(R)) &\le \sum_{i=1}^{s-1} \pi R \sin\frac{\theta_i}{2} + R\theta_s \le \sum_{i=1}^{s-1} \pi R \sin\frac{\theta_i}{2} + 2\pi R \sin\frac{\theta_s}{4} \\ &\le \pi R(s+1)\sin\frac{\pi}{s+1} \le \pi^2 R.\end{aligned} \tag{4}$$

If $s = 3$, by (4) we obtain that

$$\sum_{i=1}^{3} \ell(\Sigma_i \cap S_+^2(R)) \leq 4\pi R \sin\frac{\pi}{4} = 2\sqrt{2}\pi R \tag{5}$$

The conclusion is derived from (2), (4) and (3), (5). □

Proof of Theorem 2.1. For each minimal graph G_i, since the Gauss curvature $K_i = 0$ at infinity, it means G_i is asymptotic to a flat plane. Therefore, we can use the intersection of a plane Σ_i and $S_+^2(R)$ to approximate the curve $G_i \cap S^2(R)$. By Lemma 2.2, one has

$$\ell(G_i \cap S^2(R)) \leq \ell(\Sigma_i \cap S_+^2(R)) + o(1)R.$$

Therefore

$$\sum_{i=1}^{k} \ell(G_i \cap S^2(R)) \leq \sum_{i=1}^{k} \ell(\Sigma_i \cap S_+^2(R)) + o(1)R \leq \pi^2 R + 0(1)R. \qquad \square$$

The following area growth estimate of a minimal graph is proved using a well-known argument; one can see [Li and Wang 2001] for the details.

Lemma 2.3. *If $G = (\Omega, u)$ is a minimal graph in $\mathbb{R}^3$, the area of $G \cap B^3(R)$ satisfies*

$$A(G \cap B^3(R)) \leq 3\pi R^2.$$

Proof of Theorem 1.1. Let $B^3(R)$ be the ball of radius R in $\mathbb{R}^3$. Since

$$\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2 \leq \int_{G_i \cap \partial B^3(R))} u_i \Big(\tilde{\nabla} u_i \cdot \frac{\partial}{\partial r}\Big),$$

where $\tilde{\nabla}$ means the gradient operator on G_i, one has

$$\begin{aligned}
2\lambda_1^{1/2}(G_i \cap \partial B^3(R)) &\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2 \\
&\leq 2\lambda_1^{1/2} \int_{G_i \cap \partial B^3(R)} u_i \cdot \frac{\partial u_i}{\partial r} \\
&\leq \lambda_1 \int_{G_i \cap \partial B^3(R)} u_i^2 + \int_{G_i \cap \partial B^3(R)} \Big(\frac{\partial u_i}{\partial r}\Big)^2 \\
&\leq \int_{G_i \cap \partial B^3(R)} |\bar{\nabla} u_i|^2 + \int_{G_i \cap \partial B^3(R)} \Big(\frac{\partial u_i}{\partial r}\Big)^2 = \int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2.
\end{aligned}$$

Here $\lambda_1^{1/2}(G_i \cap \partial B^3(R))$ denotes the first Dirichlet eigenvalue on $G_i \cap \partial B^3(R)$. We know that

$$\lambda_1^{1/2}(G_i \cap \partial B^3(R)) \geq \frac{\pi^2}{\ell^2(G_i \cap \partial B^3(R))}$$

in $\mathbb{R}^3$. Therefore

$$\frac{\int_{G_i\cap\partial B^3(R)}|\tilde{\nabla}u_i|^2}{\int_{G_i\cap B^3(R)}|\tilde{\nabla}u_i|^2}\geq 2\lambda_1^{1/2}\geq\frac{2\pi}{\ell(\Gamma_i)},$$

where $\Gamma_i := G_i\cap\partial B^3(R)$. Thus we obtain

$$\sum_{i=1}^{k}\frac{\int_{G_i\cap\partial B^3(R)}|\tilde{\nabla}u_i|^2}{\int_{G_i\cap B^3(R)}|\tilde{\nabla}u_i|^2}\geq\sum_{i=1}^{k}\frac{2\pi}{\ell(\Gamma_i)}.$$

Notice that

$$k^2\leq\Bigl(\sum_{i=1}^{k}\ell(\Gamma_i)\Bigr)\Bigl(\sum_{i=1}^{k}\frac{1}{\ell(\Gamma_i)}\Bigr).$$

According to Theorem 2.1, one has

$$\sum_{i=1}^{k}\ell(\Gamma_i)\leq\pi^2R+o(1)R$$

for a sufficiently large radius R. Then it can be concluded that

$$\sum_{i=1}^{k}\frac{\int_{G_i\cap\partial B^3(R)}|\tilde{\nabla}u_i|^2}{\int_{G_i\cap B^3(R)}|\tilde{\nabla}u_i|^2}\geq\frac{2\pi k^2}{R(\pi^2+o(1))}.\tag{6}$$

Observing that

$$\int_{G_i\cap\partial B^3(r)}|\tilde{\nabla}u_i|^2=\frac{\partial}{\partial r}\int_{G_i\cap B^3(r)}|\tilde{\nabla}u_i|^2,\tag{7}$$

we obtain from (6) that

$$\ln\prod_{i=1}^{k}\frac{\int_{G_i\cap B^3(R)}|\tilde{\nabla}u_i|^2}{\int_{G_i\cap B^3(R_0)}|\tilde{\nabla}u_i|^2}\geq\frac{2\pi k^2}{\pi^2+o(1)}\ln\frac{R}{R_0}.\tag{8}$$

Let $(x, y, u_i(x, y))$ be a parametrization of G_i, so the induced metric on G_i is

$$ds^2=(1+(u_i)_x^2)\,dx^2+2(u_i)_x(u_i)_y\,dx\,dy+(1+(u_i)_y^2)\,dy^2.$$

We then have

$$|\tilde{\nabla}u_i|=\sqrt{u_{x^i}u_{x^j}g^{ij}}=\sqrt{\frac{|\nabla u_i|^2}{1+|\nabla u_i|^2}}\leq 1,$$

from which one can deduce

$$\prod_{i=1}^{k}\int_{G_i\cap B^3(R))}|\tilde{\nabla}u_i|^2\leq A^k(G_i\cap B^3(R))\leq(3\pi R^2)^k.\tag{9}$$

Combining (8) and (9) implies

$$\frac{2\pi k^2}{\pi^2+o(1)}(\ln R-\ln R_0)\le 2k\ln R+c_1.$$

Letting $R\to+\infty$ we see that $k\le\pi$; in particular, $k\le 3$.

If $k=3$, an analogous argument using the refined length estimate in Theorem 2.1 leads to $k\le 2\sqrt{2}$, which is a contradiction. Thus k has to be at most 2. □

Remark. Tkachev [2009] has already proved the number of two dimensional disjointly supported minimal graphs is at most 3. Here a different approach can lead to a better estimate if assuming the Gauss curvature vanishes at infinity.

3. Proof of Corollary 1.2

Let $\pi_i : G_i\to\mathbb{R}^2$ be the natural projective map and $B^2(R)$ be the ball of radius R in $\mathbb{R}^2$. By employing the same method in the proof of Theorem 1.1 except for using a different region of integration $\pi_i^{-1}(\Omega_i\cap B^2(R))$, one can conclude

Theorem 3.1. *Suppose $\{(\Omega_i,u_i)\}_{i=1}^k$ is a set of disjointly supported minimal graphs in R^3 where each Ω_i is an open subset in R^2. If the gradient of each u_i is bounded, say $|\nabla u_i|\le c$, then k satisfies $k\le 2\sqrt{1+c^2}$.*

Proof. By a similar argument, one can obtain that

$$\sum_{i=1}^k\frac{\int_{\pi_i^{-1}(\Omega_i\cap\partial B^2(R))}|\tilde{\nabla}u_i|^2}{\int_{\pi_i^{-1}(\Omega_i\cap B^2(R))}|\tilde{\nabla}u_i|^2}\ge\frac{2\pi k^2}{\sum_{i=1}^k\ell(\Gamma_i)}.$$

where $\Gamma_i:=\pi_i^{-1}(\Omega_i\cap\partial B^2(R))$. If one chooses the parametrization

$$(R\cos\theta,\,R\sin\theta,\,u_i(R\cos\theta,\,R\sin\theta))$$

for the curve Γ_i and assume $|\nabla u_i|\le c$, then

$$\begin{aligned}\ell(\Gamma_i)&=\int_{\theta_0}^{\theta_1}\sqrt{R^2+\big(-(u_i)_xR\sin(\theta)+(u_i)_yR\cos(\theta)\big)^2}d\theta\\&\le\int_{\theta_0}^{\theta_1}\sqrt{R^2+\big((u_i)_x^2+(u_i)_y^2\big)(R^2\sin(\theta)^2+R^2\cos(\theta)^2)}d\theta\\&\le(\theta_1-\theta_0)R\sqrt{1+c^2}.\end{aligned}$$

Since the minimal graphs are disjoint, we get

$$\sum_{i=1}^k\ell(\Gamma_i)\le 2\pi R\sqrt{1+c^2}.$$

Then it can be concluded that

$$\sum_{i=1}^{k} \frac{\int_{\pi_i^{-1}(\Omega_i \cap \partial B^2(R))} |\tilde{\nabla} u_i|^2}{\int_{\pi_i^{-1}(\Omega_i \cap B^2(R))} |\tilde{\nabla} u_i|^2} \geq \frac{k^2}{R\sqrt{1+c^2}}. \tag{10}$$

Integrating (10), one obtains

$$\ln \prod_{i=1}^{k} \frac{\int_{\pi_i^{-1}(\Omega_i \cap \partial B^2(R))} |\tilde{\nabla} u_i|^2}{\int_{\pi_i^{-1}(\Omega_i \cap B^2(R_0))} |\tilde{\nabla} u_i|^2} \geq \frac{k^2}{\sqrt{1+c^2}} \ln \frac{R}{R_0}. \tag{11}$$

On the other hand,

$$\begin{aligned} \prod_{i=1}^{k} \int_{\pi_i^{-1}(\Omega_i \cap B^2(R))} |\tilde{\nabla} u_i|^2 &\leq A^k(\pi_i^{-1}(\Omega_i \cap B^2(R))) \\ &= \left(\int_{\Omega_i \cap B^2(R)} \sqrt{1+|\nabla u|^2} \right)^k \leq (\sqrt{1+c^2}\pi R^2)^k. \end{aligned} \tag{12}$$

Combining (11) and (12), we have

$$\frac{k^2}{\sqrt{1+c^2}} (\ln R - \ln R_0) \leq 2k \ln R + c_1.$$

Letting $R \to +\infty$ yields

$$k \leq 2\sqrt{1+c^2}. \qquad \square$$

Obviously, Corollary 1.2 follows from above theorem when each graph satisfies

$$|\nabla u_i| \to 0 \quad (|x| \to +\infty).$$

Remark. J. Spruck [2002] proved Corollary 1.2 under the assumption of a certain decay rate at infinity for the Gauss curvature. However, here we do not need any restrictions on the Gauss curvature.

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