CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS IN $\mathbb{CP}^n$

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We present a method of construction of minimal and H-minimal Lagrangian submanifolds in complex projective space $\mathbb{CP}^{q+m}$ from a Legendrian submanifold in $\mathbb{S}^{2q+1}(1) \subset \mathbb{C}^{q+1}$ and a Lagrangian submanifold in $\mathbb{C}^m$ that is contained in $\mathbb{S}^{2m-1}(r)$. We also provide some explicit examples.

1. Introduction

Let $(N, J, \omega)$ be a Kähler manifold with $\dim_{\mathbb{C}} N = n$, where $J$ is the complex structure and $\omega$ is the Kähler form. An immersion $f : \Sigma \rightarrow N$ from a $q$-dimensional manifold $\Sigma$ into $N$ is called totally real if $f^* \omega = 0$. In particular, a totally real immersion $f$ is called Lagrangian if $q = n$.

We recall some definitions from Y. G. Oh’s paper [1993]. A vector field $V$ along a Lagrangian immersion $f : \Sigma \rightarrow N$ is called a Hamiltonian variation if the 1-form $\alpha_V := \left(V \cdot \omega\right)|_{\Sigma}$ is exact on $\Sigma$. A smooth family $\{f_t\}$ of immersions from $\Sigma$ into $N$ is called a Hamiltonian deformation if its derivative is Hamiltonian, and a Lagrangian immersion $f : \Sigma \rightarrow N$ is called Hamiltonian-minimal or H-minimal if it satisfies

$$\frac{d}{dt} \mid_{t=0} \text{vol } f_t(\Sigma) = 0$$

for all Hamiltonian deformations. The Euler–Lagrange equation of H-minimal Lagrangian submanifolds is

$$\delta \alpha_H = 0,$$

where $H$ is the mean curvature vector field of $f$ and $\delta$ is the codifferential operator on $\Sigma$ with respect to the induced metric. In particular, minimal Lagrangian submanifolds are trivially H-minimal.

In the past few decades, many geometers have given many methods of construction of minimal and H-minimal Lagrangian submanifolds in the complex space form. I. Castro and F. Urbano [1998] classified $\mathbb{S}^1$-invariant H-minimal Lagrangian submanifolds in $\mathbb{C}^2$, and in [Castro and Urbano 2004] they also constructed special

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Let $\mathbb{C}^m$ be the complex Euclidean space endowed with the standard Hermitian inner product $(z, w) = \sum_{j=1}^{m} z_j \bar{w}_j$ for $z = (z_1, \ldots, z_m)$, $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$ and the canonical complex structure $Jz = iz$. The real part of $(z, w)$ determines a metric $(z, w)$ on $\mathbb{C}^m$, i.e., $(z, w) = \Re(z, w)$. The Liouville 1-form on $\mathbb{C}^m$ is given by $\Omega = \frac{i}{2} \sum (z_j^i \bar{z}_j^i - \bar{z}_j^i z_j^i)$, and the Kähler form of $\mathbb{C}^m$ is $\omega_{\mathbb{C}^m} = d\Omega / 2$. Let $\mathbb{S}^{2q+1}(1)$ be the $(2q + 1)$-dimensional unit sphere in $\mathbb{C}^{q+1}$, and let $\mathcal{H} : \mathbb{S}^{2q+1}(1) \to \mathbb{CP}^q$, $Z \mapsto [Z]$, be the Hopf fibration of $\mathbb{S}^{2q+1}(1)$ over the complex projective space $\mathbb{CP}^q$. We say an immersion $\tilde{f} : \Sigma_1 \to \mathbb{S}^{2q+1}(1) \subset \mathbb{C}^{q+1}$, $p \mapsto \tilde{f}(p) = Z$, of a $q$-dimensional manifold $\Sigma_1$ into $\mathbb{S}^{2q+1}(1)$ is Legendrian if $\tilde{f}^* \Omega = 0$. In this case, $\tilde{f}$ is isotropic in $\mathbb{C}^{q+1}$, i.e., $\tilde{f}^* \omega_{\mathbb{C}^{q+1}} = 0$, and the normal bundle $T^\perp \Sigma_1$ in $T \mathbb{S}^{2q+1}(1)$ splits as $J(T \Sigma_1) \oplus \text{Span}_\mathbb{R} [JZ]$. This means that $\tilde{f}$ is horizontal with respect to the Hopf fibration $\mathcal{H}$, and hence $\tilde{f} = \mathcal{H} \circ \tilde{f} : \Sigma_1 \to \mathbb{CP}^q$ is a Lagrangian immersion and the metric induced on $\Sigma_1$ by $\tilde{f}$ and $\tilde{f}$ are the same.

In this paper we construct minimal and H-minimal Lagrangian submanifolds in $\mathbb{CP}^n$ from Legendrian submanifolds in odd-dimensional spheres and Lagrangian submanifolds in $\mathbb{C}^m$ which are contained in spheres. The basic theorem in our construction is as follows.

**Theorem 1.1.** Let $\tilde{f} : \Sigma_1 \to \mathbb{S}^{2q+1}(1)$ be a Legendrian immersion and $\hat{f} : \Sigma_2 \to \mathbb{C}^m$ a Lagrangian immersion with $\hat{f}(\Sigma_2) \subset \mathbb{S}^{2n-1}(r) \subset \mathbb{C}^m$. Write $Z = \hat{f}(p_1)$, $z = \hat{f}(p_2)$, $n = q + m$. Define a new map $\check{f} : \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2n+1}(1)$ by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(Z, z).$$

Then $f = \mathcal{H} \circ \check{f}$ is a Lagrangian immersion from $\Sigma_1 \times \Sigma_2$ into $\mathbb{CP}^n$. Moreover:

(i) The immersion $f$ is minimal if and only if $\check{f} = \mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$ is minimal and

$$(1.1) \quad \hat{H}^C - (\hat{H}^C, e_n)e_n = 0, \quad (\hat{H}^C, e_n) = \frac{i(n+1)r}{1+r^2}.$$
where $\hat{H}^C$ is the complex mean curvature vector of $\hat{f}$ and $e_n = iz/r$ defines a global vector field on $\Sigma_2$.

(ii) The immersion $f$ is $H$-minimal if and only if

$$\delta\alpha_{\hat{h}} + \delta\alpha_{\hat{h}} = r^2(\nabla h_n, e_n) - (r^2\hat{h}_n + (n + 1)r)\sum_{\lambda}(\hat{\nabla}_e e_n, e_{\lambda}),$$

where $\hat{h}_n = -\text{Im}((\hat{H}^C, e_n))$, and $\hat{\nabla}$ and $\{e_{\lambda}, e_n\}$ are respectively the connection and an orthonormal frame field on $\Sigma_2$ relative to the metric induced by $\hat{f}$.

As applications of Theorem 1.1, we have:

**Theorem 1.2.** Let $\tilde{f} : \Sigma^q \to \mathbb{S}^{2q+1}(1), \tilde{f}(p_1) = Z$, be a Legendrian immersion. If $\mathcal{H} \circ \tilde{f} : \Sigma_1 \to \mathbb{C}P^q$ is $H$-minimal, then $f = \mathcal{H} \circ \tilde{f}$ is an $H$-minimal Lagrangian immersion, where $\tilde{f} : \Sigma_1 \times T^{n-q} \to \mathbb{S}^{2n+1}(1)$ (with $T = S^1(1)$) is defined by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{n-q+1}}(Z, e^{it_1}, \ldots, e^{it_n}).$$

**Theorem 1.3.** Let $\tilde{f} : \Sigma^q \to \mathbb{S}^{2q+1}(1), \tilde{f}(p_1) = Z$, be a Legendrian immersion. Define the new map $\tilde{f} : \Sigma_1 \times S^{m-1} \times T^1 \to \mathbb{S}^{2m-1}(1)$ by

$$(p_1, x, e^{it}) \mapsto \frac{1}{\sqrt{2}}(Z, e^{it}x).$$

(i) If $q = m-1$ and $\mathcal{H} \circ \tilde{f} : \Sigma_1 \to \mathbb{C}P^{m-1}$ is minimal, then $f = \mathcal{H} \circ \tilde{f}$ is a minimal Lagrangian immersion.

(ii) If $\mathcal{H} \circ \tilde{f} : \Sigma_1 \to \mathbb{C}P^q$ is $H$-minimal, then $f = \mathcal{H} \circ \tilde{f}$ is an $H$-minimal Lagrangian immersion.

We prove these theorems in Section 3, and based on them, we give some explicit examples of minimal and $H$-minimal Lagrangian submanifolds in Section 4.

Throughout this paper, we use the following conventions for index ranges:

$$0 \leq A, B, C, \ldots \leq n; \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n;$$

$$1 \leq j, k, l, \ldots \leq q; \quad q+1 \leq \lambda, \mu, \nu, \ldots \leq n.$$

For conjugation, we use the conventions $\tilde{\omega}_{\bar{A}\bar{B}} = \omega_{\bar{A}\bar{B}}, \tilde{f}^i_\alpha = f^{\bar{A}}_i$, and so on.

2. Preliminaries

**Basic formulae of submanifolds in a Kähler manifold.** To study real submanifolds in a Kähler manifold, it is convenient to use formulae from the complex case. So, we first deduce some basic formulae that are not used frequently in the classical theory of submanifolds.
Let \( \Sigma \) be a smooth Riemannian manifold with \( \operatorname{dim} \mathbb{R} \Sigma = q \). Locally, we choose an orthonormal frame field \( \{ e_j \} \) of \( \Sigma \), and its dual \( \{ \theta^i \} \). Then the first Cartan structure equation of \( \Sigma \) is given by
\[
(2-1) \quad d\theta^j = -\theta^j_k \wedge \theta^k, \quad \theta^j + \theta^j_k = 0,
\]
where \( \theta^j_k \) are the connection forms with respect to the coframe field \( \theta^j \). Let \( N \) be a Kähler manifold with \( \operatorname{dim} \mathbb{C} N = n \). Locally, we choose a unitary frame field \( \{ e_\alpha \} \) of \((1,0)\)-type on \( N \), and denote its dual by \( \{ \phi_\alpha \} \). Then the structure equation is given by
\[
(2-2) \quad d\phi_\alpha = -\phi_\beta \wedge \phi_\beta, \quad \phi_\alpha \wedge \phi_\beta = 0,
\]
where \( \phi_\beta \) are the connection forms with respect to \( \phi_\alpha \).

Let \( f : \Sigma \to N \) be an isometric immersion. Set
\[
(2-3) \quad f^* \phi_\alpha = f^\alpha_j \theta^j.
\]
Taking the exterior derivative on both sides of (2-3), we obtain
\[
(2-4) \quad (df^\alpha_j - f^\alpha_k \theta^j + \phi_\beta \wedge f^\beta_j) \wedge \theta^j = 0
\]
by (2-1), (2-2) and (2-3). If we set
\[
(2-5) \quad Df^\alpha_j = df^\alpha_j - f^\alpha_k \theta^j + \phi_\beta \wedge f^\beta_j = f^\alpha_k \theta^k
\]
the covariant derivative of \( f^\alpha_j \), then we have \( f^\alpha_j = f^\alpha_j \) by (2-4). The tensor field \( \Pi^C = \sum_{j,k,\alpha} f^\alpha_j \theta^j \otimes \theta^k \otimes e_\alpha \) is called the complex second fundamental form of \( f \), and is a smooth section of the bundle \( T^* \Sigma \otimes T^* \Sigma \otimes T^{(1,0)} N \). The vector field \( H^C = \sum_{j,\alpha} f^\alpha_j e_\alpha \) is called the complex mean curvature vector field of \( f \).

If we split \( e_\alpha \) as \( e_\alpha = \frac{1}{2}(e_\alpha - ie_\alpha^*) \), then \( \{ e_\alpha, e_\alpha^* = J e_\alpha \} \) is an orthonormal frame field on \( N \), and its dual is denoted by \( \{ \phi^\alpha, \phi^\alpha* \} \). The first Cartan structure equation is given by
\[
(2-6) \quad d\phi^\alpha = -\phi_\beta \wedge \phi^\beta - \phi_\beta^* \wedge \phi^\beta^*, \quad d\phi^\alpha* = -\phi^\beta \wedge \phi^\beta - \phi^\beta^* \wedge \phi^\beta^*,
\]
where \( \phi_\beta, \phi_\beta^*, \phi_\beta^* \) and \( \phi_\beta^* \) are the connection forms with respect to the frame field \( \phi^\alpha, \phi^\alpha* \). Set
\[
(2-7) \quad f^* \phi^\alpha = a^\alpha_j \theta^j, \quad f^* \phi^\alpha* = a_j^\alpha \theta^j.
\]
Taking the exterior derivative of (2-7), by (2-1), (2-6) and (2-7), we obtain
\[
(2-8) \quad (da^\alpha_j - a^\alpha_k \theta^j_k + \phi_\beta^\alpha \phi_\beta^\alpha a^\beta_j + \phi^\beta \phi^\beta a^\beta_j) \wedge \theta^j = 0,
\]
\[
(2-9) \quad (da_j^\alpha - a_k^\alpha \theta^j_k + \phi^\beta \phi^\beta a^\beta_j + \phi^\beta \phi^\beta a^\beta_j) \wedge \theta^j = 0.
\]
Set
\begin{align}
\text{(2-10)} \quad Da_j^\alpha &= da_j^\alpha - a_k^\alpha \theta_j^k + \phi_\beta^\alpha a_j^\beta + \phi_{\beta*}^\alpha a_j^{\beta*} = h_j^\alpha \theta^k, \\
\text{(2-11)} \quad Da_j^{\alpha*} &= da_j^{\alpha*} - a_k^{\alpha*} \theta_j^k + \phi_{\beta*}^{\alpha*} a_j^\beta + \phi_{\beta*}^{\alpha*} a_j^{\beta*} = h_j^{\alpha*} \theta^k,
\end{align}

the covariant derivatives of $a_j^\alpha$ and $a_j^{\alpha*}$ respectively. Then, we know that $h_{jk}^\alpha = h_{kj}^\alpha$, $h_{jk}^{\alpha*} = h_{kj}^{\alpha*}$ by (2-8) and (2-9). Clearly, the tensor field
\[
\Pi = h_{jk}^\alpha \theta^j \otimes \theta^k \otimes \epsilon_\alpha + h_{jk}^{\alpha*} \theta^j \otimes \theta^k \otimes \epsilon_{\alpha*}
\]
is the real second fundamental form in the usual sense; it is a smooth section of the bundle $T^* \Sigma \otimes T^* \Sigma \otimes TN$. The vector field $H = \sum_j (h_{jj}^\alpha \epsilon_\alpha + h_{jj}^{\alpha*} \epsilon_{\alpha*})$ is the real mean curvature vector field of $f$.

The relationship between the real second fundamental form and the complex second fundamental form of $f$ is given by:

**Proposition 2.1.** With the notation above, we have
\[
\text{(2-12)} \quad h_{jk}^\alpha = \frac{1}{2} (f_{jk}^\alpha + f_{jk}^{\alpha*}), \quad h_{jk}^{\alpha*} = \frac{i}{2} (f_{jk}^{\alpha*} - f_{jk}^\alpha).
\]
Moreover, $f$ is minimal if and only if $H^C = 0$.

**Proof.** One readily checks that
\[
\text{(2-13)} \quad \varphi_\alpha = \phi_\alpha^\alpha + i \phi_\alpha^{\alpha*}.
\]
Then, from (2-3), we get
\[
\text{(2-14)} \quad f_j^\alpha = a_j^\alpha + ia_j^{\alpha*}.
\]

Since $N$ is kählerian, it’s easy to check that $\phi_\beta^\alpha = \phi_{\beta*}^{\alpha*}$ and $\phi_{\beta*}^{\alpha} = -\phi_\beta^\alpha$, which gives
\[
\text{(2-15)} \quad \varphi_{\beta*} = \phi_{\beta*} = i \phi_{\beta*}^{\alpha*},
\]
by (2-2), (2-6) and (2-13). By the definition of $f_{jk}^\alpha$ and (2-15), we have
\[
\text{(2-16)} \quad f_{jk}^\alpha \theta^k = D f_{jk}^\alpha = df_{jk}^\alpha - f_k^\alpha \theta_j^k + \varphi_{\beta*} f_j^\beta
\]
\[
= d(a_j^\alpha + ia_j^{\alpha*}) - (a_k^\alpha + ia_k^{\alpha*}) \theta_j^k + (\phi_{\beta}^\alpha - i \phi_{\beta*}^{\alpha*}) (a_j^\beta + ia_j^{\beta*})
\]
\[
= (da_j^\alpha - a_k^\alpha \theta_j^k + \phi_\beta^\alpha a_j^\beta + \phi_{\beta*}^{\alpha*} a_j^{\beta*}) + i(da_j^{\alpha*} - a_k^{\alpha*} \theta_j^k + \phi_{\beta*}^{\alpha*} a_j^\beta + \phi_\beta^\alpha a_j^{\beta*})
\]
\[
= (h_{jk}^\alpha + ih_{jk}^{\alpha*}) \theta^k,
\]
which gives (2-12). \qed
Note that the Kähler form of $N$ is $\omega_N = \frac{i}{2} \sum_{\alpha} \varphi_\alpha \wedge \varphi_\alpha^\ast$. So, for a vector field $V = v^\alpha \epsilon_\alpha + v^{\alpha^\ast} \epsilon_{\alpha^\ast}$ we have

$$V \rfloor \omega = \omega(V, \cdot) = \frac{i}{2} ((v^\alpha + iv^{\alpha^\ast}) \varphi_\alpha - (v^\alpha - iv^{\alpha^\ast}) \varphi_\alpha).$$

In particular, for the mean curvature vector field $H$ of a given isometric immersion $f : \Sigma \to N$, we have

$$\alpha_H := (H \rfloor \omega) \Sigma = h_j \theta^j, \quad h_j = \frac{i}{2} (f^a_{kk} f^a_j - f^{a^\ast}_{kk} f^{a^\ast}_j),$$

by (2-12) and (2-17). Therefore, the codifferential of $\alpha_H$ is given by

$$\delta \alpha_H = - \sum_j h_{jj},$$

where $h_{jk} \theta^k = dh_j - h_k \theta^j_k$ is the covariant derivative of $h_j$.

**Lagrangian submanifolds in $\mathbb{C}^m$ contained in a sphere.** Let $\mathbb{C}^{q+1}$ be complex Euclidean space as described in the introduction. Let $\hat{f} : \Sigma_2 \to \mathbb{C}^m$, $\hat{f}(p) = z$, be a Lagrangian immersion with $\hat{f}(\Sigma_2) \subset \mathbb{S}^{2m-1}(r)$. Locally, one can select an orthonormal frame field $e_{q+1}, \ldots, e_{n-1}, e_n = iz/r$ such that

$$dz = \sum_{\lambda=q+1}^n \hat{\theta}^\lambda e_\lambda, \quad ds^2_{\Sigma_2} = \sum_{\lambda=q+1}^n (\hat{\theta}^\lambda)^2.$$ 

Since $\hat{f}$ is Lagrangian, one readily checks that $e_\lambda$ is also a unitary frame field, i.e., $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. So, if we set

$$de_\lambda = \hat{\omega}_{\lambda\mu} e_\mu, \quad \hat{\omega}_{\lambda\bar{\mu}} = (de_\lambda, e_\mu),$$

then

$$\hat{\omega}_{\lambda\bar{\mu}} + \hat{\omega}_{\bar{\mu}\lambda} = 0,$$

because $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. Obviously, we have

$$\langle dz, e_\lambda \rangle = \hat{\theta}^\lambda, \quad \hat{\omega}_{\lambda\bar{n}} = - \hat{\omega}_{\bar{n}\lambda} = \left(\frac{i}{r} dz, e_\lambda \right) = \frac{i}{r} \hat{\theta}^\lambda.$$ 

Denote by $\hat{\theta}^\lambda_\mu$ the connection 1-forms with respect to the frame field $\hat{\theta}^\lambda$. Set

$$\hat{\theta}^\lambda_\mu = \hat{\Gamma}^\lambda_{\nu\mu} \hat{\theta}^\nu. \quad f^* \hat{\omega}_{\lambda\bar{\mu}} = \hat{\Lambda}_{\lambda\bar{\mu}}^{\lambda\bar{\nu}} \hat{\theta}^\nu.$$ 

We then obtain the complex second fundamental form of $\hat{f}$. That is:

$$\hat{f}_{\lambda\mu}^{\lambda\nu} = - \hat{\Gamma}^\lambda_{\lambda\mu} + \hat{\Lambda}_{\lambda\bar{\nu}}^{\lambda\bar{\mu}},$$

by (2-5) and the fact that $\hat{f}_{\lambda\mu}^{\lambda\nu} = \delta_{\lambda\mu}$. So, by (2-18), we obtain
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Note that $e_n$ is a globally defined vector field on $\Sigma_2$, so $(\hat{H}, e_n) = \sum_{\lambda} \hat{f}^n_{\lambda\lambda}$, which plays an important role in our main construction, is a globally defined smooth complex-valued function on $\Sigma_2$.

**Lagrangian submanifolds in $\mathbb{C}P^n$.** Complex projective space $\mathbb{C}P^n$ is the set of all one-dimensional complex lines through the origin in $\mathbb{C}^{n+1}$. It can be written as $\mathbb{C}P^n \cong U(n+1)/(U(1) \times U(n))$, where $U(n+1)$ is the unitary group; thus, $U(n+1)$ is a principal $U(1) \times U(n)$-bundle over $\mathbb{C}P^n$.

Let $Z_0, Z_1, \ldots, Z_n$ be a moving frame of $\mathbb{C}^{n+1}$. We have

\[
(2-24) \quad dZ_A = \omega_{A\bar{B}} Z_B, \quad \omega_{A\bar{B}} = (dZ_A, Z_B),
\]

where $\omega_{A\bar{B}} = (dZ_A, Z_B)$ are the Maurer–Cartan forms of $U(n+1)$. They are skew-Hermitian, i.e.,

\[
(2-25) \quad \omega_{A\bar{B}} + \omega_{\bar{B}A} = 0.
\]

Taking the exterior derivative of (2-24), we get the Maurer–Cartan equation of $U(n+1)$:

\[
(2-26) \quad d\omega_{A\bar{B}} = \sum_C \omega_{A\bar{C}} \wedge \omega_{C\bar{B}},
\]

\[
(2-27) \quad ds^2_{FS} = \sum_\alpha \omega_{0\bar{\alpha}} \omega_{0\alpha},
\]

determines a Kähler metric on $\mathbb{C}P^n$, called the Fubini–Study metric. The Kähler form of $ds^2_{FS}$ is given by

\[
\omega_{FS} = \frac{i}{2} \sum_\alpha \omega_{0\bar{\alpha}} \wedge \omega_{0\alpha}.
\]

If we set $\varphi_\alpha := \omega_{0\bar{\alpha}}$, then $\{\varphi_\alpha\}$ is a unitary frame field on $\mathbb{C}P^n$ of $(1,0)$-type (see [Griffiths 1974] for details). Therefore, by the Maurer–Cartan equation (2-26), we obtain the first structure equation:

\[
(2-28) \quad d\varphi_\alpha = -\varphi_{\beta\bar{\alpha}} \wedge \varphi_\beta, \quad \varphi_{\beta\bar{\alpha}} = \omega_{\beta\bar{\alpha}} - \omega_{0\bar{\alpha}} \delta_{\alpha\beta}, \quad \varphi_{\beta\bar{\alpha}} + \varphi_{\bar{\alpha}\bar{\beta}} = 0,
\]

where $\varphi_{\beta\bar{\alpha}}$ are the connection forms with respect to the frame field $\varphi_\alpha$.

Let $\Sigma$ be a smooth manifold with dim $\Sigma = q$, and let $f$ be an immersion from $\Sigma$ into $\mathbb{C}P^n$. Let $U \subset \Sigma$ be an open set. We say $Z : U \rightarrow U(n+1)$ is a moving frame along $f$ if $Z$ satisfies $f = \pi \circ Z$, where $\pi$ is the canonical projection. For a moving frame along a totally real immersion $f$, we have:
**Proposition 2.2.** Let \( f \) be a totally real immersion from \( \Sigma \) into \( \mathbb{C}P^n \). If \( U \) is any small enough open subset of \( \Sigma \), and the induced metric on \( U \) is given by 
\[
(f^*ds^2_{FS}) = \sum_j (\theta^j)^2,
\]
then there exists a moving frame \( Z \) along \( f \) such that 
\[
(2-29) \quad \omega_{0\bar{0}} = 0, \quad \omega_{0\bar{j}} = \theta^j, \quad \omega_{0\bar{\alpha}} = 0,
\]
where the \( \omega_{A\bar{B}} \) are the pull-backs of the Maurer–Cartan forms of \( U(n+1) \) by \( Z^* \).

**Proof.** Throughout this proof, we will assume that the neighborhoods chosen are small enough to satisfy the topological assumptions.

Without loss of generality, we may assume \( f(U) \) is contained in a small open set \( V \) of \( \mathbb{C}P^n \). Let \( e_j \) be the dual frame field of \( \theta^j \). We extend \( \varepsilon_j = \frac{1}{2}(e_j - iJ e_j) \) smoothly to \( V \) and choose \( \varepsilon_\lambda \) on \( V \) such that \( \{\varepsilon_\alpha\} \) is smooth unitary frame on \( V \). Let \( \{\varphi_\alpha\} \) be the dual of \( \{\varepsilon_\alpha\} \). Then \( \{\varphi_\alpha\} \) is a unitary coframe field of \( (1, 0) \)-type on \( V \) and satisfies \( f^*\varphi_j = \theta^j \), \( f^*\varphi_\lambda = 0 \). Notice that we have used the fact that \( f \) is totally real in choosing \( \varepsilon_i \).

Let \( \mathcal{F}_1 = (Z_0, Z_1, \ldots, Z_n)^T : V \to U(n+1) \) be a local section of the principal bundle \( \pi : U(n+1) \to \mathbb{C}P^n \). Then \( \{\mathcal{F}_1^*\omega_{0\bar{a}}\} \) is a unitary coframe field of \( (1, 0) \)-type (see [Griffiths 1974]) on \( V \). Therefore, there exists a unitary matrix \( A = (a_{\alpha\bar{\beta}})_{n \times n} \) defined on \( V \) such that \( \varphi_\alpha = \sum_\beta a_{\alpha\bar{\beta}}\mathcal{F}_1^*\omega_{0\bar{\beta}} \). If we choose another local section \( \mathcal{F}_2 = (Z_0, \tilde{Z}_1, \ldots, \tilde{Z}_n)^T : V \to U(n+1) \) such that \( \tilde{Z}_\alpha = \sum_\beta a_{\alpha\bar{\beta}}Z_\beta \), then 
\[
\varphi_\alpha = \mathcal{F}_2^*\omega_{0\bar{a}}
\]
by \( (2-24) \).

Set \( \tilde{Z} = \mathcal{F}_2^* f \). One can check that \( \tilde{Z}^*\omega_{0\bar{i}} = \theta^i \) and \( \tilde{Z}^*\omega_{0\bar{i}} = 0 \), so \( d\tilde{Z}^*\omega_{0\bar{0}} = 0 \) by the Maurer–Cartan equation \( (2-26) \), i.e., \( \tilde{Z}^*\omega_{0\bar{0}} \) is a closed 1-form on \( U \), so one can find a smooth function \( u \) defined on \( U \) such that \( i du = \tilde{Z}^*\omega_{0\bar{0}} \). Taking \( Z = e^{-iu} \tilde{Z} \), it is easily checked that the pull-back of the Maurer–Cartan form of \( U(n+1) \) by \( Z^* \) is \( (2-29) \). This completes the proof. \( \square \)

Let \( f : \Sigma \to \mathbb{C}P^n \) be a Lagrangian isometric immersion, and let \( \theta^\alpha \) be an orthonormal frame field on \( \Sigma \). By Proposition 2.2, there exists a moving frame \( Z_0, Z_1, \ldots, Z_n \) along \( f \) such that 
\[
(2-30) \quad \varphi_\alpha = \omega_{0\bar{a}} = \theta^\alpha.
\]
For later use, we set 
\[
(2-31) \quad \omega_{\alpha\bar{\beta}} = \Lambda_{\alpha\bar{\beta}, \gamma} \theta^\gamma, \quad \omega_{0\bar{0}} = \Lambda_{0\bar{0}, \gamma} \theta^\gamma,
\]
and let 
\[
(2-32) \quad \theta^\alpha_{\bar{\beta}} = \Gamma^\alpha_{\gamma\bar{\beta}} \theta^\gamma
\]
be the connection 1-forms with respect to \( \theta^\alpha \).
Note that $f^\alpha_\beta = \delta^\alpha_\beta$ by (2-30), and so the complex second fundamental form of $f$ is given by

$$f^\alpha_\beta = -\Gamma^\alpha_\beta = (2-33)$$

by (2-5), (2-28), (2-31) and (2-32). So, we obtain

$$\alpha_H = h_\beta \theta^\beta, \quad h_\beta = \frac{i}{2} (f^\alpha_\beta - f^\alpha_\beta),$$

by (2-18).

3. Proof of Theorem 1.1

Let $\tilde{f}: \Sigma_1 \to \mathbb{S}^{2q+1}(1)$, $\tilde{f}(p) = \tilde{Z}_0$, be a Legendrian isometric immersion. Then $\tilde{f} = \mathcal{H} \circ \tilde{f}: \Sigma_1 \to \mathbb{CP}^q$, $p \mapsto \tilde{f}(p) = [\tilde{Z}_0]$ is a Lagrangian isometric immersion. Since $\tilde{f}$ is a Legendrian immersion, one readily checks that

$$\tilde{\omega}_{00} = (d\tilde{Z}_0, \tilde{Z}_0) = 0.$$  

By Proposition 2.2, one can choose a pairwise Hermitian orthogonal local frame field, $\tilde{Z}_0, \ldots, \tilde{Z}_q$, such that $\tilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_q$ is a moving frame along $\tilde{f}$, and

$$\tilde{\omega}_{0j} = (d\tilde{Z}_0, \tilde{Z}_j) = \tilde{\theta}_j.$$  

are real 1-forms. As before, we set

$$\tilde{\omega}_{jk} = (d\tilde{Z}_j, \tilde{Z}_k) = \tilde{\Lambda}_{jk,l} \tilde{\theta}^l.$$  

If we denote the connection 1-forms with respect to $\tilde{\theta}^j$ by $\tilde{\theta}^j_k = \tilde{\Gamma}^j_{lk} \tilde{\theta}^l$, by similar calculations to those in Section 2, we obtain the complex fundamental form of $\tilde{f}$. That is,

$$\tilde{f}_{kl} = -\tilde{\Gamma}_{lk}^j + \tilde{\Lambda}_{jk,l},$$

by (3-1).

Define the map $\tilde{f}: \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2n+1}(1)$ by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(\tilde{f}(p_1), \tilde{f}(p_2)) = \frac{1}{\sqrt{1+r^2}}(\tilde{Z}_0, z),$$

with $\tilde{f}$ and $\tilde{z}$ as before. We will study the map $f = \mathcal{H} \circ \tilde{f}: \Sigma_1 \times \Sigma_2 \to \mathbb{CP}^n$, given by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}[\tilde{Z}_0, z].$$
We chose the moving frame $Z_0, Z_1, \ldots, Z_n$ as follows:

\[
Z_0 = \frac{1}{\sqrt{1+r^2}}(\tilde{Z}_0, z),
\]
\[
Z_j = (\tilde{Z}_j, 0),
\]
\[
Z_\lambda = (0, e_\lambda), \quad q + 1 \leq \lambda < n,
\]
\[
Z_n = \frac{1}{\sqrt{1+r^2}}(-ir\tilde{Z}_0, e_n),
\]

where $\tilde{Z}_0, \tilde{Z}_j$ and $e_\lambda, e_n$ are as they were in the context of $\tilde{f}$ and $\hat{f}$, respectively.

According to (2-24), we obtain

\[
\omega_{0j} = \frac{1}{\sqrt{1+r^2}}(d\tilde{Z}_0, \tilde{Z}_j) = \frac{1}{\sqrt{1+r^2}}\tilde{\omega}_{0j} = \frac{1}{\sqrt{1+r^2}}\hat{\theta}^j =: \theta^j,
\]
\[
\omega_{0\lambda} = \frac{1}{\sqrt{1+r^2}}(dz, e_\lambda) = \frac{1}{\sqrt{1+r^2}}\hat{\theta}^\lambda =: \theta^\lambda, \quad q + 1 \leq \lambda < n
\]
\[
\omega_{0n} = (dZ_0, Z_n) = \frac{1}{1+r^2}(dz, e_n) = \frac{1}{1+r^2}\hat{\theta}^n =: \theta^n,
\]

by (2-21) and (3-1). Similarly,

\[
\omega_{0\bar{\alpha}} = \frac{1}{1+r^2}(dz, z) = \frac{ir}{1+r^2}(dz, e_n) = ir\theta^n,
\]
\[
\omega_{\bar{\alpha}\bar{\beta}} = \hat{\omega}_{\bar{\alpha}\bar{\beta}}, \quad \omega_{\bar{\alpha}j} = 0, \quad \omega_{\bar{\alpha}n} = -ir\theta^j,
\]
\[
\omega_{\bar{\beta}n} = \frac{i}{r}\theta^\lambda, \quad \omega_{n\bar{\alpha}} = \frac{i}{r}\theta^n.
\]

where $q + 1 \leq \lambda$ and $\mu < n$.

Since $\theta^j, \theta^\lambda, \theta^n$ are real and linearly independent on $\Sigma_1 \times \Sigma_2$, so $f$ is an immersion and the induced metric is given by

\[
ds^2 = f^*ds^2_{\Sigma} = \sum_\alpha (\theta^\alpha)^2
\]
\[
= \sum_{j=1}^q \left( \frac{1}{\sqrt{1+r^2}}\hat{\theta}^j \right)^2 + \sum_{\lambda=q+1}^{n-1} \left( \frac{1}{\sqrt{1+r^2}}\hat{\theta}^\lambda \right)^2 + \left( \frac{1}{1+r^2}\hat{\theta}^n \right)^2,
\]

which is a product metric. If we choose the orthonormal frame field $\theta^\alpha$ on $\Sigma_1 \times \Sigma_2$, then

\[
f^*\omega_{0\bar{\alpha}} = \theta^\alpha, \quad f^*_\beta = \delta_{\alpha\beta}.
\]
The pull back of the Kähler form is

\[ f^* \omega_{FS} = \frac{i}{2} \sum_{\alpha} \omega_{0\bar{\alpha}} \wedge \omega_{\bar{\alpha}} = \frac{i}{2} \sum_{\alpha} \theta^\alpha \wedge \theta^\alpha = 0, \]

and thus \( f \) is a Lagrangian immersion.

**Lemma 3.1.** Let

\[ d\tilde{s}^2 = \sum_{\alpha=1}^{n} (\tilde{\theta}^\alpha)^2 \quad \text{and} \quad ds^2 = \sum_{\alpha=1}^{n} (\theta^\alpha)^2 = \sum_{\alpha=1}^{n} (a_\alpha \tilde{\theta}^\alpha)^2 \]

be two metrics, where the \( a_\alpha \) are positive constants. Let

\[ \tilde{\theta}^\alpha = \tilde{\Gamma}^\alpha_{\gamma\beta} \tilde{\theta}^\gamma \quad \text{and} \quad \theta^\alpha = \Gamma^\alpha_{\gamma\beta} \theta^\gamma \]

be the connection 1-forms with respect to \( \tilde{\theta}^\alpha \) and \( \theta^\alpha \). Then

(3-13)

\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{a} \tilde{\Gamma}^\alpha_{\beta\gamma}, \]

and if \( a_1 = \cdots = a_n = a \), then

(3-14)

\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{a} \tilde{\Gamma}^\alpha_{\beta\gamma}, \]

and if \( a_1 = \cdots = a_{n-1} = a, a_n = a^2 \), then

(3-15)

\[ \Gamma^\lambda_{\mu\mu} = \frac{1}{a} \tilde{\Gamma}^\lambda_{\mu\mu}, \quad \Gamma^\lambda_{nn} = \frac{1}{a} \tilde{\Gamma}^\lambda_{nn}, \quad \Gamma^\lambda_{\mu\mu} = \frac{1}{a^2} \tilde{\Gamma}^\lambda_{\mu\mu}, \quad \Gamma^\lambda_{nn} = \frac{1}{a^2} \tilde{\Gamma}^\lambda_{nn}, \]

where \( 1 \leq \lambda \) and \( \mu \leq n - 1 \).

**Proof.** Denote the dual of \( \{\tilde{\theta}^\alpha\} \) by \( \{\tilde{e}_\alpha\} \). Then \( \{e_\alpha = \frac{1}{a_\alpha} \tilde{e}_\alpha\} \) is the dual of \( \{\theta^\alpha\} \).

Since both \( \{\tilde{\theta}^\alpha\} \) and \( \{\theta^\alpha\} \) are orthonormal, we obtain

(3-16)

\[ \tilde{\Gamma}^\alpha_{\beta\gamma} = -\Gamma^\alpha_{\beta\gamma}, \quad \Gamma^\alpha_{\beta\gamma} = -\Gamma^\alpha_{\beta\gamma}, \]

By the structure equation (2-1), one can check that

(3-17)

\[ [\tilde{e}_\beta, \tilde{e}_\gamma] = \tilde{C}^\alpha_{\beta\gamma} \tilde{e}_\alpha, \quad \tilde{C}^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} - \tilde{\Gamma}^\alpha_{\beta\gamma}, \]

which gives

(3-18)

\[ [e_\beta, e_\gamma] = C^\alpha_{\beta\gamma} e_\alpha, \quad C^\alpha_{\beta\gamma} = \frac{a_\alpha}{a_\beta a_\gamma} \tilde{C}^\alpha_{\beta\gamma}. \]

Note that \( e_\alpha \) are orthonormal with respect to the metric \( ds^2 \), so by Koszul’s formula [Petersen 1998], we have

\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \left( C^\alpha_{\beta\gamma} - C^\beta_{\gamma\alpha} + C^\gamma_{\alpha\beta} \right), \]

which, using (3-16)–(3-18), gives (3-13). This completes the proof. \( \square \)
Next, we calculate the mean curvature of \( f \). Noting that \( ds^2 \) is a product metric, we obtain

\[
\begin{align*}
\Gamma^j_{kl} &= \sqrt{1+r^2} \tilde{\Gamma}^j_{kl}, \quad \Gamma^\lambda_{\alpha j} = \Gamma^j_{\alpha \lambda} = 0, \\
\Gamma^\lambda_{\mu \mu} &= \sqrt{1+r^2} \tilde{\Gamma}^\lambda_{\mu \mu}, \\
\Gamma^n_{\mu \mu} &= \sqrt{1+r^2} \tilde{\Gamma}^n_{\mu \mu}, \\
\Gamma^n_{\mu \mu} &= (1+r^2) \tilde{\Gamma}^n_{\mu \mu},
\end{align*}
\]

by (3-14) and (3-15) from Lemma 3.1.

From (3-8)–(3-10), one readily checks that

\[
\begin{align*}
\lambda^0 = i r \delta_{\alpha}, \\
\lambda^j_{\alpha} = \lambda^j_{\alpha} = 0, \\
\lambda^0_{\alpha} = -i r, \\
\lambda^n_{\alpha} = 0,
\end{align*}
\]

where \( q + 1 \leq \lambda < n \).

**Proof of Theorem 1.1.** According to the identities (2-33) and (3-4), we obtain

\[
\begin{align*}
f^j_{kk} &= -\Gamma^j_{kk} + \lambda^j_{k,k} = \sqrt{1+r^2}(-\tilde{\Gamma}^j_{kk} + \tilde{\lambda}^j_{k,k}) = \sqrt{1+r^2} \tilde{f}^j_{kk}, \\
\end{align*}
\]

by (3-19) and (3-22). Similarly, we obtain

\[
\begin{align*}
f^j_{\lambda \lambda} &= 0, \\
f^j_{\mu \mu} &= \sqrt{1+r^2} \tilde{f}^j_{\mu \mu}, \\
f^n_{\mu \mu} &= \sqrt{1+r^2} \tilde{f}^n_{\mu \mu}, \\
f^n_{\lambda \lambda} &= (1+r^2) \tilde{f}^n_{\lambda \lambda} - i r, \\
f^j_{jj} &= -i r, \\
f^n_{nn} &= (1+r^2) \tilde{f}^n_{nn} - 2i r.
\end{align*}
\]

So, \( f \) is minimal if and only if

\[
\begin{align*}
\sum_{k=1}^q \tilde{f}^j_{kk} &= 0, \quad 1 \leq j \leq q, \\
\sum_{\mu=q+1}^n \tilde{f}^\lambda_{\mu \mu} &= 0, \quad q + 1 \leq \mu < n, \quad \text{and} \quad \sum_{\lambda=q+1}^n \tilde{f}^n_{\lambda \lambda} = \frac{i(n+1)r}{1+r^2},
\end{align*}
\]

by Proposition 2.1. This completes the first part of Theorem 1.1.

To prove the second part, we must calculate the 1-form \( \alpha_H = (H | \omega)_{\Sigma_1 \times \Sigma_2} = \sum_{\beta=1}^n h_\beta \theta^\beta \). According to (2-34), (3-25)-(3-28), we obtain
CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS IN \( \mathbb{C}P^n \)

\[ h_j = \sqrt{1 + r^2 \tilde{h}_j}, \quad h_{\lambda} = \sqrt{1 + r^2 \hat{h}_{\lambda}}, \quad q + 1 \leq \lambda < n, \]
\[ h_n = (1 + r^2) \hat{h}_n + (n + 1)r, \]

where we have set

\[ \alpha_{\tilde{H}} = \sum_j \tilde{h}_j \tilde{\theta}^j \quad \text{and} \quad \alpha_{\hat{H}} = \sum_{\lambda} \hat{h}_{\lambda} \hat{\theta}^\lambda. \]

Next, we calculate the \( \delta \alpha_{\tilde{H}} \). The covariant derivative of \( h_\beta \) is given by

\[ Dh_\beta = h_{\beta\gamma} \theta^\gamma = dh_\beta - h_{\gamma} \theta^\gamma \]

and because \( ds^2 \) is a product metric, we have

\[ h_{j\gamma} \theta^\gamma = dh_j - h_{\gamma} \theta^\gamma = dh_j - h_k \theta^k \]
\[ = \sqrt{1 + r^2 (d\tilde{h}_j - \tilde{h}_k \theta^k)} \]
\[ = \sqrt{1 + r^2 (\tilde{h}_{j;k} \tilde{\theta}^k - \tilde{h}_l \Gamma^l_{kj} \theta^k)} \]
\[ = \sqrt{1 + r^2 (\tilde{h}_{j;k} - \tilde{h}_l \Gamma^l_{kj}) \tilde{\theta}^k} \]
\[ = (1 + r^2) \tilde{h}_{jk} \theta^k, \]

by (3-5), (3-30) and Lemma 3.1, which gives

\[ h_{jj} = (1 + r^2) \tilde{h}_{jj}. \]

Here, we write \( d\tilde{h}_j = \tilde{h}_{j;k} \tilde{\theta}^k \) and \( D\tilde{h}_j = \tilde{h}_{jk} \tilde{\theta}^k = d\tilde{h}_j - \tilde{h}_k \tilde{\theta}^k \). Similarly, we have

\[ h_{\lambda\lambda} = (1 + r^2) \hat{h}_{\lambda\lambda} - (1 + r^2) (r^2 \hat{h}_n + (n + 1)r) \hat{\Gamma}^n_{\lambda\lambda}, \quad q + 1 \leq \lambda < n, \]
\[ h_{nn} = (1 + r^2) \hat{h}_{nn} + r^2 (1 + r^2) \hat{h}_{n;n}. \]

So, by (2-19) and (3-32)–(3-33), we obtain

\[ \delta \alpha_{\tilde{H}} = (1 + r^2) (\delta \alpha_{\tilde{H}} + \delta \alpha_{\hat{H}}) + (1 + r^2) (r^2 \hat{h}_n + (n + 1)r) \hat{\Gamma}^n_{\lambda\lambda} - r^2 \hat{h}_{n;n}, \]

where \( \delta \alpha_{\tilde{H}}, \delta \alpha_{\hat{H}} \) are the codifferentials of \( \alpha_{\tilde{H}}, \alpha_{\hat{H}} \) with respect to \( \tilde{f}, \hat{f} \) respectively. This completes the proof. \[ \square \]

4. Some explicit examples

As a first example, we study the standard Lagrangian torus

\[ \hat{f}: \mathbb{S}^1(1) \times \cdots \times \mathbb{S}^1(1) \to \mathbb{C}^m, \quad z = \hat{f}(p) = (e^{it_1}, \ldots, e^{it_m}), \]

where we parametrize \( \mathbb{S}^1(1) \) by \( \mathbb{S}^1(1) = \{e^{it} : 0 \leq t \leq 2\pi\} \).
Choosing the moving frame of $\mathbb{C}^m$ along $\hat{f}$ to be

\begin{align}
\hat{e}_\lambda &= \frac{i}{\sqrt{\lambda(\lambda+1)}} (e^{it_1}, \ldots, e^{it_\lambda}, -\lambda e^{i(\lambda+1)}, 0, \ldots, 0), \quad 1 \leq \lambda < m, \\
\hat{e}_m &= \frac{i}{\sqrt{m}} (e^{it_1}, \ldots, e^{it_m}),
\end{align}

it is easy to check that the coefficients $\hat{\theta}^\lambda := (dz, e_\lambda)$ satisfy

\begin{align}
\hat{\theta}^\lambda &= \frac{1}{\sqrt{\lambda(\lambda+1)}} (dt_1 + \cdots + dt_\lambda - \lambda dt_{\lambda+1}), \quad 1 \leq \lambda < m, \\
\hat{\theta}^m &= \frac{1}{\sqrt{m}} (dt_1 + \cdots + dt_m), \\
dt_\lambda &= -(\lambda - 1) \frac{\hat{\theta}^{\lambda-1}}{\sqrt{(\lambda-1)\lambda}} + \sum_{\mu=\lambda+1}^{m-1} \frac{\hat{\theta}^\mu}{\sqrt{\mu(\mu+1)}} + \frac{\hat{\theta}^m}{\sqrt{m}}.
\end{align}

From (4-1)–(4-5), we have

\begin{align}
\hat{\omega}^{\lambda\bar{\lambda}} &= (de_\lambda, e_\lambda) = -\frac{i(\lambda-1)\hat{\theta}^\lambda}{\sqrt{\lambda(\lambda+1)}} + \sum_{\mu=\lambda+1}^{m-1} \frac{i\hat{\theta}^\mu}{\sqrt{\mu(\mu+1)}} + \frac{i\hat{\theta}^m}{\sqrt{m}}, \quad \lambda < m, \\
\hat{\omega}^{\lambda\bar{\mu}} &= -\hat{\omega}^{\mu\bar{\lambda}} = -(de_\mu, e_\lambda) = \frac{i\hat{\theta}^\lambda}{\sqrt{\mu(\mu+1)}}, \quad \lambda < \mu < m, \\
\hat{\omega}^{\lambda\bar{m}} &= -\hat{\omega}^{m\bar{\lambda}} = -(de_m, e_\lambda) = \frac{i\hat{\theta}^\lambda}{\sqrt{m}}, \quad \lambda < m, \\
\hat{\omega}^{m\bar{m}} &= i\hat{\theta}^m/\sqrt{m}.
\end{align}

The metric induced by $\hat{f}$ is flat, so we obtain, by (2-22),

\begin{align}
\hat{f}_{\mu\bar{\mu}}^{\lambda} &= \hat{\Lambda}_{\mu\bar{\lambda},\mu} = 0, \quad \lambda < \mu < m, \\
\hat{f}_{\lambda\bar{\lambda}}^{\mu} &= \hat{\Lambda}_{\lambda\bar{\lambda},\mu} = -\frac{i(\lambda-1)}{\sqrt{\lambda(\lambda+1)}}, \quad \lambda < m, \\
\hat{f}_{\mu\bar{\mu}}^{\lambda} &= \hat{\Lambda}_{\mu\bar{\alpha},\mu} = i\frac{1}{\sqrt{\lambda(\lambda+1)}}, \quad \mu < \lambda < m, \\
\hat{f}_{mm}^{\lambda} &= 0, \quad \hat{f}_{\lambda\bar{\lambda}}^{m} = \hat{f}_{m\bar{m}}^{\lambda} = i\frac{1}{\sqrt{m}}, \quad \lambda < m.
\end{align}

**Proposition 4.1.** Let $\hat{f} : \mathbb{T}^1(1) \times \cdots \times \mathbb{T}^1(1) \to \mathbb{C}^m$, $z = \hat{f}(p) = (e^{it_1}, \ldots, e^{it_m})$, be the standard Lagrangian torus in $\mathbb{C}^m$. Its complex mean curvature $\hat{H}^C$ satisfies

$$
\hat{H}^C - (\hat{H}^C, e_m)e_m = 0, \quad (\hat{H}^C, e_m) = i\sqrt{m}.
$$
Moreover, if we set \( \hat{h}_m = - \text{Im}((\hat{H}^C, e_m)) \), we have
\[
m \langle \text{grad} \hat{h}_m, e_m \rangle - (m \hat{h}_m + (n + 1) \sqrt{m}) \sum_{\lambda} \langle \hat{V}_{e_{\lambda}}, e_m, e_{\lambda} \rangle = 0.
\]

**Proof.** The first part holds because of (4-10)–(4-13). The second part is true because the induced metric is flat and \( \hat{h}_m \) is a constant. \( \square \)

**Proof of Theorem 1.2.** The theorem follows from Theorem 1.1, Proposition 4.1, and the fact that the standard torus studied above is H-minimal in \( \mathbb{C}^m \). \( \square \)

For the next example, consider
\[
S^{m-1}(1) = \{ x \in \mathbb{R}^m : |x| = 1 \}
\]
with its standard embedding in \( \mathbb{C}^m \). Take an orthonormal tangent frame field \( \hat{e}_1, \ldots, \hat{e}_{m-1} \), with respect to which the metric is expressed by
\[
dx = \sum_{\lambda=1}^{m-1} \hat{\theta}^\lambda \hat{e}_\lambda, \quad d\hat{\theta}^\lambda = -\hat{\theta}^\lambda \wedge \hat{\theta}^\mu;
\]
the coefficients \( \hat{\theta}^\lambda \) and \( \hat{\theta}^\mu \) are real. Further, set
\[
d\hat{e}_\lambda = \sum_{\mu} \hat{\omega}_{\lambda \mu} \hat{e}_\mu, \quad 1 \leq \lambda < m,
\]
where the \( \hat{\omega}_{\lambda \mu} \) are real and satisfy \( \hat{\omega}_{\lambda \mu} + \hat{\omega}_{\mu \lambda} = 0 \).

Take the immersion \( \hat{f} : S^{m-1}(1) \times T^1 \to \mathbb{C}^m \) given by \( (x, e^{it}) \mapsto z = e^{it}x \).

Choosing the moving frame of \( \mathbb{C}^m \) along \( \hat{f} \) to be
\[
e_\lambda = e^{it} \hat{e}_\lambda, \quad 1 \leq \lambda < m,
\]
\[
e_m = iz = ie^{it}x,
\]
we conclude that
\[
\theta^\lambda := (dz, e_\lambda) = \hat{\theta}^\lambda, \quad 1 \leq \lambda < m,
\]
\[
\theta^m := (dz, e_m) = dt,
\]
are real 1-forms, which implies that \( \hat{f} \) is a Lagrangian immersion. Through direct calculation, we have
\[
\omega_{\lambda \bar{\lambda}} = \omega_{m\bar{m}} = i \theta^m, \quad \omega_{\lambda \bar{m}} = i \theta^\lambda, \quad 1 \leq \lambda < m
\]
and
\[
\omega_{\lambda \bar{\mu}} = \hat{\omega}_{\lambda \mu}, \quad 1 \leq \lambda < \mu < m,
\]
which are real 1-forms. As before, we use the notation \( \omega_{\lambda \mu} = (de_{\lambda}, e_{\mu}) \).
If we denote the connection 1-forms with respect to $\theta^\lambda$ by $\theta^\lambda_\mu$, we clearly have

\[(4-20)\]

\[\theta^m_\lambda = 0, \quad \theta^\lambda_\mu = \hat{\omega}_\mu^\lambda, \quad 1 \leq \lambda, \mu < m.\]

From (4-20) and (2-33), we obtain

\[(4-21)\]

\[\hat{f}_\mu^\lambda = 0, \quad \hat{f}_\lambda^m = i, \quad 1 \leq \lambda < m, \quad 1 \leq \mu \leq m.\]

**Proposition 4.2.** The map $\hat{f} : S^{m-1}(1) \times T^1 \to \mathbb{C}^m$ given by $(x, e^{it}) \mapsto e^{it}x$ is an H-minimal Lagrangian immersion in $\mathbb{C}^m$, and its complex mean curvature $\hat{H}^C$ satisfies

\[\hat{H}^C - (\hat{H}^C, e_m)e_m = 0, \quad (\hat{H}^C, e_m) = im.\]

Moreover, if we set $\hat{h}_m = -\text{Im}((\hat{H}^C, e_m))$, we have

\[
\langle \text{grad} \hat{h}_m, e_m \rangle - (\hat{h}_m + (n + 1)) \sum_\lambda \langle \nabla e_\lambda, e_m, e_\lambda \rangle = 0.
\]

**Proof.** By the definition of $\hat{h}_\lambda$, we have $\hat{h}_\lambda = 0, 1 \leq \lambda < m$ and $\hat{h}_m = -m$, which imply $\delta \alpha \hat{h}_\mu = 0$. So, $\hat{f}$ is H-minimal. The second identity holds because $\hat{h}_m$ is a constant and $\theta^m_\lambda = 0$. □

**Proof of the Theorem 1.3.** This follows from Proposition 4.2 and Theorem 1.1. □

**Example 4.3** (Clifford torus in $\mathbb{C}P^n$). Taking $q = 0$ in Theorem 1.1, we have proved that the Clifford torus is a minimal Lagrangian submanifold in $\mathbb{C}P^n$. This is a known result; here we just provided an alternative proof.

**Example 4.4** (H-minimal $S^q(1) \times T^{n-q}$ in $\mathbb{C}P^n$). Let $\tilde{f} : S^q(1) \subset \mathbb{R}^{q+1} \hookrightarrow \mathbb{C}^{q+1}, \tilde{f}(p) = Z$, be the standard embedding. Then $\mathcal{H} \circ \tilde{f}$ is totally geodesic in $\mathbb{C}P^q$. Define $\tilde{f} : S^q(1) \times T^{n-q} \to \mathbb{S}^{2n+1}(1)$ by

\[
(Z, e^{it_{q+1}}, \ldots, e^{it_n}) \mapsto \frac{1}{\sqrt{n-q+1}}(Z, e^{it_{q+1}}, \ldots, e^{it_n}).
\]

This gives an H-minimal immersion $\mathcal{H} \circ \tilde{f}$, by Theorem 1.2.

**Example 4.5** (exotic H-minimal $S^3(1) \times T^{n-3}$ in $\mathbb{C}P^n$). Recall from [Bedulli and Gori 2008], [Chen et al. 1996], [Chiang 2004], or [Li and Tao 2006] the exotic minimal Lagrangian immersion $\hat{f} : S^3(1) \to \mathbb{C}P^3$ mapping the point $(a, b)$, where $|a|^2 + |b|^2 = 1$, to

\[
[\bar{a}^3 + 3\bar{a}b^2, \sqrt{3}(\bar{a}^2b + \bar{b}|b|^2 - 2\bar{b}|a|^2), \sqrt{3}(\bar{a}b^2 + a|a|^2 - 2a|b|^2), b^3 + 3a^2b].
\]
By Theorem 1.2, we know that the Lagrangian immersion \( f : \mathbb{S}^3(1) \times T^{n-3} \rightarrow \mathbb{C}\mathbb{P}^n \) mapping \(((a, b), (e^{it_1}, \ldots, e^{it_n}))\) to
\[
\left[ a^3 + 3ab^2, \sqrt{3}(a^2b + b|b|^2 - 2b|a|^2), \sqrt{3}(ab^2 + a|a|^2 - 2a|b|^2), b^3 + 3a^2b, e^{it_1}, \ldots, e^{it_n} \right]
\]
is H-minimal.

**Example 4.6.** Let \( S^{m-1}(1) \) be as in (4-14). The immersion
\[
f : S^{m-1}(1) \times S^{m-1}(1) \times T^1 \mapsto \mathbb{C}\mathbb{P}^{2m-1}
given by
\[(x, y, e^{it}) \mapsto [x, e^{it} y]\]
is a minimal Lagrangian immersion.

The map \( f : S^q(1) \times S^{m-1}(1) \times T^1 \mapsto \mathbb{C}\mathbb{P}^{q+m} \) given by the same formula is an H-minimal Lagrangian immersion.

**Example 4.7.** The immersion \( f : T^{m-1} \times \mathbb{S}^{m-1}(1) \times T^1 \rightarrow \mathbb{C}\mathbb{P}^{2m-1} \) given by
\[
((1, e^{it_1}, e^{it_{m-1}}), x, e^{it}) \mapsto \left[ \frac{1}{\sqrt{m}}, \frac{e^{it_1}}{\sqrt{m}}, \ldots, \frac{e^{it_{m-1}}}{\sqrt{m}}, e^{it} x \right]
\]
is a minimal Lagrangian immersion, and the map \( f : T^q \times \mathbb{S}^{m-1}(1) \times T^1 \rightarrow \mathbb{C}\mathbb{P}^{q+m} \) given by
\[
((1, e^{it_1}, \ldots, e^{it_q}), x, e^{it}) \mapsto \left[ \frac{1}{\sqrt{q+1}}, \frac{e^{it_1}}{\sqrt{q+1}}, \ldots, \frac{e^{it_q}}{\sqrt{q+1}}, e^{it} x \right]
\]
is an H-minimal Lagrangian immersion. Here, we have used the fact the Clifford torus \( T^n \rightarrow \mathbb{C}\mathbb{P}^n \), given by
\[(e^{it_1}, \ldots, e^{it_n}) \mapsto [1, e^{it_1}, \ldots, e^{it_n}],
\]
is minimal.

**Example 4.8.** The map \( \mathbb{S}^3(1) \times S^3(1) \times T^1 \rightarrow \mathbb{C}\mathbb{P}^7 \) taking \(((a, b), x, e^{it})\) to
\[
\left[ a^3 + 3\bar{a}\bar{b}^2, \sqrt{3}(\bar{a}^2b + \bar{b}|b|^2 - 2\bar{b}|a|^2), \sqrt{3}(\bar{a}b^2 + a|a|^2 - 2a|b|^2), b^3 + 3a^2b, e^{it} x \right]
\]
(where \((a, b) \in \mathbb{C}^2\) satisfies \(|a|^2 + |b|^2 = 1\) and \(x \in \mathbb{R}^4\) satisfies \(|x|^2 = 1\)) is a minimal Lagrangian immersion.

The map \( \mathbb{S}^3(1) \times S^{m-1}(1) \times T^1 \rightarrow \mathbb{C}\mathbb{P}^{m+3} \) given by the same formula (with \(x \in \mathbb{R}^m\) satisfying \(|x|^2 = 1\)) is an H-minimal Lagrangian immersion.
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References


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