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CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS IN $\mathbb{CP}^n$

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We present a method of construction of minimal and H-minimal Lagrangian submanifolds in complex projective space $\mathbb{CP}^{q+m}$ from a Legendrian submanifold in $\mathbb{S}^{2q+1}(1) \subset \mathbb{C}^{q+1}$ and a Lagrangian submanifold in $\mathbb{C}^m$ that is contained in $\mathbb{S}^{2m-1}(r)$. We also provide some explicit examples.

1. Introduction

Let (N, J, ω) be a Kähler manifold with $\dim_{\mathbb{C}} N = n$, where J is the complex structure and ω is the Kähler form. An immersion $f : \Sigma \to N$ from a q-dimensional manifold Σ into N is called *totally real* if $f^*\omega = 0$. In particular, a totally real immersion f is called *Lagrangian* if $q = n$.

We recall some definitions from Y. G. Oh's paper [1993]. A vector field V along a Lagrangian immersion $f : \Sigma \to N$ is called a *Hamiltonian variation* if the 1-form $\alpha_V := (V \rfloor \omega)|_\Sigma$ is exact on Σ. A smooth family $\{f_t\}$ of immersions from Σ into N is called a *Hamiltonian deformation* if its derivative is Hamiltonian, and a Lagrangian immersion $f : \Sigma \to N$ is called *Hamiltonian-minimal* or *H-minimal* if it satisfies

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol} f_t(\Sigma) = 0$$

for all Hamiltonian deformations. The Euler–Lagrange equation of H-minimal Lagrangian submanifolds is

$$\delta\alpha_H = 0,$$

where H is the mean curvature vector field of f and δ is the codifferential operator on Σ with respect to the induced metric. In particular, minimal Lagrangian submanifolds are trivially H-minimal.

In the past few decades, many geometers have given many methods of construction of minimal and H-minimal Lagrangian submanifolds in the complex space form. I. Castro and F. Urbano [1998] classified $\mathbb{S}^1$-invariant H-minimal Lagrangian submanifolds in $\mathbb{C}^2$, and in [Castro and Urbano 2004] they also constructed special

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Lagrangian submanifolds in $\mathbb{C}^n$. R. Schoen and J. Wolfson [1999] studied the minimal Lagrangian cones in $\mathbb{C}^2$. A. E. Mironov [2004] gave many examples of minimal and H-minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{CP}^n$, and he, jointly with D. F. Zuo [Mironov and Zuo 2008], constructed a family of flat H-minimal Lagrangian tori in $\mathbb{CP}^3$. H. Ma and M. Schmies [2006] gave a family of Hamiltonian stationary Lagrangian tori in $\mathbb{CP}^2$ with $\mathbb{S}^1$-symmetry. Castro and Urbano, together with H. Zh. Li [Castro et al. 2006] used Legendrian immersions in odd-dimensional spheres and anti–de Sitter spaces to construct minimal and H-minimal Lagrangian submanifolds in the complex space form. D. Joyce [2002] gave many examples of minimal Lagrangian submanifolds with symmetries in $\mathbb{C}^n$. L. Bedulli and A. Gori [2008] studied homogeneous Lagrangian submanifolds in $\mathbb{CP}^n$. R. Chiang [2004] gave many Lagrangian submanifolds in $\mathbb{CP}^n$ with interesting topological feature.

Let $\mathbb{C}^m$ be the complex Euclidean space endowed with the standard Hermitian inner product $(z, w) = \sum_{j=1}^m z_j \bar{w}_j$ for $z = (z_1, \dots, z_m)$, $w = (w_1, \dots, w_m) \in \mathbb{C}^m$ and the canonical complex structure $Jz = iz$. The real part of $(\ ,\ )$ determines a metric $\langle\ ,\ \rangle$ on $\mathbb{C}^m$, i.e., $\langle\ ,\ \rangle = \mathrm{Re}(\ ,\ )$. The Liouville 1-form on $\mathbb{C}^m$ is given by $\Omega = \frac{i}{2}\sum_j (z^j d\bar{z}^j - \bar{z}^j dz^j)$, and the Kähler form of $\mathbb{C}^m$ is $\omega_{\mathbb{C}^m} = d\Omega/2$. Let $\mathbb{S}^{2q+1}(1)$ be the $(2q+1)$-dimensional unit sphere in $\mathbb{C}^{q+1}$, and let $\mathcal{H} : \mathbb{S}^{2q+1}(1) \to \mathbb{CP}^q$, $Z \mapsto [Z]$, be the Hopf fibration of $\mathbb{S}^{2q+1}(1)$ over the complex projective space $\mathbb{CP}^q$. We say an immersion $\check{f} : \Sigma_1 \to \mathbb{S}^{2q+1}(1) \subset \mathbb{C}^{q+1}$, $p \mapsto \check{f}(p) = Z$, of a q-dimensional manifold Σ_1 into $\mathbb{S}^{2q+1}(1)$ is *Legendrian* if $\check{f}^*\Omega = 0$. In this case, $\check{f}$ is isotropic in $\mathbb{C}^{q+1}$, i.e., $\check{f}^*\omega_{\mathbb{C}^{q+1}} = 0$, and the normal bundle $T^\perp\Sigma_1$ in $T\mathbb{S}^{2q+1}(1)$ splits as $J(T\Sigma_1) \oplus \mathrm{Span}_{\mathbb{R}}\{JZ\}$. This means that $\check{f}$ is horizontal with respect to the Hopf fibration $\mathcal{H}$, and hence $\tilde{f} = \mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$ is a Lagrangian immersion and the metric induced on Σ_1 by $\check{f}$ and $\tilde{f}$ are the same.

In this paper we construct minimal and H-minimal Lagrangian submanifolds in $\mathbb{CP}^n$ from Legendrian submanifolds in odd-dimensional spheres and Lagrangian submanifolds in $\mathbb{C}^m$ which are contained in spheres. The basic theorem in our construction is as follows.

Theorem 1.1. *Let $\check{f} : \Sigma_1^q \to \mathbb{S}^{2q+1}(1)$ be a Legendrian immersion and $\hat{f} : \Sigma_2^m \to \mathbb{C}^m$ a Lagrangian immersion with $\hat{f}(\Sigma_2) \subset \mathbb{S}^{2m-1}(r) \subset \mathbb{C}^m$. Write $Z = \check{f}(p_1)$, $z = \hat{f}(p_2)$, $n = q + m$. Define a new map $\check{f} : \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2n+1}(1)$ by*

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(Z, z).$$

Then $f = \mathcal{H} \circ \check{f}$ is a Lagrangian immersion from $\Sigma_1 \times \Sigma_2$ into $\mathbb{CP}^n$. Moreover:

(i) *The immersion f is minimal if and only if $\tilde{f} = \mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$ is minimal and*

$$\hat{H}^{\mathbb{C}} - (\hat{H}^{\mathbb{C}}, e_n)e_n = 0, \quad (\hat{H}^{\mathbb{C}}, e_n) = \frac{i(n+1)r}{1+r^2}, \tag{1-1}$$

where $\hat{H}^{\mathbb{C}}$ is the complex mean curvature vector of $\hat{f}$ and $e_n = iz/r$ defines a global vector field on Σ_2.

(ii) *The immersion f is H-minimal if and only if*

$$\delta\alpha_{\tilde{H}} + \delta\alpha_{\hat{H}} = r^2\langle \operatorname{grad} \hat{h}_n, e_n\rangle - (r^2\hat{h}_n + (n+1)r)\sum_{\lambda}\langle \hat{\nabla}_{e_\lambda} e_n, e_\lambda\rangle, \tag{1-2}$$

where $\hat{h}_n = -\operatorname{Im}\big((\hat{H}^{\mathbb{C}}, e_n)\big)$, and $\hat{\nabla}$ and $\{e_\lambda, e_n\}$ are respectively the connection and an orthonormal frame field on Σ_2 relative to the metric induced by $\hat{f}$.

As applications of Theorem 1.1, we have:

Theorem 1.2. *Let $\check{f} : \Sigma_1^q \to \mathbb{S}^{2q+1}(1)$, $\check{f}(p_1) = Z$, be a Legendrian immersion. If $\mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$ is H-minimal, then $f = \mathcal{H} \circ \check{f}$ is an H-minimal Lagrangian immersion, where $\check{f} : \Sigma_1 \times T^{n-q} \to \mathbb{S}^{2n+1}(1)$ (with $T = S^1(1)$) is defined by*

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{n-q+1}}(Z, e^{it_{q+1}}, \dots, e^{it_n}).$$

Theorem 1.3. *Let $\check{f} : \Sigma_1^q \to \mathbb{S}^{2q+1}(1)$, $\check{f}(p_1) = Z$, be a Legendrian immersion. Define the new map $\check{f} : \Sigma_1 \times S^{m-1} \times T^1 \to \mathbb{S}^{2m-1}(1)$ by*

$$(p_1, x, e^{it}) \mapsto \frac{1}{\sqrt{2}}(Z, e^{it}x).$$

(i) *If $q = m-1$ and $\mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^{m-1}$ is minimal, then $f = \mathcal{H} \circ \check{f}$ is a minimal Lagrangian immersion.*

(ii) *If $\mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$ is H-minimal, then $f = \mathcal{H} \circ \check{f}$ is an H-minimal Lagrangian immersion.*

We prove these theorems in Section 3, and based on them, we give some explicit examples of minimal and H-minimal Lagrangian submanifolds in Section 4.

Throughout this paper, we use the following conventions for index ranges:

$$0 \le A,\ B,\ C,\ \dots \le n; \quad 1 \le \alpha,\ \beta,\ \gamma,\ \dots \le n;$$
$$1 \le j,\ k,\ l,\ \dots \le q; \quad q+1 \le \lambda,\ \mu,\ \nu,\ \dots \le n.$$

For conjugation, we use the conventions $\bar{\omega}_{A\bar{B}} = \omega_{\bar{A}B}$, $\bar{f}_i^{\alpha} = f_i^{\bar{\alpha}}$, and so on.

2. Preliminaries

Basic formulae of submanifolds in a Kähler manifold. To study real submanifolds in a Kähler manifold, it is convenient to use formulae from the complex case. So, we first deduce some basic formulae that are not used frequently in the classical theory of submanifolds.

Let Σ be a smooth Riemannian manifold with $\dim_{\mathbb{R}} \Sigma = q$. Locally, we choose an orthonormal frame field $\{e_j\}$ of Σ, and its dual $\{\theta^j\}$. Then the first Cartan structure equation of Σ is given by

$$d\theta^j = -\theta^j_k \wedge \theta^k, \quad \theta^j_k + \theta^k_j = 0, \tag{2-1}$$

where θ^j_k are the connection forms with respect to the coframe field θ^j. Let N be a Kähler manifold with $\dim_{\mathbb{C}} N = n$. Locally, we choose a unitary frame field $\{\varepsilon_\alpha\}$ of (1,0)-type on N, and denote its dual by $\{\varphi_\alpha\}$. Then the structure equation is given by

$$d\varphi_\alpha = -\varphi_{\beta\bar{\alpha}} \wedge \varphi_\beta, \quad \varphi_{\alpha\bar{\beta}} + \varphi_{\bar{\beta}\alpha} = 0, \tag{2-2}$$

where $\varphi_{\beta\bar{\alpha}}$ are the connection forms with respect to φ_α.

Let $f : \Sigma \to N$ be an isometric immersion. Set

$$f^*\varphi_\alpha = f^\alpha_j\, \theta^j. \tag{2-3}$$

Taking the exterior derivative on both sides of (2-3), we obtain

$$(df^\alpha_j - f^\alpha_k \theta^k_j + \varphi_{\beta\bar{\alpha}} f^\beta_j) \wedge \theta^j = 0 \tag{2-4}$$

by (2-1), (2-2) and (2-3). If we set

$$Df^\alpha_j = df^\alpha_j - f^\alpha_k \theta^k_j + \varphi_{\beta\bar{\alpha}} f^\beta_j = f^\alpha_{jk} \theta^k, \tag{2-5}$$

the covariant derivative of f^α_j, then we have $f^\alpha_{jk} = f^\alpha_{kj}$ by (2-4). The tensor field $\Pi^{\mathbb{C}} = \sum_{j,k,\alpha} f^\alpha_{jk}\, \theta^j \otimes \theta^k \otimes \varepsilon_\alpha$ is called the *complex second fundamental form* of f, and is a smooth section of the bundle $T^*\Sigma \otimes T^*\Sigma \otimes T^{(1,0)}N$. The vector field $H^{\mathbb{C}} = \sum_{j,\alpha} f^\alpha_{jj}\, \varepsilon_\alpha$ is called the *complex mean curvature vector field* of f.

If we split ε_α as $\varepsilon_\alpha = \frac{1}{2}(\epsilon_\alpha - i\epsilon_{\alpha^*})$, then $\{\epsilon_\alpha,\, \epsilon_{\alpha^*} = J\epsilon_\alpha\}$ is an orthonormal frame field on N, and its dual is denoted by $\{\phi^\alpha,\, \phi^{\alpha^*}\}$. The first Cartan structure equation is given by

$$d\phi^\alpha = -\phi^\alpha_\beta \wedge \phi^\beta - \phi^\alpha_{\beta^*} \wedge \phi^{\beta^*}, \quad d\phi^{\alpha^*} = -\phi^{\alpha^*}_\beta \wedge \phi^\beta - \phi^{\alpha^*}_{\beta^*} \wedge \phi^{\beta^*}, \tag{2-6}$$

where ϕ^α_β, $\phi^\alpha_{\beta^*}$, $\phi^{\alpha^*}_\beta$ and $\phi^{\alpha^*}_{\beta^*}$ are the connection forms with respect to the frame field ϕ^α, ϕ^{α^*}. Set

$$f^*\phi^\alpha = a^\alpha_j \theta^j, \quad f^*\phi^{\alpha^*} = a^{\alpha^*}_j \theta^j. \tag{2-7}$$

Taking the exterior derivative of (2-7), by (2-1), (2-6) and (2-7), we obtain

$$(da^\alpha_j - a^\alpha_k \theta^k_j + \phi^\alpha_\beta a^\beta_j + \phi^\alpha_{\beta^*} a^{\beta^*}_j) \wedge \theta^j = 0, \tag{2-8}$$

$$(da^{\alpha^*}_j - a^{\alpha^*}_k \theta^k_j + \phi^{\alpha^*}_\beta a^\beta_j + \phi^{\alpha^*}_{\beta^*} a^{\beta^*}_j) \wedge \theta^j = 0. \tag{2-9}$$

Set

$$Da_j^\alpha = da_j^\alpha - a_k^\alpha \theta_j^k + \phi_\beta^\alpha a_j^\beta + \phi_{\beta^*}^\alpha a_j^{\beta^*} = h_{jk}^\alpha \theta^k, \tag{2-10}$$

$$Da_j^{\alpha^*} = da_j^{\alpha^*} - a_k^{\alpha^*} \theta_j^k + \phi_\beta^{\alpha^*} a_j^\beta + \phi_{\beta^*}^{\alpha^*} a_j^{\beta^*} = h_{jk}^{\alpha^*} \theta^k, \tag{2-11}$$

the covariant derivatives of a_j^α and $a_j^{\alpha^*}$ respectively. Then, we know that $h_{jk}^\alpha = h_{kj}^\alpha$, $h_{jk}^{\alpha^*} = h_{kj}^{\alpha^*}$ by (2-8) and (2-9). Clearly, the tensor field

$$\Pi = h_{jk}^\alpha\, \theta^j \otimes \theta^k \otimes \epsilon_\alpha + h_{kj}^{\alpha^*}\, \theta^j \otimes \theta^k \otimes \epsilon_{\alpha *}$$

is the *real second fundamental form* in the usual sense; it is a smooth section of the bundle $T^*\Sigma \otimes T^*\Sigma \otimes TN$. The vector field $H = \sum_j (h_{jj}^\alpha \epsilon_\alpha + h_{jj}^{\alpha^*} \epsilon_{\alpha *})$ is the *real mean curvature vector field* of f.

The relationship between the real second fundamental form and the complex second fundamental form of f is given by:

Proposition 2.1. *With the notation above, we have*

$$h_{jk}^\alpha = \frac{1}{2}(f_{jk}^\alpha + f_{jk}^{\bar\alpha}), \quad h_{jk}^{\alpha^*} = \frac{i}{2}(f_{jk}^{\bar\alpha} - f_{jk}^\alpha). \tag{2-12}$$

Moreover, f is minimal if and only if $H^{\mathbb{C}} = 0$.

Proof. One readily checks that

$$\varphi_\alpha = \phi^\alpha + i\phi^{\alpha^*}. \tag{2-13}$$

Then, from (2-3), we get

$$f_j^\alpha = a_j^\alpha + i a_j^{\alpha^*}. \tag{2-14}$$

Since N is kählerian, it's easy to check that $\phi_\beta^\alpha = \phi_{\beta^*}^{\alpha^*}$ and $\phi_{\beta^*}^\alpha = -\phi_\beta^{\alpha^*}$, which gives

$$\varphi_{\beta\bar\alpha} = \phi_\beta^\alpha - i\phi_{\beta^*}^\alpha \tag{2-15}$$

by (2-2), (2-6) and (2-13). By the definition of f_{jk}^α and (2-15), we have

$$\begin{aligned} f_{jk}^\alpha \theta^k = Df_j^\alpha &= df_j^\alpha - f_k^\alpha \theta_j^k + \varphi_{\beta\bar\alpha} f_j^\beta \\ &= d(a_j^\alpha + i a_j^{\alpha^*}) - (a_k^\alpha + i a_k^{\alpha^*})\, \theta_j^k + (\phi_\beta^\alpha - i\phi_{\beta^*}^\alpha)(a_j^\beta + i a_j^{\beta^*}) \\ &= (da_j^\alpha - a_k^\alpha \theta_j^k + \phi_\beta^\alpha a_j^\beta + \phi_{\beta^*}^\alpha a_j^{\beta^*}) + i(da_j^{\alpha^*} - a_k^{\alpha^*}\theta_j^k + \phi_\beta^{\alpha^*} a_j^\beta + \phi_{\beta^*}^{\alpha^*} a_j^{\beta^*}) \\ &= (h_{jk}^\alpha + i h_{jk}^{\alpha^*})\theta^k, \end{aligned} \tag{2-16}$$

which gives (2-12). □

Note that the Kähler form of N is $\omega_N = \frac{i}{2}\sum_\alpha \varphi_\alpha \wedge \varphi_{\bar\alpha}$. So, for a vector field $V = v^\alpha \epsilon_\alpha + v^{\alpha^*}\epsilon_{\alpha^*}$ we have

$$V \rfloor \omega = \omega(V, \cdot) = \frac{i}{2}\big((v^\alpha + i v^{\alpha^*})\varphi_{\bar\alpha} - (v^\alpha - i v^{\alpha^*})\varphi_\alpha\big). \tag{2-17}$$

In particular, for the mean curvature vector field H of a given isometric immersion $f : \Sigma \to N$, we have

$$\alpha_H := (H \rfloor \omega)_\Sigma = h_j \theta^j, \quad h_j = \frac{i}{2}(f^\alpha_{kk} f^{\bar\alpha}_j - f^{\bar\alpha}_{kk} f^\alpha_j), \tag{2-18}$$

by (2-12) and (2-17). Therefore, the codifferential of α_H is given by

$$\delta\alpha_H = -\sum_j h_{jj}, \tag{2-19}$$

where $h_{jk}\theta^k = dh_j - h_k\theta^k_j$ is the covariant derivative of h_j.

Lagrangian submanifolds in $\mathbb{C}^m$ contained in a sphere. Let $\mathbb{C}^{q+1}$ be complex Euclidean space as described in the introduction. Let $\hat f : \Sigma_2 \to \mathbb{C}^m$, $\hat f(p) = z$, be a Lagrangian immersion with $\hat f(\Sigma_2) \subset \mathbb{S}^{2m-1}(r)$. Locally, one can select an orthonormal frame field $e_{q+1}, \dots, e_{n-1}, e_n = iz/r$ such that

$$dz = \sum_{\lambda=q+1}^{n} \hat\theta^\lambda e_\lambda, \quad ds^2_{\Sigma_2} = \sum_{\lambda=q+1}^{n} (\hat\theta^\lambda)^2.$$

Since $\hat f$ is Lagrangian, one readily checks that e_λ is also a unitary frame field, i.e., $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. So, if we set

$$de_\lambda = \hat\omega_{\lambda\bar\mu} e_\mu, \quad \hat\omega_{\lambda\bar\mu} = (de_\lambda, e_\mu),$$

then

$$\hat\omega_{\lambda\bar\mu} + \hat\omega_{\bar\mu\lambda} = 0, \tag{2-20}$$

because $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. Obviously, we have

$$(dz, e_\lambda) = \hat\theta^\lambda, \quad \hat\omega_{\lambda\bar n} = -\hat\omega_{\bar n\lambda} = -\overline{\Big(\frac{i}{r} dz, e_\lambda\Big)} = \frac{i}{r}\hat\theta^\lambda. \tag{2-21}$$

Denote by $\hat\theta^\lambda_\mu$ the connection 1-forms with respect to the frame field $\hat\theta^\lambda$. Set $\hat\theta^\lambda_\mu = \hat\Gamma^\lambda_{\nu\mu}\hat\theta^\nu$, $\hat f^*\hat\omega_{\lambda\bar\mu} = \hat\Lambda_{\lambda\bar\mu,\nu}\hat\theta^\nu$. We then obtain the complex second fundamental form of $\hat f$. That is:

$$\hat f^\lambda_{\mu\nu} = -\hat\Gamma^\lambda_{\lambda\mu} + \hat\Lambda_{\mu\bar\lambda,\nu}, \tag{2-22}$$

by (2-5) and the fact that $\hat f^\lambda_\mu = \delta_{\lambda\mu}$. So, by (2-18), we obtain

$$\alpha_{\hat{H}} = \hat{h}_\lambda \hat{\theta}^\lambda, \quad \hat{h}_\lambda = \frac{i}{2}(\hat{f}^\lambda_{\mu\mu} - \hat{f}^{\bar{\lambda}}_{\mu\mu}). \tag{2-23}$$

Note that e_n is a globally defined vector field on Σ_2, so $(\hat{H}^{\mathbb{C}}, e_n) = \sum_\lambda \hat{f}^n_{\lambda\lambda}$, which plays an important role in our main construction, is a globally defined smooth complex-valued function on Σ_2.

Lagrangian submanifolds in $\mathbb{CP}^n$**.** Complex projective space $\mathbb{CP}^n$ is the set of all one-dimensional complex lines through the origin in $\mathbb{C}^{n+1}$. It can be written as $\mathbb{CP}^n \cong U(n+1)/(U(1)\times U(n))$, where $U(n+1)$ is the unitary group; thus, $U(n+1)$ is a principal $U(1)\times U(n)$-bundle over $\mathbb{CP}^n$.

Let $Z_0, Z_1, \dots, Z_n$ be a moving frame of $\mathbb{C}^{n+1}$. We have

$$dZ_A = \omega_{A\bar{B}} Z_B, \quad \omega_{A\bar{B}} = (dZ_A, Z_B), \tag{2-24}$$

where $\omega_{A\bar{B}} = (dZ_A, Z_B)$ are the Maurer–Cartan forms of $U(n+1)$. They are skew-Hermitian, i.e.,

$$\omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{2-25}$$

Taking the exterior derivative of (2-24), we get the Maurer–Cartan equation of $U(n+1)$:

$$d\omega_{A\bar{B}} = \sum_C \omega_{A\bar{C}} \wedge \omega_{C\bar{B}}, \tag{2-26}$$

$$ds^2_{FS} = \sum_\alpha \omega_{0\bar{\alpha}} \omega_{\bar{0}\alpha}, \tag{2-27}$$

determines a Kähler metric on $\mathbb{CP}^n$, called the Fubini–Study metric. The Kähler form of ds^2_{FS} is given by

$$\omega_{FS} = \frac{i}{2}\sum_\alpha \omega_{0\bar{\alpha}} \wedge \omega_{\bar{0}\alpha}.$$

If we set $\varphi_\alpha := \omega_{0\bar{\alpha}}$, then $\{\varphi_\alpha\}$ is a unitary frame field on $\mathbb{CP}^n$ of (1,0)-type (see [Griffiths 1974] for details). Therefore, by the Maurer–Cartan equation (2-26), we obtain the first structure equation:

$$d\varphi_\alpha = -\varphi_{\beta\bar{\alpha}} \wedge \varphi_\beta, \quad \varphi_{\beta\bar{\alpha}} = \omega_{\beta\bar{\alpha}} - \omega_{0\bar{0}}\delta_{\alpha\beta}, \quad \varphi_{\beta\bar{\alpha}} + \varphi_{\bar{\alpha}\beta} = 0, \tag{2-28}$$

where $\varphi_{\beta\bar{\alpha}}$ are the connection forms with respect to the frame field φ_α.

Let Σ be a smooth manifold with $\dim \Sigma = q$, and let f be an immersion from Σ into $\mathbb{CP}^n$. Let $U \subset \Sigma$ be an open set. We say $Z : U \to U(n+1)$ is a *moving frame* along f if Z satisfies $f = \pi \circ Z$, where π is the canonical projection. For a moving frame along a totally real immersion f, we have:

Proposition 2.2. *Let f be a totally real immersion from Σ into $\mathbb{CP}^n$. If U is any small enough open subset of Σ, and the induced metric on U is given by $f^*ds^2_{FS} = \sum_j (\theta^j)^2$, then there exists a moving frame Z along f such that*

$$\omega_{0\bar{0}} = 0, \quad \omega_{0\bar{j}} = \theta^j, \quad \omega_{0\bar{\lambda}} = 0, \tag{2-29}$$

where the $\omega_{A\bar{B}}$ are the pull-backs of the Maurer–Cartan forms of $U(n+1)$ by Z^*.

Proof. Throughout this proof, we will assume that the neighborhoods chosen are small enough to satisfy the topological assumptions.

Without loss of generality, we may assume $f(U)$ is contained in a small open set V of $\mathbb{CP}^n$. Let e_j be the dual frame field of θ^j. We extend $\varepsilon_j = \frac{1}{2}(e_j - iJe_j)$ smoothly to V and choose ε_λ on V such that $\{\varepsilon_\alpha\}$ is smooth unitary frame on V. Let $\{\varphi_\alpha\}$ be the dual of $\{\varepsilon_\alpha\}$. Then $\{\varphi_\alpha\}$ is a unitary coframe field of $(1, 0)$-type on V and satisfies $f^*\varphi_j = \theta^j$, $f^*\varphi_\lambda = 0$. Notice that we have used the fact that f is totally real in choosing ε_i.

Let $\mathcal{S}_1 = (Z_0, Z_1, \dots, Z_n)^T : V \to U(n+1)$ be a local section of the principal bundle $\pi : U(n+1) \to \mathbb{CP}^n$. Then $\{\mathcal{S}_1^*\omega_{0\bar{\alpha}}\}$ is a unitary coframe field of $(1, 0)$-type (see [Griffiths 1974]) on V. Therefore, there exists a unitary matrix $A = (a_{\alpha\bar{\beta}})_{n\times n}$ defined on V such that $\varphi_\alpha = \sum_\beta a_{\alpha\bar{\beta}}\mathcal{S}_1^*\omega_{0\bar{\alpha}}$. If we choose another local section $\mathcal{S}_2 = (Z_0, \tilde{Z}_1, \dots, \tilde{Z}_n)^T : V \to U(n+1)$ such that $\tilde{Z}_\alpha = \sum_\beta a_{\bar{\alpha}\beta}Z_\beta$, then

$$\varphi_\alpha = \mathcal{S}_2^*\omega_{0\bar{\alpha}}$$

by (2-24).

Set $\tilde{Z} = \mathcal{S}_2 \circ f$. One can check that $\tilde{Z}^*\omega_{0\bar{i}} = \theta^i$ and $\tilde{Z}^*\omega_{0\bar{\lambda}} = 0$, so $d\tilde{Z}^*\omega_{0\bar{0}} = 0$ by the Maurer–Cartan equation (2-26), i.e., $\tilde{Z}^*\omega_{0\bar{0}}$ is a closed 1-form on U, so one can find a smooth function u defined on U such that $idu = \tilde{Z}^*\omega_{0\bar{0}}$. Taking $Z = e^{-iu}\tilde{Z}$, it is easily checked that the pull-back of the Maurer–Cartan form of $U(n+1)$ by Z^* is (2-29). This completes the proof. □

Let $f : \Sigma \to \mathbb{CP}^n$ be a Lagrangian isometric immersion, and let θ^α be an orthonormal frame field on Σ. By Proposition 2.2, there exists a moving frame $Z_0, Z_1, \dots, Z_n$ along f such that

$$\varphi_\alpha = \omega_{0\bar{\alpha}} = \theta^\alpha. \tag{2-30}$$

For later use, we set

$$\omega_{\alpha\bar{\beta}} = \Lambda_{\alpha\bar{\beta},\gamma}\,\theta^\gamma, \quad \omega_{0\bar{0}} = \Lambda_{0\bar{0},\gamma}\,\theta^\gamma, \tag{2-31}$$

and let

$$\theta^\alpha_\beta = \Gamma^\alpha_{\gamma\beta}\,\theta^\gamma \tag{2-32}$$

be the connection 1-forms with respect to θ^α.

Note that $f^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ by (2-30), and so the complex second fundamental form of f is given by

$$f^{\alpha}_{\beta\gamma} = -\Gamma^{\alpha}_{\gamma\beta} + \Lambda_{\beta\bar{\alpha},\gamma} - \delta_{\alpha\beta}\Lambda_{0\bar{0},\gamma}, \tag{2-33}$$

by (2-5), (2-28), (2-31) and (2-32). So, we obtain

$$\alpha_H = h_{\beta}\,\theta^{\beta}, \quad h_{\beta} = \frac{i}{2}(f^{\beta}_{\gamma\gamma} - f^{\bar{\beta}}_{\gamma\gamma}), \tag{2-34}$$

by (2-18).

3. Proof of Theorem 1.1

Let $\check{f} : \Sigma_1 \to \mathbb{S}^{2q+1}(1)$, $\check{f}(p) = \tilde{Z}_0$, be a Legendrian isometric immersion. Then $\tilde{f} = \mathcal{H} \circ \check{f} : \Sigma_1 \to \mathbb{CP}^q$, $p \mapsto \tilde{f}(p) = [\tilde{Z}_0]$ is a Lagrangian isometric immersion. Since $\check{f}$ is a Legendrian immersion, one readily checks that

$$\tilde{\omega}_{0\bar{0}} = (d\tilde{Z}_0, \tilde{Z}_0) = 0. \tag{3-1}$$

By Proposition 2.2, one can choose a pairwise Hermitian orthogonal local frame field, $\tilde{Z}_1, \dots, \tilde{Z}_q$, such that $\tilde{Z}_0, \tilde{Z}_1, \dots, \tilde{Z}_q$ is a moving frame along $\tilde{f}$, and

$$\tilde{\omega}_{0\bar{j}} = (d\tilde{Z}_0, \tilde{Z}_j) = \tilde{\theta}^j. \tag{3-2}$$

are real 1-forms. As before, we set

$$\tilde{\omega}_{j\bar{k}} = (d\tilde{Z}_j, \tilde{Z}_k) = \tilde{\Lambda}_{j\bar{k},l}\,\tilde{\theta}^l. \tag{3-3}$$

If we denote the connection 1-forms with respect to $\tilde{\theta}^j$ by $\tilde{\theta}^j_k = \tilde{\Gamma}^j_{lk}\,\tilde{\theta}^l$, by similar calculations to those in Section 2, we obtain the complex fundamental form of $\tilde{f}$. That is,

$$\tilde{f}^j_{kl} = -\tilde{\Gamma}^j_{lk} + \tilde{\Lambda}_{k\bar{j},l}, \tag{3-4}$$

by (3-1).

Define the map $\check{f} : \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2n+1}(1)$ by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(\check{f}(p_1), \hat{f}(p_2)) = \frac{1}{\sqrt{1+r^2}}(\tilde{Z}_0, z),$$

with $\check{f}$ and $\hat{f}$ as before. We will study the map $f = \mathcal{H} \circ \check{f} : \Sigma_1 \times \Sigma_2 \to \mathbb{CP}^n$, given by

$$(p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}[\tilde{Z}_0, z].$$

We chose the moving frame $Z_0, Z_1, \ldots, Z_n$ as follows:

$$\begin{aligned} Z_0 &= \frac{1}{\sqrt{1+r^2}}(\tilde{Z}_0, z),\\ Z_j &= (\tilde{Z}_j, 0),\\ Z_\lambda &= (0, e_\lambda), \quad q+1 \le \lambda < n,\\ Z_n &= \frac{1}{\sqrt{1+r^2}}(-ir\tilde{Z}_0, e_n), \end{aligned}$$

where $\tilde{Z}_0$, $\tilde{Z}_j$ and e_λ, e_n are as they were in the context of $\check{f}$ and $\hat{f}$, respectively.

According to (2-24), we obtain

$$\omega_{0\bar{j}} = \frac{1}{\sqrt{1+r^2}}(d\tilde{Z}_0, \tilde{Z}_j) = \frac{1}{\sqrt{1+r^2}}\tilde{\omega}_{0\bar{j}} = \frac{1}{\sqrt{1+r^2}}\tilde{\theta}^j =: \theta^j, \tag{3-5}$$

$$\omega_{0\bar{\lambda}} = \frac{1}{\sqrt{1+r^2}}(dz, e_\lambda) = \frac{1}{\sqrt{1+r^2}}\hat{\theta}^\lambda =: \theta^\lambda, \quad q+1 \le \lambda < n \tag{3-6}$$

$$\omega_{0\bar{n}} = (dZ_0, Z_n) = \frac{1}{1+r^2}(dz, e_n) = \frac{1}{1+r^2}\hat{\theta}^n =: \theta^n, \tag{3-7}$$

by (2-21) and (3-1). Similarly,

$$\omega_{0\bar{0}} = \frac{1}{1+r^2}(dz, z) = \frac{ir}{1+r^2}(dz, e_n) = ir\theta^n, \tag{3-8}$$

$$\omega_{j\bar{k}} = \tilde{\omega}_{j\bar{k}}, \quad \omega_{j\bar{\lambda}} = 0, \qquad \omega_{j\bar{n}} = -ir\theta^j, \tag{3-9}$$

$$\omega_{\lambda\bar{\mu}} = \hat{\omega}_{\lambda\bar{\mu}}, \quad \omega_{\lambda\bar{n}} = \frac{i}{r}\theta^\lambda, \quad \omega_{n\bar{n}} = \frac{i}{r}\theta^n. \tag{3-10}$$

where $q+1 \le \lambda$ and $\mu < n$.

Since θ^j, θ^λ, θ^n are real and linearly independent on $\Sigma_1 \times \Sigma_2$, so f is an immersion and the induced metric is given by

$$\begin{aligned} ds^2 = f^* ds^2_{FS} &= \sum_\alpha (\theta^\alpha)^2 \\ &= \sum_{j=1}^{q}\left(\frac{1}{\sqrt{1+r^2}}\tilde{\theta}^j\right)^2 + \sum_{\lambda=q+1}^{n-1}\left(\frac{1}{\sqrt{1+r^2}}\hat{\theta}^\lambda\right)^2 + \left(\frac{1}{1+r^2}\hat{\theta}^n\right)^2, \end{aligned} \tag{3-11}$$

which is a product metric. If we choose the orthonormal frame field θ^α on $\Sigma_1 \times \Sigma_2$, then

$$f^*\omega_{0\bar{\alpha}} = \theta^\alpha, \quad f^\alpha_\beta = \delta_{\alpha\beta}. \tag{3-12}$$

The pull back of the Kähler form is

$$f^*\omega_{FS} = \frac{i}{2}\sum_\alpha \omega_{0\bar\alpha}\wedge\omega_{\bar 0\alpha} = \frac{i}{2}\sum_\alpha \theta^\alpha\wedge\theta^\alpha = 0,$$

and thus f is a Lagrangian immersion.

Lemma 3.1. *Let*

$$d\tilde s^2 = \sum_{\alpha=1}^n (\tilde\theta^\alpha)^2 \quad \textit{and} \quad ds^2 = \sum_{\alpha=1}^n (\theta^\alpha)^2 = \sum_{\alpha=1}^n (a_\alpha\tilde\theta^\alpha)^2$$

be two metrics, where the a_α are positive constants. Let

$$\tilde\theta^\alpha_\beta = \tilde\Gamma^\alpha_{\gamma\beta}\,\tilde\theta^\gamma \quad \textit{and} \quad \theta^\alpha_\beta = \Gamma^\alpha_{\gamma\beta}\,\theta^\gamma$$

be the connection 1-forms with respect to $\tilde\theta^\alpha$ and θ^α. Then

(3-13)
$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\biggl(\Bigl(\frac{a_\alpha}{a_\beta a_\gamma}+\frac{a_\gamma}{a_\alpha a_\beta}\Bigr)\tilde\Gamma^\alpha_{\beta\gamma}+\Bigl(\frac{a_\beta}{a_\alpha a_\gamma}-\frac{a_\alpha}{a_\beta a_\gamma}\Bigr)\tilde\Gamma^\alpha_{\gamma\beta}+\Bigl(\frac{a_\gamma}{a_\alpha a_\beta}-\frac{a_\beta}{a_\alpha a_\gamma}\Bigr)\tilde\Gamma^\gamma_{\alpha\beta}\biggr).$$

In particular, if $a_1 = \cdots = a_n = a$, then

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{a}\tilde\Gamma^\alpha_{\beta\gamma}, \tag{3-14}$$

and if $a_1 = \cdots = a_{n-1} = a$, $a_n = a^2$, then

$$\Gamma^\lambda_{\mu\mu} = \frac{1}{a}\tilde\Gamma^\lambda_{\mu\mu},\quad \Gamma^\lambda_{nn} = \frac{1}{a}\tilde\Gamma^\lambda_{nn},\quad \Gamma^n_{\mu\mu} = \frac{1}{a^2}\tilde\Gamma^n_{\mu\mu},\quad \Gamma^n_{nn} = \frac{1}{a^2}\tilde\Gamma^n_{nn}, \tag{3-15}$$

where $1 \le \lambda$ and $\mu \le n-1$.

Proof. Denote the dual of $\{\tilde\theta^\alpha\}$ by $\{\tilde e_\alpha\}$. Then $\{e_\alpha = \frac{1}{a_\alpha}\tilde e_\alpha\}$ is the dual of $\{\theta^\alpha\}$. Since both $\{\tilde\theta^\alpha\}$ and $\{\theta^\alpha\}$ are orthonormal, we obtain

$$\tilde\Gamma^\alpha_{\beta\gamma} = -\tilde\Gamma^\gamma_{\beta\alpha},\quad \Gamma^\alpha_{\beta\gamma} = -\Gamma^\gamma_{\beta\alpha}. \tag{3-16}$$

By the structure equation (2-1), one can check that

$$[\tilde e_\beta, \tilde e_\gamma] = \tilde C^\alpha_{\beta\gamma}\,\tilde e_\alpha,\quad \tilde C^\alpha_{\beta\gamma} = \tilde\Gamma^\alpha_{\beta\gamma} - \tilde\Gamma^\alpha_{\gamma\beta}, \tag{3-17}$$

which gives

$$[e_\beta, e_\gamma] = C^\alpha_{\beta\gamma}\,e_\alpha,\quad C^\alpha_{\beta\gamma} = \frac{a_\alpha}{a_\beta a_\gamma}\tilde C^\alpha_{\beta\gamma}. \tag{3-18}$$

Note that e_α are orthonormal with respect to the metric ds^2, so by Koszul's formula [Petersen 1998], we have

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}(C^\alpha_{\beta\gamma} - C^\beta_{\gamma\alpha} + C^\gamma_{\alpha\beta}),$$

which, using (3-16)–(3-18), gives (3-13) . This completes the proof. □

Next, we calculate the mean curvature of f. Noting that ds^2 is a product metric, we obtain

$$\Gamma^j_{kl} = \sqrt{1+r^2}\,\tilde{\Gamma}^j_{kl}, \qquad \Gamma^\lambda_{\alpha j} = \Gamma^j_{\alpha\lambda} = 0, \tag{3-19}$$

$$\Gamma^\lambda_{\mu\mu} = \sqrt{1+r^2}\,\hat{\Gamma}^\lambda_{\mu\mu}, \quad \Gamma^\lambda_{nn} = \sqrt{1+r^2}\,\hat{\Gamma}^\lambda_{nn}, \quad q+1 \le \lambda, \mu < n, \tag{3-20}$$

$$\Gamma^n_{nn} = (1+r^2)\hat{\Gamma}^n_{nn}, \qquad \Gamma^n_{\mu\mu} = (1+r^2)\hat{\Gamma}^n_{\mu\mu}, \qquad q+1 \le \mu < n, \tag{3-21}$$

by (3-14) and (3-15) from Lemma 3.1.

From (3-8)–(3-10), one readily checks that

$$\Lambda_{0\bar{0},\alpha} = ir\delta_{n\alpha}, \qquad \Lambda_{j\bar{k},l} = \sqrt{1+r^2}\,\tilde{\Lambda}_{j\bar{k},l}, \tag{3-22}$$

$$\Lambda_{\lambda\bar{j},\alpha} = \Lambda_{j\bar{\lambda},\alpha} = 0, \quad \Lambda_{j\bar{n},j} = -ir, \quad \Lambda_{\lambda\bar{\mu},\nu} = \sqrt{1+r^2}\,\hat{\lambda}_{\lambda\bar{\mu},\nu}, \tag{3-23}$$

$$\Lambda_{\lambda\bar{n},\lambda} = \hat{\Lambda}_{\lambda\bar{n},\lambda} = \frac{i}{r}, \quad \Lambda_{n\bar{\lambda},n} = 0, \qquad \Lambda_{n\bar{n},n} = \hat{\Lambda}_{n\bar{n},n} = \frac{i}{r}, \tag{3-24}$$

where $q+1 \le \lambda < n$.

Proof of Theorem 1.1. According to the identities (2-33) and (3-4), we obtain

$$f^j_{kk} = -\Gamma^j_{kk} + \Lambda_{k\bar{j},k} = \sqrt{1+r^2}(-\tilde{\Gamma}^j_{kk} + \tilde{\Lambda}_{k\bar{j},k}) = \sqrt{1+r^2}\,\tilde{f}^j_{kk}, \tag{3-25}$$

by (3-19) and (3-22). Similarly, we obtain

$$f^j_{\lambda\lambda} = 0, \qquad f^\lambda_{jj} = 0, \qquad q+1 \le \lambda < n, \tag{3-26}$$

$$f^\lambda_{\mu\mu} = \sqrt{1+r^2}\,\hat{f}^\lambda_{\mu\mu}, \quad f^\lambda_{nn} = \sqrt{1+r^2}\,\hat{f}^\lambda_{nn}, \qquad q+1 \le \lambda, \mu < n, \tag{3-27}$$

$$f^n_{\lambda\lambda} = (1+r^2)\hat{f}^n_{\lambda\lambda} - ir, \qquad q+1 \le \lambda < n, \tag{3-28}$$

$$f^n_{jj} = -ir, \qquad f^n_{nn} = (1+r^2)\hat{f}^n_{nn} - 2ir. \tag{3-29}$$

So, f is minimal if and only if

$$\sum_{k=1}^{q} \tilde{f}^j_{kk} = 0, \quad 1 \le j \le q,$$

$$\sum_{\mu=q+1}^{n} \hat{f}^\lambda_{\mu\mu} = 0, \quad q+1 \le \mu < n, \quad \text{and} \quad \sum_{\lambda=q+1}^{n} \hat{f}^n_{\lambda\lambda} = \frac{i(n+1)r}{1+r^2},$$

by Proposition 2.1. This completes the first part of Theorem 1.1.

To prove the second part, we must calculate the 1-form $\alpha_H = (H\rfloor\omega)_{\Sigma_1\times\Sigma_2} = \sum_{\beta=1}^{n} h_\beta\,\theta^\beta$. According to (2-34), (3-25)-(3-28), we obtain

$$h_j = \sqrt{1+r^2}\,\tilde{h}_j, \quad h_\lambda = \sqrt{1+r^2}\,\hat{h}_\lambda, \quad q+1 \le \lambda < n, \tag{3-30}$$

$$h_n = (1+r^2)\,\hat{h}_n + (n+1)r, \tag{3-31}$$

where we have set

$$\alpha_{\tilde{H}} = \sum_j \tilde{h}_j\,\tilde{\theta}^j \quad \text{and} \quad \alpha_{\hat{H}} = \sum_\lambda \hat{h}_\lambda\,\hat{\theta}^\lambda.$$

Next, we calculate the $\delta\alpha_H$. The covariant derivative of h_β is given by

$$Dh_\beta = h_{\beta\gamma}\theta^\gamma = dh_\beta - h_\gamma\theta^\gamma_\beta,$$

and because ds^2 is a product metric, we have

$$\begin{aligned} h_{j\gamma}\theta^\gamma &= dh_j - h_\gamma\,\theta^\gamma_j = dh_j - h_k\,\theta^k_j \\ &= \sqrt{1+r^2}(d\tilde{h}_j - \tilde{h}_k\,\theta^k_j) \\ &= \sqrt{1+r^2}(\tilde{h}_{j;k}\,\tilde{\theta}^k - \tilde{h}_l\,\Gamma^l_{kj}\,\theta^k) \\ &= \sqrt{1+r^2}(\tilde{h}_{j;k} - \tilde{h}_l\,\tilde{\Gamma}^l_{kj})\,\tilde{\theta}^k \\ &= (1+r^2)\tilde{h}_{jk}\,\theta^k, \end{aligned}$$

by (3-5), (3-30) and Lemma 3.1, which gives

$$h_{jj} = (1+r^2)\tilde{h}_{jj}. \tag{3-32}$$

Here, we write $d\tilde{h}_j = \tilde{h}_{j;k}\,\tilde{\theta}^k$ and $D\tilde{h}_j = \tilde{h}_{jk}\,\tilde{\theta}^k = d\tilde{h}_j - \tilde{h}_k\,\tilde{\theta}^k_j$. Similarly, we have

$$h_{\lambda\lambda} = (1+r^2)\hat{h}_{\lambda\lambda} - (1+r^2)\big(r^2\hat{h}_n + (n+1)r\big)\hat{\Gamma}^n_{\lambda\lambda}, \quad q+1 \le \lambda < n, \tag{3-33}$$

$$h_{nn} = (1+r^2)\hat{h}_{nn} + r^2(1+r^2)\hat{h}_{n;n}. \tag{3-34}$$

So, by (2-19) and (3-32)–(3-33), we obtain

$$\delta\alpha_H = (1+r^2)(\delta\alpha_{\tilde{H}} + \delta\alpha_{\hat{H}}) + (1+r^2)\big((r^2\hat{h}_n + (n+1)r)\hat{\Gamma}^n_{\lambda\lambda} - r^2\hat{h}_{n;n}\big), \tag{3-35}$$

where $\delta\alpha_{\tilde{H}}$, $\delta\alpha_{\hat{H}}$ are the codifferentials of $\alpha_{\tilde{H}}$, $\alpha_{\hat{H}}$ with respect to $\tilde{f}$, $\hat{f}$ respectively. This completes the proof. □

4. Some explicit examples

As a first example, we study the standard Lagrangian torus

$$\hat{f} : \mathbb{S}^1(1) \times \cdots \times \mathbb{S}^1(1) \to \mathbb{C}^m, \quad z = \hat{f}(p) = (e^{it_1}, \ldots, e^{it_m}),$$

where we parametrize $\mathbb{S}^1(1)$ by $\mathbb{S}^1(1) = \{e^{it} : 0 \le t \le 2\pi\}$.

Choosing the moving frame of $\mathbb{C}^m$ along $\hat{f}$ to be

$$e_\lambda = \frac{i}{\sqrt{\lambda(\lambda+1)}}(e^{it_1}, \dots, e^{it_\lambda}, -\lambda e^{it_{\lambda+1}}, 0, \dots, 0), \quad 1 \le \lambda < m, \tag{4-1}$$

$$e_m = \frac{i}{\sqrt{m}}(e^{it_1}, \dots, e^{it_m}), \tag{4-2}$$

it is easy to check that the coefficients $\hat{\theta}^\lambda := (dz, e_\lambda)$ satisfy

$$\hat{\theta}^\lambda = \frac{1}{\sqrt{\lambda(\lambda+1)}}(dt_1 + \cdots + dt_\lambda - \lambda\, dt_{\lambda+1}), \quad 1 \le \lambda < m, \tag{4-3}$$

$$\hat{\theta}^m = \frac{1}{\sqrt{m}}(dt_1 + \cdots + dt_m), \tag{4-4}$$

$$dt_\lambda = -(\lambda-1)\frac{\hat{\theta}^{\lambda-1}}{\sqrt{(\lambda-1)\lambda}} + \sum_{\mu=\lambda}^{m-1} \frac{\hat{\theta}^\mu}{\sqrt{\mu(\mu+1)}} + \frac{\hat{\theta}^m}{\sqrt{m}}. \tag{4-5}$$

From (4-1)–(4-5), we have

$$\hat{\omega}_{\lambda\bar{\lambda}} = (de_\lambda, e_\lambda) = -\frac{i(\lambda-1)\hat{\theta}^\lambda}{\sqrt{\lambda(\lambda+1)}} + \sum_{\mu=\lambda+1}^{m-1} \frac{i\hat{\theta}^\mu}{\sqrt{\mu(\mu+1)}} + \frac{i\hat{\theta}^m}{\sqrt{m}}, \quad \lambda < m, \tag{4-6}$$

$$\hat{\omega}_{\lambda\bar{\mu}} = -\hat{\omega}_{\bar{\mu}\lambda} = -\overline{(de_\mu, e_\lambda)} = \frac{i\hat{\theta}^\lambda}{\sqrt{\mu(\mu+1)}}, \quad \lambda < \mu < m, \tag{4-7}$$

$$\hat{\omega}_{\lambda\bar{m}} = -\hat{\omega}_{\bar{m}\lambda} = -\overline{(de_m, e_\lambda)} = \frac{i\hat{\theta}^\lambda}{\sqrt{m}}, \quad \lambda < m, \tag{4-8}$$

$$\hat{\omega}_{m\bar{m}} = \frac{i\hat{\theta}^m}{\sqrt{m}}. \tag{4-9}$$

The metric induced by $\hat{f}$ is flat, so we obtain, by (2-22),

$$\hat{f}^\lambda_{\mu\mu} = \hat{\Lambda}_{\mu\bar{\lambda},\mu} = 0, \quad \lambda < \mu < m, \tag{4-10}$$

$$\hat{f}^\lambda_{\lambda\lambda} = \hat{\Lambda}_{\lambda\bar{\lambda},\lambda} = -\frac{i(\lambda-1)}{\sqrt{\lambda(\lambda+1)}}, \quad \lambda < m, \tag{4-11}$$

$$\hat{f}^\lambda_{\mu\mu} = \hat{\Lambda}_{\mu\bar{\lambda},\mu} = \frac{i}{\sqrt{\lambda(\lambda+1)}}, \quad \mu < \lambda < m, \tag{4-12}$$

$$\hat{f}^\lambda_{mm} = 0, \quad \hat{f}^m_{\lambda\lambda} = \hat{f}^m_{mm} = \frac{i}{\sqrt{m}}, \quad \lambda < m. \tag{4-13}$$

Proposition 4.1. *Let $\hat{f} : \mathbb{S}^1(1) \times \cdots \times \mathbb{S}^1(1) \to \mathbb{C}^m$, $z = \hat{f}(p) = (e^{it_1}, \dots, e^{it_m})$, be the standard Lagrangian torus in $\mathbb{C}^m$. Its complex mean curvature $H^{\mathbb{C}}$ satisfies*

$$\hat{H}^{\mathbb{C}} - (\hat{H}^{\mathbb{C}}, e_m)e_m = 0, \quad (\hat{H}^{\mathbb{C}}, e_m) = i\sqrt{m}.$$

Moreover, if we set $\hat{h}_m = -\operatorname{Im}\big((\hat{H}^{\mathbb{C}}, e_m)\big)$, *we have*

$$m\langle \operatorname{grad}\hat{h}_m, e_m\rangle - (m\hat{h}_m + (n+1)\sqrt{m})\sum_{\lambda}\langle \hat{\nabla}_{e_\lambda} e_m, e_\lambda\rangle = 0.$$

Proof. The first part holds because of (4-10)–(4-13). The second part is true because the induced metric is flat and $\hat{h}_m$ is a constant. □

Proof of Theorem 1.2. The theorem follows from Theorem 1.1, Proposition 4.1, and the fact that the standard torus studied above is H-minimal in $\mathbb{C}^m$. □

For the next example, consider

$$S^{m-1}(1) = \{x \in \mathbb{R}^m : |x| = 1\} \tag{4-14}$$

with its standard embedding in $\mathbb{C}^m$. Take an orthonormal tangent frame field $\hat{e}_1, \dots, \hat{e}_{m-1}$, with respect to which the metric is expressed by

$$dx = \sum_{\lambda=1}^{m-1} \hat{\theta}^\lambda\, \hat{e}_\lambda, \quad d\hat{\theta}^\lambda = -\hat{\theta}^\lambda_\mu \wedge \hat{\theta}^\mu;$$

the coefficients $\hat{\theta}^\lambda$ and $\hat{\theta}^\lambda_\mu$ are real. Further, set

$$d\hat{e}_\lambda = \sum_{\mu} \hat{\omega}_{\lambda\mu}\, \hat{e}_\mu, \quad 1 \le \lambda, \mu < m, \tag{4-15}$$

where the $\hat{\omega}_{\lambda\mu}$ are real and satisfy $\hat{\omega}_{\lambda\mu} + \hat{\omega}_{\mu\lambda} = 0$.

Take the immersion $\hat{f} : S^{m-1}(1) \times T^1 \to \mathbb{C}^m$ given by $(x, e^{it}) \mapsto z = e^{it}x$. Choosing the moving frame of $\mathbb{C}^m$ along $\hat{f}$ to be

$$\begin{aligned} e_\lambda &= e^{it}\hat{e}_\lambda, \qquad 1 \le \lambda < m,\\ e_m &= iz = ie^{it}x, \end{aligned} \tag{4-16}$$

we conclude that

$$\begin{aligned} \theta^\lambda &:= (dz, e_\lambda) = \hat{\theta}^\lambda, \quad 1 \le \lambda < m,\\ \theta^m &:= (dz, e_m) = dt, \end{aligned} \tag{4-17}$$

are real 1-forms, which implies that $\hat{f}$ is a Lagrangian immersion. Through direct calculation, we have

$$\omega_{\lambda\bar{\lambda}} = \omega_{m\bar{m}} = i\theta^m, \quad \omega_{\lambda\bar{m}} = i\theta^\lambda, \quad 1 \le \lambda < m \tag{4-18}$$

and

$$\omega_{\lambda\bar{\mu}} = \hat{\omega}_{\lambda\mu}, \quad 1 \le \lambda < \mu < m, \tag{4-19}$$

which are real 1-forms. As before, we use the notation $\omega_{\lambda\mu} = (de_\lambda, e_\mu)$.

If we denote the connection 1-forms with respect to θ^λ by θ^λ_μ, we clearly have

$$\theta^m_\lambda = 0, \quad \theta^\lambda_\mu = \hat{\theta}^\lambda_\mu = \hat{\omega}_{\mu\lambda}, \quad 1 \le \lambda, \mu < m. \tag{4-20}$$

From (4-20) and (2-33), we obtain

$$\hat{f}^\lambda_{\mu\mu} = 0, \quad \hat{f}^m_{\lambda\lambda} = i, \quad 1 \le \lambda < m, \ 1 \le \mu \le m. \tag{4-21}$$

Proposition 4.2. *The map $\hat{f} : \mathbb{S}^{m-1}(1) \times T^1 \to \mathbb{C}^m$ given by $(x, e^{it}) \mapsto e^{it}x$ is an H-minimal Lagrangian immersion in $\mathbb{C}^m$, and its complex mean curvature $H^{\mathbb{C}}$ satisfies*

$$\hat{H}^{\mathbb{C}} - (\hat{H}^{\mathbb{C}}, e_m)e_m = 0, \quad (\hat{H}^{\mathbb{C}}, e_m) = im.$$

Moreover, if we set $\hat{h}_m = -\operatorname{Im}\big((\hat{H}^{\mathbb{C}}, e_m)\big)$, we have

$$\langle \operatorname{grad} \hat{h}_m, e_m \rangle - (\hat{h}_m + (n+1)) \sum_\lambda \langle \hat{\nabla}_{e_\lambda} e_m, e_\lambda \rangle = 0.$$

Proof. By the definition of $\hat{h}_\lambda$, we have $\hat{h}_\lambda = 0$, $1 \le \lambda < m$ and $\hat{h}_m = -m$, which imply $\delta\alpha_{\hat{H}} = 0$. So, $\hat{f}$ is H-minimal. The second identity holds because $\hat{h}_m$ is a constant and $\theta^m_\lambda = 0$. □

Proof of the Theorem 1.3. This follows from Proposition 4.2 and Theorem 1.1. □

Example 4.3 (Clifford torus in $\mathbb{CP}^n$). Taking $q = 0$ in Theorem 1.1, we have proved that the Clifford torus is a minimal Lagrangian submanifold in $\mathbb{CP}^n$. This is a known result; here we just provided an alternative proof.

Example 4.4 (H-minimal $S^q(1) \times T^{n-q}$ in $\mathbb{CP}^n$). Let $\check{f} : S^q(1) \subset \mathbb{R}^{q+1} \hookrightarrow \mathbb{C}^{q+1}$, $f(p) = Z$, be the standard embedding. Then $\mathcal{H} \circ \check{f}$ is totally geodesic in $\mathbb{CP}^q$. Define $\check{f} : S^q(1) \times T^{n-q} \to \mathbb{S}^{2n+1}(1)$ by

$$(Z, e^{it_{q+1}}, \dots, e^{it_n}) \mapsto \frac{1}{\sqrt{n-q+1}} (Z, e^{it_{q+1}}, \dots, e^{it_n}).$$

This gives an H-minimal immersion $\mathcal{H} \circ \check{f}$, by Theorem 1.2.

Example 4.5 (exotic H-minimal $\mathbb{S}^3(1) \times T^{n-3}$ in $\mathbb{CP}^n$). Recall from [Bedulli and Gori 2008], [Chen et al. 1996], [Chiang 2004], or [Li and Tao 2006] the exotic minimal Lagrangian immersion $\check{f} : \mathbb{S}^3(1) \to \mathbb{CP}^3$ mapping the point (a, b), where $|a|^2 + |b|^2 = 1$, to

$$\big[\bar{a}^3 + 3\bar{a}\bar{b}^2,\ \sqrt{3}(\bar{a}^2 b + \bar{b}|b|^2 - 2\bar{b}|a^2|),\ \sqrt{3}(\bar{a}b^2 + a|a|^2 - 2a|b|^2),\ b^3 + 3a^2 b\big].$$

By Theorem 1.2, we know that the Lagrangian immersion $f : \mathbb{S}^3(1) \times T^{n-3} \to \mathbb{CP}^n$ mapping $\big((a, b), (e^{it_4}, \dots, e^{it_n})\big)$ to

$$\big[\bar{a}^3 + 3\bar{a}\bar{b}^2,\ \sqrt{3}(\bar{a}^2 b + \bar{b}|b|^2 - 2\bar{b}|a^2|),\ \sqrt{3}(\bar{a}b^2 + a|a|^2 - 2a|b|^2),\ b^3 + 3a^2 b,\\ e^{it_4}, \dots, e^{it_n}\big]$$

is H-minimal.

Example 4.6. Let $S^{m-1}(1)$ be as in (4-14). The immersion

$$f : S^{m-1}(1) \times S^{m-1}(1) \times T^1 \mapsto \mathbb{CP}^{2m-1}$$

given by

$$(x, y, e^{it}) \mapsto [x, e^{it} y]$$

is a minimal Lagrangian immersion.

The map $f : S^q(1) \times S^{m-1}(1) \times T^1 \mapsto \mathbb{CP}^{q+m}$ given by the same formula is an H-minimal Lagrangian immersion.

Example 4.7. The immersion $f : T^{m-1} \times \mathbb{S}^{m-1}(1) \times T^1 \to \mathbb{CP}^{2m-1}$ given by

$$\big((1, e^{it_1}, e^{it_{m-1}}), x, e^{it}\big) \mapsto \left[\frac{1}{\sqrt{m}}, \frac{e^{it_1}}{\sqrt{m}}, \dots, \frac{e^{it_{m-1}}}{\sqrt{m}}, e^{it} x\right]$$

is a minimal Lagrangian immersion, and the map $f : T^q \times \mathbb{S}^{m-1}(1) \times T^1 \to \mathbb{CP}^{q+m}$ given by

$$\big((1, e^{it_1}, \dots, e^{it_q}), x, e^i t\big) \mapsto \left[\frac{1}{\sqrt{q+1}}, \frac{e^{it_1}}{\sqrt{q+1}}, \dots, \frac{e^{it_q}}{\sqrt{q+1}}, e^{it} x\right]$$

is an H-minimal Lagrangian immersion. Here, we have used the fact the Clifford torus $T^n \to \mathbb{CP}^n$, given by

$$(e^{it_1}, \dots, e^{it_n}) \mapsto [1, e^{it_1}, \dots, e^{it_n}],$$

is minimal.

Example 4.8. The map $\mathbb{S}^3(1) \times S^3(1) \times T^1 \to \mathbb{CP}^7$ taking $\big((a, b), x, e^{it}\big)$ to

$$\big[\bar{a}^3 + 3\bar{a}\bar{b}^2, \sqrt{3}(\bar{a}^2 b + \bar{b}|b|^2 - 2\bar{b}|a^2|), \sqrt{3}(\bar{a}b^2 + a|a|^2 - 2a|b|^2), b^3 + 3a^2 b, e^{it} x\big]$$

(where $(a, b) \in \mathbb{C}^2$ satisfies $|a|^2 + |b|^2 = 1$ and $x \in \mathbb{R}^4$ satisfies $|x|^2 = 1$) is a minimal Lagrangian immersion.

The map $\mathbb{S}^3(1) \times S^{m-1}(1) \times T^1 \to \mathbb{CP}^{m+3}$ given by the same formula (with $x \in \mathbb{R}^m$ satisfying $|x|^2 = 1$) is an H-minimal Lagrangian immersion.

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References

[Bedulli and Gori 2008] L. Bedulli and A. Gori, "Homogeneous Lagrangian submanifolds", *Comm. Anal. Geom.* **16**:3 (2008), 591–615. MR 2009k:53209 Zbl 1152.53060

[Castro and Urbano 1998] I. Castro and F. Urbano, "Examples of unstable Hamiltonian-minimal Lagrangian tori in $\mathbb{C}^2$", *Compositio Math.* **111**:1 (1998), 1–14. MR 98m:53075 Zbl 0896.53039

[Castro and Urbano 2004] I. Castro and F. Urbano, "On a new construction of special Lagrangian immersions in complex Euclidean space", *Q. J. Math.* **55**:3 (2004), 253–265. MR 2005g:53087 Zbl 1086.53074

[Castro et al. 2006] I. Castro, H. Li, and F. Urbano, "Hamiltonian-minimal Lagrangian submanifolds in complex space forms", *Pacific J. Math.* **227**:1 (2006), 43–63. MR 2007k:53092 Zbl 1129.53039

[Chen et al. 1996] B.-Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, "An exotic totally real minimal immersion of S^3 in $\mathbf{CP}^3$ and its characterisation", *Proc. Roy. Soc. Edinburgh Sect. A* **126**:1 (1996), 153–165. MR 97e:53111 Zbl 0855.53011

[Chiang 2004] R. Chiang, "New Lagrangian submanifolds of $\mathbb{CP}^n$", *Int. Math. Res. Not.* **2004**:45 (2004), 2437–2441. MR 2005f:53140 Zbl 1075.53080

[Griffiths 1974] P. Griffiths, "On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry", *Duke Math. J.* **41** (1974), 775–814. MR 53 #14355 Zbl 0294.53034

[Joyce 2002] D. Joyce, "Special Lagrangian m-folds in $\mathbb{C}^m$ with symmetries", *Duke Math. J.* **115**:1 (2002), 1–51. MR 2003m:53083 Zbl 1023.53033

[Li and Tao 2006] Z. Q. Li and Y. Q. Tao, "Equivariant Lagrangian minimal S^3 in $\mathbb{C}\mathrm{P}^3$", *Acta Math. Sin.* (*Engl. Ser.*) **22**:4 (2006), 1215–1220. MR 2007c:53075 Zbl 1110.53046

[Ma and Schmies 2006] H. Ma and M. Schmies, "Examples of Hamiltonian stationary Lagrangian tori in $\mathbb{C}\mathrm{P}^2$", *Geom. Dedicata* **118** (2006), 173–183. MR 2007b:53124 Zbl 1096.53048

[Mironov 2004] A. E. Mironov, "New examples of Hamilton-minimal and minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}\mathrm{P}^n$", *Mat. Sb.* **195**:1 (2004), 89–102. In Russian; translated in *Sb. Math* **195**:1 (2004), 85–96. MR 2005e:53127 Zbl 1078.53079

[Mironov and Zuo 2008] A. E. Mironov and D. Zuo, "On a family of conformally flat Hamiltonian-minimal Lagrangian tori in $\mathbb{CP}^3$", *Int. Math. Res. Not.* **2008** (2008), ID rnn 078. MR 2009g:53120 Zbl 1153.53043

[Oh 1993] Y.-G. Oh, "Volume minimization of Lagrangian submanifolds under Hamiltonian deformations", *Math. Z.* **212**:2 (1993), 175–192. MR 94a:58040 Zbl 0791.53050

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics **171**, Springer, New York, 1998. MR 98m:53001 Zbl 0914.53001

[Schoen and Wolfson 1999] R. Schoen and J. Wolfson, "Minimizing volume among Lagrangian submanifolds", pp. 181–199 in *Differential equations* (Florence, 1996), edited by M. Giaquinta et al., Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999. MR 99k:53130 Zbl 1031.53112

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