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Let k be a field, let G be a finite group and let $k(x_g : g \in G)$ be the rational function field over k, on which G acts by the k-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Noether's problem asks whether the fixed subfield $k(G) := k(x_g : g \in G)^G$ is k-rational, that is, purely transcendental over k. If $\widehat{S}_n$ is the double cover of the symmetric group S_n, in which the liftings of transpositions and products of disjoint transpositions are of order 4, Serre shows that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not $\mathbb{Q}$-rational. We will prove that if k is a field such that $\operatorname{char} k \neq 2, 3$, and $k(\zeta_8)$ is a cyclic extension of k, then $k(\widehat{S}_4)$ is k-rational. If it is assumed furthermore that $\operatorname{char} k = 0$, then $k(\widehat{S}_5)$ is also k-rational.

1. Introduction

Let k be a field, and L be a finitely generated field extension of k. L is called k-rational (or rational over k) if L is purely transcendental over k; that is, L is isomorphic to some rational function field over k. L is called stably k-rational if $L(y_1, \dots, y_m)$ is k-rational for some $y_1, \dots, y_m$ that are algebraically independent over L. L is called k-unirational if L is k-isomorphic to a subfield of some k-rational field extension of k. It is easy to see that

$$k\text{-rational} \Rightarrow \text{stably } k\text{-rational} \Rightarrow k\text{-unirational}.$$

A notion of retract rationality was introduced in [Saltman 1984] (see also [Kang 2012]). It is known that if k is an infinite field, then

$$\text{stably } k\text{-rational} \Rightarrow \text{retract } k\text{-rational} \Rightarrow k\text{-unirational}.$$

Let k be a field and G a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$.

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Denote by $k(G)$ the fixed subfield, that is, $k(G) = k(x_g : g \in G)^G$. Noether's problem asks under what conditions is the field $k(G)$ k-rational.

Noether's problem is related to the inverse Galois problem and the existence of generic G-Galois extensions over k. For the details, see Swan's survey paper [Swan 1983]. The purpose of this paper is to study Noether's problem for some double covers of the symmetric group S_n.

It is known that there are four different double covers of S_n when $n \geq 4$, that is, groups G that fit into a short exact sequence $1 \to C_2 \to G \to S_n \to 1$; see, for example, [Serre 1984, p. 653].

Definition 1.1 [Garibaldi et al. 2003, pp. 58, 90; Hoffman and Humphreys 1992, p. 18; Karpilovsky 1985, pp. 177–181]. Let $C_2 = \{\pm 1\}$ be the cyclic group of order 2. When $n \geq 4$, the group $\widehat{S}_n$ is the unique central extension of S_n by C_2, that is,

$$1 \to C_2 \to \widehat{S}_n \to S_n \to 1,$$

satisfying the condition that the transpositions and the product of two disjoint transpositions in S_n lift to elements of order 4 in $\widehat{S}_n$. On the other hand, the group $\widetilde{S}_n$ is the central extension

$$1 \to C_2 \to \widetilde{S}_n \to S_n \to 1,$$

such that a transposition in S_n lifts to an element of order 2 of $\widetilde{S}_n$, but a product of two disjoint transpositions in S_n lifts to an element of order 4.

Note that we follow the notation of $\widehat{S}_n$ and $\widetilde{S}_n$ adopted by Serre.

Theorem 1.2 (Serre [Garibaldi et al. 2003, p. 90]). *Both $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract $\mathbb{Q}$-rational. In particular, they are not $\mathbb{Q}$-rational.*

Serre proves this using cohomological invariants and trace forms over $\mathbb{Q}$ — the e-invariant method, in short. In pp. 89–90 of the same book, he proves that $\mathrm{Rat}(G/\mathbb{Q})$ is false for $G = \widehat{S}_4$ and $\widehat{S}_5$. Actually he proves a bit more. From Serre's proof it is easy to find that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract $\mathbb{Q}$-rational (see [Kang 2012, Section 1] for the relationship of the property $\mathrm{Rat}(G/k)$ and the retract k-rationality of $k(G)$). This is the reason why we formulate Serre's theorem in the version above. In fact, Theorem 1.2 can be perceived also from Serre's own remark in [Garibaldi et al. 2003, p. 13, Remark 5.8].

We don't know whether Theorem 1.2 is valid for fields k other than the field $\mathbb{Q}$; for example, the field k satisfying the condition that $k(\zeta_8)$ is not cyclic over k. In fact, in a private communication, Serre told us that the e-invariant method remains valid (under the assumption that $k(\zeta_8)$ is not cyclic over k) if k is an algebraic number field of odd degree over $\mathbb{Q}$, or if $k = \mathbb{Q}(\sqrt{n})$, where $n \equiv 1 \pmod 8$. However, if $k = \mathbb{Q}(x, y)$ with $x^2 + y^2 = -1$, the assumption that $k(\zeta_8)$

is not cyclic over k is valid while the e-invariant method doesn't work any more [Serre 2011].

On the other hand, we have:

Theorem 1.3 [Plans 2007; 2009]. (1) *For any field k, $k(\widetilde{S}_4)$ is k-rational. Thus, if k is a field with* $\operatorname{char} k = 0$, *$k(\widetilde{S}_5)$ is also k-rational.*

(2) *For any infinite field k with* $\operatorname{char} k \neq 2$ *such that $\sqrt{-1} \in k$, both $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$ are k-rational.*

The main result of this article is the following rationality criterion for $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$.

Theorem 1.4. *Let k be a field with* $\operatorname{char} k \neq 2$ *or* 3, *and ζ_8 be a primitive eighth root of unity in some extension field of k. If $k(\zeta_8)$ is a cyclic extension of k, then $k(\widehat{S}_4)$ is k-rational*; *if it is assumed furthermore that* $\operatorname{char} k = 0$, *then $k(\widehat{S}_5)$ is also k-rational.*

When k is a field with $\operatorname{char} k = p > 0$ and $p \neq 2$, the assumption that $k(\zeta_8)$ is a cyclic extension of k is satisfied automatically. Thus $k(\widehat{S}_4)$ is k-rational provided that k is any field with $\operatorname{char} k \neq 2$ or 3.

Besides the groups $\widehat{S}_4$ and $\widehat{S}_5$, Serre shows that $\mathbb{Q}(G)$ is not retract $\mathbb{Q}$-rational if G is any one of the groups $\mathrm{SL}_2(\mathbb{F}_7)$, $\mathrm{SL}_2(\mathbb{F}_9)$ and the generalized quaternion group of order 16; see [Garibaldi et al. 2003, p. 90, Example 33.27]. In case G is the generalized quaternion group of order 16 and $k(\zeta_8)$ is cyclic over k, it is known that $k(G)$ is k-rational [Kang 2005]. We don't know whether analogous results as Theorem 1.4 are valid when the groups are $\mathrm{SL}_2(\mathbb{F}_7)$ and $\mathrm{SL}_2(\mathbb{F}_9)$.

The main idea of the proof of Theorem 1.4 is to use the method of Galois descent, namely we first enlarge the field k to $k(\zeta_8)$, solve the rationality of $k(\zeta_8)(\widehat{S}_4)$, and then descend the ground field to k.

The proof that $k(\zeta_8)(\widehat{S}_4)$ is $k(\zeta_8)$-rational requires at least two techniques. In order to decrease the number of variables (by applying Theorem 2.2), we will construct a 4-dimensional faithful representation V of $\widehat{S}_4$ defined over the field k. It seems the representation and the idea to find it are not well-known. Once we have this representation, we adjoin ζ_8 to the field k and write $\pi = \mathrm{Gal}(k(\zeta_8)/k)$. We will prove that $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k-rational.

The rationality problem of $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is not straightforward. In several steps of computations we use computers to facilitate the process of symbolic computation. However, we emphasize that computers play only a minor role; we don't use particular codes of data bases such as GAP.

On the other hand, we point out that the first several steps in proving that $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k-rational are rather similar to those in [Kang and Zhou 2012, Section 5]. This seems unsurprising because the group $\widetilde{S}_4$ considered in [Kang and Zhou 2012, Section 5] and the group $\widehat{S}_4$ here have a common subgroup $\widetilde{A}_4$.

For the rationality problem of $k(\widehat{S}_5)$, we apply Theorem 2.5 of Plans, which asserts that $k(\widehat{S}_5)$ is a rational extension of $k(\widehat{S}_4)$, whence the result.

We organize this paper as follows. We recall some preliminaries in Section 2 that will be used in the proof of Theorem 1.4. In Section 3, several low-dimensional faithful representations of $\widehat{S}_4$ over a field k with $\operatorname{char} k \neq 2$ will be constructed (the reader may find another explicit construction in [Karpilovsky 1985, p. 177–179]). Theorem 1.4 will be proved in Section 4. In Section 5 we will consider the rationality problem of $k(G_n)$ (see Definition 5.1 for the group G_n).

Throughout this article, whenever we write $k(x_1, x_2, x_3, x_4)$ or $k(x, y)$ without explanation, it is understood that it is a rational function field over k. We will denote by ζ_8 (or simply by ζ) a primitive eighth root of unity.

2. Preliminaries

We recall several results that will be used in tackling the rationality problem.

Theorem 2.1 [Ahmad et al. 2000, Theorem 3.1]. *Let L be any field, $L(x)$ the rational function field of one variable over L and G a finite group acting on $L(x)$. Suppose that for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G, \deg g(x) \geq 1\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property that $L(x)^G = L^G(f)$.*

Theorem 2.2 [Hajja and Kang 1995, Theorem 1]. *Let G be a finite group acting on the rational function field $L(x_1, \ldots, x_n)$ of n variables over a field L. Suppose that*:

(i) *For any $\sigma \in G$, $\sigma(L) \subset L$.*

(ii) *The restriction of the action of G to L is faithful.*

(iii) *For any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in \mathrm{GL_n(L)}$ and $B(\sigma)$ is a $n \times 1$ matrix over L.

Then there exist elements $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ that are algebraically independent over L and satisfy $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \leq i \leq n$.

Theorem 2.3 [Yamasaki 2009]. *Let k be a field with* $\operatorname{char} k \neq 2$, *let $a \in k\backslash\{0\}$, and define a k-automorphism σ of the rational function field $k(x, y)$ by $\sigma(x) = a/x$ and $\sigma(y) = a/y$. Then $k(x, y)^{\langle\sigma\rangle} = k(u, v)$, where $u = (x - y)/(a - xy)$ and $v = (x + y)/(a + xy)$.*

Theorem 2.4 [Masuda 1955, Theorem 3; Hoshi and Kang 2010, Theorem 2.2]. *Let k be a field and let σ be the k-automorphism of the rational function field $k(x, y, z)$ defined by $\sigma : x \mapsto y \mapsto z \mapsto x$. Then $k(x, y, z)^{\langle\sigma\rangle} = k(s_1, u, v) = k(s_3, u, v)$, where s_1, s_2, s_3 are the elementary symmetric functions of degree one, two and three in x, y, z and u and v are defined by*

$$u = \frac{x^2y + y^2z + z^2x - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx} \quad \textit{and} \quad v = \frac{xy^2 + yz^2 + zx^2 - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$

Theorem 2.5 [Plans 2009, Theorem 11]. *Let $n \geq 5$ be an odd integer and let k be a field with* $\operatorname{char} k = 0$. *Then $k(\widehat{S}_n)$ is rational over $k(\widehat{S}_{n-1})$.*

Theorem 2.6 [Kang and Plans 2009, Theorem 1.9]. *Let k be a field and let G_1 and G_2 be two finite groups. If both $k(G_1)$ and $k(G_2)$ are k-rational, so is $k(G_1 \times G_2)$.*

3. Faithful representations of $\widehat{S}_4$

In this and the next section, the field k we consider is of $\operatorname{char} k \neq 2$ or 3. We will denote by $\zeta_8 = (1 + \sqrt{-1})/\sqrt{2}$ a primitive eighth root of unity.

In [Springer 1977, p. 92] a generating set of $\widehat{S}_4$ is given (where the group is called the binary octahedral group): $\widehat{S}_4 = \langle a', b, c\rangle$ with relations $a'^8 = b^4 = c^6 = 1$, $ba'b^{-1} = a'^{-1}$, $cbc^{-1} = a'^2$ and $(a'c)^2 = -a'^2b$ (here -1 is the element that is equal to $a'^4 = b^2 = c^3$). Note that we have a short exact sequence of groups

$$1 \to \{\pm 1\} \to \widehat{S}_4 \xrightarrow{p} S_4 \to 1,$$

and that $p(a') = (1, 2, 3, 4)$, $p(b) = (1, 4)(2, 3)$ and $p(c) = (1, 2, 3)$. Note that $p(ba') = (1, 4)(2, 3)(1, 2, 3, 4) = (1, 3)$.

If $\zeta_8 \in k$, a faithful 2-dimensional representation $\Phi : \widehat{S}_4 \to \mathrm{GL}_2(\mathrm{k})$ is given in [Springer 1977, p. 92] as follows (we write $\zeta = \zeta_8$),

$$\Phi(a') = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^7 \end{pmatrix}, \quad \Phi(b) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \Phi(c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^7 & \zeta^7 \\ \zeta^5 & \zeta \end{pmatrix}. \tag{3-1}$$

Suppose that $\sqrt{2} \in k$ (but not necessarily that $\sqrt{-1} \in k$). We may obtain a 4-dimensional representation $\widehat{S}_4 \to \mathrm{GL}_4(\mathrm{k})$ by making in (3-1) the substitutions

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{2})$. This process is an easy application of Weil's restriction [Weil 1956; Voskresenskii 1998, p. 38]. Thus we get

$$
\text{(3-2)}\quad a' \mapsto \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 1 & -1 & & \\ 1 & 1 & & \\ \hline & & 1 & 1 \\ & & -1 & 1 \end{array}\right),\ b \mapsto \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},\ c \mapsto \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.
$$

Similarly, when $\sqrt{-2}$ is in k (but possibly $\sqrt{-1}$ is not in k), write $\sqrt{-2} = \sqrt{-1}\cdot\sqrt{2}$. Thus represent $\sqrt{2}$ as $-\sqrt{-1}\cdot\sqrt{-2}$ and $\zeta = (1+\sqrt{-1})/\sqrt{2}$ becomes $\sqrt{-2}(1-\sqrt{-1})/2$. Make in (3-1) the substitutions

$$
\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$

where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{-2})$. We get

$$
\text{(3-3)}\quad a' \mapsto \frac{\sqrt{-2}}{2}\left(\begin{array}{cc|cc} 1 & 1 & & \\ -1 & 1 & & \\ \hline & & -1 & 1 \\ & & -1 & -1 \end{array}\right),\ b \mapsto \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},\ c \mapsto \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.
$$

The same way, if $\sqrt{-1} \in k$ (but possibly $\sqrt{2} \notin k$), make in (3-1) the substitutions

$$
\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$

where k_0 is the prime filed of k and $\alpha \in k_0(\sqrt{-1})$. We get

$$
\text{(3-4)}\quad
\begin{gathered}
a' \mapsto \left(\begin{array}{cc|cc} 0 & 1+\sqrt{-1} & & \\ \frac{1+\sqrt{-1}}{2} & 0 & & \\ \hline & & 0 & 1-\sqrt{-1} \\ & & \frac{1-\sqrt{-1}}{2} & 0 \end{array}\right), \\
b \mapsto \left(\begin{array}{cc|cc} & & \sqrt{-1} & 0 \\ & & 0 & \sqrt{-1} \\ \hline \sqrt{-1} & 0 & & \\ 0 & \sqrt{-1} & & \end{array}\right), \quad
c \mapsto \begin{pmatrix} \frac{1-\sqrt{-1}}{2} & 0 & \frac{1-\sqrt{-1}}{2} & 0 \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 & \frac{1-\sqrt{-1}}{2} \\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ 0 & \frac{-1-\sqrt{-1}}{2} & 0 & \frac{1+\sqrt{-1}}{2} \end{pmatrix}.
\end{gathered}
$$

Finally, from (3-2) we may get a faithful 8-dimensional representation of $\widehat{S}_4$ into $\mathrm{GL}_8(\mathrm{k}_0)$, where k_0 is the prime field of k. Explicitly, make in (3-2) the substitutions

$$
\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$

where $\alpha \in k_0$. We get

$$
a' \mapsto \frac{1}{2}\left(\begin{array}{cccc|cccc}
0 & 2 & 0 & -2 & & & & \\
1 & 0 & -1 & 0 & & & & \\
0 & 2 & 0 & 2 & & & & \\
1 & 0 & 1 & 0 & & & & \\
\hline
 & & & & 0 & 2 & 0 & 2 \\
 & & & & 1 & 0 & 1 & 0 \\
 & & & & 0 & -2 & 0 & 2 \\
 & & & & -1 & 0 & 1 & 0
\end{array}\right),
$$

$$
b \mapsto \left(\begin{array}{cccc|cccc}
 & & & & & & -1 & 0 \\
 & & & & & & 0 & -1 \\
 & & & & 1 & 0 & & \\
 & & & & 0 & 1 & & \\
\hline
 & & -1 & 0 & & & & \\
 & & 0 & -1 & & & & \\
1 & 0 & & & & & & \\
0 & 1 & & & & & &
\end{array}\right), \tag{3-5}
$$

$$
c \mapsto \frac{1}{2}\left(\begin{array}{cccc|cccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
\hline
-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\
-1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1
\end{array}\right).
$$

4. Proof of Theorem 1.4

By Theorem 2.5, in case $\operatorname{char} k = 0$ and it is known that $k(\widehat{S}_4)$ is k-rational, it follows immediately that $k(\widehat{S}_5)$ is also k-rational. Hence, in proving Theorem 1.4, it suffices to prove the rationality of $k(\widehat{S}_4)$.

By assumption, $k(\zeta_8)$ is a cyclic extension of k. Hence at least one of $\sqrt{-1}$, $\sqrt{2}$ or $\sqrt{-2}$ belongs to k.

Case 1: $\zeta_8 \in k$. Since $\operatorname{char} k \neq 2$ or 3, the group algebra $k[\widehat{S}_4]$ is semisimple. Hence the 2-dimensional faithful representation provided by Equation (3-1) can be embedded into the regular representation whose dual space is $V_{\mathrm{reg}} = \oplus_{g \in \widehat{S}_4} k \cdot x(g)$, where $\widehat{S}_4$ acts on V_{reg} by $h \cdot x(g) = x(hg)$ for any $g, h \in \widehat{S}_4$. By Theorem 2.2, we find that $k(\widehat{S}_4) = k(x(g) : g \in \widehat{S}_4)^{\widehat{S}_4}$ is rational over $k(x, y)^{\widehat{S}_4}$, where the actions given by Equation (3-1) are

$$\begin{aligned} a' &: x \mapsto \zeta x, && y \mapsto \zeta^7 y,\\ b &: x \mapsto \sqrt{-1}y, && y \mapsto \sqrt{-1}x,\\ c &: x \mapsto (\zeta^7 x + \zeta^5 y)/\sqrt{2}, && y \mapsto (\zeta^7 x + \zeta y)/\sqrt{2}. \end{aligned}$$

Set $z = x/y$. Then $k(x, y) = k(z, x)$. By applying Theorem 2.1 we get that $k(z, x)^{\widehat{S}_4} = k(z)^{\widehat{S}_4}(t)$ for some element t fixed by $\widehat{S}_4$. The field $k(z)^{\widehat{S}_4}$ is k-rational by Lüroth's theorem. Hence $k(z, x)^{\widehat{S}_4}$ and $k(\widehat{S}_4)$ are k-rational.

Case 2: $\sqrt{2} \in k$ but $\sqrt{-1} \notin k$. We will use the 4-dimensional faithful representation of $\widehat{S}_4$ over k provided by Equation (3-2). This representation provides an action of $\widehat{S}_4$ on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{aligned} a' &: x_1 \mapsto (x_1 + x_2)/\sqrt{2}, && x_2 \mapsto (-x_1 + x_2)/\sqrt{2},\\ &\quad x_3 \mapsto (x_3 - x_4)/\sqrt{2}, && x_4 \mapsto (x_3 + x_4)/\sqrt{2},\\ b &: x_1 \mapsto x_4 \mapsto -x_1, && x_2 \mapsto -x_3,\\ &\quad x_3 \mapsto x_2,\\ c &: x_1 \mapsto (x_1 - x_2 - x_3 - x_4)/2, && x_2 \mapsto (x_1 + x_2 + x_3 - x_4)/2,\\ &\quad x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, && x_4 \mapsto (x_1 + x_2 - x_3 + x_4)/2. \end{aligned} \tag{4-1}$$

• Step 1. Apply Theorem 2.2 and use the arguments of Case 1. We find that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. It remains to show that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k-rational.

• Step 2. Write $\pi = \mathrm{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$, where $\rho(\sqrt{-1}) = -\sqrt{-1}$. Extend the actions of π and $\widehat{S}_4$ on $k(\sqrt{-1})$ and $k(x_1, x_2, x_3, x_4)$ to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by requiring that $\rho(x_i) = x_i$ for $1 \le i \le 4$ and $g(\sqrt{-1}) = \sqrt{-1}$ for all $g \in \widehat{S}_4$. It follows that

$$\begin{aligned} k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} &= \{k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle \rho \rangle}\}^{\langle a', b, c \rangle}\\ &= k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle a', b, c, \rho \rangle}. \end{aligned}$$

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned} y_1 &= \sqrt{-1}x_1 + \sqrt{-1}x_2 - x_3 + x_4, && y_2 = -\sqrt{-1}x_1 + \sqrt{-1}x_2 + x_3 + x_4,\\ y_3 &= x_1 - x_2 - \sqrt{-1}x_3 - \sqrt{-1}x_4, && y_4 = x_1 + x_2 - \sqrt{-1}x_3 + \sqrt{-1}x_4. \end{aligned}$$

Then

$$k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$$

and the actions in (4-1) become

$$
\begin{aligned}
a' :\ & y_1 \mapsto (y_1+y_2)/\sqrt{2}, && y_2 \mapsto (-y_1+y_2)/\sqrt{2},\\
& y_3 \mapsto (y_3+y_4)/\sqrt{2}, && y_4 \mapsto (-y_3+y_4)/\sqrt{2},\\
b :\ & y_1 \mapsto \sqrt{-1}\,y_1, && y_2 \mapsto -\sqrt{-1}\,y_2,\\
& y_3 \mapsto \sqrt{-1}\,y_3, && y_4 \mapsto -\sqrt{-1}\,y_4,\\
c :\ & y_1 \mapsto \frac{y_1-\sqrt{-1}\,y_2}{1+\sqrt{-1}}, && y_2 \mapsto \frac{y_1+\sqrt{-1}\,y_2}{1+\sqrt{-1}},\\
& y_3 \mapsto \frac{y_3-\sqrt{-1}\,y_4}{1+\sqrt{-1}}, && y_4 \mapsto \frac{y_3+\sqrt{-1}\,y_4}{1+\sqrt{-1}},\\
\rho :\ & y_1 \mapsto -\sqrt{-1}\,y_4, && y_2 \mapsto \sqrt{-1}\,y_3,\\
& y_3 \mapsto \sqrt{-1}\,y_2, && y_4 \mapsto -\sqrt{-1}\,y_1.
\end{aligned} \tag{4-2}
$$

Note that the action of a'^2 is given by

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

The reader might find interesting to compare the actions in (4-2) with those in [Kang and Zhou 2012, Section 4]. It turns out that the formulae for b, a'^2, c^2 are completely the same as those for $\lambda_1, \lambda_2, \sigma$ in [Kang and Zhou 2012, Formula (4.3)]. As mentioned before, both the subgroups $\langle b, a'^2, c^2\rangle$ and $\langle \lambda_1, \lambda_2, \sigma\rangle$ are isomorphic to $\widetilde{A}_4$ (where $\widetilde{A}_4 = p^{-1}(A_4)$ in the notation of Section 3) as abstract groups.

• Step 3. Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. By Theorem 2.1, we find that

$$
\begin{aligned}
k(\sqrt{-1})(y_1,y_2,y_3,y_4)^{\langle \widehat{S}_4,\pi\rangle} &= k(\sqrt{-1})(z_1,z_2,z_3)(y_4)^{\langle \widehat{S}_4,\pi\rangle}\\
&= k(\sqrt{-1})(z_1,z_2,z_3)^{\langle \widehat{S}_4,\pi\rangle}(z_0),
\end{aligned}
$$

where z_0 is fixed by the actions of $\widehat{S}_4$ and π. There remains to show the k-rationality of $k(\sqrt{-1})(z_1,z_2,z_3)^{\langle \widehat{S}_4,\pi\rangle}$ is

Before we find $k(\sqrt{-1})(z_1,z_2,z_3)^{\langle \widehat{S}_4,\pi\rangle}$, we will find $k(\sqrt{-1})(z_1,z_2,z_3)^{\langle b,a'^2\rangle}$. The method is the same as in Steps 3 and 4 in [Kang and Zhou 2012, Section 4]. We will write down the details for the convenience of the reader.

Define $u_1 = z_1/z_2$, $u_2 = z_1z_2$, $u_3 = z_3$. Then

$$k(\sqrt{-1})(z_1,z_2,z_3)^{\langle b\rangle} = k(\sqrt{-1})(u_1,u_2,u_3).$$

The action of a'^2 is given by

$$a'^2 : u_1 \mapsto 1/u_1, \quad u_2 \mapsto 1/u_2, \quad u_3 \mapsto u_3/u_1.$$

Define

$$v_1=\frac{u_1-u_2}{1-u_1u_2},\quad v_2=\frac{u_1+u_2}{1+u_1u_2},\quad v_3=u_3\Big(1+\frac{1}{u_1}\Big).$$

Then

$$k(\sqrt{-1})(u_1,u_2,u_3)^{\langle a'^2\rangle}=k(\sqrt{-1})(u_1,u_2,v_3)^{\langle a'^2\rangle}=k(\sqrt{-1})(v_1,v_2,v_3)$$

by Theorem 2.3 (note that $a'^2(v_3)=v_3$). In summary,

$$k(\sqrt{-1})(z_1,z_2,z_3)^{\langle b,a'^2\rangle}=k(\sqrt{-1})(v_1,v_2,v_3).$$

• Step 4. The action of c on v_1, v_2, v_3 is given by

$$c: v_1\mapsto 1/v_2,\quad v_2\mapsto v_1/v_2,\quad v_3\mapsto v_3(v_1+v_2)/[v_2(1+v_1)].$$

Define $X_3=v_3(1+v_1+v_2)/[(1+v_1)(1+v_2)]$. Then $c(X_3)=X_3$ and

$$k(\sqrt{-1})(v_1,v_2,v_3)=k(\sqrt{-1})(v_1,v_2,X_3).$$

Thus we may apply Theorem 2.4 (regarding v_1, $1/v_2$, v_2/v_1 as x, y, z in its statement). More precisely, define

$$X_1=(v_1^3v_2^3+v_1^3+v_2^3-3v_1^2v_2^2)/(v_1^4v_2^2+v_2^4+v_1^2-v_1^2v_2^3-v_1v_2^2-v_1^3v_2),$$
$$X_2=(v_1v_2^4+v_1v_2+v_1^4v_2-3v_1^2v_2^2)/(v_1^4v_2^2+v_2^4+v_1^2-v_1^2v_2^3-v_1v_2^2-v_1^3v_2).$$

By Theorem 2.4 we get $k(\sqrt{-1})(v_1,v_2,X_3)^{\langle c\rangle}=k(\sqrt{-1})(X_1,X_2,X_3)$.

• Step 5. With the aid of computers, we find that the actions of a' and ρ on X_1, X_2, X_3 are given by

$$a': X_1\mapsto \frac{X_1}{X_1^2-X_1X_2+X_2^2},\quad X_2\mapsto\frac{X_2}{X_1^2-X_1X_2+X_2^2},\quad X_3\mapsto X_3,$$
$$\rho\ : X_1\mapsto \frac{X_2}{X_1^2-X_1X_2+X_2^2},\quad X_2\mapsto\frac{X_1}{X_1^2-X_1X_2+X_2^2},\quad X_3\mapsto -2A/X_3,$$

where $A=g_1g_2g_3^{-1}$ and

$$g_1=(1+X_1)^2-X_2(1+X_1)+X_2^2,$$
$$g_2=(1+X_2)^2-X_1(1+X_2)+X_1^2,$$
$$g_3=1+X_1+X_2+X_1^3+X_2^3+X_1X_2(3X_1X_2-2X_1^2-2X_2^2+2)+X_1^4+X_2^4.$$

Note that $\rho(g_1)=g_2/(X_1^2-X_1X_2+X_2^2)$ and $a'(g_1)=g_1/(X_1^2-X_1X_2+X_2^2)$. Define $Y_1=X_1/X_2$, $Y_2=X_1$, $Y_3=X_1X_3/g_1$. We find that

$$a': Y_1\mapsto Y_1,\quad Y_2\mapsto Y_1^2/(Y_2(1-Y_1+Y_1^2)),\quad Y_3\mapsto Y_3.$$

Thus

$$k(\sqrt{-1})(X_1, X_2, X_3)^{\langle a'\rangle} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{\langle a'\rangle} = k(\sqrt{-1})(Z_1, Z_2, Z_3),$$

where $Z_1 = Y_1$, $Z_2 = Y_2 + a'(Y_2)$, $Z_3 = Y_3$.

• Step 6. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto -2Z_1^3/(A'Z_3),$$

where A' is defined to be

$$-2Z_1^2+Z_1Z_2+Z_2^2+4Z_1^3-2Z_1Z_2^2-2Z_1^4+3Z_1^2Z_2^2+Z_1^4Z_2-2Z_1^3Z_2^2+Z_1^4Z_2^2.$$

Define $U_1 = Z_2+\rho(Z_2)$, $U_2 = \sqrt{-1}(Z_2-\rho(Z_2))$, $U_3 = Z_3+\rho(Z_3)$ and $U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. We see that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{\langle\rho\rangle} = k(U_1, U_2, U_3, U_4)$ with a relation

$$U_3^2 + U_4^2 + 32(U_1^2 + U_2^2)/B = 0,$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Dividing this relation by $16(U_1^2 + U_2^2)^2/B^2$, we get

$$\big(BU_3/(4U_1^2 + 4U_2^2)\big)^2 + \big(BU_4/(4U_1^2 + 4U_2^2)\big)^2 + 2B/(U_1^2 + U_2^2) = 0.$$

Multiply this relation by $U_1^2 + U_2^2$ and use the identity

$$(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) = (\alpha\delta + \beta\gamma)^2 + (\alpha\gamma - \beta\delta)^2$$

to obtain the simplification

$$V_3^2 + V_4^2 + 2B = 0, \tag{4-3}$$

where

$$V_3 = B\frac{U_1U_3 + U_2U_4}{4U_1^2 + 4U_2^2} \quad \text{and} \quad V_4 = B\frac{U_1U_4 - U_2U_3}{4U_1^2 + 4U_2^2}.$$

Note that $k(U_1, U_2, U_3, U_4) = K(U_1, U_2, V_3, V_4)$.

Define $w_1 = 8U_1/(U_1^2-3U_2^2)$, $w_2 = 8U_2/(U_1^2-3U_2^2)$, $w_3 = V_3/(U_1^2-3U_2^2)$, $w_4 = V_4/(U_1^2 - 3U_2^2)$. Then $k(U_1, U_2, V_3, V_4) = k(w_1, w_2, w_3, w_4)$ and the relation (4-3) becomes

$$w_3^2 + w_4^2 + 2 + w_1 + w_2^2 = 0.$$

Hence $w_1 \in k(w_2, w_3, w_4)$. Thus $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{\langle\rho\rangle} = k(w_2, w_3, w_4)$ is k-rational.

Case 3: $\sqrt{-2} \in k$ *but* $\sqrt{-1} \notin k$. We use the 4-dimensional faithful representation of $\widehat{S}_4$ over k provided by (3-3). This representation provides an action of $\widehat{S}_4$ on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{aligned}
a' &: x_1 \mapsto \sqrt{-2}(x_1 - x_2)/2, && x_2 \mapsto \sqrt{-2}(x_1 + x_2)/2,\\
&\ x_3 \mapsto \sqrt{-2}(-x_3 - x_4)/2, && x_4 \mapsto \sqrt{-2}(x_3 - x_4)/2,\\
b &: x_1 \mapsto x_4 \mapsto -x_1, && x_2 \mapsto -x_3,\\
&\ x_3 \mapsto x_2,\\
c &: x_1 \mapsto (x_1 - x_2 - x_3 - x_4)/2, && x_2 \mapsto (x_1 + x_2 + x_3 - x_4)/2,\\
&\ x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, && x_4 \mapsto (x_1 + x_2 - x_3 + x_4)/2.
\end{aligned}$$

The proof of this case is very similar to that of Case 2.

• Step 1. Apply Theorem 2.2. We see that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. Hence the proof is reduced to proving that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k-rational.

• Step 2. Write $\pi = \mathrm{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$, where $\rho(\sqrt{-1}) = -\sqrt{-1}$. Extend the actions of π and $\widehat{S}_4$ to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ as in Step 2 of Case 2. We find that

$$k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle a', b, c, \rho \rangle}.$$

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned}
y_1 &= -x_1 - \sqrt{-1}x_2 + x_3 + \sqrt{-1}x_4, & y_2 &= \sqrt{-1}x_1 - x_2 + \sqrt{-1}x_3 - x_4,\\
y_3 &= x_1 - \sqrt{-1}x_2 + x_3 - \sqrt{-1}x_4, & y_4 &= \sqrt{-1}x_1 + x_2 - \sqrt{-1}x_3 - x_4.
\end{aligned}$$

We get $k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$ and the actions are

$$\begin{aligned}
a' &: y_1 \mapsto (-y_1 - y_2)/\sqrt{2}, && y_2 \mapsto (y_1 - y_2)/\sqrt{2},\\
&\ y_3 \mapsto (y_3 + y_4)/\sqrt{2}, && y_4 \mapsto (-y_3 + y_4)/\sqrt{2},\\
b &: y_1 \mapsto \sqrt{-1}y_1, && y_2 \mapsto -\sqrt{-1}y_2,\\
&\ y_3 \mapsto \sqrt{-1}y_3, && y_4 \mapsto -\sqrt{-1}y_4,\\
c &: y_1 \mapsto \frac{y_1 - \sqrt{-1}y_2}{1 + \sqrt{-1}}, && y_2 \mapsto \frac{y_1 + \sqrt{-1}y_2}{1 + \sqrt{-1}},\\
&\ y_3 \mapsto \frac{y_3 - \sqrt{-1}y_4}{1 + \sqrt{-1}}, && y_4 \mapsto \frac{y_3 + \sqrt{-1}y_4}{1 + \sqrt{-1}},\\
\rho &: y_1 \mapsto \sqrt{-1}y_4, && y_2 \mapsto -\sqrt{-1}y_3,\\
&\ y_3 \mapsto -\sqrt{-1}y_2, && y_4 \mapsto \sqrt{-1}y_1.
\end{aligned} \tag{4-4}$$

Note that the action of a'^2 is

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

(Compare with (4-2) and (4-4).) The actions of a'^2, b, c in both cases are the same.

• Step 3. Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. As in Step 3 of Case 2, it suffices to prove that $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle \widehat{S}_4, \pi\rangle}$ is k-rational.

Define u_1, u_2, u_3, v_1, v_2, v_3, X_1, X_2, X_3 by the same formulae as in Step 3 and Step 4 of Case 2. We find that $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b, a'^2, c\rangle} = k(\sqrt{-1})(X_1, X_2, X_3)$.

• Step 4. The actions of a', ρ on X_1, X_2, X_3 are slightly different from Step 5 of Case 2. In the present case, we have

$$\begin{aligned} a' : X_1 &\mapsto \frac{X_1}{X_1^2 - X_1X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_2}{X_1^2 - X_1X_2 + X_2^2}, \quad X_3 \mapsto -X_3, \\ \rho\ : X_1 &\mapsto \frac{X_2}{X_1^2 - X_1X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_1}{X_1^2 - X_1X_2 + X_2^2}, \quad X_3 \mapsto -2A/X_3, \end{aligned}$$

where $A = g_1 g_2 g_3^{-1}$ and

$$\begin{aligned} g_1 &= (1+X_1)^2 - X_2(1+X_1) + X_2^2, \\ g_2 &= (1+X_2)^2 - X_1(1+X_2) + X_1^2, \\ g_3 &= 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1X_2(3X_1X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4. \end{aligned}$$

Note that the action of ρ is the same as in Step 5 of Case 2.

Define $Y_1 = X_1/X_2$, $Y_2 = X_1$, $Y_3 = X_1X_3/g_1$. We get

$$a' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_1^2/\big(Y_2(1 - Y_1 + Y_1^2)\big), \quad Y_3 \mapsto -Y_3.$$

Thus $k(\sqrt{-1})(X_1, X_2, X_3)^{\langle a'\rangle} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{\langle a'\rangle} = k(\sqrt{-1})(Z_1, Z_2, Z_3)$, where $Z_1 = Y_1$, $Z_2 = Y_2 + a'(Y_2)$, $Z_3 = Y_3(Y_2 - a'(Y_2))$.

• Step 5. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto C/Z_3,$$

where C is defined to be

$$\frac{2Z_1^2(-4Z_1^2 + Z_2^2 - Z_1Z_2^2 + Z_1^2Z_2^2)/(1 - Z_1 + Z_1^2)}{-2Z_1^2 + Z_1Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1Z_2^2 - 2Z_1^4 + 3Z_1^2Z_2^2 + Z_1^4Z_2 - 2Z_1^3Z_2^2 + Z_1^4Z_2^2}.$$

Define $U_1 = Z_2 + \rho(Z_2)$, $U_2 = \sqrt{-1}(Z_2 - \rho(Z_2))$, $U_3 = Z_3 + \rho(Z_3)$ and $U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{\langle\rho\rangle} = k(U_1, U_2, U_3, U_4)$ with a relation

$$U_3^2 + U_4^2 = 8(U_1^2 + U_2^2)^2(-16 + U_1^2 - 3U_2^2)/B(U_1^2 - 3U_2^2), \tag{4-5}$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Note that the above formula of B is identically the same as that in Step 6 of Case 2. It remains to simplify the relation (4-5). Dividing both sides by $(U_1^2 + U_2^2)^2$, we get

$$\big(U_3/(U_1^2+U_2^2)\big)^2+\big(U_4/(U_1^2+U_2^2)\big)^2=8(-16+U_1^2-3U_2^2)/B(U_1^2-3U_2^2).$$

Divide both sides of the above identity by $(2(U_1^2-3U_2^2)/B)^2$. We get a relation

$$V_3^2+V_4^2=2(1-V_1^2+3V_2^2)(1+V_1+2V_2^2), \tag{4-6}$$

where

$$V_1=\frac{4U_1}{U_1^2-3U_2^2},\quad V_3=\frac{BU_3}{(U_1^2-3U_2^2)(2U_1^2+2U_2^2)},$$
$$V_2=\frac{4U_2}{U_1^2-3U_2^2},\quad V_4=\frac{BU_4}{(U_1^2-3U_2^2)(2U_1^2+2U_2^2)}.$$

Note that $k(U_1,U_2,U_3,U_4)=k(V_1,V_2,V_3,V_4)$.

Define $w_1 = 1/(1+V_1)$, $w_2 = V_2/(1+V_1)$, $w_3 = V_3/(1+V_1)^2$, $w_4 = V_4/(1+V_1)^2$. We get $k(V_1,V_2,V_3,V_4) = k(w_1,w_2,w_3,w_3)$ and the relation (4-6) becomes

$$w_3^2+w_4^2=2(-1+2w_1+3w_2^2)(w_1+2w_2^2).$$

Divide the above identity by $(w_1+2w_2^2)^2$. We get

$$\big(w_3/(w_1+2w_2^2)\big)^2+\big(w_4/(w_1+2w_2^2)\big)^2=2(-1+2w_1+3w_2^2)/(w_1+2w_2^2).$$

Since $2(-1+2w_1+3w_2^2)/(w_1+2w_2^2)$ is a "fractional linear transformation" of w_1 and it belongs to $k(w_2, w_3/(w_1+2w_2^2), w_4/(w_1+2w_2^2))$, we find that w_1 is in $k(w_2, w_3/(w_1+2w_2^2), w_4/(w_1+2w_2^2))$. Thus

$$k(w_1,w_2,w_3,w_4)=k(w_2,w_3/(w_1+2w_2^2),w_4/(w_1+2w_2^2)).$$

We find that $k(\sqrt{-1})(Z_1,Z_2,Z_3)^{\langle\rho\rangle}$ is k-rational.

Case 4: $\sqrt{-1}\in k$ but $\sqrt{2}\notin k$. This is similar to Cases 2 or 3, so the detailed proof is omitted. In the case $\operatorname{char} k = 0$, we may apply Plans' result, Theorem 1.3. □

5. Other double covers of S_n

In this section we consider the rationality problem of G_n, which is a double cover of the symmetric group and different from both $\widehat{S}_n$ and $\widetilde{S}_n$.

There are four double covers of the symmetric group S_n when $n \geq 4$. The trivial case is the split group $S_n \times C_2$. The rationality problem of the group $S_n \times C_2$ is easy because we may apply Theorem 2.6. It remains to consider the non split cases: they are $\widehat{S}_n$, $\widetilde{S}_n$, and the group G_n defined below.

Definition 5.1. For $n \geq 3$, consider the group G_n such that the short exact sequence $1 \to \{\pm 1\} \to G_n \xrightarrow{p} S_n \to 1$ is induced by the cup product $\varepsilon_n \cup \varepsilon_n \in H^2(S_n, \{\pm 1\})$, (see, for example, [Serre 1984, page 654]) where $\varepsilon_n : S_n \to \{\pm 1\}$ is the signed map, that is, $\varepsilon_n(\sigma) = -1$ if and only if $\sigma \in S_n$ is an odd permutation. Note that the group G_n is denoted by $\overline{S}_n$ in [Plans 2009].

The group G_n can be constructed explicitly as follows. Let

$$1 \to \{\pm 1\} \to C_4 = \{\pm\sqrt{-1}, \pm 1\} \xrightarrow{p_0} \{\pm 1\} \to 1$$

be the short exact sequence defined by $p_0(\sqrt{-1}) = -1$. The group G_n can be realized as the pullback of the diagram

$$\begin{array}{ccc} & & S_n \\ & & \downarrow \varepsilon_n \\ C_4 & \xrightarrow{\rho_0} & \{\pm 1\}. \end{array}$$

Explicitly, as a subgroup of $S_n \times C_4$,

$$\begin{aligned} G_n &= \{(\sigma, (\sqrt{-1})^i) \in S_n \times C_4 : \varepsilon_n(\sigma) = p_0((\sqrt{-1})^i)\} \\ &= (A_n \times \{\pm 1\}) \cup \{(\sigma, \pm\sqrt{-1}) \in S_n \times C_4 : \sigma \notin A_n\}. \end{aligned}$$

If k is a field with $\operatorname{char} k \neq 2$, a faithful $2n$-dimensional representation can be defined as follows. Let $X = \left(\bigoplus_{1 \leq i \leq n} k \cdot x_i\right) \oplus \left(\bigoplus_{1 \leq i \leq n} k \cdot y_i\right)$ and let G_n act on X by, for $1 \leq i \leq n$,

$$\begin{array}{lll} t &: x_i \mapsto -x_i, & y_i \mapsto -y_i, \\ \tau &: x_i \mapsto x_{\tau(i)}, & y_i \mapsto y_{\sigma^{-1}\tau\sigma(i)}, \\ \overline{\sigma} &: x_i \mapsto y_i \mapsto -x_i, & \end{array} \tag{5-1}$$

where $t = (1, -1) \in G_n \subset S_n \times C_4$, $\tau \in A_n$ and τ is identified with $(\tau, 1) \in G_n$, $\sigma = (1, 2) \in S_n$ and $\overline{\sigma} = (\sigma, \sqrt{-1}) \in G_n$.

The next result was proved in [Plans 2009, Theorem 14(b)] under the assumptions that $\operatorname{char} k = 0$ and $\sqrt{-1} \in k$. Our proof is different from Plans' even in the situation when $\operatorname{char} k = 0$.

Theorem 5.2. *Assume that k is a field that satisfies*:

(i) *Either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k = p > 0$ *with* $p \nmid 2n$.

(ii) $\sqrt{-1} \in k$.

Then $k(G_n)$ is k-rational for $n \geq 3$.

Proof. The reader will find that (i) the assumption $\operatorname{char} k \neq 2$ is used throughout the proof; (ii) the assumption $\operatorname{char} k \nmid n$ is used in Step 2; (iii) the assumption $\sqrt{-1} \in k$ is used in Step 3.

• Step 1. Apply Theorem 2.2. We find that $k(G_n)$ is rational over

$$k(x_i, y_i : 1 \le i \le n)^{G_n},$$

where G_n acts on the rational function field $k(x_i, y_i : 1 \le i \le n)$ by (5-1).

• Step 2. Define $u_0 = \sum_{1\le i\le n} x_i$, $v_0 = \sum_{1\le i\le n} y_i$ and $u_i = x_i/u_0$, $v_i = y_i/v_0$ for $1 \le i \le n$. Note that $k(x_i, y_i : 1 \le i \le n) = k(u_j, v_j : 0 \le j \le n)$ with the relations $\sum_{1\le i\le n} u_i = \sum_{1\le i\le n} v_i = 1$. The action of G_n is given by

$$\begin{array}{lllll} t & : & u_0 \mapsto -u_0, & v_0 \mapsto -v_0, & u_i \mapsto u_i, \quad v_i \mapsto v_i, \\ \tau & : & u_0 \mapsto u_0, & v_0 \mapsto v_0, & u_i \mapsto u_{\tau(i)}, \quad v_i \mapsto v_{\sigma^{-1}\tau\sigma(i)}, \\ \overline{\sigma} & : & u_0 \mapsto v_0 \mapsto -v_0, & u_i \mapsto v_i \mapsto u_i, & \end{array}$$

where $1 \le i \le n$ and t, τ, $\overline{\sigma}$ are defined in (5-1).

Define $w_1 = u_0 v_0$, $w_2 = u_0/v_0$. Then

$$k(u_j, v_j : 0 \le j \le n)^{\langle t\rangle} = k(u_i, v_i : 1 \le i \le n)(w_1, w_2).$$

Note that $\tau(w_i) = w_i$ for $1 \le i \le 2$, $\overline{\sigma}(w_1) = -w_1$, $\overline{\sigma}(w_2) = -1/w_2$. By Theorem 2.1,

$$k(u_i, v_i : 1 \le i \le n)(w_1, w_2)^{G_n/\langle t\rangle} = k(u_i, v_i : 1 \le i \le n)(w_2)^{G_n/\langle t\rangle}(w')$$

for some w' fixed by the action of $G_n/\langle t\rangle$. Moreover, we may identify $G_n/\langle t\rangle$ with S_n and identify $\overline{\sigma}$ (modulo $\langle t\rangle$) with σ.

Define $U_i = u_i - (1/n)$, $V_i = v_i - (1/n)$ for $1 \le i \le n$. We find that

$$\sum_{1\le i\le n} U_i = \sum_{1\le i\le n} V_i = 0$$

and the action of S_n on $k(U_i, V_i : 1 \le i \le n)$ becomes linear. We will consider $k(U_i, V_i : 1 \le i \le n)(w_2)^{S_n}$. The action of S_n is given by

$$\text{(5-2)} \qquad \begin{array}{lll} \tau & : & U_i \mapsto U_{\tau(i)}, \quad V_i \mapsto V_{\sigma^{-1}\tau\sigma(i)}, \quad w_2 \mapsto w_2, \\ \sigma & : & U_i \mapsto V_i \mapsto U_i, \quad w_2 \mapsto -1/w_2, \end{array}$$

where $1 \le i \le n$, $\tau \in A_n$, $\sigma = (1, 2)$ and $\sum_{1\le i\le n} U_i = \sum_{1\le i\le n} V_i = 0$.

• Step 3. Since $\sqrt{-1} \in k$, define $w = (\sqrt{-1} - w_2)/(\sqrt{-1} + w_2)$. We find that $\tau(w) = w$ for $\tau \in A_n$ and $\sigma(w) = -w$. Apply Theorem 2.1. We find that

$$k(u_i, v_i : 1 \le i \le n)(w_2)^{S_n} = k(U_i, V_i : 1 \le i \le n)^{S_n}(w'')$$

for some w'' fixed by the action of S_n.

It remains to show that $k(U_i, V_i : 1 \le i \le n)^{S_n}$ is k-rational. The following proof of this fact is due to the referee.

Define $W_i^{\pm} = U_i \pm V_{\sigma(i)}$. It is easy to verify that for $\tau(W_i^{\pm}) = W_{\tau(i)}^{\pm}$ for $\tau \in A_n$; and that for $\sigma = (1, 2)$, $\sigma(W_i^{+}) = W_{\sigma(i)}^{+}$ and $\sigma(W_i^{-}) = -W_{\sigma(i)}^{-}$.

Define subspaces W and W' by $W = \sum_{1\le i\le n} k\cdot W_i^{+}$ and $W' = \sum_{1\le i\le n} k\cdot W_i^{-}$. Note that

$$\sum_{1\le i\le n} k \cdot U_i \oplus \sum_{1\le i\le n} k \cdot V_i = W \oplus W'.$$

Moreover, W is the standard representation of S_n, that is, $W \simeq \sum_{1\le i\le n} k \cdot s_i$ with $\sum_{1\le i\le n} s_i = 0$ and $\lambda(s_i) = s_{\lambda(i)}$ for all $\lambda \in S_n$, for all $1 \le i \le n$. On the other hand, W' is the representation space of the tensor product of the standard representation and the linear character $\varepsilon_n : S_n \to \{\pm 1\}$.

• Step 4. Apply Theorem 2.2 to $k(U_i, V_i : 1 \le i \le n)^{S_n}$. We find that

$$k(U_i, V_i : 1 \le i \le n)^{S_n} = k(W \oplus W')^{S_n} = k(W_i^{+} : 1 \le i \le n-1)^{S_n}(t_1, \dots, t_{n-1}),$$

where each t_i is fixed by S_n. Obviously the field $k(W_i^{+} : 1 \le i \le n-1)^{S_n}$ is k-rational, whence the result. □

In the following theorem the assumption $\sqrt{-1} \in k$ from Theorem 5.2 will be dropped. The first part of the following theorem was proved by Plans [2009, Theorem 14 (b)]; there he assumed that $\operatorname{char} k = 0$.

Theorem 5.3. (1) *If k is a field with* $\operatorname{char} k \ne 2$ *or* 3, *then $k(G_3)$ is k-rational.*

(2) *If k is a field with* $\operatorname{char} k \ne 2$, *then $k(G_4)$ is k-rational. Moreover, if* $\operatorname{char} k = 0$, *then $k(G_5)$ is also k-rational.*

Proof. Case 1: $n = 3$. By Step 2 in the proof of Theorem 5.2, it suffices to consider $k(U_i, V_i : 1 \le i \le 3)(w_2)^{S_3}$, where $\sum_{1\le i\le 3} U_i = \sum_{1\le i\le 3} V_i = 0$. Define $\tau = (1, 2, 3) \in S_3$. The actions are given by

$$\begin{aligned} \tau &: U_1 \mapsto U_2 \mapsto -U_1 - U_2, \quad V_2 \mapsto V_1 \mapsto -V_1 - V_2, \\ \sigma &: U_1 \leftrightarrow V_1, \quad U_2 \leftrightarrow V_2. \end{aligned}$$

Define $w_3 = U_1/V_2$, $w_4 = U_2/V_1$, $w_5 = V_1/V_2$. It follows that

$$k(U_i, V_i : 1 \le i \le 3)(w_2)^{S_3} = k(w_j : 2 \le j \le 5)(V_1)^{S_3} = k(w_j : 2 \le j \le 5)^{S_3}(w_0)$$

for some w_0 by Theorem 2.1.

It remains to show that $k(w_j : 2 \le j \le 5)^{S_3}$ is k-rational. Note that

$$\begin{aligned} \tau &: w_2 \mapsto w_2, \quad w_3 \mapsto w_4 \mapsto (w_3 + w_4 w_5)/(1 + w_5), \\ \sigma &: w_2 \mapsto -1/w_2, \quad w_3 \mapsto 1/w_4, \quad w_4 \mapsto 1/w_3, \quad w_5 \mapsto w_3/(w_4 w_5). \end{aligned}$$

Define $w_6 = (w_3 + w_4 w_5)/(1 + w_5)$. Note that $k(w_3, w_4, w_5) = k(w_3, w_4, w_6)$ and

$$\tau : w_3 \mapsto w_4 \mapsto w_6 \mapsto w_3 \quad \text{and} \quad \sigma : w_6 \mapsto 1/w_6.$$

Define $w_7 = (1-w_3)/(1+w_3)$, $w_8 = (1-w_4)/(1+w_4)$, $w_9 = (1-w_6)/(1+w_6)$. Then $k(w_3, w_4, w_6) = k(w_7, w_8, w_9)$ and

$$\tau : w_7 \mapsto w_8 \mapsto w_9 \mapsto w_7,$$
$$\sigma : w_7 \mapsto -w_8, \quad w_8 \mapsto -w_7, \quad w_9 \mapsto -w_9.$$

By Theorem 2.4 we find that $k(w_2, w_3, w_4, w_5)^{\langle\tau\rangle} = k(w_2, X_1, X_2, X_3)$, where $X_1 = w_7 + w_8 + w_9$ and

$$X_2 = \frac{w_7^2 w_8 + w_8^2 w_9 + w_9^2 w_7 - 3w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2 - w_7 w_8 - w_7 w_9 - w_8 w_9},$$
$$X_3 = \frac{w_7 w_8^2 + w_8 w_9^2 + w_9 w_7^2 - 3w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2 - w_7 w_8 - w_7 w_9 - w_8 w_9}.$$

Moreover, the action of σ is given by

$$\sigma : w_2 \mapsto -1/w_2, \quad X_1 \mapsto -X_1, \quad X_2 \mapsto -X_3, \quad X_3 \mapsto -X_2.$$

Apply Theorem 2.2. We find that $k(w_2, X_1, X_2, X_3)^{\langle\sigma\rangle} = k(w_2)^{\langle\sigma\rangle}(Y_1, Y_2, Y_3)$ for some Y_1, Y_2, Y_3 fixed by σ. Since $k(w_2)^{\langle\sigma\rangle}$ is k-rational, it follows that $k(w_2, X_1, X_2, X_3)^{\langle\sigma\rangle}$ is k-rational.

Case 2: $n = 4$. Once again we use Step 2 in the proof of Theorem 5.2. It suffices to consider $k(U_i, V_i : 1 \le i \le 4)(w_2)^{S_4}$, where $\sum_{1\le i\le 4} U_i = \sum_{1\le i\le 4} V_i = 0$. Set $\lambda_1 = (1,2)(3,4)$, $\lambda_2 = (1,3)(2,4)$, $\rho = (1,2,3)$ and $\sigma = (1,2)$ as before. Then S_4 is generated by λ_1, λ_2, ρ and σ.

Define $t_1 = U_1 + U_2$, $t_2 = V_1 + V_2$, $t_3 = U_1 + U_3$, $t_4 = V_2 + V_3$, $t_5 = U_2 + U_3$ and $t_6 = V_1 + V_3$. The action of S_4 is given by

$$\lambda_1 : t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_3 \mapsto -t_3, \quad t_4 \mapsto -t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6,$$
$$\lambda_2 : t_1 \mapsto -t_1, \quad t_2 \mapsto -t_2, \quad t_3 \mapsto t_3, \quad t_4 \mapsto t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6,$$
$$\rho : t_1 \mapsto t_5 \mapsto t_3 \mapsto t_1, \quad t_2 \mapsto t_6 \mapsto t_4 \mapsto t_2,$$
$$\sigma : t_1 \leftrightarrow t_2, \quad t_3 \leftrightarrow t_6, \quad t_4 \leftrightarrow t_5.$$

It follows that $k(t_i : 1 \le i \le 6)(w_2)^{<\lambda_1,\lambda_2>} = k(T_i : 1 \le i \le 6)(w_2)$, where $T_1 = t_1/t_2$, $T_2 = t_3/t_4$, $T_3 = t_5/t_6$, $T_4 = t_2 t_6/t_4$, $T_5 = t_4 t_6/t_2$, $T_6 = t_2 t_4/t_6$.

Moreover, the actions of ρ and σ are given by

$$\begin{aligned}
&\rho : T_1 \mapsto T_3 \mapsto T_2 \mapsto T_1, \quad T_4 \mapsto T_5 \mapsto T_6 \mapsto T_4,\\
&\sigma : T_1 \mapsto 1/T_1, \quad T_2 \mapsto 1/T_3, \quad T_3 \mapsto 1/T_2,\\
&\qquad T_4 \mapsto (T_1T_2/T_3)T_6, \quad T_5 \mapsto (T_2T_3/T_1)T_5, \quad T_6 \mapsto (T_1T_3/T_2)T_4.
\end{aligned}$$

By Theorem 2.2, it suffices to show that $k(T_i : 1 \le i \le 3)(w_2)^{<\rho,\sigma>}$ is k-rational. Define $w_3 = (1-T_1)/(1+T_1)$, $w_4 = (1-T_2)/(1+T_2)$, $w_5 = (1-T_3)/(1+T_3)$. Then we find

$$\begin{aligned}
&\rho : w_2 \mapsto w_2, \quad w_3 \mapsto w_5 \mapsto w_4 \mapsto w_3,\\
&\sigma : w_2 \mapsto -1/w_2, \quad w_3 \mapsto -w_3, \quad w_4 \mapsto -w_5, \quad w_5 \mapsto -w_4.
\end{aligned}$$

Use Theorem 2.4 to find that $k(T_i : 1 \le i \le 3)(w_2)^{<\rho>}$. The remaining part of the proof is very similar to the last part of Case 1. The details are omitted.

Case 3: $n = 5$. By [Plans 2009, Theorem 11], $k(G_5)$ is rational over $k(G_4)$. Since $k(G_4)$ is k-rational by Case 2, we are done. □

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References

[Ahmad et al. 2000] H. Ahmad, M. Hajja, and M.-C. Kang, "Rationality of some projective linear actions", *J. Algebra* **228**:2 (2000), 643–658. MR 2001e:12003 Zbl 0993.12003

[Garibaldi et al. 2003] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, Providence, RI, 2003. MR 2004f:11034 Zbl 1159.12311

[Hajja and Kang 1995] M. Hajja and M.-C. Kang, "Some actions of symmetric groups", *J. Algebra* **177**:2 (1995), 511–535. MR 96i:20013 Zbl 0837.20054

[Hoffman and Humphreys 1992] P. N. Hoffman and J. F. Humphreys, *Projective representations of the symmetric groups: Q-functions and shifted tableaux*, Clarendon, New York, 1992. MR 94f:20027 Zbl 0777.20005

[Hoshi and Kang 2010] A. Hoshi and M.-C. Kang, "Twisted symmetric group actions", *Pacific J. Math.* **248**:2 (2010), 285–304. MR 2011k:12006 Zbl 1203.14018

[Kang 2005] M.-C. Kang, "Noether's problem for dihedral 2-groups, II", *Pacific J. Math.* **222**:2 (2005), 301–316. MR 2007a:12003 Zbl 1100.12003

[Kang 2012] M.-C. Kang, "Retract rational fields", *J. Algebra* **349** (2012), 22–37. MR 2853623 Zbl 06037563

[Kang and Plans 2009] M.-C. Kang and B. Plans, “Reduction theorems for Noether’s problem”, *Proc. Amer. Math. Soc.* **137**:6 (2009), 1867–1874. MR 2010c:12004 Zbl 1177.12006

[Kang and Zhou 2012] M.-C. Kang and J. Zhou, “The rationality problem for finite subgroups of $GL_4(\mathbb{Q})$”, *J. Algebra* **368**:1 (2012), 53–69.

[Karpilovsky 1985] G. Karpilovsky, *Projective representations of finite groups*, Pure and Applied Mathematics **94**, Dekker, New York, 1985. MR 86m:20014 Zbl 0568.20016

[Masuda 1955] K. Masuda, “On a problem of Chevalley”, *Nagoya Math. J.* **8** (1955), 59–63. MR 16:993c Zbl 0065.02601

[Plans 2007] B. Plans, “Noether’s problem for $\mathrm{GL}(2, 3)$”, *Manuscripta Math.* **124**:4 (2007), 481–487. MR 2008i:12009 Zbl 1146.12004

[Plans 2009] B. Plans, “On Noether’s problem for central extensions of symmetric and alternating groups”, *J. Algebra* **321**:12 (2009), 3704–3713. MR 2010e:12007 Zbl 1177.12007

[Saltman 1984] D. J. Saltman, “Retract rational fields and cyclic Galois extensions”, *Israel J. Math.* **47**:2-3 (1984), 165–215. MR 85j:13008 Zbl 0546.14013

[Serre 1984] J.-P. Serre, “L’invariant de Witt de la forme $\mathrm{Tr}(x^2)$”, *Comment. Math. Helv.* **59**:4 (1984), 651–676. MR 86k:11067 Zbl 0565.12014

[Serre 2011] J.-P. Serre, April 23 2011. Private email to M. Kang.

[Springer 1977] T. A. Springer, *Invariant theory*, Lecture Notes in Mathematics **585**, Springer, Berlin, 1977. MR 56 #5740 Zbl 0346.20020

[Swan 1983] R. G. Swan, “Noether’s problem in Galois theory”, pp. 21–40 in *Emmy Noether in Bryn Mawr* (Bryn Mawr, PA, 1982), edited by B. Srinivasan and J. D. Sally, Springer, New York, 1983. MR 84k:12013 Zbl 0538.12012

[Voskresenskii 1998] V. E. Voskresenskii, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs **179**, American Mathematical Society, Providence, RI, 1998. MR 99g:20090 Zbl 0974.14034

[Weil 1956] A. Weil, “The field of definition of a variety”, *Amer. J. Math.* **78** (1956), 509–524. MR 18,601a Zbl 0072.16001

[Yamasaki 2009] A. Yamasaki, “Negative solutions to three-dimensional monomial Noether problem”, preprint, 2009. arXiv 0909.0586v1

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