NOETHER’S PROBLEM FOR $\hat{S}_4$ AND $\hat{S}_5$

Ming-chang Kang and Jian Zhou
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Let $k$ be a field, let $G$ be a finite group and let $k(x_g : g \in G)$ be the rational function field over $k$, on which $G$ acts by the $k$-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Noether’s problem asks whether the fixed subfield $k(G) := k(x_g : g \in G)^G$ is $k$-rational, that is, purely transcendental over $k$. If $\hat{S}_n$ is the double cover of the symmetric group $S_n$, in which the liftings of transpositions and products of disjoint transpositions are of order 4, Serre shows that $\mathbb{Q}(\hat{S}_4)$ and $\mathbb{Q}(\hat{S}_5)$ are not $\mathbb{Q}$-rational. We will prove that if $k$ is a field such that $\text{char } k \neq 2, 3$, and $k(\zeta_8)$ is a cyclic extension of $k$, then $k(\hat{S}_4)$ is $k$-rational. If it is assumed furthermore that $\text{char } k = 0$, then $k(\hat{S}_5)$ is also $k$-rational.

1. Introduction

Let $k$ be a field, and $L$ be a finitely generated field extension of $k$. $L$ is called $k$-rational (or rational over $k$) if $L$ is purely transcendental over $k$; that is, $L$ is isomorphic to some rational function field over $k$. $L$ is called stably $k$-rational if $L(y_1, \ldots, y_m)$ is $k$-rational for some $y_1, \ldots, y_m$ that are algebraically independent over $L$. $L$ is called $k$-unirational if $L$ is $k$-isomorphic to a subfield of some $k$-rational field extension of $k$. It is easy to see that

$$k \text{-rational} \Rightarrow \text{stably } k \text{-rational} \Rightarrow \text{ } k \text{-unirational}.$$ 

A notion of retract rationality was introduced in [Saltman 1984] (see also [Kang 2012]). It is known that if $k$ is an infinite field, then

$$\text{stably } k \text{-rational} \Rightarrow \text{retract } k \text{-rational} \Rightarrow \text{ } k \text{-unirational}.$$ 

Let $k$ be a field and $G$ a finite group. Let $G$ act on the rational function field $k(x_g : g \in G)$ by $k$-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. 

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Denote by $k(G)$ the fixed subfield, that is, $k(G) = k(x_g : g \in G)^G$. Noether’s problem asks under what conditions is the field $k(G)$ $k$-rational.

Noether’s problem is related to the inverse Galois problem and the existence of generic $G$-Galois extensions over $k$. For the details, see Swan’s survey paper [Swan 1983]. The purpose of this paper is to study Noether’s problem for some double covers of the symmetric group $S_n$.

It is known that there are four different double covers of $S_n$ when $n \geq 4$, that is, groups $G$ that fit into a short exact sequence $1 \to C_2 \to G \to S_n \to 1$; see, for example, [Serre 1984, p. 653].

**Definition 1.1** [Garibaldi et al. 2003, pp. 58, 90; Hoffman and Humphreys 1992, p. 18; Karpilovsky 1985, pp. 177–181]. Let $C_2 = \{ \pm 1 \}$ be the cyclic group of order 2. When $n \geq 4$, the group $\hat{S}_n$ is the unique central extension of $S_n$ by $C_2$, that is,

$$1 \to C_2 \to \hat{S}_n \to S_n \to 1,$$

satisfying the condition that the transpositions and the product of two disjoint transpositions in $S_n$ lift to elements of order 4 in $\hat{S}_n$. On the other hand, the group $\bar{S}_n$ is the central extension

$$1 \to C_2 \to \bar{S}_n \to S_n \to 1,$$

such that a transposition in $S_n$ lifts to an element of order 2 of $\bar{S}_n$, but a product of two disjoint transpositions in $S_n$ lifts to an element of order 4.

Note that we follow the notation of $\hat{S}_n$ and $\bar{S}_n$ adopted by Serre.

**Theorem 1.2** (Serre [Garibaldi et al. 2003, p. 90]). Both $\mathbb{Q} (\hat{S}_4)$ and $\mathbb{Q} (\bar{S}_5)$ are not retract $\mathbb{Q}$-rational. In particular, they are not $\mathbb{Q}$-rational.

Serre proves this using cohomological invariants and trace forms over $\mathbb{Q}$ — the $e$-invariant method, in short. In pp. 89–90 of the same book, he proves that Rat$(G/\mathbb{Q})$ is false for $G = \hat{S}_4$ and $\bar{S}_5$. Actually he proves a bit more. From Serre’s proof it is easy to find that $\mathbb{Q} (\hat{S}_4)$ and $\mathbb{Q} (\bar{S}_5)$ are not retract $\mathbb{Q}$-rational (see [Kang 2012, Section 1] for the relationship of the property Rat$(G/\mathbb{Q})$ and the retract $k$-rationality of $k(G)$). This is the reason why we formulate Serre’s theorem in the version above. In fact, Theorem 1.2 can be perceived also from Serre’s own remark in [Garibaldi et al. 2003, p. 13, Remark 5.8].

We don’t know whether Theorem 1.2 is valid for fields $k$ other than the field $\mathbb{Q}$; for example, the field $k$ satisfying the condition that $k(\zeta_8)$ is not cyclic over $k$. In fact, in a private communication, Serre told us that the $e$-invariant method remains valid (under the assumption that $k(\zeta_8)$ is not cyclic over $k$) if $k$ is an algebraic number field of odd degree over $\mathbb{Q}$, or if $k = \mathbb{Q}(\sqrt{n})$, where $n \equiv 1$ (mod 8). However, if $k = \mathbb{Q}(x, y)$ with $x^2 + y^2 = -1$, the assumption that $k(\zeta_8)$
is not cyclic over \( k \) is valid while the \( e \)-invariant method doesn’t work any more [Serre 2011].

On the other hand, we have:

**Theorem 1.3** [Plans 2007; 2009]. (1) For any field \( k \), \( k(\hat{S}_4) \) is \( k \)-rational. Thus, if \( k \) is a field with \( \text{char } k = 0 \), \( k(\hat{S}_5) \) is also \( k \)-rational.

(2) For any infinite field \( k \) with \( \text{char } k \neq 2 \) such that \( \sqrt{-1} \in k \), both \( k(\hat{S}_4) \) and \( k(\hat{S}_5) \) are \( k \)-rational.

The main result of this article is the following rationality criterion for \( k(\hat{S}_4) \) and \( k(\hat{S}_5) \).

**Theorem 1.4.** Let \( k \) be a field with \( \text{char } k \neq 2 \) or 3, and \( \xi_8 \) be a primitive eighth root of unity in some extension field of \( k \). If \( k(\xi_8) \) is a cyclic extension of \( k \), then \( k(\hat{S}_4) \) is \( k \)-rational; if it is assumed furthermore that \( \text{char } k = 0 \), then \( k(\hat{S}_5) \) is also \( k \)-rational.

When \( k \) is a field with \( \text{char } k = p > 0 \) and \( p \neq 2 \), the assumption that \( k(\xi_8) \) is a cyclic extension of \( k \) is satisfied automatically. Thus \( k(\hat{S}_4) \) is \( k \)-rational provided that \( k \) is any field with \( \text{char } k \neq 2 \) or 3.

Besides the groups \( \hat{S}_4 \) and \( \hat{S}_5 \), Serre shows that \( \mathbb{Q}(G) \) is not retract \( \mathbb{Q} \)-rational if \( G \) is any one of the groups \( \text{SL}_2(\mathbb{F}_7) \), \( \text{SL}_2(\mathbb{F}_9) \) and the generalized quaternion group of order 16; see [Garibaldi et al. 2003, p. 90, Example 33.27]. In case \( G \) is the generalized quaternion group of order 16 and \( k(\xi_8) \) is cyclic over \( k \), it is known that \( k(G) \) is \( k \)-rational [Kang 2005]. We don’t know whether analogous results as Theorem 1.4 are valid when the groups are \( \text{SL}_2(\mathbb{F}_7) \) and \( \text{SL}_2(\mathbb{F}_9) \).

The main idea of the proof of Theorem 1.4 is to use the method of Galois descent, namely we first enlarge the field \( k \) to \( k(\xi_8) \), solve the rationality of \( k(\xi_8)(\hat{S}_4) \), and then descend the ground field to \( k \).

The proof that \( k(\xi_8)(\hat{S}_4) \) is \( k(\xi_8) \)-rational requires at least two techniques. In order to decrease the number of variables (by applying Theorem 2.2), we will construct a 4-dimensional faithful representation \( V \) of \( \hat{S}_4 \) defined over the field \( k \). It seems the representation and the idea to find it are not well-known. Once we have this representation, we adjoin \( \xi_8 \) to the field \( k \) and write \( \pi = \text{Gal}(k(\xi_8)/k) \). We will prove that \( k(\xi_8)(V)^{(\hat{S}_4, \pi)} \) is \( k \)-rational.

The rationality problem of \( k(\xi_8)(V)^{(\hat{S}_4, \pi)} \) is not straightforward. In several steps of computations we use computers to facilitate the process of symbolic computation. However, we emphasize that computers play only a minor role; we don’t use particular codes of data bases such as GAP.

On the other hand, we point out that the first several steps in proving that \( k(\xi_8)(V)^{(\hat{S}_4, \pi)} \) is \( k \)-rational are rather similar to those in [Kang and Zhou 2012,
Section 5). This seems unsurprising because the group $\tilde{S}_4$ considered in [Kang and Zhou 2012, Section 5] and the group $\tilde{S}_4$ here have a common subgroup $A_4$.

For the rationality problem of $k(\tilde{S}_5)$, we apply Theorem 2.5 of Plans, which asserts that $k(\tilde{S}_5)$ is a rational extension of $k(\tilde{S}_4)$, whence the result.

We organize this paper as follows. We recall some preliminaries in Section 2 that will be used in the proof of Theorem 1.4. In Section 3, several low-dimensional faithful representations of $\tilde{S}_4$ over a field $k$ with char $k \neq 2$ will be constructed (the reader may find another explicit construction in [Karpilovsky 1985, p. 177–179]). Theorem 1.4 will be proved in Section 4. In Section 5 we will consider the rationality problem of $k(G_n)$ (see Definition 5.1 for the group $G_n$).

Throughout this article, whenever we write $k(x_1, x_2, x_3, x_4)$ or $k(x, y)$ without explanation, it is understood that it is a rational function field over $k$. We will denote by $\zeta_8$ (or simply by $\zeta$) a primitive eighth root of unity.

2. Preliminaries

We recall several results that will be used in tackling the rationality problem.

**Theorem 2.1** [Ahmad et al. 2000, Theorem 3.1]. Let $L$ be any field, $L(x)$ the rational function field of one variable over $L$ and $G$ a finite group acting on $L(x)$. Suppose that for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G, \deg g(x) \geq 1\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property that $L(x)^G = L^G(f)$.

**Theorem 2.2** [Hajja and Kang 1995, Theorem 1]. Let $G$ be a finite group acting on the rational function field $L(x_1, \ldots, x_n)$ of $n$ variables over a field $L$. Suppose that:

(i) For any $\sigma \in G$, $\sigma(L) \subset L$.

(ii) The restriction of the action of $G$ to $L$ is faithful.

(iii) For any $\sigma \in G$,

$$
\begin{pmatrix}
\sigma(x_1) \\
\sigma(x_2) \\
\vdots \\
\sigma(x_n)
\end{pmatrix} = A(\sigma) \cdot
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} + B(\sigma),
$$

where $A(\sigma) \in \text{GL}_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over $L$.

Then there exist elements $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ that are algebraically independent over $L$ and satisfy $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \leq i \leq n$. 
Theorem 2.3 [Yamasaki 2009]. Let $k$ be a field with char $k \neq 2$, let $a \in k \setminus \{0\}$, and define a $k$-automorphism $\sigma$ of the rational function field $k(x, y)$ by $\sigma(x) = a/x$ and $\sigma(y) = a/y$. Then $k(x, y)^{\sigma} = k(u, v)$, where $u = (x - y)/(a - xy)$ and $v = (x + y)/(a + xy)$.

Theorem 2.4 [Masuda 1955, Theorem 3; Hoshi and Kang 2010, Theorem 2.2]. Let $k$ be a field and let $\sigma$ be the $k$-automorphism of the rational function field $k(x, y, z)$ defined by $\sigma : x \mapsto y \mapsto z \mapsto x$. Then $k(x, y, z)^{\sigma} = k(s_1, u, v) = k(s_3, u, v)$, where $s_1, s_2, s_3$ are the elementary symmetric functions of degree one, two and three in $x, y, z$ and $u$ and $v$ are defined by

$$u = \frac{x^2 y + y^2 z + z^2 x - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx} \quad \text{and} \quad v = \frac{xy^2 + yz^2 + zx^2 - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$ 

Theorem 2.5 [Plans 2009, Theorem 11]. Let $n \geq 5$ be an odd integer and let $k$ be a field with char $k = 0$. Then $k(\hat{S}_n)$ is rational over $k(\hat{S}_{n-1})$.

Theorem 2.6 [Kang and Plans 2009, Theorem 1.9]. Let $k$ be a field and let $G_1$ and $G_2$ be two finite groups. If both $k(G_1)$ and $k(G_2)$ are $k$-rational, so is $k(G_1 \times G_2)$.

3. Faithful representations of $\hat{S}_4$

In this and the next section, the field $k$ we consider is of char $k \neq 2$ or 3. We will denote by $\xi_8 = (1 + \sqrt{-1})/\sqrt{2}$ a primitive eighth root of unity.

In [Springer 1977, p. 92] a generating set of $\hat{S}_4$ is given (where the group is called the binary octahedral group): $\hat{S}_4 = \langle a', b, c \rangle$ with relations $a'^8 = b^4 = c^6 = 1$, $ba'b^{-1} = a'^{-1}$, $cbbc^{-1} = a'^2$ and $(a'c)^2 = -a'^2b$ (here $-1$ is the element that is equal to $a'^4 = b^2 = c^3$). Note that we have a short exact sequence of groups

$$1 \to \{\pm 1\} \to \hat{S}_4 \to S_4 \to 1,$$

and that $p(a') = (1, 2, 3, 4)$, $p(b) = (1, 4)(2, 3)$ and $p(c) = (1, 2, 3)$. Note that $p(ba') = (1, 4)(2, 3)(1, 2, 3, 4) = (1, 3)$.

If $\xi_8 \in k$, a faithful 2-dimensional representation $\Phi : \hat{S}_4 \to \text{GL}_2(k)$ is given in [Springer 1977, p. 92] as follows (we write $\xi = \xi_8$).

$$\Phi(a') = \begin{pmatrix} \xi & 0 \\ 0 & \xi^7 \end{pmatrix}, \quad \Phi(b) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \Phi(c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^7 & \xi^7 \\ \xi^7 & \xi \end{pmatrix}.$$ 

Suppose that $\sqrt{2} \in k$ (but not necessarily that $\sqrt{-1} \in k$). We may obtain a 4-dimensional representation $\hat{S}_4 \to \text{GL}_4(k)$ by making in (3-1) the substitutions

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$
where $k_0$ is the prime field of $k$ and $\alpha \in k_0(\sqrt{2})$. This process is an easy application of Weil’s restriction [Weil 1956; Voskresenskii 1998, p. 38]. Thus we get

$$
(3-2) \quad a' \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.
$$

Similarly, when $\sqrt{-2}$ is in $k$ (but possibly $\sqrt{-1}$ is not in $k$), write $\sqrt{-2} = \sqrt{-1} \cdot \sqrt{2}$. Thus represent $\sqrt{2}$ as $-\sqrt{-1} \cdot \sqrt{-2}$ and $\zeta = (1 + \sqrt{-1})/\sqrt{2}$ becomes $\sqrt{-2}(1 - \sqrt{-1})/2$. Make in (3-1) the substitutions

$$
\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$

where $k_0$ is the prime field of $k$ and $\alpha \in k_0(\sqrt{-2})$. We get

$$
(3-3) \quad a' \mapsto \frac{-\sqrt{-2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.
$$

The same way, if $\sqrt{-1} \in k$ (but possibly $\sqrt{2} \notin k$), make in (3-1) the substitutions

$$
\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$

where $k_0$ is the prime filed of $k$ and $\alpha \in k_0(\sqrt{-1})$. We get

$$
a' \mapsto \begin{pmatrix} 0 & 1 + \sqrt{-1} & 1 + \sqrt{-1} \\ 1 + \sqrt{-1} & 0 & 1 - \sqrt{-1} \\ 0 & 1 - \sqrt{-1} & 0 \end{pmatrix},
$$

$$
(3-4) \quad b \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \\ 0 & \sqrt{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 - \sqrt{-1} & 0 & 1 - \sqrt{-1} & 0 \\ 0 & 1 - \sqrt{-1} & 0 & 1 - \sqrt{-1} \\ 1 - \sqrt{-1} & 0 & 1 + \sqrt{-1} & 0 \\ 0 & 1 + \sqrt{-1} & 0 & 1 + \sqrt{-1} \end{pmatrix}.
$$

Finally, from (3-2) we may get a faithful 8-dimensional representation of $\hat{S}_4$ into $GL_8(k_0)$, where $k_0$ is the prime field of $k$. Explicitly, make in (3-2) the substitutions

$$
\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},
$$
where $\alpha \in k_0$. We get

$$a' \mapsto \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$b \mapsto \begin{pmatrix} 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

$$c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

4. Proof of Theorem 1.4

By Theorem 2.5, in case $\text{char } k = 0$ and it is known that $k(\hat{S}_4)$ is $k$-rational, it follows immediately that $k(\hat{S}_5)$ is also $k$-rational. Hence, in proving Theorem 1.4, it suffices to prove the rationality of $k(\hat{S}_4)$.

By assumption, $k(\xi_8)$ is a cyclic extension of $k$. Hence at least one of $\sqrt{-1}$, $\sqrt{2}$ or $\sqrt{-2}$ belongs to $k$.

Case 1: $\xi_8 \in k$. Since $\text{char } k \neq 2$ or 3, the group algebra $k[\hat{S}_4]$ is semisimple. Hence the 2-dimensional faithful representation provided by Equation (3-1) can be embedded into the regular representation whose dual space is $V_{\text{reg}} = \bigoplus_{g \in \hat{S}_4} k \cdot x(g)$, where $\hat{S}_4$ acts on $V_{\text{reg}}$ by $h \cdot x(g) = x(hg)$ for any $g, h \in \hat{S}_4$. By Theorem 2.2, we find that $k(\hat{S}_4) = k(x(g) : g \in \hat{S}_4)\hat{S}_4$ is rational over $k(x, y)\hat{S}_4$, where the actions
given by Equation (3-1) are

\[
\begin{align*}
    a' & : x \mapsto \xi x, & y \mapsto \xi^7 y, \\
    b & : x \mapsto \sqrt{-1} y, & y \mapsto \sqrt{-1} x, \\
    c & : x \mapsto (\xi^7 x + \xi^5 y)/\sqrt{2}, & y \mapsto (\xi^7 x + \xi y)/\sqrt{2}.
\end{align*}
\]

Set \( z = x/y \). Then \( k(x, y) = k(z, x) \). By applying Theorem 2.1 we get that \( k(z, x)^{\hat{S}_4} = k(z)^{\hat{S}_4}(t) \) for some element \( t \) fixed by \( \hat{S}_4 \). The field \( k(z)^{\hat{S}_4} \) is \( k \)-rational by Lüroth’s theorem. Hence \( k(z, x)^{\hat{S}_4} \) and \( k(\hat{S}_4) \) are \( k \)-rational.

**Case 2:** \( \sqrt{2} \in k \) but \( \sqrt{-1} \notin k \). We will use the 4-dimensional faithful representation of \( \hat{S}_4 \) over \( k \) provided by Equation (3-2). This representation provides an action of \( \hat{S}_4 \) on \( k(x_1, x_2, x_3, x_4) \) given by

\[
\begin{align*}
    a' & : x_1 \mapsto (x_1 + x_2)/\sqrt{2}, & x_2 \mapsto (-x_1 + x_2)/\sqrt{2}, \\
    & x_3 \mapsto (x_3 - x_4)/\sqrt{2}, & x_4 \mapsto (x_3 + x_4)/\sqrt{2}, \\
    b & : x_1 \mapsto x_4 \mapsto -x_1, & x_2 \mapsto -x_3, \\
    & x_3 \mapsto x_2, & \\
    c & : x_1 \mapsto (x_1 - x_2 - x_3 - x_4)/2, & x_2 \mapsto (x_1 + x_2 + x_3 - x_4)/2, \\
    & x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, & x_4 \mapsto (x_1 + x_2 - x_3 + x_4)/2.
\end{align*}
\]

**Step 1.** Apply Theorem 2.2 and use the arguments of Case 1. We find that \( k(\hat{S}_4) \) is rational over \( k(x_1, x_2, x_3, x_4)^{\hat{S}_4} \). It remains to show that \( k(x_1, x_2, x_3, x_4)^{\hat{S}_4} \) is \( k \)-rational.

**Step 2.** Write \( \pi = \text{Gal}(k(\sqrt{-1})/k) = (\rho) \), where \( \rho(\sqrt{-1}) = -\sqrt{-1} \). Extend the actions of \( \pi \) and \( \hat{S}_4 \) on \( k(\sqrt{-1}) \) and \( k(x_1, x_2, x_3, x_4) \) to \( k(\sqrt{-1})(x_1, x_2, x_3, x_4) \) by requiring that \( \rho(x_i) = x_i \) for \( 1 \leq i \leq 4 \) and \( g(\sqrt{-1}) = \sqrt{-1} \) for all \( g \in \hat{S}_4 \). It follows that

\[
k(x_1, x_2, x_3, x_4)^{\hat{S}_4} = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{(\rho)} (a', b, c)
\]

Define \( y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4) \) by

\[
\begin{align*}
    y_1 & = \sqrt{-1}x_1 + \sqrt{-1}x_2 - x_3 + x_4, & y_2 & = -\sqrt{-1}x_1 + \sqrt{-1}x_2 + x_3 + x_4, \\
    y_3 & = x_1 - x_2 - \sqrt{-1}x_3 - \sqrt{-1}x_4, & y_4 & = x_1 + x_2 - \sqrt{-1}x_3 + \sqrt{-1}x_4.
\end{align*}
\]

Then

\[
k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)
\]
and the actions in (4-1) become

\[
\begin{align*}
    a' : & \quad y_1 \mapsto (y_1 + y_2)/\sqrt{2}, \quad y_2 \mapsto (-y_1 + y_2)/\sqrt{2}, \\
    & \quad y_3 \mapsto (y_3 + y_4)/\sqrt{2}, \quad y_4 \mapsto (-y_3 + y_4)/\sqrt{2}, \\
    b : & \quad y_1 \mapsto \sqrt{-1}y_1, \quad y_2 \mapsto -\sqrt{-1}y_2, \\
    & \quad y_3 \mapsto \sqrt{-1}y_3, \quad y_4 \mapsto -\sqrt{-1}y_4, \\
    c : & \quad y_1 \mapsto \frac{y_1 - \sqrt{-1}y_2}{1 + \sqrt{-1}}, \quad y_2 \mapsto \frac{y_1 + \sqrt{-1}y_2}{1 + \sqrt{-1}}, \\
    & \quad y_3 \mapsto \frac{y_3 - \sqrt{-1}y_4}{1 + \sqrt{-1}}, \quad y_4 \mapsto \frac{y_3 + \sqrt{-1}y_4}{1 + \sqrt{-1}}, \\
    \rho : & \quad y_1 \mapsto -\sqrt{-1}y_4, \quad y_2 \mapsto \sqrt{-1}y_3, \\
    & \quad y_3 \mapsto \sqrt{-1}y_2, \quad y_4 \mapsto -\sqrt{-1}y_1.
\end{align*}
\]

(4-2)

Note that the action of \(a'^2\) is given by

\[
a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.
\]

The reader might find interesting to compare the actions in (4-2) with those in [Kang and Zhou 2012, Section 4]. It turns out that the formulae for \(b, a'^2, c^2\) are completely the same as those for \(\lambda_1, \lambda_2, \sigma\) in [Kang and Zhou 2012, Formula (4.3)]. As mentioned before, both the subgroups \(\langle b, a'^2, c^2 \rangle\) and \(\langle \lambda_1, \lambda_2, \sigma \rangle\) are isomorphic to \(\tilde{A}_4\) (where \(\tilde{A}_4 = p^{-1}(A_4)\) in the notation of Section 3) as abstract groups.

• Step 3. Define \(z_1 = y_1/y_2, z_2 = y_3/y_4, z_3 = y_1/y_3\). By Theorem 2.1, we find that

\[
k(\sqrt{-1})(y_1, y_2, y_3, y_4)^{(\hat{S}_4, \pi)} = k(\sqrt{-1})(z_1, z_2, z_3)(y_4)^{(\hat{S}_4, \pi)} = k(\sqrt{-1})(z_1, z_2, z_3)^{(\hat{S}_4, \pi)}(z_0),
\]

where \(z_0\) is fixed by the actions of \(\hat{S}_4\) and \(\pi\). There remains to show the \(k\)-rationality of \(k(\sqrt{-1})(z_1, z_2, z_3)^{(\hat{S}_4, \pi)}\) is

Before we find \(k(\sqrt{-1})(z_1, z_2, z_3)^{(\hat{S}_4, \pi)}\), we will find \(k(\sqrt{-1})(z_1, z_2, z_3)^{(b,a'^2)}\).

The method is the same as in Steps 3 and 4 in [Kang and Zhou 2012, Section 4]. We will write down the details for the convenience of the reader.

Define \(u_1 = z_1/z_2, u_2 = z_1z_2, u_3 = z_3\). Then

\[
k(\sqrt{-1})(z_1, z_2, z_3)^{(b)} = k(\sqrt{-1})(u_1, u_2, u_3).
\]

The action of \(a'^2\) is given by

\[
a'^2 : u_1 \mapsto 1/u_1, \quad u_2 \mapsto 1/u_2, \quad u_3 \mapsto u_3/u_1.
\]
Define
\[ v_1 = \frac{u_1 - u_2}{1 - u_1 u_2}, \quad v_2 = \frac{u_1 + u_2}{1 + u_1 u_2}, \quad v_3 = u_3 \left(1 + \frac{1}{u_1}\right). \]

Then
\[ k\left(\sqrt{-1}\right)(u_1, u_2, u_3)^{(a'^2)} = k\left(\sqrt{-1}\right)(u_1, u_2, v_3)^{(a'^2)} = k\left(\sqrt{-1}\right)(v_1, v_2, v_3) \]

by Theorem 2.3 (note that \(a'^2(v_3) = v_3\)). In summary,
\[ k\left(\sqrt{-1}\right)(z_1, z_2, z_3)^{(b, a'^2)} = k\left(\sqrt{-1}\right)(v_1, v_2, v_3). \]

• Step 4. The action of \(c\) on \(v_1, v_2, v_3\) is given by
\[ c : v_1 \mapsto 1/v_2, \quad v_2 \mapsto v_1/v_2, \quad v_3 \mapsto v_3(v_1 + v_2)/[v_2(1 + v_1)]. \]

Define \(X_3 = v_3(1 + v_1 + v_2)/[(1 + v_1)(1 + v_2)]\). Then \(c(X_3) = X_3\) and
\[ k\left(\sqrt{-1}\right)(v_1, v_2, v_3) = k\left(\sqrt{-1}\right)(v_1, v_2, X_3). \]

Thus we may apply Theorem 2.4 (regarding \(v_1, 1/v_2, v_2/v_1\) as \(x, y, z\) in its statement). More precisely, define
\[
X_1 = (v_1^3 v_2^3 + v_1^3 + v_2^3 - 3v_1^2 v_2^2)/(v_1^4 v_2^2 + v_2^4 + v_1^2 v_2^2 - v_1^3 v_2 - v_1^2 v_2^3 - v_1^2 v_2^3),
\]
\[
X_2 = (v_1 v_2^4 + v_1 v_2 + v_1^4 v_2 - 3v_1^2 v_2^2)/(v_1^4 v_2^2 + v_2^4 + v_1^2 v_2^2 - v_1^3 v_2 - v_1^2 v_2^3 - v_1^2 v_2^3).
\]

By Theorem 2.4 we get \(k\left(\sqrt{-1}\right)(v_1, v_2, X_3)^{(c)} = k\left(\sqrt{-1}\right)(X_1, X_2, X_3). \)

• Step 5. With the aid of computers, we find that the actions of \(a'\) and \(\rho\) on \(X_1, X_2, X_3\) are given by
\[
da' : X_1 \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_3 \mapsto X_3,
\]
\[
\rho : X_1 \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_3 \mapsto -2A/X_3,
\]

where \(A = g_1 g_2 g_3^{-1}\) and
\[
g_1 = (1 + X_1)^2 - X_2(1 + X_1) + X_2^2,
\]
\[
g_2 = (1 + X_2)^2 - X_1(1 + X_2) + X_1^2,
\]
\[
g_3 = 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1 X_2(3X_1 X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4.
\]

Note that \(\rho(g_1) = g_2/(X_1^2 - X_1 X_2 + X_2^2)\) and \(a'(g_1) = g_1/(X_1^2 - X_1 X_2 + X_2^2)\).

Define \(Y_1 = X_1/X_2, Y_2 = X_1, Y_3 = X_1 X_3/g_1\). We find that
\[
da' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_2^2/(Y_2(1 - Y_1 + Y_1^2)), \quad Y_3 \mapsto Y_3.
\]
Thus
\[ k(\sqrt{-1})(X_1, X_2, X_3)^{(a')} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{(a')} = k(\sqrt{-1})(Z_1, Z_2, Z_3), \]
where \( Z_1 = Y_1, Z_2 = Y_2 + a'(Y_2), Z_3 = Y_3. \)

- Step 6. Using computers, we find that the action of \( \rho \) is given by
\[ \rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto -2Z_1^2/(A'Z_3), \]
where \( A' \) is defined to be
\[-2Z_1^2 + Z_1Z_2 + Z_2^2 + 4Z_1^2 - 2Z_1Z_2^2 - 2Z_1^2Z_2 + Z_1^2Z_2^2 + Z_1^4Z_2 - 2Z_1^3Z_2^2 + Z_1^4Z_2^2.\]

Define \( U_1 = Z_2 + \rho(Z_2), U_2 = \sqrt{-1}(Z_2 - \rho(Z_2)), U_3 = Z_3 + \rho(Z_3) \) and \( U_4 = \sqrt{-1}(Z_3 - \rho(Z_3)). \) We see that \( k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(U_1, U_2, U_3, U_4) \) with a relation
\[ U_1^2 + U_2^2 + 32(U_1^2 + U_2^2)/B = 0, \]
where \( B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2. \)

Dividing this relation by \( 16(U_1^2 + U_2^2)^2/B^2, \) we get
\[ (BU_1/(4U_1^2 + 4U_2^2))^2 + (BU_4/(4U_1^2 + 4U_2^2))^2 + 2B/(U_1^2 + U_2^2) = 0. \]

Multiply this relation by \( U_1^2 + U_2^2 \) and use the identity
\[ (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) = (\alpha \delta + \beta \gamma)^2 + (\alpha \gamma - \beta \delta)^2 \]
to obtain the simplification
\[ (4-3) \quad V_3^2 + V_4^2 + 2B = 0, \]
where
\[ V_3 = B \frac{U_1U_3 + U_2U_4}{4U_1^2 + 4U_2^2} \quad \text{and} \quad V_4 = B \frac{U_1U_4 - U_2U_3}{4U_1^2 + 4U_2^2}. \]

Note that \( k(U_1, U_2, U_3, U_4) = K(U_1, U_2, V_3, V_4). \)

Define \( w_1 = 8U_1/(U_1^2 - 3U_2^2), \) \( w_2 = 8U_2/(U_1^2 - 3U_2^2), \) \( w_3 = V_3/(U_1^2 - 3U_2^2), \) \( w_4 = V_4/(U_1^2 - 3U_2^2). \) Then \( k(U_1, U_2, V_3, V_4) = k(w_1, w_2, w_3, w_4) \) and the relation (4-3) becomes
\[ w_1^2 + w_2^2 + 2 + w_1 + w_2^2 = 0. \]

Hence \( w_1 \in k(w_2, w_3, w_4). \) Thus \( k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(w_2, w_3, w_4) \) is \( k \)-rational.
Case 3: $\sqrt{-2} \in k$ but $\sqrt{-1} \notin k$. We use the 4-dimensional faithful representation of $\hat{S}_4$ over $k$ provided by (3-3). This representation provides an action of $\hat{S}_4$ on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{align*}
a' & : x_1 \mapsto \frac{\sqrt{-2}(x_1 - x_2)}{2}, & x_2 & \mapsto \frac{\sqrt{-2}(x_1 + x_2)}{2}, \\
x_3 & \mapsto \frac{\sqrt{-2}(-x_3 - x_4)}{2}, & x_4 & \mapsto \frac{\sqrt{-2}(x_3 - x_4)}{2}, \\
b & : x_1 \mapsto x_4 \mapsto -x_1, & x_2 & \mapsto -x_3, \\
x_3 & \mapsto x_2, \\
c & : x_1 \mapsto \frac{(x_1 - x_2 - x_3 - x_4)}{2}, & x_2 & \mapsto \frac{(x_1 + x_2 + x_3 - x_4)}{2}, \\
x_3 & \mapsto \frac{(x_1 - x_2 + x_3 + x_4)}{2}, & x_4 & \mapsto \frac{(x_1 + x_2 - x_3 + x_4)}{2}.
\end{align*}$$

The proof of this case is very similar to that of Case 2.

- **Step 1.** Apply Theorem 2.2. We see that $\hat{k} = k(\hat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\hat{S}_4}$. Hence the proof is reduced to proving that $k(x_1, x_2, x_3, x_4)^{\hat{S}_4}$ is $k$-rational.

- **Step 2.** Write $\pi = \text{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$, where $\rho(\sqrt{-1}) = -\sqrt{-1}$. Extend the actions of $\pi$ and $\hat{S}_4$ to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ as in Step 2 of Case 2. We find that

$$\hat{k}(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{(a', b, c, \rho)}.$$  

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{align*}
y_1 & = -x_1 - \sqrt{-1}x_2 + x_3 + \sqrt{-1}x_4, & y_2 & = \sqrt{-1}x_1 - x_2 + \sqrt{-1}x_3 - x_4, \\
y_3 & = x_1 - \sqrt{-1}x_2 + x_3 - \sqrt{-1}x_4, & y_4 & = \sqrt{-1}x_1 + x_2 - \sqrt{-1}x_3 - x_4.
\end{align*}$$

We get $k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$ and the actions are

$$\begin{align*}
a' & : y_1 \mapsto (-y_1 - y_2)/\sqrt{2}, & y_2 & \mapsto (y_1 - y_2)/\sqrt{2}, \\
y_3 & \mapsto (y_3 + y_4)/\sqrt{2}, & y_4 & \mapsto (-y_3 + y_4)/\sqrt{2}, \\
b & : y_1 \mapsto \sqrt{-1}y_1, & y_2 & \mapsto -\sqrt{-1}y_2, \\
y_3 & \mapsto \sqrt{-1}y_3, & y_4 & \mapsto -\sqrt{-1}y_4, \\
c & : y_1 \mapsto \frac{y_1 - \sqrt{-1}y_2}{1 + \sqrt{-1}}, & y_2 & \mapsto \frac{y_1 + \sqrt{-1}y_2}{1 + \sqrt{-1}}, \\
y_3 & \mapsto \frac{y_3 - \sqrt{-1}y_4}{1 + \sqrt{-1}}, & y_4 & \mapsto \frac{y_3 + \sqrt{-1}y_4}{1 + \sqrt{-1}}, \\
(4-4) \quad \rho & : y_1 \mapsto \sqrt{-1}y_4, & y_2 & \mapsto -\sqrt{-1}y_3, \\
y_3 & \mapsto -\sqrt{-1}y_2, & y_4 & \mapsto \sqrt{-1}y_1.
\end{align*}$$

Note that the action of $a'^2$ is

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$
(Compare with (4-2) and (4-4).) The actions of \( a'^2, b, c \) in both cases are the same.

- Step 3. Define \( z_1 = y_1/y_2, z_2 = y_3/y_4, z_3 = y_1/y_3 \). As in Step 3 of Case 2, it suffices to prove that \( k(\sqrt{-1})(z_1, z_2, z_3)_{(\mathfrak{S}_4, \pi)} \) is \( k \)-rational.

Define \( u_1, u_2, u_3, v_1, v_2, v_3, X_1, X_2, X_3 \) by the same formulae as in Step 3 and Step 4 of Case 2. We find that \( k(\sqrt{-1})(z_1, z_2, z_3)^{(b, a'^2, c)} = k(\sqrt{-1})(X_1, X_2, X_3) \).

- Step 4. The actions of \( a', \rho \) on \( X_1, X_2, X_3 \) are slightly different from Step 5 of Case 2. In the present case, we have

\[
\begin{align*}
a' : X_1 & \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, & X_2 & \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, & X_3 & \mapsto -X_3, \\
\rho : X_1 & \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, & X_2 & \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, & X_3 & \mapsto -2A/X_3,
\end{align*}
\]

where \( A = g_1 g_2 g_3^{-1} \) and

\[
\begin{align*}
g_1 &= (1 + X_1)^2 - X_2(1 + X_1) + X_2^2, \\
g_2 &= (1 + X_2)^2 - X_1(1 + X_2) + X_1^2, \\
g_3 &= 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1 X_2 (3X_1 X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4.
\end{align*}
\]

Note that the action of \( \rho \) is the same as in Step 5 of Case 2.

Define \( Y_1 = X_1/X_2, Y_2 = X_1, Y_3 = X_1 X_3/g_1. \) We get

\[
\begin{align*}
a' : Y_1 & \mapsto Y_1, & Y_2 & \mapsto Y_2^2/(Y_2(1 - Y_1 + Y_1^2)), & Y_3 & \mapsto -Y_3.
\end{align*}
\]

Thus \( k(\sqrt{-1})(X_1, X_2, X_3)^{(a')} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{(a')} = k(\sqrt{-1})(Z_1, Z_2, Z_3), \)

where \( Z_1 = Y_1, Z_2 = Y_2 + a'(Y_2), Z_3 = Y_3(Y_2 - a'(Y_2)). \)

- Step 5. Using computers, we find that the action of \( \rho \) is given by

\[
\begin{align*}
\rho : Z_1 & \mapsto 1/Z_1, & Z_2 & \mapsto Z_2/Z_1, & Z_3 & \mapsto C/Z_3,
\end{align*}
\]

where \( C \) is defined to be

\[
\frac{2Z_1^2(-4Z_1^2 + Z_2^2 - Z_1 Z_2 + Z_1^2 Z_2^2)/(1 - Z_1 + Z_1^2)}{-2Z_1^2 + Z_1 Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1 Z_2^2 - 2Z_1^2 + 3Z_1^2 Z_2^2 + Z_1^4 Z_2 - 2Z_1^3 Z_2^2 + Z_1^4 Z_2^2}.
\]

Define \( U_1 = Z_2 + \rho(Z_2), U_2 = \sqrt{-1}(Z_2 - \rho(Z_2)), U_3 = Z_3 + \rho(Z_3) \) and \( U_4 = \sqrt{-1}(Z_3 - \rho(Z_3)). \) We find that \( k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(U_1, U_2, U_3, U_4) \)

with a relation

\[
(4-5) \quad U_3^2 + U_4^2 = 8(U_1^2 + U_2^2)^2(-16 + U_1^2 - 3U_2^2)/B(U_1^2 - 3U_2^2),
\]

where \( B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2. \)
Note that the above formula of $B$ is identically the same as that in Step 6 of Case 2. It remains to simplify the relation (4-5). Dividing both sides by $(U_1^2 + U_2^2)^2$, we get

$$(U_3/(U_1^2 + U_2^2))^2 + (U_4/(U_1^2 + U_2^2))^2 = 8(-16 + U_1^2 - 3U_2^2)/B(U_1^2 - 3U_2^2).$$

Divide both sides of the above identity by $(2(U_1^2 - 3U_2^2)/B)^2$. We get a relation

$$V_3^2 + V_4^2 = 2(1 - V_1^2 + 3V_2^2)(1 + V_1 + 2V_2^2),$$

where

$$V_1 = \frac{4U_1}{U_1^2 - 3U_2^2}, \quad V_3 = \frac{BU_3}{(U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2)},$$
$$V_2 = \frac{4U_2}{U_1^2 - 3U_2^2}, \quad V_4 = \frac{BU_4}{(U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2)}.$$ 

Note that $k(U_1, U_2, U_3, U_4) = k(V_1, V_2, V_3, V_4)$.

Define $w_1 = 1/(1 + V_1)$, $w_2 = V_2/(1 + V_1)$, $w_3 = V_3/(1 + V_1)^2$, $w_4 = V_4/(1 + V_1)^2$. We get $k(V_1, V_2, V_3, V_4) = k(w_1, w_2, w_3, w_3)$ and the relation (4-6) becomes

$$w_3^2 + w_4^2 = 2(-1 + 2w_1 + 3w_2^2)(w_1 + 2w_2^2).$$

Divide the above identity by $(w_1 + 2w_2^2)^2$. We get

$$(w_3/(w_1 + 2w_2^2))^2 + (w_4/(w_1 + 2w_2^2))^2 = 2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2).$$

Since $2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2)$ is a “fractional linear transformation” of $w_1$ and it belongs to $k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$, we find that $w_1$ is in $k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$. Thus

$$k(w_1, w_2, w_3, w_4) = k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2)).$$

We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)}$ is $k$-rational.

Case 4: $\sqrt{-1} \in k$ but $\sqrt{2} \notin k$. This is similar to Cases 2 or 3, so the detailed proof is omitted. In the case char $k = 0$, we may apply Plans’ result, Theorem 1.3. \hfill $\square$

5. Other double covers of $S_n$

In this section we consider the rationality problem of $G_n$, which is a double cover of the symmetric group and different from both $\widetilde{S}_n$ and $\widetilde{S}_n$.

There are four double covers of the symmetric group $S_n$ when $n \geq 4$. The trivial case is the split group $S_n \times C_2$. The rationality problem of the group $S_n \times C_2$ is
easy because we may apply Theorem 2.6. It remains to consider the non split cases: they are $\tilde{S}_n$, $\tilde{S}_5$, and the group $G_n$ defined below.

**Definition 5.1.** For $n \geq 3$, consider the group $G_n$ such that the short exact sequence $1 \rightarrow \{\pm 1\} \rightarrow G_n \rightarrow S_n \rightarrow 1$ is induced by the cup product $\varepsilon_n \cup \varepsilon_n \in H^2(S_n, \{\pm 1\})$, (see, for example, [Serre 1984, page 654]) where $\varepsilon_n : S_n \rightarrow \{\pm 1\}$ is the signed map, that is, $\varepsilon_n(\sigma) = -1$ if and only if $\sigma \in S_n$ is an odd permutation. Note that the group $G_n$ is denoted by $\tilde{S}_n$ in [Plans 2009].

The group $G_n$ can be constructed explicitly as follows. Let

$$1 \rightarrow \{\pm 1\} \rightarrow C_4 = \{\pm \sqrt{-1}, \pm 1\} \xrightarrow{p_0} \{\pm 1\} \rightarrow 1$$

be the short exact sequence defined by $p_0(\sqrt{-1}) = -1$. The group $G_n$ can be realized as the pullback of the diagram

$$\begin{array}{ccc}
S_n & \xrightarrow{\varepsilon_n} & C_4 \\
\downarrow & & \xrightarrow{p_0} \\
C_4 & \rightarrow & \{\pm 1\}.
\end{array}$$

Explicitly, as a subgroup of $S_n \times C_4$,

$$G_n = \{(\sigma, (\sqrt{-1})^i) \in S_n \times C_4 : \varepsilon_n(\sigma) = p_0((\sqrt{-1})^i)\}$$

$$= (A_n \times \{\pm 1\}) \cup \{(\sigma, \pm \sqrt{-1}) \in S_n \times C_4 : \sigma \notin A_n\}.$$

If $k$ is a field with $\text{char } k \neq 2$, a faithful $2n$-dimensional representation can be defined as follows. Let $X = \left(\bigoplus_{1 \leq i \leq n} k \cdot x_i\right) \oplus \left(\bigoplus_{1 \leq i \leq n} k \cdot y_i\right)$ and let $G_n$ act on $X$ by, for $1 \leq i \leq n$,

\begin{align*}
t & : x_i \mapsto -x_i, & \quad y_i \mapsto -y_i, \\
\tau & : x_i \mapsto x_{\tau(i)}, & \quad y_i \mapsto y_{\sigma^{-1} \tau \sigma(i)}, \\
\overline{\sigma} & : x_i \mapsto y_i \mapsto -x_i,
\end{align*}

where $t = (1, -1) \in G_n \subset S_n \times C_4$, $\tau \in A_n$ and $\tau$ is identified with $(\tau, 1) \in G_n$, $\sigma = (1, 2) \in S_n$ and $\overline{\sigma} = (\sigma, \sqrt{-1}) \in G_n$.

The next result was proved in [Plans 2009, Theorem 14(b)] under the assumptions that $\text{char } k = 0$ and $\sqrt{-1} \in k$. Our proof is different from Plans’ even in the situation when $\text{char } k = 0$.

**Theorem 5.2.** Assume that $k$ is a field that satisfies:

(i) Either $\text{char } k = 0$ or $\text{char } k = p > 0$ with $p \nmid 2n$.

(ii) $\sqrt{-1} \in k$.

Then $k(G_n)$ is $k$-rational for $n \geq 3$. 
Theorem 2.1, and the action of $S$ for some $\sigma$. The reader will find that (i) the assumption $\text{char } k \neq 2$ is used throughout the proof; (ii) the assumption $\text{char } k \nmid n$ is used in Step 2; (iii) the assumption $\sqrt{-1} \in k$ is used in Step 3.

- Step 1. Apply Theorem 2.2. We find that $k(G_n)$ is rational over

$$k(x_i, y_i : 1 \leq i \leq n)^{G_n},$$

where $G_n$ acts on the rational function field $k(x_i, y_i : 1 \leq i \leq n)$ by (5-1).

- Step 2. Define $u_0 = \sum_{1 \leq i \leq n} x_i, v_0 = \sum_{1 \leq i \leq n} y_i$ and $u_i = x_i/u_0$, $v_i = y_i/v_0$ for $1 \leq i \leq n$. Note that $k(x_i, y_i : 1 \leq i \leq n) = k(u_j, v_j : 0 \leq j \leq n)$ with the relations $\sum_{1 \leq i \leq n} u_i = \sum_{1 \leq i \leq n} v_i = 1$. The action of $G_n$ is given by

$$t : u_0 \mapsto -u_0, \quad v_0 \mapsto -v_0, \quad u_i \mapsto u_i, \quad v_i \mapsto v_i,$$

$$v \mapsto v_0 \mapsto v, \quad u_i \mapsto u_i, \quad v_i \mapsto v_{\sigma^{-1} \tau \sigma(i)}.$$ 

where $1 \leq i \leq n$ and $t, \tau, \sigma$ are defined in (5-1).

Define $w_1 = u_0 v_0, w_2 = u_0/v_0$. Then

$$k(u_j, v_j : 0 \leq j \leq n)^{(t)} = k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2).$$

Note that $\tau(w_i) = w_i$ for $1 \leq i \leq 2$, $\sigma(w_1) = -w_1, \sigma(w_2) = -1/w_2$. By Theorem 2.1,

$$k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2)^{G_n/(t)} = k(u_i, v_i : 1 \leq i \leq n)(w_2)^{G_n/(t)}(w')$$

for some $w'$ fixed by the action of $G_n/(t)$. Moreover, we may identify $G_n/(t)$ with $S_n$ and identify $\sigma$ (modulo $(t)$) with $\sigma$.

Define $U_i = u_i - (1/n), V_i = v_i - (1/n)$ for $1 \leq i \leq n$. We find that

$$\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$$

and the action of $S_n$ on $k(U_i, V_i : 1 \leq i \leq n)$ becomes linear. We will consider $k(U_i, V_i : 1 \leq i \leq n)(w_2)^{S_n}$. The action of $S_n$ is given by

$$\tau : U_i \mapsto U_{\tau(i)}, \quad V_i \mapsto V_{\sigma^{-1} \tau \sigma(i)}, \quad w_2 \mapsto w_2,$$

$$\sigma : U_i \mapsto V_i \mapsto U_i, \quad w_2 \mapsto -1/w_2,$$

where $1 \leq i \leq n, \tau \in A_n, \sigma = (1, 2)$ and $\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$.

- Step 3. Since $\sqrt{-1} \in k$, define $w = (\sqrt{-1} - w_2)/(-\sqrt{-1} + w_2)$. We find that $\tau(w) = w$ for $\tau \in A_n$ and $\sigma(w) = -w$. Apply Theorem 2.1. We find that

$$k(u_i, v_i : 1 \leq i \leq n)(w_2)^{S_n} = k(U_i, V_i : 1 \leq i \leq n)^{S_n}(w'')$$

for some $w''$ fixed by the action of $S_n$. 

It remains to show that \( k(U_i, V_i : 1 \leq i \leq n)^{S_n} \) is \( k \)-rational. The following proof of this fact is due to the referee.

Define \( W_i^\pm = U_i \pm V_{\sigma(i)} \). It is easy to verify that for \( \tau(W_i^\pm) = W_{\tau(i)}^\pm \) for \( \tau \in A_n \); and that for \( \sigma = (1, 2) \), \( \sigma(W_i^+) = W_{\sigma(i)}^- \) and \( \sigma(W_i^-) = -W_{\sigma(i)}^- \).

Define subspaces \( W \) and \( W' \) by \( W = \sum_{1 \leq i \leq n} k \cdot W_i^+ \) and \( W' = \sum_{1 \leq i \leq n} k \cdot W_i^- \). Note that
\[
\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i = W \oplus W'.
\]
Moreover, \( W \) is the standard representation of \( S_n \), that is, \( W \cong \sum_{1 \leq i \leq n} k \cdot s_i \) with \( \sum_{1 \leq i \leq n} s_i = 0 \) and \( \lambda(s_i) = s_{\lambda(i)} \) for all \( \lambda \in S_n \), for all \( 1 \leq i \leq n \). On the other hand, \( W' \) is the representation space of the tensor product of the standard representation and the linear character \( \varepsilon_n : S_n \to \{ \pm 1 \} \).

- Step 4. Apply Theorem 2.2 to \( k(U_i, V_i : 1 \leq i \leq n)^{S_n} \). We find that
\[
k(U_i, V_i : 1 \leq i \leq n)^{S_n} = k(W \oplus W')^{S_n} = k(W_i^+ : 1 \leq i \leq n-1)^{S_n} (t_1, \ldots, t_{n-1}),
\]
where each \( t_i \) is fixed by \( S_n \). Obviously the field \( k(W_i^+ : 1 \leq i \leq n-1)^{S_n} \) is \( k \)-rational, whence the result.

In the following theorem the assumption \( \sqrt{-1} \notin k \) from Theorem 5.2 will be dropped. The first part of the following theorem was proved by Plans [2009, Theorem 14 (b)]; there he assumed that \( \text{char} \, k = 0 \).

**Theorem 5.3.** (1) If \( k \) is a field with \( \text{char} \, k \neq 2 \) or \( 3 \), then \( k(G_3) \) is \( k \)-rational.

(2) If \( k \) is a field with \( \text{char} \, k \neq 2 \), then \( k(G_4) \) is \( k \)-rational. Moreover, if \( \text{char} \, k = 0 \), then \( k(G_5) \) is also \( k \)-rational.

**Proof.** Case 1: \( n = 3 \). By Step 2 in the proof of Theorem 5.2, it suffices to consider \( k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3} \), where \( \sum_{1 \leq i \leq 3} U_i = \sum_{1 \leq i \leq 3} V_i = 0 \). Define \( \tau = (1, 2, 3) \in S_3 \). The actions are given by
\[
\tau : U_1 \mapsto U_2 \mapsto -U_1 - U_2, \quad V_2 \mapsto V_1 \mapsto -V_1 - V_2,
\]
\[
\sigma : U_1 \leftrightarrow V_1, \quad U_2 \leftrightarrow V_2.
\]
Define \( w_3 = U_1/V_2, w_4 = U_2/V_1, w_5 = V_1/V_2 \). It follows that
\[
k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3} = k(w_j : 2 \leq j \leq 5)(V_1)^{S_3} = k(w_j : 2 \leq j \leq 5)^{S_3}(w_0)
\]
for some \( w_0 \) by Theorem 2.1.

It remains to show that \( k(w_j : 2 \leq j \leq 5)^{S_3} \) is \( k \)-rational. Note that
\[
\tau : w_2 \mapsto w_2, \quad w_3 \mapsto w_4 \mapsto (w_3 + w_4w_5)/(1 + w_5),
\]
\[
\sigma : w_2 \mapsto -1/w_2, \quad w_3 \mapsto 1/w_4, \quad w_4 \mapsto 1/w_3, \quad w_5 \mapsto w_3/(w_4w_5).
\]
Define $w_6 = (w_3 + w_4 w_5)/(1 + w_5)$. Note that $k(w_3, w_4, w_5) = k(w_3, w_4, w_6)$ and

$$\tau : w_3 \mapsto w_4 \mapsto w_6 \mapsto w_3 \quad \text{and} \quad \sigma : w_6 \mapsto 1/w_6.$$  

Define $w_7 = (1-w_3)/(1+w_3)$, $w_8 = (1-w_4)/(1+w_4)$, $w_9 = (1-w_6)/(1+w_6)$. Then $k(w_3, w_4, w_6) = k(w_7, w_8, w_9)$ and

$$\tau : w_7 \mapsto w_8 \mapsto w_9 \mapsto w_7,$$

$$\sigma : w_7 \mapsto -w_8, \quad w_8 \mapsto -w_7, \quad w_9 \mapsto -w_9.$$  

By Theorem 2.4 we find that $k(w_2, w_3, w_4, w_5)^{(\tau)} = k(w_2, X_1, X_2, X_3)$, where $X_1 = w_7 + w_8 + w_9$ and

$$X_2 = \frac{w_7^2 w_8^2 + w_8^2 w_9^2 + w_9^2 w_7^2 - 3 w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2},$$

$$X_3 = \frac{w_7 w_8^2 + w_8 w_9^2 + w_9 w_7^2 - 3 w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2}.$$  

Moreover, the action of $\sigma$ is given by

$$\sigma : w_2 \mapsto -1/w_2, \quad X_1 \mapsto -X_1, \quad X_2 \mapsto -X_2, \quad X_3 \mapsto -X_3.$$  

Apply Theorem 2.2. We find that $k(w_2, X_1, X_2, X_3)^{(\sigma)} = k(w_2)^{(\sigma)}(Y_1, Y_2, Y_3)$ for some $Y_1$, $Y_2$, $Y_3$ fixed by $\sigma$. Since $k(w_2)^{(\sigma)}$ is $k$-rational, it follows that $k(w_2, X_1, X_2, X_3)^{(\sigma)}$ is $k$-rational.

Case 2: $n = 4$. Once again we use Step 2 in the proof of Theorem 5.2. It suffices to consider $k(U_i, V_i : 1 \leq i \leq 4)(w_2)^{S_4}$, where $\sum_{1 \leq i \leq 4} U_i = \sum_{1 \leq i \leq 4} V_i = 0$. Set $\lambda_1 = (1, 2)(3, 4)$, $\lambda_2 = (1, 3)(2, 4)$, $\rho = (1, 2, 3)$ and $\sigma = (1, 2)$ as before. Then $S_4$ is generated by $\lambda_1, \lambda_2, \rho$ and $\sigma$.

Define $t_1 = U_1 + U_2, t_2 = V_1 + V_2, t_3 = U_1 + U_3, t_4 = V_2 + V_3, t_5 = U_2 + U_3$ and $t_6 = V_1 + V_3$. The action of $S_4$ is given by

$$\lambda_1 : t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_3 \mapsto -t_3, \quad t_4 \mapsto -t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6,$$

$$\lambda_2 : t_1 \mapsto -t_1, \quad t_2 \mapsto -t_2, \quad t_3 \mapsto t_3, \quad t_4 \mapsto t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6,$$

$$\rho : t_1 \mapsto t_5 \mapsto t_3 \mapsto t_1, \quad t_2 \mapsto t_6 \mapsto t_4 \mapsto t_2,$$

$$\sigma : t_1 \mapsto t_2, \quad t_3 \mapsto t_6, \quad t_4 \mapsto t_5.$$  

It follows that $k(t_i : 1 \leq i \leq 6)(w_2)^{<\lambda_1, \lambda_2>} = k(T_i : 1 \leq i \leq 6)(w_2)$, where $T_1 = t_1/t_2$, $T_2 = t_3/t_4$, $T_3 = t_5/t_6$, $T_4 = t_2 t_6/t_4$, $T_5 = t_4 t_6/t_2$, $T_6 = t_2 t_4/t_6$. 

Moreover, the actions of $\rho$ and $\sigma$ are given by

\[
\begin{align*}
\rho : T_1 &\mapsto T_3 \mapsto T_2 \mapsto T_1, & T_4 &\mapsto T_5 \mapsto T_6 \mapsto T_4, \\
\sigma : T_1 &\mapsto 1/T_1, & T_2 &\mapsto 1/T_3, & T_3 &\mapsto 1/T_2, \\
T_4 &\mapsto (T_1 T_2/T_3)T_6, & T_5 &\mapsto (T_2 T_3/T_1)T_5, & T_6 &\mapsto (T_1 T_3/T_2)T_4.
\end{align*}
\]

By Theorem 2.2, it suffices to show that $k(T_i : 1 \leq i \leq 3)(w_2)^{<\rho,\sigma>}$ is $k$-rational. Define $w_3 = (1 - T_1)/(1 + T_1)$, $w_4 = (1 - T_2)/(1 + T_2)$, $w_5 = (1 - T_3)/(1 + T_3)$. Then we find

\[
\begin{align*}
\rho : w_2 &\mapsto w_2, & w_3 &\mapsto w_5 \mapsto w_4 \mapsto w_3, \\
\sigma : w_2 &\mapsto -1/w_2, & w_3 &\mapsto -w_3, & w_4 &\mapsto -w_5, & w_5 &\mapsto -w_4.
\end{align*}
\]

Use Theorem 2.4 to find that $k(T_i : 1 \leq i \leq 3)(w_2)^{<\rho>}$ is $k$-rational. The remaining part of the proof is very similar to the last part of Case 1. The details are omitted.

Case 3: $n = 5$. By [Plans 2009, Theorem 11], $k(G_5)$ is rational over $k(G_4)$. Since $k(G_4)$ is $k$-rational by Case 2, we are done. \qed

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References


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