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MING-CHANG KANG AND JIAN ZHOU

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Let k be a field, let G be a finite group and let $k(x_g : g \in G)$ be the rational function field over k , on which G acts by the k -automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Noether's problem asks whether the fixed subfield $k(\widehat{G}) := k(x_g : g \in G)^G$ is k -rational, that is, purely transcendental over k . If \widehat{S}_n is the double cover of the symmetric group S_n , in which the liftings of transpositions and products of disjoint transpositions are of order 4, Serre shows that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not \mathbb{Q} -rational. We will prove that if k is a field such that $\text{char } k \neq 2, 3$, and $k(\zeta_8)$ is a cyclic extension of k , then $k(\widehat{S}_4)$ is k -rational. If it is assumed furthermore that $\text{char } k = 0$, then $k(\widehat{S}_5)$ is also k -rational.

1. Introduction

Let k be a field, and L be a finitely generated field extension of k . L is called k -rational (or rational over k) if L is purely transcendental over k ; that is, L is isomorphic to some rational function field over k . L is called stably k -rational if $L(y_1, \dots, y_m)$ is k -rational for some y_1, \dots, y_m that are algebraically independent over L . L is called k -unirational if L is k -isomorphic to a subfield of some k -rational field extension of k . It is easy to see that

$$k\text{-rational} \Rightarrow \text{stably } k\text{-rational} \Rightarrow k\text{-unirational}.$$

A notion of retract rationality was introduced in [Saltman 1984] (see also [Kang 2012]). It is known that if k is an infinite field, then

$$\text{stably } k\text{-rational} \Rightarrow \text{retract } k\text{-rational} \Rightarrow k\text{-unirational}.$$

Let k be a field and G a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k -automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$.

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Denote by $k(G)$ the fixed subfield, that is, $k(G) = k(x_g : g \in G)^G$. Noether's problem asks under what conditions is the field $k(G)$ k -rational.

Noether's problem is related to the inverse Galois problem and the existence of generic G -Galois extensions over k . For the details, see Swan's survey paper [Swan 1983]. The purpose of this paper is to study Noether's problem for some double covers of the symmetric group S_n .

It is known that there are four different double covers of S_n when $n \geq 4$, that is, groups G that fit into a short exact sequence $1 \rightarrow C_2 \rightarrow G \rightarrow S_n \rightarrow 1$; see, for example, [Serre 1984, p. 653].

Definition 1.1 [Garibaldi et al. 2003, pp. 58, 90; Hoffman and Humphreys 1992, p. 18; Karpilovsky 1985, pp. 177–181]. Let $C_2 = \{\pm 1\}$ be the cyclic group of order 2. When $n \geq 4$, the group \widehat{S}_n is the unique central extension of S_n by C_2 , that is,

$$1 \rightarrow C_2 \rightarrow \widehat{S}_n \rightarrow S_n \rightarrow 1,$$

satisfying the condition that the transpositions and the product of two disjoint transpositions in S_n lift to elements of order 4 in \widehat{S}_n . On the other hand, the group \widetilde{S}_n is the central extension

$$1 \rightarrow C_2 \rightarrow \widetilde{S}_n \rightarrow S_n \rightarrow 1,$$

such that a transposition in S_n lifts to an element of order 2 of \widetilde{S}_n , but a product of two disjoint transpositions in S_n lifts to an element of order 4.

Note that we follow the notation of \widehat{S}_n and \widetilde{S}_n adopted by Serre.

Theorem 1.2 (Serre [Garibaldi et al. 2003, p. 90]). *Both $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract \mathbb{Q} -rational. In particular, they are not \mathbb{Q} -rational.*

Serre proves this using cohomological invariants and trace forms over \mathbb{Q} —the e -invariant method, in short. In pp. 89–90 of the same book, he proves that $\text{Rat}(G/\mathbb{Q})$ is false for $G = \widehat{S}_4$ and \widehat{S}_5 . Actually he proves a bit more. From Serre's proof it is easy to find that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract \mathbb{Q} -rational (see [Kang 2012, Section 1] for the relationship of the property $\text{Rat}(G/k)$ and the retract k -rationality of $k(G)$). This is the reason why we formulate Serre's theorem in the version above. In fact, Theorem 1.2 can be perceived also from Serre's own remark in [Garibaldi et al. 2003, p. 13, Remark 5.8].

We don't know whether Theorem 1.2 is valid for fields k other than the field \mathbb{Q} ; for example, the field k satisfying the condition that $k(\zeta_8)$ is not cyclic over k . In fact, in a private communication, Serre told us that the e -invariant method remains valid (under the assumption that $k(\zeta_8)$ is not cyclic over k) if k is an algebraic number field of odd degree over \mathbb{Q} , or if $k = \mathbb{Q}(\sqrt[n]{n})$, where $n \equiv 1 \pmod{8}$. However, if $k = \mathbb{Q}(x, y)$ with $x^2 + y^2 = -1$, the assumption that $k(\zeta_8)$

is not cyclic over k is valid while the e -invariant method doesn't work any more [Serre 2011].

On the other hand, we have:

Theorem 1.3 [Plans 2007; 2009]. (1) For any field k , $k(\widetilde{S}_4)$ is k -rational. Thus, if k is a field with $\text{char } k = 0$, $k(\widetilde{S}_5)$ is also k -rational.

(2) For any infinite field k with $\text{char } k \neq 2$ such that $\sqrt{-1} \in k$, both $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$ are k -rational.

The main result of this article is the following rationality criterion for $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$.

Theorem 1.4. Let k be a field with $\text{char } k \neq 2$ or 3 , and ζ_8 be a primitive eighth root of unity in some extension field of k . If $k(\zeta_8)$ is a cyclic extension of k , then $k(\widehat{S}_4)$ is k -rational; if it is assumed furthermore that $\text{char } k = 0$, then $k(\widehat{S}_5)$ is also k -rational.

When k is a field with $\text{char } k = p > 0$ and $p \neq 2$, the assumption that $k(\zeta_8)$ is a cyclic extension of k is satisfied automatically. Thus $k(\widehat{S}_4)$ is k -rational provided that k is any field with $\text{char } k \neq 2$ or 3 .

Besides the groups \widehat{S}_4 and \widehat{S}_5 , Serre shows that $\mathbb{Q}(G)$ is not retract \mathbb{Q} -rational if G is any one of the groups $\text{SL}_2(\mathbb{F}_7)$, $\text{SL}_2(\mathbb{F}_9)$ and the generalized quaternion group of order 16; see [Garibaldi et al. 2003, p. 90, Example 33.27]. In case G is the generalized quaternion group of order 16 and $k(\zeta_8)$ is cyclic over k , it is known that $k(G)$ is k -rational [Kang 2005]. We don't know whether analogous results as Theorem 1.4 are valid when the groups are $\text{SL}_2(\mathbb{F}_7)$ and $\text{SL}_2(\mathbb{F}_9)$.

The main idea of the proof of Theorem 1.4 is to use the method of Galois descent, namely we first enlarge the field k to $k(\zeta_8)$, solve the rationality of $k(\zeta_8)(\widehat{S}_4)$, and then descend the ground field to k .

The proof that $k(\zeta_8)(\widehat{S}_4)$ is $k(\zeta_8)$ -rational requires at least two techniques. In order to decrease the number of variables (by applying Theorem 2.2), we will construct a 4-dimensional faithful representation V of \widehat{S}_4 defined over the field k . It seems the representation and the idea to find it are not well-known. Once we have this representation, we adjoin ζ_8 to the field k and write $\pi = \text{Gal}(k(\zeta_8)/k)$. We will prove that $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k -rational.

The rationality problem of $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is not straightforward. In several steps of computations we use computers to facilitate the process of symbolic computation. However, we emphasize that computers play only a minor role; we don't use particular codes of data bases such as GAP.

On the other hand, we point out that the first several steps in proving that $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k -rational are rather similar to those in [Kang and Zhou 2012,

Section 5]. This seems unsurprising because the group \tilde{S}_4 considered in [Kang and Zhou 2012, Section 5] and the group \hat{S}_4 here have a common subgroup \tilde{A}_4 .

For the rationality problem of $k(\hat{S}_5)$, we apply Theorem 2.5 of Plans, which asserts that $k(\hat{S}_5)$ is a rational extension of $k(\hat{S}_4)$, whence the result.

We organize this paper as follows. We recall some preliminaries in Section 2 that will be used in the proof of Theorem 1.4. In Section 3, several low-dimensional faithful representations of \hat{S}_4 over a field k with $\text{char } k \neq 2$ will be constructed (the reader may find another explicit construction in [Karpilovsky 1985, p. 177–179]). Theorem 1.4 will be proved in Section 4. In Section 5 we will consider the rationality problem of $k(G_n)$ (see Definition 5.1 for the group G_n).

Throughout this article, whenever we write $k(x_1, x_2, x_3, x_4)$ or $k(x, y)$ without explanation, it is understood that it is a rational function field over k . We will denote by ζ_8 (or simply by ζ) a primitive eighth root of unity.

2. Preliminaries

We recall several results that will be used in tackling the rationality problem.

Theorem 2.1 [Ahmad et al. 2000, Theorem 3.1]. *Let L be any field, $L(x)$ the rational function field of one variable over L and G a finite group acting on $L(x)$. Suppose that for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G, \deg g(x) \geq 1\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property that $L(x)^G = L^G(f)$.*

Theorem 2.2 [Hajja and Kang 1995, Theorem 1]. *Let G be a finite group acting on the rational function field $L(x_1, \dots, x_n)$ of n variables over a field L . Suppose that:*

- (i) For any $\sigma \in G$, $\sigma(L) \subset L$.
- (ii) The restriction of the action of G to L is faithful.
- (iii) For any $\sigma \in G$,

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in \text{GL}_n(L)$ and $B(\sigma)$ is a $n \times 1$ matrix over L .

Then there exist elements $z_1, \dots, z_n \in L(x_1, \dots, x_n)$ that are algebraically independent over L and satisfy $L(x_1, \dots, x_n) = L(z_1, \dots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \leq i \leq n$.

Theorem 2.3 [Yamasaki 2009]. *Let k be a field with $\text{char } k \neq 2$, let $a \in k \setminus \{0\}$, and define a k -automorphism σ of the rational function field $k(x, y)$ by $\sigma(x) = a/x$ and $\sigma(y) = a/y$. Then $k(x, y)^{\langle \sigma \rangle} = k(u, v)$, where $u = (x - y)/(a - xy)$ and $v = (x + y)/(a + xy)$.*

Theorem 2.4 [Masuda 1955, Theorem 3; Hoshi and Kang 2010, Theorem 2.2]. *Let k be a field and let σ be the k -automorphism of the rational function field $k(x, y, z)$ defined by $\sigma : x \mapsto y \mapsto z \mapsto x$. Then $k(x, y, z)^{\langle \sigma \rangle} = k(s_1, u, v) = k(s_3, u, v)$, where s_1, s_2, s_3 are the elementary symmetric functions of degree one, two and three in x, y, z and u and v are defined by*

$$u = \frac{x^2y + y^2z + z^2x - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx} \quad \text{and} \quad v = \frac{xy^2 + yz^2 + zx^2 - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$

Theorem 2.5 [Plans 2009, Theorem 11]. *Let $n \geq 5$ be an odd integer and let k be a field with $\text{char } k = 0$. Then $k(\widehat{S}_n)$ is rational over $k(\widehat{S}_{n-1})$.*

Theorem 2.6 [Kang and Plans 2009, Theorem 1.9]. *Let k be a field and let G_1 and G_2 be two finite groups. If both $k(G_1)$ and $k(G_2)$ are k -rational, so is $k(G_1 \times G_2)$.*

3. Faithful representations of \widehat{S}_4

In this and the next section, the field k we consider is of $\text{char } k \neq 2$ or 3. We will denote by $\zeta_8 = (1 + \sqrt{-1})/\sqrt{2}$ a primitive eighth root of unity.

In [Springer 1977, p. 92] a generating set of \widehat{S}_4 is given (where the group is called the binary octahedral group): $\widehat{S}_4 = \langle a', b, c \rangle$ with relations $a'^8 = b^4 = c^6 = 1$, $ba'b^{-1} = a'^{-1}$, $cbc^{-1} = a'^2$ and $(a'c)^2 = -a'^2b$ (here -1 is the element that is equal to $a'^4 = b^2 = c^3$). Note that we have a short exact sequence of groups

$$1 \rightarrow \{\pm 1\} \rightarrow \widehat{S}_4 \xrightarrow{p} S_4 \rightarrow 1,$$

and that $p(a') = (1, 2, 3, 4)$, $p(b) = (1, 4)(2, 3)$ and $p(c) = (1, 2, 3)$. Note that $p(ba') = (1, 4)(2, 3)(1, 2, 3, 4) = (1, 3)$.

If $\zeta_8 \in k$, a faithful 2-dimensional representation $\Phi : \widehat{S}_4 \rightarrow \text{GL}_2(k)$ is given in [Springer 1977, p. 92] as follows (we write $\zeta = \zeta_8$),

$$(3-1) \quad \Phi(a') = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^7 \end{pmatrix}, \quad \Phi(b) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \Phi(c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^7 & \zeta^7 \\ \zeta^5 & \zeta \end{pmatrix}.$$

Suppose that $\sqrt{2} \in k$ (but not necessarily that $\sqrt{-1} \in k$). We may obtain a 4-dimensional representation $\widehat{S}_4 \rightarrow \text{GL}_4(k)$ by making in (3-1) the substitutions

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{2})$. This process is an easy application of Weil’s restriction [Weil 1956; Voskresenskii 1998, p. 38]. Thus we get

$$(3-2) \quad a' \mapsto \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & & \\ 1 & 1 & & \\ \hline & & 1 & 1 \\ & & -1 & 1 \end{array} \right), \quad b \mapsto \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \\ 1 & & \end{pmatrix}, \quad c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Similarly, when $\sqrt{-2}$ is in k (but possibly $\sqrt{-1}$ is not in k), write $\sqrt{-2} = \sqrt{-1} \cdot \sqrt{2}$. Thus represent $\sqrt{2}$ as $-\sqrt{-1} \cdot \sqrt{-2}$ and $\zeta = (1 + \sqrt{-1})/\sqrt{2}$ becomes $\sqrt{-2}(1 - \sqrt{-1})/2$. Make in (3-1) the substitutions

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{-2})$. We get

$$(3-3) \quad a' \mapsto \frac{\sqrt{-2}}{2} \left(\begin{array}{cc|cc} 1 & 1 & & \\ -1 & 1 & & \\ \hline & & -1 & 1 \\ & & -1 & -1 \end{array} \right), \quad b \mapsto \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \\ 1 & & \end{pmatrix}, \quad c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

The same way, if $\sqrt{-1} \in k$ (but possibly $\sqrt{2} \notin k$), make in (3-1) the substitutions

$$\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{-1})$. We get

$$(3-4) \quad a' \mapsto \left(\begin{array}{cc|cc} 0 & 1+\sqrt{-1} & & \\ \frac{1+\sqrt{-1}}{2} & 0 & & \\ \hline & & 0 & 1-\sqrt{-1} \\ \frac{1-\sqrt{-1}}{2} & & & 0 \end{array} \right),$$

$$b \mapsto \left(\begin{array}{cc|cc} & & \sqrt{-1} & 0 \\ & & 0 & \sqrt{-1} \\ \hline \sqrt{-1} & 0 & & \\ 0 & \sqrt{-1} & & \end{array} \right), \quad c \mapsto \begin{pmatrix} \frac{1-\sqrt{-1}}{2} & 0 & \frac{1-\sqrt{-1}}{2} & 0 \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 & \frac{1-\sqrt{-1}}{2} \\ -\frac{1-\sqrt{-1}}{2} & 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ 0 & \frac{-1-\sqrt{-1}}{2} & 0 & \frac{1+\sqrt{-1}}{2} \end{pmatrix}.$$

Finally, from (3-2) we may get a faithful 8-dimensional representation of \widehat{S}_4 into $GL_8(k_0)$, where k_0 is the prime field of k . Explicitly, make in (3-2) the substitutions

$$\sqrt{2} \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha \in k_0$. We get

$$(3-5) \quad \begin{aligned} a' &\mapsto \frac{1}{2} \left(\begin{array}{cccc|cccc} 0 & 2 & 0 & -2 & & & & \\ 1 & 0 & -1 & 0 & & & & \\ 0 & 2 & 0 & 2 & & & & \\ 1 & 0 & 1 & 0 & & & & \\ \hline & & & & 0 & 2 & 0 & 2 \\ & & & & 1 & 0 & 1 & 0 \\ & & & & 0 & -2 & 0 & 2 \\ & & & & -1 & 0 & 1 & 0 \end{array} \right), \\ b &\mapsto \left(\begin{array}{cccc|cccc} & & & & & -1 & 0 & \\ & & & & & 0 & -1 & \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \\ \hline & & -1 & 0 & & & & \\ & & 0 & -1 & & & & \\ 1 & 0 & & & & & & \\ 0 & 1 & & & & & & \end{array} \right), \\ c &\mapsto \frac{1}{2} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{array} \right). \end{aligned}$$

4. Proof of Theorem 1.4

By Theorem 2.5, in case $\text{char } k = 0$ and it is known that $k(\widehat{S}_4)$ is k -rational, it follows immediately that $k(\widehat{S}_5)$ is also k -rational. Hence, in proving Theorem 1.4, it suffices to prove the rationality of $k(\widehat{S}_4)$.

By assumption, $k(\zeta_8)$ is a cyclic extension of k . Hence at least one of $\sqrt{-1}$, $\sqrt{2}$ or $\sqrt{-2}$ belongs to k .

Case 1: $\zeta_8 \in k$. Since $\text{char } k \neq 2$ or 3 , the group algebra $k[\widehat{S}_4]$ is semisimple. Hence the 2-dimensional faithful representation provided by Equation (3-1) can be embedded into the regular representation whose dual space is $V_{\text{reg}} = \bigoplus_{g \in \widehat{S}_4} k \cdot x(g)$, where \widehat{S}_4 acts on V_{reg} by $h \cdot x(g) = x(hg)$ for any $g, h \in \widehat{S}_4$. By Theorem 2.2, we find that $k(\widehat{S}_4) = k(x(g) : g \in \widehat{S}_4)^{\widehat{S}_4}$ is rational over $k(x, y)^{\widehat{S}_4}$, where the actions

given by [Equation \(3-1\)](#) are

$$\begin{aligned} a' &: x \mapsto \zeta x, & y &\mapsto \zeta^7 y, \\ b &: x \mapsto \sqrt{-1}y, & y &\mapsto \sqrt{-1}x, \\ c &: x \mapsto (\zeta^7 x + \zeta^5 y)/\sqrt{2}, & y &\mapsto (\zeta^7 x + \zeta y)/\sqrt{2}. \end{aligned}$$

Set $z = x/y$. Then $k(x, y) = k(z, x)$. By applying [Theorem 2.1](#) we get that $k(z, x)^{\widehat{S}_4} = k(z)^{\widehat{S}_4}(t)$ for some element t fixed by \widehat{S}_4 . The field $k(z)^{\widehat{S}_4}$ is k -rational by Lüroth's theorem. Hence $k(z, x)^{\widehat{S}_4}$ and $k(\widehat{S}_4)$ are k -rational.

Case 2: $\sqrt{2} \in k$ but $\sqrt{-1} \notin k$. We will use the 4-dimensional faithful representation of \widehat{S}_4 over k provided by [Equation \(3-2\)](#). This representation provides an action of \widehat{S}_4 on $k(x_1, x_2, x_3, x_4)$ given by

$$(4-1) \quad \begin{aligned} a' &: x_1 \mapsto (x_1 + x_2)/\sqrt{2}, & x_2 &\mapsto (-x_1 + x_2)/\sqrt{2}, \\ & x_3 \mapsto (x_3 - x_4)/\sqrt{2}, & x_4 &\mapsto (x_3 + x_4)/\sqrt{2}, \\ b &: x_1 \mapsto x_4 \mapsto -x_1, & x_2 &\mapsto -x_3, \\ & x_3 \mapsto x_2, \\ c &: x_1 \mapsto (x_1 - x_2 - x_3 - x_4)/2, & x_2 &\mapsto (x_1 + x_2 + x_3 - x_4)/2, \\ & x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, & x_4 &\mapsto (x_1 + x_2 - x_3 + x_4)/2. \end{aligned}$$

• Step 1. Apply [Theorem 2.2](#) and use the arguments of Case 1. We find that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. It remains to show that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k -rational.

• Step 2. Write $\pi = \text{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$, where $\rho(\sqrt{-1}) = -\sqrt{-1}$. Extend the actions of π and \widehat{S}_4 on $k(\sqrt{-1})$ and $k(x_1, x_2, x_3, x_4)$ to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by requiring that $\rho(x_i) = x_i$ for $1 \leq i \leq 4$ and $g(\sqrt{-1}) = \sqrt{-1}$ for all $g \in \widehat{S}_4$. It follows that

$$\begin{aligned} k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} &= \{k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle \rho \rangle}\}^{\langle a', b, c \rangle} \\ &= k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle a', b, c, \rho \rangle}. \end{aligned}$$

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned} y_1 &= \sqrt{-1}x_1 + \sqrt{-1}x_2 - x_3 + x_4, & y_2 &= -\sqrt{-1}x_1 + \sqrt{-1}x_2 + x_3 + x_4, \\ y_3 &= x_1 - x_2 - \sqrt{-1}x_3 - \sqrt{-1}x_4, & y_4 &= x_1 + x_2 - \sqrt{-1}x_3 + \sqrt{-1}x_4. \end{aligned}$$

Then

$$k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$$

and the actions in (4-1) become

$$(4-2) \quad \begin{aligned} a' &: y_1 \mapsto (y_1 + y_2)/\sqrt{2}, & y_2 &\mapsto (-y_1 + y_2)/\sqrt{2}, \\ & y_3 \mapsto (y_3 + y_4)/\sqrt{2}, & y_4 &\mapsto (-y_3 + y_4)/\sqrt{2}, \\ b &: y_1 \mapsto \sqrt{-1}y_1, & y_2 &\mapsto -\sqrt{-1}y_2, \\ & y_3 \mapsto \sqrt{-1}y_3, & y_4 &\mapsto -\sqrt{-1}y_4, \\ c &: y_1 \mapsto \frac{y_1 - \sqrt{-1}y_2}{1 + \sqrt{-1}}, & y_2 &\mapsto \frac{y_1 + \sqrt{-1}y_2}{1 + \sqrt{-1}}, \\ & y_3 \mapsto \frac{y_3 - \sqrt{-1}y_4}{1 + \sqrt{-1}}, & y_4 &\mapsto \frac{y_3 + \sqrt{-1}y_4}{1 + \sqrt{-1}}, \\ \rho &: y_1 \mapsto -\sqrt{-1}y_4, & y_2 &\mapsto \sqrt{-1}y_3, \\ & y_3 \mapsto \sqrt{-1}y_2, & y_4 &\mapsto -\sqrt{-1}y_1. \end{aligned}$$

Note that the action of a'^2 is given by

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

The reader might find interesting to compare the actions in (4-2) with those in [Kang and Zhou 2012, Section 4]. It turns out that the formulae for b , a'^2 , c^2 are completely the same as those for λ_1 , λ_2 , σ in [Kang and Zhou 2012, Formula (4.3)]. As mentioned before, both the subgroups $\langle b, a'^2, c^2 \rangle$ and $\langle \lambda_1, \lambda_2, \sigma \rangle$ are isomorphic to \widetilde{A}_4 (where $\widetilde{A}_4 = p^{-1}(A_4)$ in the notation of Section 3) as abstract groups.

• Step 3. Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. By Theorem 2.1, we find that

$$\begin{aligned} k(\sqrt{-1})(y_1, y_2, y_3, y_4)^{(\widehat{S}_4, \pi)} &= k(\sqrt{-1})(z_1, z_2, z_3)(y_4)^{(\widehat{S}_4, \pi)} \\ &= k(\sqrt{-1})(z_1, z_2, z_3)^{(\widehat{S}_4, \pi)}(z_0), \end{aligned}$$

where z_0 is fixed by the actions of \widehat{S}_4 and π . There remains to show the k -rationality of $k(\sqrt{-1})(z_1, z_2, z_3)^{(\widehat{S}_4, \pi)}$ is

Before we find $k(\sqrt{-1})(z_1, z_2, z_3)^{(\widehat{S}_4, \pi)}$, we will find $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b, a'^2 \rangle}$. The method is the same as in Steps 3 and 4 in [Kang and Zhou 2012, Section 4]. We will write down the details for the convenience of the reader.

Define $u_1 = z_1/z_2$, $u_2 = z_1z_2$, $u_3 = z_3$. Then

$$k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b \rangle} = k(\sqrt{-1})(u_1, u_2, u_3).$$

The action of a'^2 is given by

$$a'^2 : u_1 \mapsto 1/u_1, \quad u_2 \mapsto 1/u_2, \quad u_3 \mapsto u_3/u_1.$$

Define

$$v_1 = \frac{u_1 - u_2}{1 - u_1 u_2}, \quad v_2 = \frac{u_1 + u_2}{1 + u_1 u_2}, \quad v_3 = u_3 \left(1 + \frac{1}{u_1}\right).$$

Then

$$k(\sqrt{-1})(u_1, u_2, u_3)^{(a'^2)} = k(\sqrt{-1})(u_1, u_2, v_3)^{(a'^2)} = k(\sqrt{-1})(v_1, v_2, v_3)$$

by [Theorem 2.3](#) (note that $a'^2(v_3) = v_3$). In summary,

$$k(\sqrt{-1})(z_1, z_2, z_3)^{(b, a'^2)} = k(\sqrt{-1})(v_1, v_2, v_3).$$

• Step 4. The action of c on v_1, v_2, v_3 is given by

$$c : v_1 \mapsto 1/v_2, \quad v_2 \mapsto v_1/v_2, \quad v_3 \mapsto v_3(v_1 + v_2)/[v_2(1 + v_1)].$$

Define $X_3 = v_3(1 + v_1 + v_2)/[(1 + v_1)(1 + v_2)]$. Then $c(X_3) = X_3$ and

$$k(\sqrt{-1})(v_1, v_2, v_3) = k(\sqrt{-1})(v_1, v_2, X_3).$$

Thus we may apply [Theorem 2.4](#) (regarding $v_1, 1/v_2, v_2/v_1$ as x, y, z in its statement). More precisely, define

$$X_1 = (v_1^3 v_2^3 + v_1^3 + v_2^3 - 3v_1^2 v_2^2)/(v_1^4 v_2^2 + v_2^4 + v_1^2 - v_1^2 v_2^3 - v_1 v_2^2 - v_1^3 v_2),$$

$$X_2 = (v_1 v_2^4 + v_1 v_2 + v_1^4 v_2 - 3v_1^2 v_2^2)/(v_1^4 v_2^2 + v_2^4 + v_1^2 - v_1^2 v_2^3 - v_1 v_2^2 - v_1^3 v_2).$$

By [Theorem 2.4](#) we get $k(\sqrt{-1})(v_1, v_2, X_3)^{(c)} = k(\sqrt{-1})(X_1, X_2, X_3)$.

• Step 5. With the aid of computers, we find that the actions of a' and ρ on X_1, X_2, X_3 are given by

$$a' : X_1 \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_3 \mapsto X_3,$$

$$\rho : X_1 \mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_2 \mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, \quad X_3 \mapsto -2A/X_3,$$

where $A = g_1 g_2 g_3^{-1}$ and

$$g_1 = (1 + X_1)^2 - X_2(1 + X_1) + X_2^2,$$

$$g_2 = (1 + X_2)^2 - X_1(1 + X_2) + X_1^2,$$

$$g_3 = 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1 X_2(3X_1 X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4.$$

Note that $\rho(g_1) = g_2/(X_1^2 - X_1 X_2 + X_2^2)$ and $a'(g_1) = g_1/(X_1^2 - X_1 X_2 + X_2^2)$. Define $Y_1 = X_1/X_2, Y_2 = X_1, Y_3 = X_1 X_3/g_1$. We find that

$$a' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_1^2/(Y_2(1 - Y_1 + Y_1^2)), \quad Y_3 \mapsto Y_3.$$

Thus

$$k(\sqrt{-1})(X_1, X_2, X_3)^{(a')} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{(a')} = k(\sqrt{-1})(Z_1, Z_2, Z_3),$$

where $Z_1 = Y_1$, $Z_2 = Y_2 + a'(Y_2)$, $Z_3 = Y_3$.

• Step 6. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto -2Z_1^3/(A'Z_3),$$

where A' is defined to be

$$-2Z_1^2 + Z_1Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1Z_2^2 - 2Z_1^4 + 3Z_1^2Z_2^2 + Z_1^4Z_2 - 2Z_1^3Z_2^2 + Z_1^4Z_2^2.$$

Define $U_1 = Z_2 + \rho(Z_2)$, $U_2 = \sqrt{-1}(Z_2 - \rho(Z_2))$, $U_3 = Z_3 + \rho(Z_3)$ and $U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. We see that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(U_1, U_2, U_3, U_4)$ with a relation

$$U_3^2 + U_4^2 + 32(U_1^2 + U_2^2)/B = 0,$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Dividing this relation by $16(U_1^2 + U_2^2)^2/B^2$, we get

$$(BU_3/(4U_1^2 + 4U_2^2))^2 + (BU_4/(4U_1^2 + 4U_2^2))^2 + 2B/(U_1^2 + U_2^2) = 0.$$

Multiply this relation by $U_1^2 + U_2^2$ and use the identity

$$(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) = (\alpha\delta + \beta\gamma)^2 + (\alpha\gamma - \beta\delta)^2$$

to obtain the simplification

$$(4-3) \quad V_3^2 + V_4^2 + 2B = 0,$$

where

$$V_3 = B \frac{U_1U_3 + U_2U_4}{4U_1^2 + 4U_2^2} \quad \text{and} \quad V_4 = B \frac{U_1U_4 - U_2U_3}{4U_1^2 + 4U_2^2}.$$

Note that $k(U_1, U_2, U_3, U_4) = K(U_1, U_2, V_3, V_4)$.

Define $w_1 = 8U_1/(U_1^2 - 3U_2^2)$, $w_2 = 8U_2/(U_1^2 - 3U_2^2)$, $w_3 = V_3/(U_1^2 - 3U_2^2)$, $w_4 = V_4/(U_1^2 - 3U_2^2)$. Then $k(U_1, U_2, V_3, V_4) = k(w_1, w_2, w_3, w_4)$ and the relation (4-3) becomes

$$w_3^2 + w_4^2 + 2 + w_1 + w_2 = 0.$$

Hence $w_1 \in k(w_2, w_3, w_4)$. Thus $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(w_2, w_3, w_4)$ is k -rational.

Case 3: $\sqrt{-2} \in k$ but $\sqrt{-1} \notin k$. We use the 4-dimensional faithful representation of \widehat{S}_4 over k provided by (3-3). This representation provides an action of \widehat{S}_4 on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{aligned} a' : x_1 &\mapsto \sqrt{-2}(x_1 - x_2)/2, & x_2 &\mapsto \sqrt{-2}(x_1 + x_2)/2, \\ &x_3 \mapsto \sqrt{-2}(-x_3 - x_4)/2, & x_4 &\mapsto \sqrt{-2}(x_3 - x_4)/2, \\ b : x_1 &\mapsto x_4 \mapsto -x_1, & x_2 &\mapsto -x_3, \\ &x_3 \mapsto x_2, \\ c : x_1 &\mapsto (x_1 - x_2 - x_3 - x_4)/2, & x_2 &\mapsto (x_1 + x_2 + x_3 - x_4)/2, \\ &x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, & x_4 &\mapsto (x_1 + x_2 - x_3 + x_4)/2. \end{aligned}$$

The proof of this case is very similar to that of Case 2.

• Step 1. Apply Theorem 2.2. We see that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. Hence the proof is reduced to proving that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k -rational.

• Step 2. Write $\pi = \text{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$, where $\rho(\sqrt{-1}) = -\sqrt{-1}$. Extend the actions of π and \widehat{S}_4 to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ as in Step 2 of Case 2. We find that

$$k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{(a', b, c, \rho)}.$$

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned} y_1 &= -x_1 - \sqrt{-1}x_2 + x_3 + \sqrt{-1}x_4, & y_2 &= \sqrt{-1}x_1 - x_2 + \sqrt{-1}x_3 - x_4, \\ y_3 &= x_1 - \sqrt{-1}x_2 + x_3 - \sqrt{-1}x_4, & y_4 &= \sqrt{-1}x_1 + x_2 - \sqrt{-1}x_3 - x_4. \end{aligned}$$

We get $k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$ and the actions are

$$(4-4) \quad \begin{aligned} a' : y_1 &\mapsto (-y_1 - y_2)/\sqrt{2}, & y_2 &\mapsto (y_1 - y_2)/\sqrt{2}, \\ &y_3 \mapsto (y_3 + y_4)/\sqrt{2}, & y_4 &\mapsto (-y_3 + y_4)/\sqrt{2}, \\ b : y_1 &\mapsto \sqrt{-1}y_1, & y_2 &\mapsto -\sqrt{-1}y_2, \\ &y_3 \mapsto \sqrt{-1}y_3, & y_4 &\mapsto -\sqrt{-1}y_4, \\ c : y_1 &\mapsto \frac{y_1 - \sqrt{-1}y_2}{1 + \sqrt{-1}}, & y_2 &\mapsto \frac{y_1 + \sqrt{-1}y_2}{1 + \sqrt{-1}}, \\ &y_3 \mapsto \frac{y_3 - \sqrt{-1}y_4}{1 + \sqrt{-1}}, & y_4 &\mapsto \frac{y_3 + \sqrt{-1}y_4}{1 + \sqrt{-1}}, \\ \rho : y_1 &\mapsto \sqrt{-1}y_4, & y_2 &\mapsto -\sqrt{-1}y_3, \\ &y_3 \mapsto -\sqrt{-1}y_2, & y_4 &\mapsto \sqrt{-1}y_1. \end{aligned}$$

Note that the action of a'^2 is

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

(Compare with (4-2) and (4-4).) The actions of a'^2 , b , c in both cases are the same.

• Step 3. Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. As in Step 3 of Case 2, it suffices to prove that $k(\sqrt{-1})(z_1, z_2, z_3)^{(\widehat{S}_4, \pi)}$ is k -rational.

Define $u_1, u_2, u_3, v_1, v_2, v_3, X_1, X_2, X_3$ by the same formulae as in Step 3 and Step 4 of Case 2. We find that $k(\sqrt{-1})(z_1, z_2, z_3)^{(b, a'^2, c)} = k(\sqrt{-1})(X_1, X_2, X_3)$.

• Step 4. The actions of a' , ρ on X_1, X_2, X_3 are slightly different from Step 5 of Case 2. In the present case, we have

$$\begin{aligned} a' : X_1 &\mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, & X_2 &\mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, & X_3 &\mapsto -X_3, \\ \rho : X_1 &\mapsto \frac{X_2}{X_1^2 - X_1 X_2 + X_2^2}, & X_2 &\mapsto \frac{X_1}{X_1^2 - X_1 X_2 + X_2^2}, & X_3 &\mapsto -2A/X_3, \end{aligned}$$

where $A = g_1 g_2 g_3^{-1}$ and

$$g_1 = (1 + X_1)^2 - X_2(1 + X_1) + X_2^2,$$

$$g_2 = (1 + X_2)^2 - X_1(1 + X_2) + X_1^2,$$

$$g_3 = 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1 X_2(3X_1 X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4.$$

Note that the action of ρ is the same as in Step 5 of Case 2.

Define $Y_1 = X_1/X_2$, $Y_2 = X_1$, $Y_3 = X_1 X_3/g_1$. We get

$$a' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_1^2 / (Y_2(1 - Y_1 + Y_1^2)), \quad Y_3 \mapsto -Y_3.$$

Thus $k(\sqrt{-1})(X_1, X_2, X_3)^{(a')} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{(a')} = k(\sqrt{-1})(Z_1, Z_2, Z_3)$, where $Z_1 = Y_1$, $Z_2 = Y_2 + a'(Y_2)$, $Z_3 = Y_3(Y_2 - a'(Y_2))$.

• Step 5. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto C/Z_3,$$

where C is defined to be

$$\frac{2Z_1^2(-4Z_1^2 + Z_2^2 - Z_1 Z_2^2 + Z_1^2 Z_2^2)/(1 - Z_1 + Z_1^2)}{-2Z_1^2 + Z_1 Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1 Z_2^2 - 2Z_1^4 + 3Z_1^2 Z_2^2 + Z_1^4 Z_2 - 2Z_1^3 Z_2^2 + Z_1^4 Z_2^2}.$$

Define $U_1 = Z_2 + \rho(Z_2)$, $U_2 = \sqrt{-1}(Z_2 - \rho(Z_2))$, $U_3 = Z_3 + \rho(Z_3)$ and $U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(U_1, U_2, U_3, U_4)$ with a relation

$$(4-5) \quad U_3^2 + U_4^2 = 8(U_1^2 + U_2^2)^2(-16 + U_1^2 - 3U_2^2)/B(U_1^2 - 3U_2^2),$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Note that the above formula of B is identically the same as that in Step 6 of Case 2. It remains to simplify the relation (4-5). Dividing both sides by $(U_1^2 + U_2^2)^2$, we get

$$(U_3/(U_1^2 + U_2^2))^2 + (U_4/(U_1^2 + U_2^2))^2 = 8(-16 + U_1^2 - 3U_2^2)/B(U_1^2 - 3U_2^2).$$

Divide both sides of the above identity by $(2(U_1^2 - 3U_2^2)/B)^2$. We get a relation

$$(4-6) \quad V_3^2 + V_4^2 = 2(1 - V_1^2 + 3V_2^2)(1 + V_1 + 2V_2^2),$$

where

$$V_1 = \frac{4U_1}{U_1^2 - 3U_2^2}, \quad V_3 = \frac{BU_3}{(U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2)},$$

$$V_2 = \frac{4U_2}{U_1^2 - 3U_2^2}, \quad V_4 = \frac{BU_4}{(U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2)}.$$

Note that $k(U_1, U_2, U_3, U_4) = k(V_1, V_2, V_3, V_4)$.

Define $w_1 = 1/(1 + V_1)$, $w_2 = V_2/(1 + V_1)$, $w_3 = V_3/(1 + V_1)^2$, $w_4 = V_4/(1 + V_1)^2$. We get $k(V_1, V_2, V_3, V_4) = k(w_1, w_2, w_3, w_4)$ and the relation (4-6) becomes

$$w_3^2 + w_4^2 = 2(-1 + 2w_1 + 3w_2^2)(w_1 + 2w_2^2).$$

Divide the above identity by $(w_1 + 2w_2^2)^2$. We get

$$(w_3/(w_1 + 2w_2^2))^2 + (w_4/(w_1 + 2w_2^2))^2 = 2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2).$$

Since $2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2)$ is a “fractional linear transformation” of w_1 and it belongs to $k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$, we find that w_1 is in $k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$. Thus

$$k(w_1, w_2, w_3, w_4) = k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2)).$$

We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)}$ is k -rational.

Case 4: $\sqrt{-1} \in k$ but $\sqrt{2} \notin k$. This is similar to Cases 2 or 3, so the detailed proof is omitted. In the case $\text{char } k = 0$, we may apply Plans’ result, Theorem 1.3. □

5. Other double covers of S_n

In this section we consider the rationality problem of G_n , which is a double cover of the symmetric group and different from both \widehat{S}_n and \widetilde{S}_n .

There are four double covers of the symmetric group S_n when $n \geq 4$. The trivial case is the split group $S_n \times C_2$. The rationality problem of the group $S_n \times C_2$ is

easy because we may apply [Theorem 2.6](#). It remains to consider the non split cases: they are \widehat{S}_n , \widetilde{S}_n , and the group G_n defined below.

Definition 5.1. For $n \geq 3$, consider the group G_n such that the short exact sequence $1 \rightarrow \{\pm 1\} \rightarrow G_n \xrightarrow{p} S_n \rightarrow 1$ is induced by the cup product $\varepsilon_n \cup \varepsilon_n \in H^2(S_n, \{\pm 1\})$, (see, for example, [[Serre 1984](#), page 654]) where $\varepsilon_n : S_n \rightarrow \{\pm 1\}$ is the signed map, that is, $\varepsilon_n(\sigma) = -1$ if and only if $\sigma \in S_n$ is an odd permutation. Note that the group G_n is denoted by \overline{S}_n in [[Plans 2009](#)].

The group G_n can be constructed explicitly as follows. Let

$$1 \rightarrow \{\pm 1\} \rightarrow C_4 = \{\pm\sqrt{-1}, \pm 1\} \xrightarrow{p_0} \{\pm 1\} \rightarrow 1$$

be the short exact sequence defined by $p_0(\sqrt{-1}) = -1$. The group G_n can be realized as the pullback of the diagram

$$\begin{array}{ccc} & S_n & \\ & \downarrow \varepsilon_n & \\ C_4 & \xrightarrow{p_0} & \{\pm 1\}. \end{array}$$

Explicitly, as a subgroup of $S_n \times C_4$,

$$\begin{aligned} G_n &= \{(\sigma, (\sqrt{-1})^i) \in S_n \times C_4 : \varepsilon_n(\sigma) = p_0((\sqrt{-1})^i)\} \\ &= (A_n \times \{\pm 1\}) \cup \{(\sigma, \pm\sqrt{-1}) \in S_n \times C_4 : \sigma \notin A_n\}. \end{aligned}$$

If k is a field with $\text{char } k \neq 2$, a faithful $2n$ -dimensional representation can be defined as follows. Let $X = (\bigoplus_{1 \leq i \leq n} k \cdot x_i) \oplus (\bigoplus_{1 \leq i \leq n} k \cdot y_i)$ and let G_n act on X by, for $1 \leq i \leq n$,

$$(5-1) \quad \begin{aligned} t &: x_i \mapsto -x_i, & y_i &\mapsto -y_i, \\ \tau &: x_i \mapsto x_{\tau(i)}, & y_i &\mapsto y_{\sigma^{-1}\tau\sigma(i)}, \\ \bar{\sigma} &: x_i \mapsto y_i \mapsto -x_i, \end{aligned}$$

where $t = (1, -1) \in G_n \subset S_n \times C_4$, $\tau \in A_n$ and τ is identified with $(\tau, 1) \in G_n$, $\sigma = (1, 2) \in S_n$ and $\bar{\sigma} = (\sigma, \sqrt{-1}) \in G_n$.

The next result was proved in [[Plans 2009](#), Theorem 14(b)] under the assumptions that $\text{char } k = 0$ and $\sqrt{-1} \in k$. Our proof is different from Plans' even in the situation when $\text{char } k = 0$.

Theorem 5.2. *Assume that k is a field that satisfies:*

- (i) *Either $\text{char } k = 0$ or $\text{char } k = p > 0$ with $p \nmid 2n$.*
- (ii) *$\sqrt{-1} \in k$.*

Then $k(G_n)$ is k -rational for $n \geq 3$.

Proof. The reader will find that (i) the assumption $\text{char } k \neq 2$ is used throughout the proof; (ii) the assumption $\text{char } k \nmid n$ is used in Step 2; (iii) the assumption $\sqrt{-1} \in k$ is used in Step 3.

• Step 1. Apply [Theorem 2.2](#). We find that $k(G_n)$ is rational over

$$k(x_i, y_i : 1 \leq i \leq n)^{G_n},$$

where G_n acts on the rational function field $k(x_i, y_i : 1 \leq i \leq n)$ by [\(5-1\)](#).

• Step 2. Define $u_0 = \sum_{1 \leq i \leq n} x_i$, $v_0 = \sum_{1 \leq i \leq n} y_i$ and $u_i = x_i/u_0$, $v_i = y_i/v_0$ for $1 \leq i \leq n$. Note that $k(x_i, y_i : 1 \leq i \leq n) = k(u_j, v_j : 0 \leq j \leq n)$ with the relations $\sum_{1 \leq i \leq n} u_i = \sum_{1 \leq i \leq n} v_i = 1$. The action of G_n is given by

$$\begin{aligned} t &: u_0 \mapsto -u_0, & v_0 &\mapsto -v_0, & u_i &\mapsto u_i, & v_i &\mapsto v_i, \\ \tau &: u_0 \mapsto u_0, & v_0 &\mapsto v_0, & u_i &\mapsto u_{\tau(i)}, & v_i &\mapsto v_{\sigma^{-1}\tau\sigma(i)}, \\ \bar{\sigma} &: u_0 \mapsto v_0 \mapsto -v_0, & u_i &\mapsto v_i \mapsto u_i, \end{aligned}$$

where $1 \leq i \leq n$ and $t, \tau, \bar{\sigma}$ are defined in [\(5-1\)](#).

Define $w_1 = u_0 v_0$, $w_2 = u_0/v_0$. Then

$$k(u_j, v_j : 0 \leq j \leq n)^{\langle t \rangle} = k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2).$$

Note that $\tau(w_i) = w_i$ for $1 \leq i \leq 2$, $\bar{\sigma}(w_1) = -w_1$, $\bar{\sigma}(w_2) = -1/w_2$. By [Theorem 2.1](#),

$$k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2)^{G_n/\langle t \rangle} = k(u_i, v_i : 1 \leq i \leq n)(w_2)^{G_n/\langle t \rangle}(w')$$

for some w' fixed by the action of $G_n/\langle t \rangle$. Moreover, we may identify $G_n/\langle t \rangle$ with S_n and identify $\bar{\sigma}$ (modulo $\langle t \rangle$) with σ .

Define $U_i = u_i - (1/n)$, $V_i = v_i - (1/n)$ for $1 \leq i \leq n$. We find that

$$\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$$

and the action of S_n on $k(U_i, V_i : 1 \leq i \leq n)$ becomes linear. We will consider $k(U_i, V_i : 1 \leq i \leq n)(w_2)^{S_n}$. The action of S_n is given by

$$(5-2) \quad \begin{aligned} \tau &: U_i \mapsto U_{\tau(i)}, & V_i &\mapsto V_{\sigma^{-1}\tau\sigma(i)}, & w_2 &\mapsto w_2, \\ \sigma &: U_i \mapsto V_i \mapsto U_i, & w_2 &\mapsto -1/w_2, \end{aligned}$$

where $1 \leq i \leq n$, $\tau \in A_n$, $\sigma = (1, 2)$ and $\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$.

• Step 3. Since $\sqrt{-1} \in k$, define $w = (\sqrt{-1} - w_2)/(\sqrt{-1} + w_2)$. We find that $\tau(w) = w$ for $\tau \in A_n$ and $\sigma(w) = -w$. Apply [Theorem 2.1](#). We find that

$$k(u_i, v_i : 1 \leq i \leq n)(w_2)^{S_n} = k(U_i, V_i : 1 \leq i \leq n)^{S_n}(w'')$$

for some w'' fixed by the action of S_n .

It remains to show that $k(U_i, V_i : 1 \leq i \leq n)^{S_n}$ is k -rational. The following proof of this fact is due to the referee.

Define $W_i^\pm = U_i \pm V_{\sigma(i)}$. It is easy to verify that for $\tau(W_i^\pm) = W_{\tau(i)}^\pm$ for $\tau \in A_n$; and that for $\sigma = (1, 2)$, $\sigma(W_i^+) = W_{\sigma(i)}^+$ and $\sigma(W_i^-) = -W_{\sigma(i)}^-$.

Define subspaces W and W' by $W = \sum_{1 \leq i \leq n} k \cdot W_i^+$ and $W' = \sum_{1 \leq i \leq n} k \cdot W_i^-$. Note that

$$\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i = W \oplus W'.$$

Moreover, W is the standard representation of S_n , that is, $W \simeq \sum_{1 \leq i \leq n} k \cdot s_i$ with $\sum_{1 \leq i \leq n} s_i = 0$ and $\lambda(s_i) = s_{\lambda(i)}$ for all $\lambda \in S_n$, for all $1 \leq i \leq n$. On the other hand, W' is the representation space of the tensor product of the standard representation and the linear character $\varepsilon_n : S_n \rightarrow \{\pm 1\}$.

• Step 4. Apply [Theorem 2.2](#) to $k(U_i, V_i : 1 \leq i \leq n)^{S_n}$. We find that

$$k(U_i, V_i : 1 \leq i \leq n)^{S_n} = k(W \oplus W')^{S_n} = k(W_i^+ : 1 \leq i \leq n-1)^{S_n}(t_1, \dots, t_{n-1}),$$

where each t_i is fixed by S_n . Obviously the field $k(W_i^+ : 1 \leq i \leq n-1)^{S_n}$ is k -rational, whence the result. \square

In the following theorem the assumption $\sqrt{-1} \in k$ from [Theorem 5.2](#) will be dropped. The first part of the following theorem was proved by [Plans \[2009, Theorem 14 \(b\)\]](#); there he assumed that $\text{char } k = 0$.

Theorem 5.3. (1) *If k is a field with $\text{char } k \neq 2$ or 3, then $k(G_3)$ is k -rational.*

(2) *If k is a field with $\text{char } k \neq 2$, then $k(G_4)$ is k -rational. Moreover, if $\text{char } k = 0$, then $k(G_5)$ is also k -rational.*

Proof. Case 1: $n = 3$. By Step 2 in the proof of [Theorem 5.2](#), it suffices to consider $k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3}$, where $\sum_{1 \leq i \leq 3} U_i = \sum_{1 \leq i \leq 3} V_i = 0$. Define $\tau = (1, 2, 3) \in S_3$. The actions are given by

$$\begin{aligned} \tau : U_1 &\mapsto U_2 \mapsto -U_1 - U_2, & V_2 &\mapsto V_1 \mapsto -V_1 - V_2, \\ \sigma : U_1 &\leftrightarrow V_1, & U_2 &\leftrightarrow V_2. \end{aligned}$$

Define $w_3 = U_1/V_2$, $w_4 = U_2/V_1$, $w_5 = V_1/V_2$. It follows that

$$k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3} = k(w_j : 2 \leq j \leq 5)(V_1)^{S_3} = k(w_j : 2 \leq j \leq 5)^{S_3}(w_0)$$

for some w_0 by [Theorem 2.1](#).

It remains to show that $k(w_j : 2 \leq j \leq 5)^{S_3}$ is k -rational. Note that

$$\begin{aligned} \tau : w_2 &\mapsto w_2, & w_3 &\mapsto w_4 \mapsto (w_3 + w_4 w_5)/(1 + w_5), \\ \sigma : w_2 &\mapsto -1/w_2, & w_3 &\mapsto 1/w_4, & w_4 &\mapsto 1/w_3, & w_5 &\mapsto w_3/(w_4 w_5). \end{aligned}$$

Define $w_6 = (w_3 + w_4w_5)/(1 + w_5)$. Note that $k(w_3, w_4, w_5) = k(w_3, w_4, w_6)$ and

$$\tau : w_3 \mapsto w_4 \mapsto w_6 \mapsto w_3 \quad \text{and} \quad \sigma : w_6 \mapsto 1/w_6.$$

Define $w_7 = (1-w_3)/(1+w_3)$, $w_8 = (1-w_4)/(1+w_4)$, $w_9 = (1-w_6)/(1+w_6)$. Then $k(w_3, w_4, w_6) = k(w_7, w_8, w_9)$ and

$$\begin{aligned} \tau : w_7 \mapsto w_8 \mapsto w_9 \mapsto w_7, \\ \sigma : w_7 \mapsto -w_8, \quad w_8 \mapsto -w_7, \quad w_9 \mapsto -w_9. \end{aligned}$$

By [Theorem 2.4](#) we find that $k(w_2, w_3, w_4, w_5)^{(\tau)} = k(w_2, X_1, X_2, X_3)$, where $X_1 = w_7 + w_8 + w_9$ and

$$\begin{aligned} X_2 &= \frac{w_7^2w_8 + w_8^2w_9 + w_9^2w_7 - 3w_7w_8w_9}{w_7^2 + w_8^2 + w_9^2 - w_7w_8 - w_7w_9 - w_8w_9}, \\ X_3 &= \frac{w_7w_8^2 + w_8w_9^2 + w_9w_7^2 - 3w_7w_8w_9}{w_7^2 + w_8^2 + w_9^2 - w_7w_8 - w_7w_9 - w_8w_9}. \end{aligned}$$

Moreover, the action of σ is given by

$$\sigma : w_2 \mapsto -1/w_2, \quad X_1 \mapsto -X_1, \quad X_2 \mapsto -X_3, \quad X_3 \mapsto -X_2.$$

Apply [Theorem 2.2](#). We find that $k(w_2, X_1, X_2, X_3)^{(\sigma)} = k(w_2)^{(\sigma)}(Y_1, Y_2, Y_3)$ for some Y_1, Y_2, Y_3 fixed by σ . Since $k(w_2)^{(\sigma)}$ is k -rational, it follows that $k(w_2, X_1, X_2, X_3)^{(\sigma)}$ is k -rational.

Case 2: $n = 4$. Once again we use Step 2 in the proof of [Theorem 5.2](#). It suffices to consider $k(U_i, V_i : 1 \leq i \leq 4)(w_2)^{S_4}$, where $\sum_{1 \leq i \leq 4} U_i = \sum_{1 \leq i \leq 4} V_i = 0$. Set $\lambda_1 = (1, 2)(3, 4)$, $\lambda_2 = (1, 3)(2, 4)$, $\rho = (1, 2, 3)$ and $\sigma = (1, 2)$ as before. Then S_4 is generated by $\lambda_1, \lambda_2, \rho$ and σ .

Define $t_1 = U_1 + U_2, t_2 = V_1 + V_2, t_3 = U_1 + U_3, t_4 = V_2 + V_3, t_5 = U_2 + U_3$ and $t_6 = V_1 + V_3$. The action of S_4 is given by

$$\begin{aligned} \lambda_1 : t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_3 \mapsto -t_3, \quad t_4 \mapsto -t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6, \\ \lambda_2 : t_1 \mapsto -t_1, \quad t_2 \mapsto -t_2, \quad t_3 \mapsto t_3, \quad t_4 \mapsto t_4, \quad t_5 \mapsto -t_5, \quad t_6 \mapsto -t_6, \\ \rho : t_1 \mapsto t_5 \mapsto t_3 \mapsto t_1, \quad t_2 \mapsto t_6 \mapsto t_4 \mapsto t_2, \\ \sigma : t_1 \leftrightarrow t_2, \quad t_3 \leftrightarrow t_6, \quad t_4 \leftrightarrow t_5. \end{aligned}$$

It follows that $k(t_i : 1 \leq i \leq 6)(w_2)^{\langle \lambda_1, \lambda_2 \rangle} = k(T_i : 1 \leq i \leq 6)(w_2)$, where $T_1 = t_1/t_2, T_2 = t_3/t_4, T_3 = t_5/t_6, T_4 = t_2t_6/t_4, T_5 = t_4t_6/t_2, T_6 = t_2t_4/t_6$.

Moreover, the actions of ρ and σ are given by

$$\begin{aligned}\rho : T_1 &\mapsto T_3 \mapsto T_2 \mapsto T_1, & T_4 &\mapsto T_5 \mapsto T_6 \mapsto T_4, \\ \sigma : T_1 &\mapsto 1/T_1, & T_2 &\mapsto 1/T_3, & T_3 &\mapsto 1/T_2, \\ & & T_4 &\mapsto (T_1 T_2 / T_3) T_6, & T_5 &\mapsto (T_2 T_3 / T_1) T_5, & T_6 &\mapsto (T_1 T_3 / T_2) T_4.\end{aligned}$$

By [Theorem 2.2](#), it suffices to show that $k(T_i : 1 \leq i \leq 3)(w_2)^{\langle \rho, \sigma \rangle}$ is k -rational. Define $w_3 = (1 - T_1)/(1 + T_1)$, $w_4 = (1 - T_2)/(1 + T_2)$, $w_5 = (1 - T_3)/(1 + T_3)$. Then we find

$$\begin{aligned}\rho : w_2 &\mapsto w_2, & w_3 &\mapsto w_5 \mapsto w_4 \mapsto w_3, \\ \sigma : w_2 &\mapsto -1/w_2, & w_3 &\mapsto -w_3, & w_4 &\mapsto -w_5, & w_5 &\mapsto -w_4.\end{aligned}$$

Use [Theorem 2.4](#) to find that $k(T_i : 1 \leq i \leq 3)(w_2)^{\langle \rho \rangle}$. The remaining part of the proof is very similar to the last part of Case 1. The details are omitted.

Case 3: $n = 5$. By [[Plans 2009](#), Theorem 11], $k(G_5)$ is rational over $k(G_4)$. Since $k(G_4)$ is k -rational by Case 2, we are done. \square

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MING-CHANG KANG
 DEPARTMENT OF MATHEMATICS AND
 TAIDA INSTITUTE OF MATHEMATICAL SCIENCES
 NATIONAL TAIWAN UNIVERSITY
 TAIPEI 106
 TAIWAN
kang@math.ntu.edu.tw

JIAN ZHOU
 SCHOOL OF MATHEMATICAL SCIENCES
 PEKING UNIVERSITY
 BEIJING, 100871
 CHINA
zhjn@math.pku.edu.cn

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pacific@math.ucla.edu

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University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

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Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

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Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jonr@math.ucla.edu

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