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CLASSIFICATION OF SINGULAR $\mathbb{Q}$-HOMOLOGY PLANES II: $\mathbb{C}^1$- AND $\mathbb{C}^*$-RULINGS

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A $\mathbb{Q}$-homology plane is a normal complex algebraic surface having trivial rational homology. We classify singular $\mathbb{Q}$-homology planes that are $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled. We analyze their completions, the number of different rulings they have, and the number of affine lines on them; and we give constructions. Together with previously known results, this completes the classification of $\mathbb{Q}$-homology planes with smooth locus of nongeneral type. We show also that the dimension of a family of homeomorphic but nonisomorphic singular $\mathbb{Q}$-homology planes having the same weighted boundary, singularities and Kodaira dimension can be arbitrarily big.

We work with complex algebraic varieties.

1. Main results

A $\mathbb{Q}$-homology plane is a normal surface whose rational cohomology is the same as that of $\mathbb{C}^2$. This paper is the last piece of the classification of $\mathbb{Q}$-homology planes having smooth locus of nongeneral type. The classification is built on the work of many authors; for a summary of what is known about smooth and singular $\mathbb{Q}$-homology planes, see [Miyanishi 2001, §3.4] and [Palka 2011b]. In [Palka 2008], we classified singular $\mathbb{Q}$-homology planes with nonquotient singularities, showing in particular that they are quotients of affine cones over projective curves by actions of finite groups that respect the set of lines through the vertex. In [Palka 2011a], we classified singular $\mathbb{Q}$-homology planes whose smooth locus is of nongeneral type and admits no $\mathbb{C}^1$- or $\mathbb{C}^*$-ruling (*exceptional planes*). Here we classify singular $\mathbb{Q}$-homology planes that admit a $\mathbb{C}^1$- or a $\mathbb{C}^*$-ruling. We analyze completions and boundaries rather than the open surfaces themselves. To deal with nonuniqueness of these, we use the notions of a *balanced* and a *strongly balanced* weighted boundary and completion of an open surface (see Definitions 2.7 and 2.10).

We classify $\mathbb{C}^1$- and $\mathbb{C}^*$-ruled $\mathbb{Q}$-homology planes by giving necessary and sufficient conditions for a $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled open surface to be a $\mathbb{Q}$-homology plane

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(see Lemmas 2.12, 3.2 and 4.4 and the remarks before the latter) and then giving general constructions (see Construction 3.3 and Section 4D). We compute the Kodaira dimension of a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane and of its smooth locus (Theorem 4.9) in terms of properties of singular fibers, and we list the planes with smooth locus of Kodaira dimension zero (Section 4C). As a corollary of the classification, we obtain the following result.

Theorem 1.1. *Let S' be a singular $\mathbb{Q}$-homology plane, and let S_0 be its smooth locus. Assume that S' is not affine-ruled and that $\bar{\kappa}(S_0) \neq 2$.*

(1) *Either S' has a unique balanced completion up to isomorphism, or it admits an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and more than one singular fiber. In the latter case, S' has exactly two strongly balanced completions.*

(2) *If S' has more than one singular point, then it has exactly two singular points, both of Dynkin type A_1, and there is a twisted $\mathbb{C}^*$-ruling of S' such that both singular points are contained in a unique fiber isomorphic to $\mathbb{C}^1$.*

(3) *If S' contains a quotient noncyclic singularity, then either $S' \cong \mathbb{C}^2/G$ for a small finite noncyclic subgroup of* $\mathrm{GL}(2, \mathbb{C})$, *or S' has a twisted $\mathbb{C}^*$-ruling. In the latter case, the unique fiber isomorphic to $\mathbb{C}^1$ is of type* (*A*)(*iv*) (*see Theorem 4.9*) *and contains a singular point of Dynkin type D_k for some $k \geq 4$.*

We now comment on other corollaries of the classification. First, the case can occur when S' has exactly one singular point and it is a cyclic singularity. Second, we show that if S' is affine-ruled, then its strongly balanced weighted boundary is unique unless it is a chain, but that even if it is unique, there still may be infinitely many strongly balanced completions (see Example 3.6). Third, the singularities of affine-ruled S' are necessarily cyclic, but there may be arbitrarily many of them (see [Miyanishi and Sugie 1991] or Section 3). Regarding the remaining case $\bar{\kappa}(S_0) = 2$, which we do not analyze here, let us mention that it follows from the logarithmic Bogomolov–Miyaoka–Yau inequality (see [Palka 2008], for example) that S' has only one singular point and it is of quotient type.

It is known that smooth $\mathbb{Q}$-homology planes can have moduli [Flenner and Zaĭdenberg 1994]. The same is true for singular ones. We prove the following result.

Theorem 1.2. *There exist arbitrarily high-dimensional families of nonisomorphic singular $\mathbb{Q}$-homology planes having smooth locus of negative Kodaira dimension and having the same singularities, same homeomorphism type, and same weighted strongly balanced boundary.*

An important property of any $\mathbb{Q}$-homology plane with smooth locus of general type is that it does not contain topologically contractible curves. In fact, the number of contractible curves on a $\mathbb{Q}$-homology plane is known except in the case when the surface is singular and the smooth locus has Kodaira dimension zero (see Section 6).

In Theorem 6.1, we compute the number of different $\mathbb{C}^*$-rulings a $\mathbb{Q}$-homology plane can have. The computation of the number of contractible curves follows from it.

Theorem 1.3. *If a singular $\mathbb{Q}$-homology plane has smooth locus of Kodaira dimension zero, then it contains one or two irreducible topologically contractible curves if the smooth locus admits a $\mathbb{C}^*$-ruling, and no such curves otherwise.*

The notion of a balanced weighted boundary of an open surface (see Definition 2.10) is a more flexible version of the notion of a *standard graph* from [Flenner et al. 2007], which has its origin in [Daigle 2008]. It follows from above that every $\mathbb{Q}$-homology plane admits up to isomorphism one or two strongly balanced boundaries, but this is not so for the standard ones. The set of such boundaries is a useful invariant of the surface.

Integral homology groups and necessary conditions for singular fibers of $\mathbb{C}^1$- and $\mathbb{C}^*$-ruled $\mathbb{Q}$-homology planes have already been analyzed in [Miyanishi and Sugie 1991]. For $\mathbb{C}^*$-rulings, however, these conditions are not sufficient (see Examples 4.2 and 4.3), and a more detailed analysis is necessary. Also, some formulas for the Kodaira dimension in terms of singular fibers from [Miyanishi and Sugie 1991] require nontrivial corrections (see Section 4B).

2. Preliminaries

We follow the notational conventions and terminology of [Miyanishi 2001], [Fujita 1982] and [Palka 2008]. We recall some of them for the convenience of the reader.

2A. ***Divisors and normal pairs.*** Let $T=\sum t_iT_i$ be an snc-divisor on a smooth complete surface with distinct irreducible components T_i. Then $\underline{T}=\sum T_i$, where the sum runs over i with $t_i\neq 0$, is the reduced divisor with the same support as T, and $\beta_T(T_i)=\underline{T}\cdot(\underline{T}-T_i)$ is the *branching number* of T_i. A *tip* has $\beta_T(T_i)\leq 1$. By $Q(T)$ we denote the intersection matrix of T; we put $d(0)=1$ and $d(T)=\det(-Q(T))$ for $T\neq 0$. The symbol "$\equiv$" denotes numerical equivalence of divisors.

If T is reduced and its dual graph is linear, it is called a *chain*, and in writing it as a sum of irreducible components $T=T_1+\cdots+T_n$, we assume that $T_i\cdot T_{i+1}=1$ for $1\leq i\leq n-1$. We put $T^t=T_n+\cdots+T_1$. If T is a rational chain, then we write $T=[-T_1^2,\ldots,-T_n^2]$. A rational chain with all $T_i^2\leq -2$ is called *admissible*. A *fork* is a rational tree for which the branching component is unique and has $\beta=3$.

Let D be some reduced snc-divisor that is not an admissible chain. A rational chain with support contained in D, not containing branching components of D and containing one of its tips, is called a *twig* of D. For an admissible (ordered) chain, we put

$$e(T)=\frac{d(T-T_1)}{d(T)}\quad\text{and}\quad \tilde{e}(T)=e(T^t).$$

In general, $e(T)$ and $\tilde{e}(T)$ are defined as the sums of respective numbers computed for all maximal admissible twigs of T. Here we use the convention that the tip of the twig is the first component.

If X is a complete surface and D is a reduced snc-divisor contained in the smooth part of X, then we call (X, D) an *snc-pair* and we write $X - D$ for $X \setminus D$. The pair is *normal* (resp. *smooth*) if X is normal (resp. smooth). If X is a normal surface, then an embedding $\iota\colon X \to \overline{X}$, where $(\overline{X}, \overline{X} \setminus X)$ is a normal pair, is called a *normal completion* of X. If X is smooth, then $\overline{X}$ is smooth and $(\overline{X}, D, \iota)$ is called a *smooth completion* of X. A morphism of two completions $\iota_j : X \to \overline{X}_j$, with $j = 1, 2$, of a given surface X is a morphism $f\colon \overline{X}_1 \to \overline{X}_2$ such that $\iota_2 = f \circ \iota_1$.

Let $\pi : (X, D) \to (X', D')$ be a birational morphism of normal pairs. We put $\pi^{-1}D' = \underline{\pi^* D'}$; that is, $\pi^{-1}D'$ is the reduced total transform of D'. Assume $\pi^{-1}D' = D$. If π is a blow-up, then we call it *subdivisional* (resp. *sprouting*) for D' if its center belongs to two (resp. one) components of D'. In general, we say that π is *subdivisional* for D' (and for D) if for any component T of D' we have $\beta_{D'}(T) = \beta_D(\pi^{-1}T)$. The exceptional locus of a birational morphism between two surfaces $\eta : X \to X'$, denoted by $\mathrm{Exc}(\eta)$, is defined as the locus of points in X for which η is not a local isomorphism.

A b-curve is a smooth rational curve with self-intersection b. A divisor is snc-minimal if all of its (-1)-curves are branching. We write K_X for the canonical divisor on a complete surface X.

Definition 2.1. A birational morphism of surfaces $\pi : X \to X'$ is a *connected modification* if it is proper, $\pi(\mathrm{Exc}(\pi))$ is a smooth point on X', and $\mathrm{Exc}(\pi)$ contains a unique (-1)-curve. If π is a morphism of pairs $\pi : (X, D) \to (X', D')$ such that $\pi^{-1}(D') = D$ and $\pi(\mathrm{Exc}(\pi)) \in D'$, we call it a *connected modification over D'*.

A sequence of blow-downs (and its reversing sequence of blow-ups) whose composition is a connected modification is called a *connected sequence of blow-downs* (*blow-ups*).

2B. ***Rational rulings.*** A surjective morphism $p_0 : X_0 \to B_0$ of a normal surface onto a smooth curve is a *rational ruling* if general fibers are rational curves. By a *completion of* p_0, we mean a triple (X, D, p), where (X, D) is a normal completion of X_0 and $p\colon X \to B$ is an extension of p_0 to a $\mathbb{P}^1$-ruling, with B being a smooth completion of B_0. We say that p is a *minimal completion of* p_0 if p does not dominate any other completion of p_0. In this case we also say that D is *p-minimal*. It is easy to check that D is p-minimal if and only if all of its nonbranching (-1)-curves are horizontal. Let F be a fiber of p. An irreducible curve $G \subseteq X$ is an *n-section* of p if $G \cdot F = n$. A *section* is a 1-section. We call p_0 a $\mathbb{C}^{(n*)}$-ruling if $F \cdot D = n + 1$ for $n \geq 1$. In the case $n = 0$, we call p_0 a $\mathbb{C}^1$-ruling or an *affine ruling*; the arithmetic genus of F ($p_a(F) = \frac{1}{2}F \cdot (K_X + F) + 1$) vanishes and

$F^2 = 0$. Conversely, it is well-known that an effective divisor with these properties on a complete surface is a fiber of such a ruling [Barth et al. 2004, V.4.3]. If J is a component of F, then we denote by $\mu_F(J)$ the multiplicity of J; that is, $F = \mu_F(J)J + F'$, where F' is effective and $J \not\subseteq F'$. The structure of fibers of a $\mathbb{P}^1$-ruling is well known [Fujita 1982, §4].

Lemma 2.2. *Let F be a singular fiber of a $\mathbb{P}^1$-ruling of a smooth complete surface. Then F is a tree of rational curves and it contains a (-1)-curve. Each (-1)-curve of F meets at most two other components. If F contains a unique (-1)-curve C, then*:

(i) $\mu(C) > 1$. *There are exactly two components of F with multiplicity one*, *and they are tips of the fiber.*

(ii) *If $\mu(C) = 2$, then either $F = [2, 1, 2]$ or C is a tip of F*; *in the latter case either $\underline{F} - C = [2, 2, 2]$ or $\underline{F} - C$ is a (-2)-fork of type $(2, 2, n)$.*

(iii) *If $\underline{F}$ is not a chain*, *then the connected component of $\underline{F} - C$ not containing curves of multiplicity one is a chain* (*possibly empty*).

We define

$$\Sigma_{X-D} = \sum_{F \not\subseteq D} (\sigma(F) - 1),$$

where $\sigma(F)$ is the number of $(X - D)$-components of a fiber F [Fujita 1982, 4.16]. If p is a $\mathbb{P}^1$-ruling as above, then we call an irreducible curve G *vertical* (*for p*) if $p_*G = 0$; otherwise it is *horizontal*. A divisor is vertical (resp. horizontal) if all of its components are vertical (resp. horizontal). We decompose D as $D = D_h + (D - D_h)$, where D_h is horizontal and $D - D_h$ is vertical. The numbers h and ν are defined respectively as the number of irreducible components of D_h and as the number of fibers contained in D. We have [Fujita 1982, §4]

$$\Sigma_{X-D} = h + \nu + b_2(X) - b_2(D) - 2.$$

We call a connected component of $F \cap D$ a D-*rivet* (or *rivet* if this causes no confusion) if it meets D_h at more than one point or if it is a node of D_h.

Definition 2.3. Suppose (X, D, p) is a completion of a $\mathbb{C}^*$-ruling of a normal surface X. We say that the original ruling $p_0 = p_{|X-D}$ is *twisted* if D_h is a 2-section. If D_h consists of two sections, we say that p_0 is *untwisted.* Let F be a singular fiber of p that does not contain singular points of X. We say that F is *columnar* if $\underline{F}$ is a chain that can be written as

$$\underline{F} = A_n + \cdots + A_1 + C + B_1 + \cdots + B_m,$$

where C is a unique (-1)-curve and D_h meets F exactly in A_n and B_m. The chains $A = A_1 + \cdots + A_n$ and $B = B_1 + \cdots + B_m$ are called *adjoint chains.*

Remark. By expansion properties of determinants (see [Koras and Russell 2007, 2.1.1], for example) and the fact that $d(A)$ and $d(A - A_1)$ are coprime, we have $e(A)+e(B)=1$ and $d(A)=d(B)=\mu_F(C)$. In fact, we have also $\tilde{e}(B)+\tilde{e}(A)=1$ [Fujita 1982, 3.7].

2C. *Balanced completions.*

Definition 2.4. A pair (D, w) consisting of a complete curve D and a rationally valued function w defined on the set of irreducible components of D is called a *weighted curve*. If (X, D) is a normal pair, then (D, w) with w defined by $w(D_i) = D_i^2$ is a *weighted boundary* of $X - D$.

Definition 2.5. Let (X, D) be a normal pair.

(i) Let L be a 0-curve that is a nonbranching component of D, and let $c \in L$ be chosen so that if L intersects two other components of D, then c is one of the points of intersection. Make a blow-up of c and contract the proper transform of L. The resulting pair (X', D'), where D' is the reduced direct image of the total transform of D, is called an *elementary transform of* (X, D). The pair $\Phi = (\Phi^\circ, \Phi^\bullet)$ consisting of an assignment $\Phi^\circ\colon (X, D) \mapsto (X', D')$ together with the resulting rational mapping $\Phi^\bullet\colon X \dashrightarrow X'$ is called an *elementary transformation over* D. Φ is *inner* (*for* D) if $\beta_D(L) = 2$, and *outer* (*for* D) if $\beta_D(L) = 1$. The point $c \in L$ is the *center* of Φ.

(ii) For a sequence of (inner) elementary transformations

$$\Phi_i^\circ\colon (X_i, D_i) \mapsto (X_{i+1}, D_{i+1}),$$

with $i = 1, \dots, n-1$, we put $\Phi^\circ = (\Phi_1^\circ, \dots, \Phi_{n-1}^\circ)$, $\Phi^\bullet = (\Phi_1^\bullet, \dots, \Phi_{n-1}^\bullet)$ and we call $\Phi = (\Phi^\circ, \Phi^\bullet)$ *an* (*inner*) *flow in* D_1. We denote it by $\Phi\colon (X_1, D_1) \rightsquigarrow (X_n, D_n)$.

$\Phi^\bullet = (\Phi_1^\bullet, \dots, \Phi_{n-1}^\bullet)$ induces a rational mapping $X_1 \dashrightarrow X_n$, which we also denote by $\Phi^\bullet$. There exists the largest open subset of X_1 on which $\Phi_1^\bullet$ is a morphism; the complement of this subset is called the *support of* Φ. Clearly, $\operatorname{Supp}\Phi_1 \subseteq D_1$. If $\operatorname{Supp}\Phi = \varnothing$, then Φ is a *trivial flow*.

A weighted curve (D, w) determines the weighted dual graph of D. If (D, w) is a weighted boundary coming from a fixed normal pair (X, D), we omit the weight function w from the notation. For Φ as above, D_1 and D_n are isomorphic as curves. They have the same dual graphs, but usually different weights of components.

Example 2.6. Let $T = [0, 0, a_1, \dots, a_n]$. Each chain of type $[0, b, a_1, \dots, a_n]$, $[a_1, \dots, a_{k-1}, a_k - b, 0, b, a_{k+1}, \dots, a_n]$ or $[a_1, \dots, a_n, b, 0]$, where $1 \le k \le n$ and $b \in \mathbb{Z}$, can be obtained from T by a flow. This follows from the observation that an elementary transformation interchanges the chains $[w, x, 0, y-1, z]$ and $[w, x-1, 0, y, z]$. Looking at the dual graph, we see the weights can “flow” from one side of a 0-curve to another, and possibly vanish ($b = 0$ or $b = a_k$). If they do, then again the weights can flow through the new zero.

Definition 2.7. A rational chain $D = [a_1, \dots, a_n]$ is *balanced* if $a_1, \dots, a_n \in \{0, 2, 3, \dots\}$ or if $D = [1]$. A reduced snc-divisor whose dual graph contains no loops (snc-forest) is *balanced* if all rational chains contained in D that do not contain branching components of the divisor are balanced. A normal pair (X, D) is *balanced* if D is balanced.

Recall that if (X_i, D_i) for $i = 1, 2$ are normal pairs such that $X_1 - D_1 \cong X_2 - D_2$, then D_1 is a forest if and only if D_2 is a forest.

Proposition 2.8. *A normal surface that admits a normal completion with a forest as a boundary has a balanced completion. Two such completions differ by a flow.*

As we discovered after completing the proof, a more general version of this proposition was proved in a graph-theoretic context in [Flenner et al. 2007, Theorem 3.1 and Corollary 3.36]. We therefore leave our more direct arguments to be published elsewhere. In fact, some key observations were made earlier in [Daigle 2008, 4.23.1, 3.2, 5.2]. Let us restate some definitions from [Flenner et al. 2007] on the level of pairs.

Definition 2.9. Let (X, D) be a normal pair and assume D is an snc-forest.

(i) Connected components of the divisor that remains after subtracting all nonrational and all branching components of D are called the *segments of* D.

(ii) D is *standard* if for each of its connected components, either the component is equal to $[1]$ or all of its segments are of types $[0]$, $[0, 0, 0]$ or $[0^{2k}, a_1, \dots, a_n]$, with $k \in \{0, 1\}$ and $a_1, \dots, a_n \geq 2$.

(iii) Let $D_0 = [0, 0, a_1, \dots, a_n]$, with $a_i \geq 2$ for $i = 1, \dots, n$, be a segment of D. A *reversion of* D_0 is a nontrivial flow $\Phi : (X, D) \rightsquigarrow (X', D')$ that is supported in D_0, is inner for D_0, and satisfies $D' - (\Phi^\bullet)_*(D - D_0) = [a_1, a_2, \dots, a_n, 0, 0]$.

The condition that Φ be nontrivial is introduced for the following reason: we want the reversion to transform the two zeros to the other end of the chain, and the condition in necessary to force this in case D is symmetric, that is, when $[a_1, \dots, a_n]^t = [a_1, \dots, a_n]$. Standard chains are called *canonical* in [Daigle 2008]. The Hodge index theorem implies that if (X, D) is a smooth pair and D is a forest, then it cannot have segments of type $[0^{2k+1}]$ or $[0^{2k}, a_1, \dots, a_n]$ for $k > 1$, and can have at most one such segment for $k = 1$.

Clearly, not every balanced forest is standard, but by a flow one can easily make it so. It follows from Proposition 2.8 that if D and D' are two standard boundaries of the same surface and D is a chain, then either D and D' are isomorphic as weighted curves or D' is the reversion of D. Unfortunately, the notion of a standard boundary is not as restrictive as one may imagine, and the difference between two standard boundaries can be more than just a reversion of some segments. An additional ambiguity is related to the existence of segments of type $[0^{2k+1}]$. Specifically, if $[0^{2k+1}]$ is a segment of D, then one can change by a flow the self-intersections of the components of D intersecting the segment. For example, consider a surface whose standard boundary is a rational fork with a dual graph

$$\begin{tikzcd} -2 \arrow[r, dash] & b \arrow[r, dash] \arrow[d, dash] & -2 \\ & 0 & \end{tikzcd}$$

for some $b \in \mathbb{Z}$. Then for any $b \in \mathbb{Z}$, there is a completion of this surface for which the boundary is standard and has the dual graph as above.[1] We therefore introduce the following more restrictive conditions.

Definition 2.10. A balanced snc-forest D is *strongly balanced* if it is standard and either D contains no segments of type $[0]$ or $[0, 0, 0]$, or for at least one such segment there is a component $B \subseteq D$ intersecting it such that $B^2 = 0$. A normal pair (X, D) for which D is a forest is *strongly balanced* if D is strongly balanced.

2D. *Basic properties of $\mathbb{Q}$-homology planes.* We assume that S' is a *singular $\mathbb{Q}$-homology plane*, that is, a normal nonsmooth complex algebraic surface with $H^*(S', \mathbb{Q}) \cong \mathbb{Q}$. Let $\epsilon\colon S \to S'$ be a resolution such that the inverse image of the singular locus is an snc-divisor, and let $(\bar{S}, D)$ be a smooth completion of S. Denote the singular points of S' by $p_1, \ldots, p_q$ and the smooth locus by S_0. We put $\widehat{E}_i = \epsilon^{-1}(p_i)$ and assume that $\widehat{E} = \widehat{E}_1 + \widehat{E}_2 + \cdots + \widehat{E}_q$ is snc-minimal. Recall that S' is called *logarithmic* if and only if every singular point of S' is locally analytically isomorphic to $\mathbb{C}^2/G$ for some finite subgroup $G < \mathrm{GL}(2, \mathbb{C})$ (a *quotient singularity*). In [Palka 2008], we classified nonlogarithmic $\mathbb{Q}$-homology planes. In particular, it is known that they do not admit $\mathbb{C}^1$- or $\mathbb{C}^*$-rulings. Therefore, from now on we assume that S' is logarithmic. It follows that each $\widehat{E}_i$ is either an admissible chain or an admissible fork (that is, an snc-minimal fork with negative definite intersection matrix). By [Gurjar et al. 1997], S' is rational. By the argument in [Fujita 1982, 2.4], it is affine.

Proposition 2.11. *Let the notation be as above.*

(i) *D is a rational tree with $d(D) = -d(\widehat{E}) \cdot |H_1(S', \mathbb{Z})|^2$.*

(ii) *The embedding $D \cup \widehat{E} \to \bar{S}$ induces an isomorphism on $H_2(-, \mathbb{Q})$.*

[1]This observation was missed in [Flenner et al. 2007], whose Corollary 3.33 is false. See [Flenner et al. 2011] for corrections. In [Daigle 2008, Solution to problem 5] this ambiguity is implicitly taken into account without restricting to balanced divisors.

(iii) $\pi_1(S') \cong \pi_1(S)$ *and* $H_k(S', \mathbb{Z}) = 0$ *for* $k > 1$.

(iv) $b_i(S_0) = 0$ *for* $i = 1, 2, 4$ *and* $b_3(S_0) = q$.

(v) $\Sigma_{S_0} = h + \nu - 2$ *and* $\nu \leq 1$.

Proof. See [Palka 2008, 3.1, 3.2] and [Miyanishi and Sugie 1991, 2.2]. □

Lemma 2.12. *Let* $(\bar{S}, T)$ *be a smooth pair and let* $p\colon \bar{S} \to B$ *be a* $\mathbb{P}^1$*-ruling. Assume that*

(i) *there exists a unique connected component* D *of* T *that is not vertical*,

(ii) D *is a rational tree*,

(iii) $\Sigma_{\bar{S}-T} = h + \nu - 2$, *and*

(iv) $d(D) \neq 0$.

Then the surface S' *defined as the image of* $\bar{S} - D$ *after contraction of connected components of* $T - D$ *to points is a rational* $\mathbb{Q}$*-homology plane*, *and* p *induces a rational ruling of* S'. *Conversely*, *if* $p'\colon S' \to B$ *is a rational ruling of a rational* $\mathbb{Q}$*-homology plane* S', *then any completion* $(\bar{S}, T, p)$ *of the restriction of* p' *to the smooth locus of* S' *has the above properties.*

Proof. Since the base of p has some component of D as a branched cover, it is rational, and hence $\bar{S}$ is rational. We may assume that T is p-minimal. Put $\widehat{E} = T - D$. Since $\widehat{E}$ is vertical and since $\widehat{E} \cap D = \varnothing$, $Q(\widehat{E})$ is negative definite and $b_1(\widehat{E}) = 0$. Fujita's equation

$$\Sigma_{\bar{S}-T} = h + \nu - 2 + b_2(\bar{S}) - b_2(D + \widehat{E})$$

gives $b_2(\bar{S}) = b_2(T)$, so by (iv), the inclusion $T \to \bar{S}$ induces an isomorphism on $H_2(-, \mathbb{Q})$. By [Palka 2008, 2.6], S' is normal and affine, and in particular $b_4(S') = b_3(S') = 0$. Since $b_1(D) = 0$, the exact sequence of the pair $(\bar{S}, D)$ together with the Lefschetz duality give

$$b_2(S) = b_2(\bar{S}, D) = b_2(\bar{S}) - b_2(D) = b_2(\widehat{E}).$$

Since $b_1(\widehat{E}) = 0$, we get from the exact sequence of the pair $(S, \widehat{E})$ that $b_2(S') = b_2(S, \widehat{E}) = b_2(S) - b_2(\widehat{E}) = 0$. Now

$$\chi(S') = \chi(\bar{S}) - \chi(D \cup \widehat{E}) + b_0(\widehat{E}) = b_0(D) = 1,$$

so we obtain $b_1(S') = b_2(S') = 0$, and hence S' is $\mathbb{Q}$-acyclic.

Conversely, if p' is as above, then let $\widehat{E}$ be an exceptional divisor of a resolution of singularities of S', and let $D = T - \widehat{E}$. Since $\widehat{E}$ is vertical for the $\mathbb{P}^1$-ruling p, we have $b_1(\widehat{E}) = 0$. Then the necessity of the above conditions follows from [ibid., 3.1 and 3.2]. □

3. Smooth locus of negative Kodaira dimension

Here we assume that the smooth locus S_0 of the logarithmic $\mathbb{Q}$-homology plane S' has negative Kodaira dimension, implying that the Kodaira dimension of S' is also negative. This case was analyzed and a structure theorem given in [Miyanishi and Sugie 1991, 2.5–2.8]. We recover these results in Lemma 3.2 and Proposition 3.1, but we concentrate on analyzing possible completions and boundaries instead of S' itself. This gives more information, allowing us to give a construction and to answer the question of uniqueness of an affine ruling of S_0 (if it exists). The information about completions is also used in the analysis of an example where moduli occur.

Proposition 3.1. *If a singular $\mathbb{Q}$-homology plane has smooth locus of negative Kodaira dimension, then it is affine-ruled or isomorphic to $\mathbb{C}^2/G$ for some small finite, noncyclic subgroup $G < \mathrm{GL}(2, \mathbb{C})$. The surfaces $\mathbb{C}^2/G$ and $\mathbb{C}^2/G'$ are isomorphic if and only if G and G' are conjugate in* GL(2, $\mathbb{C}$). *The minimal normal completion of $\mathbb{C}^2/G$ is unique and the boundary is a nonadmissible rational fork with admissible twigs.*

Proof. For the first part of the statement, we follow the arguments of [Koras and Russell 2007, §3]. Assume that S' is not affine-ruled. Then S_0 is not affine-ruled. Since S' is affine, $D + \widehat{E}$ is not negative definite, so by [Miyanishi 2001, 2.5.1], S_0 contains a platonically $\mathbb{C}^*$-fibered open subset U, which is its almost minimal model. Also, $\chi(U) \le \chi(S_0)$ (see [Palka 2011a, 2.8]). The algorithm of construction of an almost minimal model [Miyanishi 2001, 2.3.8, 2.3.11] implies that $S_0 - U$ is a disjoint sum of s curves isomorphic to $\mathbb{C}$ and s' curves isomorphic to $\mathbb{C}^*$, for some $s, s' \in \mathbb{N}$. It follows that

$$0 = \chi(U) = \chi(S_0) - s = \chi(S') - q - s = 1 - q - s,$$

so $s = 0,\ q = 1$, and $s' \le 1$. If $s' \neq 0$, then the boundary divisor of U is connected, and hence U and S_0 are affine-ruled. Thus $s' = 0$ and $S_0 = U$, and by [Miyanishi and Tsunoda 1984], $S' \cong \mathbb{C}^2/G$, where G is a small finite noncyclic subgroup of $\mathrm{GL}(2, \mathbb{C})$.

Suppose G and G' are two subgroups of $\mathrm{GL}(2, \mathbb{C})$ such that $\mathbb{C}^2/G \cong \mathbb{C}^2/G'$. Then $\widehat{\mathcal{O}}_{\mathbb{C}^2/G,(0)} \cong \widehat{\mathcal{O}}_{\mathbb{C}^2/G',(0)}$, so if G and G' are small then they are conjugate, by [Prill 1967, Theorem 2]. The $\mathbb{C}^*$-ruling of S_0 does not extend to a ruling of S', so by [Palka 2008, 4.5], its boundary is a rational fork with admissible maximal twigs and its minimal normal completion is unique up to isomorphism. (For the description of the boundary, one could also use a more general result [Miyanishi 2001, 2.5.2.14].) $\square$

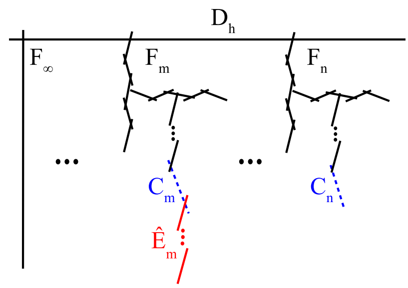

Figure 1. Affine-ruled S'.

3A. *Affine-ruled planes.* By Proposition 3.1, we may assume that S' is affine-ruled. This gives an affine ruling of S_0. We assume that $(\bar{S}, D+\widehat{E}, p)$ is a minimal completion of the latter. This weakens our initial snc-minimality assumption on D; that is, D is now p-minimal, but the unique section contained in D may be a nonbranching (-1)-curve. The base of p is rational because it is isomorphic to a section contained in $D+\widehat{E}$.

Lemma 3.2. *If S' is affine-ruled, then there exists exactly one fiber of p contained in D* (see Figure 1). *Each other singular fiber has a unique* (-1)*-curve, which is an S_0-component. The singularities of S' are cyclic.*

Proof. We have $\Sigma_{S_0} = \nu - 1$ and $\nu \le 1$ by Proposition 2.11, so $\Sigma_{S_0} = 0$ and there is exactly one fiber F_∞ contained in D. The fiber is smooth by the p-minimality of D. Each singular fiber F of p contains exactly one (-1)-curve. Indeed, if $D_0 \subseteq D$ is a vertical (-1)-curve, then by the p-minimality of D, it meets D_h and two D-components, so $\mu(D_0) > 1$. This is impossible because $D_h \cdot F = 1$. The (-1)-curve, say C, has $\mu(C) > 1$ and is the unique S_0-component of F. There are exactly two components of multiplicity one in F; they are tips of F and D_h intersects one of them. Thus the connected component of $\underline{F} - C$ not contained in D is a chain, so S' has only cyclic singularities. □

Remark. In Lemma 3.2, it was assumed (as in the whole paper — see Section 2D) that S' is logarithmic, but there is in fact no need for this. In any case $\widehat{E}$ is vertical, so it is a rational forest. Then D is a rational tree, and $\bar{S}$ and the base of p are rational by [Palka 2008, 3.4(i)]. The rest of the argument goes through.

Construction 3.3. Let $\mathbb{F}_1 = \mathbb{P}(\mathbb{O}_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1}(-1))$ be the first Hirzebruch surface with a (unique) projection $\tilde{p}\colon \mathbb{F}_1 \to \mathbb{P}^1$. Denote the section coming from the inclusion of the first summand by D'_h; then $D'^2_h = -1$. Choose $n+1$ distinct points $x_\infty, x_1, \ldots, x_n \in D'_h$, and let F_∞ be the fiber containing x_∞. For each $i = 1, \ldots, n$ starting from a blow-up of x_i, create a fiber F_i over $\tilde{p}(x_i)$ containing a unique (-1)-curve C_i. Let D_i be the connected component of $\underline{F_i} - C_i$ intersecting D_h, the proper transform of D'_h. By renumbering, we may assume there is $m \leq n$ such that C_i is a tip of F_i if and only if $i > m$. Assume also that $m \geq 1$ (for $m = 0$ we would get a smooth surface). For $i \leq m$, put $\widehat{E}_i = \underline{F_i} - D_i - C_i$. Clearly, each $\widehat{E}_i$ is a chain. Let $\bar{S}$ be the resulting surface and let $p\colon \bar{S} \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-ruling. Put $D = F_\infty + D_h + \sum_{i=1}^n D_i$, $S = \bar{S} - D$ and $\widehat{E} = \sum_{i=1}^m \widehat{E}_i$. We define $\epsilon\colon S \to S'$ as the morphism contracting $\widehat{E}_i$'s.

Remark 3.4. Let $p\colon \bar{S} \to \mathbb{P}^1$ be as in 3.3, and for a fiber F denote the greatest common divisor of multiplicities of all S-components of F by $\mu_S(F)$. By Proposition 2.11, we have $H_1(S', \mathbb{Z}) = H_1(S, \mathbb{Z})$. By [Fujita 1982, 4.19, 5.9],

$$H_1(S, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}_{\mu_S(F_i)},$$

so $H_1(S', \mathbb{Z})$ can be any finite abelian group. It is easy to see that $\mu_S(F_i) = \mu(C_i)/d(\widehat{E}_i)$, where $d(\widehat{E}_i) = d(0) = 1$ if $i > m$. In particular, S' is a $\mathbb{Z}$-homology plane if and only if $m = n$ and each F_i is a chain. In fact in the latter case $\pi_1(S)$ vanishes and so S' is contractible.

Theorem 3.5. *The surface S' in Construction 3.3 is an affine-ruled singular $\mathbb{Q}$-homology plane. Conversely, each singular $\mathbb{Q}$-homology plane admitting an affine ruling can be obtained by Construction 3.3. Its strongly balanced boundary is unique if it is branched and is unique up to reversion if it is a chain. The affine ruling of S' is unique if and only if its strongly balanced boundary is not a chain.*

Proof. By definition, $\widehat{E}_i$'s are admissible chains, so S' is normal and has only cyclic singularities. We have $d(D) = -\prod_i d(D_i)$ [Koras and Russell 1999, 2.1.1], so $d(D) \neq 0$, and hence S' is a singular $\mathbb{Q}$-homology plane by Lemma 2.12. The last part of the statement almost follows from Lemma 3.2. It remains to note that by a flow (see Example 2.6), we can freely change the self-intersection of the horizontal boundary component without changing the rest of D, so we can assume that the construction starts with a negative section on $\mathbb{F}_1$. (We could, for instance, start with D'_h equal to the negative section on $\mathbb{F}_n$, so that the resulting boundary would be strongly balanced; see Definition 2.10). The uniqueness of a strongly balanced boundary follows from Proposition 2.8.

We now consider the uniqueness of an affine ruling. Let (V_i, D_i, p_i) be two minimal completions of two affine rulings of S' (see Section 2B). By Lemma 3.2, both D_i contain a 0-curve $F_{\infty,i}$ as a tip. By flows with supports in $F_{\infty,i}$, we may assume both D_i are standard (see Definition 2.9).

Assume that D_1 is not a chain. Then D_1 and D_2 are isomorphic as weighted curves (see Proposition 2.8). Let T_i be the unique maximal twig of D_i containing a 0-curve. Then either $T_i = F_{\infty,i} = [0]$, or we can write $T_i = [0, 0, a_1, \ldots, a_n]$ with $[a_1, \ldots, a_n]$ admissible. Then there is a flow $\Phi : (V_1, D_1) \rightsquigarrow (V_2, D_2)$ by

Proposition 2.8. Because D_1 is branched, $\operatorname{Supp}\Phi^\bullet \subseteq T_1$. Also, it follows from Proposition 2.8 and Example 2.6 that $\operatorname{Supp}\Phi^\bullet \subseteq F_{\infty,i}$. For $i = 1, 2$, let f_i be some fiber of p_i other than $F_{\infty,i}$. Since $\Phi^\bullet(f_1)$ is disjoint from $F_{\infty,2}$, we get $\Phi^\bullet(f_1)\cdot f_2 = 0$, so p_1 and p_2 agree on S'.

Let (V_1, D_1) be a standard completion of S' with $D_1 = [0, 0, a_1, \dots, a_n]$. We may assume that $[a_1, \dots, a_n]$ is admissible and nonempty; if it is empty, then $S' \cong \mathbb{C}^2$ is smooth, and if it is nonadmissible, then by the Hodge index theorem we necessarily have $D_1 = [0, 0, 0]$, which disagrees with Proposition 2.11(i). Let (V_2, D_2) be another completion of S', with D_2 being a reversion of D_1. The 0-tip T_i of each D_i induces an affine ruling on S'. Let (V, D) be a minimal normal pair dominating both (V_i, D_i), such that both affine rulings extend to $\mathbb{P}^1$-rulings of V. We argue that these affine rulings are different by proving that $\sigma_1^*T_1\cdot\sigma_2^*T_2 \neq 0$, where $\sigma_i : (V, D) \to (V_i, D_i)$ are the dominations. Suppose $\sigma_1^*T_1\cdot\sigma_2^*T_2 = 0$. Let H be an ample divisor on V and let $(\lambda_1, \lambda_2) \neq (0, 0)$ be such that $\widetilde{T}\cdot H = 0$ for $\widetilde{T} = \lambda_1\sigma_1^*T_1 + \lambda_2\sigma_2^*T_2$. We have $(\sigma_i^*T_i)^2 = T_i^2 = 0$, so

$$\widetilde{T}^2 = 2\lambda_1\lambda_2\sigma_1^*T_1\cdot\sigma_2^*T_2 = 0,$$

and hence $\widetilde{T} \equiv 0$ by the Hodge index theorem. But D has a nondegenerate intersection matrix, because $d(D) = d(D_1) \neq 0$, so $\widetilde{T}$ is a zero divisor. Then $\sigma_1^*T_1 = [0]$, for otherwise $\sigma_1^*T_1$ and $\sigma_2^*T_2$ would contain a common (-1)-curve, which contradicts the minimality of (V, D). It follows that σ_1 (and σ_2) are identities. This contradicts the fact that the reversion for nonempty $[a_1, \dots, a_n]$ is a nontrivial transformation of the completion (even if $[a_1, \dots, a_n]^t = [a_1, \dots, a_n]$). □

The following example shows that even if the strongly balanced boundary is unique, there might be infinitely many strongly balanced completions.

Example 3.6. Let (V, D, ι) be an snc-minimal completion (ι is the embedding; see Section 2A) of an affine-ruled singular $\mathbb{Q}$-homology plane S' as above. Assume that D_h is branched and that $D_h^2 = -1$. The only change of D that can be made by a flow is a change of the weight of D_h. If we now make an elementary transformation $(V, D) \mapsto (V_x, D_x)$ with a center $x \in F_\infty \setminus D_h$, then D becomes strongly balanced (see Definition 2.10). Denote the resulting completion by (V_x, D_x, ι_x) and let $F_{\infty,x}$ be the new fiber at infinity. The isomorphism type of the weighted boundary D_x does not depend on x, but for different x the completions (triples) are clearly different. In general, even the isomorphism type of the pair (V_x, D_x) depends on x. To see this, let $(V_x, D_x) \cong (V_y, D_y)$. Because the isomorphism maps $F_{\infty,x}$ to $F_{\infty,y}$, we get an automorphism of (V, D) mapping x to y. Taking a minimal resolution $\bar{S} \to V$, contracting all singular fibers to smooth fibers without touching D_h, and contracting D_h, we see that for $x \neq y$, this automorphism descends to a nontrivial automorphism of $\mathbb{P}^2$ fixing points that are images of contracted S_0-components and of D_h. In general such an automorphism does not exist.

3B. ***Moduli.*** Repeating Construction 3.3 in a special case, we obtain arbitrarily high-dimensional families of nonisomorphic singular $\mathbb{Q}$-homology planes with negative Kodaira dimension of the smooth locus and the same homeomorphism type. Example 3.7 gives a proof of Theorem 1.2. For smooth $\mathbb{Q}$-homology planes, a similar example was considered in [Flenner and Zaĭdenberg 1994, 4.16].

Example 3.7. Put $m = 2$ and $n = N + 2$ for some $N > 0$, and let $\overline{S}$, D, $\widehat{E}$, etc. be created as in the construction above, so that $D_1 = [3]$, $D_2 = [2]$ and $D_i = [2, 2, 2]$ for $3 \le i \le n$. Then $\widehat{E}_1 = [2, 2]$ and $\widehat{E}_2 = [2]$ (see Figure 2).

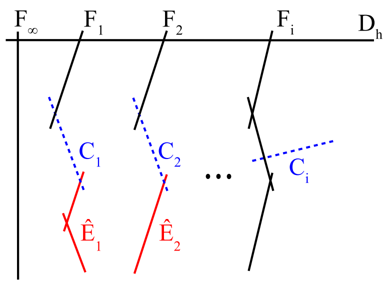

Figure 2. Singular fibers in Example 3.7.

Denoting the contraction of $\sum_{i=3}^{n} C_i$ by $\sigma : \overline{S} \to V$, we can factor the contraction $\overline{S} \to \mathbb{F}_1$ (which reverses the construction) as the composition $\overline{S} \xrightarrow{\sigma} V \xrightarrow{\sigma'} \mathbb{F}_1$. Put $y_i = \sigma(C_i)$ and $y = (y_3, \dots, y_n)$. While σ'^{-1} is determined uniquely by the choice of $(x_1, \dots, x_n)$, σ^{-1} and the resulting surface $\overline{S}$ (and hence S') can depend on the choice of y. Let us write $\overline{S}_y$ and S'_y to indicate this dependence. For $3 \le i \le n$, let D_i^0 be the open subset of the middle component of D_i remaining after subtracting two points belonging to other components of D_i. Put

$$U = D_4^0 \times \cdots \times D_n^0 \cong \mathbb{C}^{N-1}.$$

The family

$$\{S'_y\}_{y \in D_3^0 \times U} \to D_3^0 \times U$$

is N-dimensional. Since there is a compactly supported autodiffeomorphism of the pair $(\mathbb{C}^2, \mathbb{C}^* \times \{0\})$ mapping $(p, 0)$ to $(q, 0)$ for any $p, q \neq 0$, the choice of $y \in D_3^0 \times U$ is unique up to a diffeomorphism fixing irreducible components of $\sigma_*(D + \widehat{E} + C_1 + C_2)$. Thus all S'_y are homeomorphic.

Let $\pi : \mathfrak{X} \to U$ be the subfamily over $\{y_3^0\} \times U$. We show that the fibers of π are nonisomorphic. Suppose that $S'_y \cong S'_z$ for $y, z \in \{y_3^0\} \times U$. The isomorphism extends to snc-minimal resolutions. There is a flow $\Phi^\bullet : \overline{S}_y \dashrightarrow \overline{S}_z$ by Proposition 2.8, which is an isomorphism outside F_∞. Clearly, $\Phi^\bullet$ fixes $D_h \setminus \{x_\infty\}$, F_1 and F_2, and hence restricts to an identity on $D_h \setminus \{x_\infty\}$ and respects fibers. Since the C_i are unique (-1)-curves of the fibers, they are fixed by $\Phi^\bullet$. Therefore $\Phi^\bullet_{|\overline{S}-F_\infty-D_h}$ descends to an automorphism Φ_V of $V - F_\infty - D_h$ fixing the fibers, such that

$$\Phi_V(y_i) = z_i.$$

Also, Φ_V descends to an automorphism $\Phi_{\mathbb{F}_1}$ of $\mathbb{F}_1 - F_\infty - D'_h$ fixing fibers. If (x, y) are coordinates on $\mathbb{F}_1 - F_\infty - D'_h \cong \mathbb{C}^2$ such that x is a fiber coordinate, then

$$\Phi_{\mathbb{F}_1}(x, y) = (x, \lambda y + P(x))$$

for some $P \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}$. Introducing successive affine maps for the blow-ups, one can check that in some coordinates Φ_V acts on D_i^0 as $t \to \lambda^{\mu(C_i)} t$. Now the requirement $y_3 = y_3^0$ fixes $\lambda^2 = 1$, so because $\mu(C_i) = 2$ for each $3 \le i \le n$, we get that $y = z$.

Remark. By [Fujita 1982, 4.19 and 5.9], for S' as above, $\pi_1(S')$ is the N-fold free product of $\mathbb{Z}_2$. It follows from Remark 3.4 that given a weighted boundary, there exist only finitely many affine-ruled singular $\mathbb{Z}$-homology planes with this boundary. That is why in Example 3.7 we use branched fibers F_i for $3 \le i \le n$; so that the resulting surfaces are $\mathbb{Q}$-, but not $\mathbb{Z}$-homology planes.

4. $\mathbb{C}^*$-ruled $\mathbb{Q}$-homology planes

By [Palka 2008, 1.1(2) and 1.2] and Section 3A, to finish the classification of singular $\mathbb{Q}$-homology planes with smooth locus of nongeneral type, one needs to classify $\mathbb{Q}$-homology planes that are $\mathbb{C}^*$-ruled. Therefore, we assume here that S' is $\mathbb{C}^*$-ruled (and logarithmic; see Section 2D). The first homology group of S' and some necessary conditions for singular fibers of such rulings are analyzed in [Miyanishi and Sugie 1991, 2.9 and 2.10]. As before, we concentrate on completions rather than the affine part itself, because this gives more information and allows us to give a general method of construction. It also allows us to compute the number of different $\mathbb{C}^*$-rulings, and as a consequence the number of affine lines on S'.

4A. ***Properties of*** $\mathbb{C}^*$***-rulings.*** We can lift the $\mathbb{C}^*$-ruling of S' to a $\mathbb{C}^*$-ruling of the resolution and extend it to a $\mathbb{P}^1$-ruling $p\colon \overline{S} \to \mathbb{P}^1$ of a smooth completion. Assume that $D + \widehat{E}$ is p-minimal. By Proposition 2.11(v), $\Sigma_{S_0} = h + \nu - 2$ and $\nu \le 1$, so $(h, \nu, \Sigma_{S_0}) = (1, 1, 0)$, $(2, 1, 1)$ or $(2, 0, 0)$. The original $\mathbb{C}^*$-ruling of S' is twisted with base $\mathbb{C}^1$ in the first case, untwisted with base $\mathbb{C}^1$ in the second case, and untwisted with base $\mathbb{P}^1$ in the third case.

Lemma 4.1. *Denote by $F_1, \dots, F_n$ all the columnar fibers of $p : \overline{S} \to \mathbb{P}^1$ (see Definition 2.3). Let F_∞ be the fiber contained in D if $\nu = 1$. There is exactly one more singular fiber F_0; it contains $\widehat{E}$. Moreover*:

(i) *If* $(h, \nu) = (1, 1)$, *then* $F_\infty = [2, 1, 2]$, $\sigma(F_0) = 1$, *and* F_0 *and* F_∞ *contain branching points of* $p_{|D_h}$.

(ii) *If* $(h, \nu) = (2, 1)$, *then* F_∞ *is smooth and* $\sigma(F_0) = 2$.

(iii) *If* $(h, \nu) = (2, 0)$, *then* $\sigma(F_0) = 1$ *and* F_0 *contains a* D*-rivet.*

(iv) *If* $h = 2$, *then the components of* D_h *are disjoint.*

Proof. Let $(h, \nu) = (1, 1)$. Then $\Sigma_{S_0} = 0$, so by [Fujita 1982, 7.6], every singular fiber other than F_∞ is either columnar or contains a branching point of $p_{|D_h}$. Now D_h is rational and $p_{|D_h}$ has two branching points, one of them contained in F_∞, because D is a tree. Thus F_0 is unique. The p-minimality of D implies that $F_\infty = [2, 1, 2]$. Now let $h = 2$. We have $\Sigma_{S_0} = \nu \in \{0, 1\}$, and the p-minimality of D gives (ii), (iii) and the uniqueness of F_0. Suppose the components of D_h have a common point. D is a tree, so in this case $\nu = 0$, which gives $\sigma(F_0) = 1$. Because D is a simple normal crossing divisor, the common point belongs to the unique S_0-component of F_0, which therefore has multiplicity one. The connectedness of D implies that F_0 contains no D-components. But then F_0 has a unique (-1)-curve, which is impossible by Lemma 2.2. □

Lemma 4.1 is essentially [Miyanishi and Sugie 1991, 2.10]. While the conditions stated above are necessary, they are not sufficient. In the following examples the $\mathbb{C}^*$-ruling satisfies them, but the $\mathbb{C}^*$-ruled surface one obtains is not a $\mathbb{Q}$-homology plane.

Example 4.2. For $n \geq 0$, let $\mathbb{F}_n$ be the n-th Hirzebruch surface, and let D_0, D_∞ be sections with $D_0^2 = n$ and $D_\infty^2 = -n$. Let F_∞ be a fiber and put $D = D_0 + D_\infty + F_\infty$. Pick a point not belonging to D and make a connected sequence of blow-ups over it. Let C_0 be the unique (-1)-curve in the inverse image of the point, and let F_0 and C_1 be the reduced total and the proper transform of the fiber. Denote the resulting surface by $\overline{S}$, put $S = \overline{S} - D$ and $\widehat{E} = F_0 - C_0 - C_1$, and let $S \to S'$ be the morphism contracting $\widehat{E}$. In particular, $\widehat{E}$ might be any admissible chain, in which case S' has a unique cyclic singular point. S' is not a $\mathbb{Q}$-homology plane because $d(D) = 0$; see Lemma 2.12(iv).

Example 4.3. Take the pair $(\mathbb{F}_1, D_0 + D_\infty)$, where $\mathbb{F}_1$ is the first Hirzebruch surface and D_0 and D_∞ are sections with $D_0^2 = 1$ and $D_\infty^2 = -1$. Pick two points on D_0 and blow up over it to create two singular fibers $F_1 = [2, 1, 2]$, $F_2 = [2, 1, 2]$. Denote their (-1)-curves by C_1, C_2. These (-1)-curves separate two chains $T_0 = [2, 1, 2]$ and $T_\infty = [2, 1, 2]$, where the middle (-1)-curves are D_0 and D_∞, respectively. We have $d(T_0) = d(T_\infty) = 0$. Pick a point on some C_i, say C_1, that does not belong to $T_0 + T_\infty$, and make a connected sequence of blow-ups over it. Let C_0 be the unique (-1)-curve in the inverse image of the point, and let F_0 be the total reduced transform of the fiber. Denote the resulting complete surface by $\overline{S}$. If C_0 is not a tip of F_0, then denote the connected component of $F_0 - C_0$ not meeting $D_0 + D_\infty$ by $\widehat{E}$. Let D be the reduced divisor with support $T_0 \cup T_\infty \cup (F_0 - C_0 - \widehat{E})$. Put $S = \bar{S} - D$ and $\widehat{E} = F_0 - C_0 - C_1$, and let $S \to S'$ be the morphism contracting $\widehat{E}$ (which is necessarily an admissible chain). Again, S' is not a $\mathbb{Q}$-homology plane because $d(D) = 0$.

Theoretically, if X is a normal surface and $p' : X \to B$ is a $\mathbb{C}^*$-ruling, then by taking a completion of X and an extension of p' to a $\mathbb{P}^1$-ruling, with Lemma 2.12 we can recognize when X is a $\mathbb{Q}$-homology plane (B has to be rational). However, to give constructions we need to reformulate the condition $d(D) \neq 0$ in a way that is easier to verify by looking at the geometry of singular fibers. Recall that for a family of subsets $(A_i)_{i \in I}$ of a topological space Y, a subset $X \subseteq Y$ *separates* the subsets $(A_i)_{i \in I}$ (inside Y) if and only if each A_i is contained in a closure of some connected component of $Y \setminus X$ and none of these closures contains more than one A_i. Recall also that by convention, a twig of a fixed divisor is ordered so that its tip is the first component.

Lemma 4.4. *Let $(\bar{S}, T, p)$ be a triple satisfying conditions* (i)–(iii) *of Lemma 2.12. Assume also that T is p-minimal and that $f \cdot T = 2$ for a general fiber f of p. When $(h, \nu) = (2, 0)$, let D_0 be some horizontal component of D, let F_0 be a unique fiber containing a D-rivet, let B be a unique component of D separating D_0, $D_h - D_0$ and $\widehat{E}$ inside $D \cup F_0$, and let $\widetilde{D}_0$ be a connected component of $D - B$ containing D_0. Then $d(D) \neq 0$ if and only if the following conditions hold*:

(i) *The base of the fibration is $\mathbb{P}^1$ or $\mathbb{C}^1$ (that is, $\nu \leq 1$).*

(ii) *If $(h, \nu) = (2, 1)$, both $\bar{S} - T$-components of the fiber with $\sigma = 2$ intersect D.*

(iii) *If $(h, \nu) = (2, 0)$, then $d(\widetilde{D}_0) \neq 0$.*

The advantage of condition (iii) over $d(D) \neq 0$ is that $\widetilde{D}_0$ is simpler than D, containing at most one branching component.

Proof. Clearly, if $d(D) \neq 0$, then S' is a $\mathbb{Q}$-homology plane by Lemma 2.12, which implies (i) and (ii) (D meets each curve not contained in $D + \widehat{E}$ because S' is affine). Suppose now that (i) and (ii) are satisfied. We show that $d(D) \neq 0$ is equivalent to (iii) (which is an empty condition if $(h, \nu) \neq (2, 0)$). Note that $d(D) \neq 0$ is equivalent to $d(T) \neq 0$, because $T - D$ is negative definite.

Consider the case $h = 1$. We have $\Sigma_{\bar{S}-T} = \nu - 1$, and hence $\nu = 1$ and $\Sigma = 0$. The horizontal component D_h meets the unique fiber F_∞ contained in T in one point, because T is a forest. Let T_∞ be the component meeting D_h. We have $d(F_\infty) = 0$, so by [Koras and Russell 1999, 2.1.1(i)],

$$d(D) = d(F_\infty) d(D - F_\infty) - d(F_\infty - T_\infty) d(D - F_\infty - D_h),$$

and we obtain

$$d(D) = -d(F_\infty - T_\infty)d(D - F_\infty - D_h).$$

Since $F_\infty - T_\infty$ and $D - F_\infty - D_h$ are vertical and do not contain whole fibers, they are negative definite, and hence $d(D) < 0$.

We may now assume $h = 2$. Then $\Sigma = \nu \in \{0, 1\}$. Put $\widehat{E} = T - D$. When $\nu = 1$, let F_∞ be the unique fiber contained in D, and let F_0 be the unique singular fiber with $\sigma(F_0) = 2$. When $\nu = 0$, let F_0 be the unique fiber containing a D-rivet. All other singular fibers are columnar by [Fujita 1982, 7.6], so they contain no components of $\widehat{E}$. We need to prepare some tools to proceed. Recall that the Neron–Severi group of $\bar{S} - T$ is defined as the quotient of $\mathrm{NS}(\bar{S})$ by the subgroup generated by components of T. We put $\rho(\bar{S} - T) = \dim \mathrm{NS}(\bar{S} - T) \otimes \mathbb{Q}$.

Let (X, R) be a smooth pair with X rational. Suppose $R = R_1 + R_2$, where R_1 and R_2 meet in unique components $C_1 \subseteq R_1$, $C_2 \subseteq R_2$ respectively. If at least one of R_i is negative definite for $i = 1, 2$, then we call $R - C_1$ a *swap* of $R - C_2$ and vice versa. Similarly, $(X, R - C_i)$ are by definition swaps of each other, and so are $X - (R - C_i)$, for $i = 1, 2$. The basic property of this operation that we need is that

$$\rho(X - (R - C_1)) = \rho(X - (R - C_2)).$$

To see this, it is enough to show that C_1, C_2 do not belong to the subspace V of $\mathrm{NS}(X) \otimes \mathbb{Q}$ generated by components of $R_1 - C_1 + R_2 - C_2$. By symmetry, we can assume that R_2 is negative definite. Suppose that $C_1 \in V$ and write

$$C_1 \equiv U_1 + U_2,$$

where U_i is in the subspace generated by components of $R_i - C_i$. Then $0 = C_1 \cdot U_2 = U_1 \cdot U_2 + U_2^2 = U_2^2$, and hence $U_2 \equiv 0$ by the negative definiteness of R_2. Then $0 < C_1 \cdot C_2 = U_1 \cdot C_2 = 0$, a contradiction. Suppose $C_2 \in V$ and write $C_2 \equiv U_1 + U_2$ as above. Then $(C_2 - U_2)^2 = (C_2 - U_2) \cdot U_1 = 0$, so $C_2 \equiv U_2$ by the negative definiteness of R_2. Then $0 < C_1 \cdot C_2 = C_1 \cdot U_2 = 0$, a contradiction. Thus, swapping preserves ρ. Though the definition is of general use, we use only a special kind of swapping, when C_2 is a (-1)-curve and it is absorbed into the boundary (keeping the assumption that R_2 is negative definite); that is, we do the swap one way, changing $(X, R - C_2)$ to $(X, R - C_1)$.

Take $(\bar{S}, T)$ and interchangeably perform contractions of (-1)-curves in F_0 (and its images) that are nonbranching components of the boundary and swaps absorbing vertical (-1)-curves in F_0 (and its images) into the boundary. Denote the resulting smooth pair by (X, T'). By the properties of swaps and blow-ups, the rank of the Neron–Severi group of the open part and the difference between b_2 of the complete surface and the number of components in the boundary remain constant. Also, T' is a rational forest. Crucially, $d(T) = 0$ if and only if $d(T') = 0$. To see this, we may assume that (X, T') is simply a swap of $(\bar{S}, T)$ as above. Since the number of components of T equals $b_2(\bar{S})$, we know $d(T) \neq 0$ if and only if $\rho(\bar{S} - T) = 0$, which is equivalent to $\rho(X - T') = 0$ and then to $d(T') \neq 0$.

Consider the case $\Sigma = \nu = 0$. At some point, the process of swapping and contracting makes B into a 0-curve or a (-1)-curve. It is easy to see that the divisor $\widetilde{D}_0 + \widetilde{D}_\infty$ is not affected by the process, so we have $d(D) \neq 0$ if and only if $d(\widetilde{D}_0) \cdot d(\widetilde{D}_\infty) \neq 0$. All singular fibers of the induced $\mathbb{P}^1$-ruling at this stage are columnar, so they can be written as $R_{i,0} + C_i + R_{i,\infty}$, where $i = 1, \dots, n'$ enumerates these fibers, C_i^2 equals -1, and $R_{i,0}$ and $R_{i,\infty}$ are chains whose last components meet D_0 and D_∞, respectively. For $j = 0, \infty$, put $\tilde{e}_j = \tilde{e}(\widetilde{D}_j)$ (see Section 2A). Then $\tilde{e}_j = \sum_i \tilde{e}(R_{i,j})$. We have $d(\widetilde{D}_j) = (-D_j^2 - \tilde{e}_j) \cdot \prod_i d(R_{i,j})$. By the properties of columnar fibers,

$$d(\widetilde{D}_0) + d(\widetilde{D}_\infty) = -(D_0^2 + D_\infty^2 + n') \cdot \prod_i d(R_{i,0}).$$

When contracting singular fibers to smooth ones, $D_0 + D_\infty$ is touched n' times and its image consists of two disjoint sections on a Hirzebruch surface. It follows that $D_0^2 + D_\infty^2 + n' = 0$, and hence $d(\widetilde{D}_\infty) + d(\widetilde{D}_0) = 0$. Thus $d(D) \neq 0$ if and only if $d(\widetilde{D}_0) \neq 0$.

Consider the case $\Sigma = \nu = 1$. We show that T' has at most one horizontal component. Suppose that it has two. Then $\sigma(\widetilde{F}_0) = \sigma(F_0) = 2$, so $\widetilde{F}_0$ contains a (-1)-curve, say C_1. Because T' is p-minimal, $C_1 \not\subseteq T$. Because we assumed that every $\bar{S} - T$-component meets D, by the properties of swaps, every $X - T'$-component meets T'. By the definition of X, absorbing the (-1)-curve by a swap into the boundary is impossible. In particular, if $\widetilde{F}_0$ has no more (-1)-curves, then C_1 is not a tip of $\widetilde{F}_0$, so $\widetilde{F}_0$ is a chain. However, since $\sigma(\widetilde{F}_0) = 2$, a swap absorbing C_1 into the boundary is possible, which is a contradiction. Thus, $\widetilde{F}_0$ has two (-1)-curves, C_1 and C_2. One of them meets some horizontal component of T'; otherwise, either C_1 or C_2 is a tip or $\widetilde{F}_0 \cap T'$ has three connected components, and in either case a swap absorbing one of the C_i's into the boundary would be possible. A similar argument shows that the second (-1)-curve also meets a horizontal component of T'. Thus, $\widetilde{F}'_0$ is a chain with C_1 and C_2 as tips, and again a swap is possible, a contradiction. So T' has at most one horizontal component. But after the first swap where σ of the image of F_0 drops, the fiber has only one (-1)-curve, which therefore has multiplicity greater than one, so no more swaps of this kind are possible. Thus, T' has a unique horizontal component T'_h. Then

$$d(T') = d(F_\infty) d(T' - F_\infty) - d(T' - F_\infty - D_\infty) = -d(T' - F_\infty - D_\infty).$$

Now $T' - F_\infty - D_\infty$ is vertical and does not contain whole fibers, so it is negative definite and we obtain $d(T') = d(T' - F_\infty - D_\infty) \neq 0$. □

Remark. By Proposition 2.11, for any $\mathbb{Q}$-homology plane, we have $H_i(S', \mathbb{Z}) = 0$ for $i > 1$ and

$$|H_1(S', \mathbb{Z})|^2 = \frac{d(D)}{d(\widehat{E})},$$

and hence S' is a $\mathbb{Z}$-homology plane if and only if $d(D) = d(\widehat{E})$. For a $\mathbb{C}^*$-ruled S', more explicit computations are done in [Miyanishi and Sugie 1991], which we do not repeat here. For example, by [ibid., 2.17], if a $\mathbb{Z}$-homology plane with $\bar{\kappa}(S_0) \neq -\infty$ is $\mathbb{C}^*$-ruled, then $\bar{\kappa}(S_0) = 1$ and the ruling is untwisted with base $\mathbb{P}^1$. The conditions for S' having such a ruling to be contractible are given in [ibid., 2.11] (in particular $n = 2$).

4B. ***The Kodaira dimension.*** In [Miyanishi and Sugie 1991, 2.9–2.17] one can find formulas for the Kodaira dimension of the smooth locus $\bar{\kappa}(S_0)$ in terms of properties of singular fibers of the $\mathbb{C}^*$-ruling (there, $\bar{\kappa}(S')$ is by definition equal to $\bar{\kappa}(S_0)$). Unfortunately, their formulas 2.14(4), 2.15(2), and 2.16(2) are incorrect. The corrections require splitting into cases depending on additional properties of singular fibers. We also compute the Kodaira dimension of S'. We keep the notation for singular fibers as in Lemma 4.1. When $\nu = 0$, put $F_\infty = 0$. Let J be the reduced divisor with support equal to $D \cup F_0$. For $i = 1, \dots, n$, denote the (-1)-curve of the columnar fiber F_i by C_i and the multiplicity of C_i by μ_i. Put $J^+ = J + C_1 + \cdots + C_n$.

Lemma 4.5. *The divisor J^+ has simple normal crossings. Contract vertical (-1)-curves in J^+ and its images as long as the image is an snc-divisor. Let*

$$\zeta : (\bar{S}, J^+) \to (W, \zeta_* J^+)$$

be the composition of these contractions. Then the $\zeta_* F_i$ are smooth for $i = 1, \dots, n$; moreover:

(i) *If $h = 1$, then $\zeta_* F_0 = [2, 1, 2]$, $(\zeta_* D_h)^2 = 0$, and one can further contract $\zeta_* F_0$ and F_∞ to smooth fibers so that W maps to $\mathbb{F}_1$ and $\zeta_* D_h$ maps to a smooth 2-section of the $\mathbb{P}^1$-ruling of $\mathbb{F}_1$ disjoint from the negative section.*

(ii) *If $h = 2$, then $\zeta_* F_0$ is smooth, W is a Hirzebruch surface, and the components of $\zeta_* D_h$ are disjoint. Also, at least one of the components of D_h has negative self-intersection, and by changing ζ if necessary, one can assume that it is not affected by ζ.*

Proof. Suppose the crossings of J^+ at x are not simple normal. By Lemma 4.1, this only happens if $h = 2$. Also, x belongs to $D_h \cap F_0$ and is a branching point of $p_{|D_h}$, and two components of F_0 of multiplicity one meet at x. Because D has normal crossings, one of them is the unique S_0-component of F_0. By the p-minimality of D, it has to be a unique (-1)-curve of F_0 too, which is impossible by Lemma 2.2(i). Thus, J^+ is an snc-divisor. Because F_i for $i = 1, \dots, n$ are columnar, $\zeta_* F_i$ are smooth.

Suppose $h = 2$. Write $D_h = H + H'$. By Lemma 4.1, H and H' are disjoint. Since H and H' meet F_0 only in the components of multiplicity one, it follows from the definition of ζ that the images of H' and H intersect the same component of $\zeta_* F_0$. But this is possible only if $\zeta_* F_0$ is smooth. Since $\zeta_* J^+$ is snc, these images are disjoint. Say $H'^2 \leq H^2$. Choosing the contracted (-1)-curves correctly, we may assume that H' is not affected by ζ. Since $\zeta_* D_h$ consists of two disjoint sections on a Hirzebruch surface, we have $(\zeta_* D_h)^2 = 0$, so $D_h^2 \leq 0$. Suppose $H^2 = H'^2 = 0$. Then ζ does not affect D_h, so $n = 0$ and H and H' intersect the same component B of F_∞. If $\nu = 1$, then B is an S_0-component and the second S_0-component of F_0 does not intersect D, a contradiction with the affineness of S'. Thus $\nu = 0$ and Lemma 4.4 is not satisfied (in other words, $d(D) = 0$), a contradiction.

Suppose $h = 1$. By the definition of ζ, the image of D_h intersects the unique (-1)-curve of $\zeta_* F_0$. It follows that $\zeta_* F_0 = [2, 1, 2]$. Now after the contraction of F_0 and F_∞ to smooth fibers, the image of W is a Hirzebruch surface $\mathbb{F}_N$, where $N \geq 0$, and the image D'_h of D_h is a smooth 2-section. Write $D'_h \equiv \alpha f + 2H$, where H is a section with $H^2 = -N$ and f is a fiber of the induced $\mathbb{P}^1$-ruling of $\mathbb{F}_N$. We compute

$$p_a(\alpha f + 2H) = \alpha - N - 1,$$

so because D'_h is smooth, its arithmetic genus vanishes and $\alpha = N + 1$. Also, $D'_h \cdot H = \alpha - 2N$, and hence $D'_h \cdot H + N = 1$. Now if $N = 0$, then $\mathbb{F}_N = \mathbb{P}^1 \times \mathbb{P}^1$, and an elementary transformation with center equal to the point of tangency of D'_h and the image of F_∞ (which corresponds to a different choice of components to be contracted in F_∞) leads to $N = 1$ and $D'_h \cdot H = 0$. $\square$

Remark 4.6. Let (X, D) be a smooth pair, and let L be the exceptional divisor of a blow-up $\sigma : X' \to X$ of a point in D. Then

$$K_{X'} + \sigma^{-1} D = \sigma^*(K_X + D)$$

if σ is subdivisional for D, and

$$K_{X'} + \sigma^{-1} D = \sigma^*(K_X + D) + L$$

if σ is sprouting for D.

Decompose ζ into a sequence of blow-downs $\zeta = \sigma_k \circ \cdots \circ \sigma_1$, and let $m \leq k$ be the minimal number such that for $j > m$, the blow-up σ_j is subdivisional for $(\sigma_j \circ \cdots \circ \sigma_1)_* J^+$. Define $\eta \colon \bar{S} \to \widetilde{S}$ and $\theta \colon \widetilde{S} \to W$ as

$$\eta = \sigma_m \circ \cdots \circ \sigma_1 \quad \text{and} \quad \theta = \sigma_k \circ \cdots \circ \sigma_{m+1}.$$

Clearly, η is the identity outside F_0. We denote a general fiber of a $\mathbb{P}^1$-ruling by f.

Lemma 4.7. *Let $\eta : \bar{S} \to \widetilde{S}$ and $\theta \colon \widetilde{S} \to W$ be as above. Then*

$$K_{\widetilde{S}} + \eta_* J \equiv \Big(n + \nu - 1 - \sum_{i=1}^{n} \frac{1}{\mu_i}\Big) f + G + \theta^* \tfrac{1}{2}(U + U'),$$

where G is a negative definite effective divisor with support contained in the support of $F_\infty + \sum_{i=1}^{n} F_i$ and U, U' are the (-2)-tips of $\zeta_* F_0$ if p is twisted and are zero otherwise.

Proof. Let $V \subseteq W$ be defined as the sum of (four) (-2)-tips of $\underline{F_\infty} + \zeta_* \underline{F_0}$ if p is twisted and as zero otherwise. We check easily that

$$K_W + D_h + \underline{F_\infty} + \zeta_* \underline{F_0} \equiv (\nu - 1) f + \tfrac{1}{2} V.$$

Indeed, if p is untwisted, this is just $K_W + D_h + 2f \equiv 0$ on a Hirzebruch surface, and if p is twisted, then it follows from the numerical equivalences $K_W + D_h + f \equiv 0$ and $\underline{F_\infty} + \zeta_* \underline{F_0} - \frac{1}{2} V \equiv f$. By Remark 4.6,

$$K_{\widetilde{S}} + \eta_* J^+ \equiv (n + \nu - 1) f + \theta^* \tfrac{1}{2} V.$$

For every $i = 1, \dots, n$, the divisor $G_i = (1/\mu_i) F_i - C_i$ is effective and negative definite because C_i is not contained in its support. We get

$$K_{\widetilde{S}} + \eta_* J \equiv (n + \nu - 1) f + \sum_{i=1}^{n} \Big(G_i - \frac{1}{\mu_i} F_i\Big) + \theta^* \tfrac{1}{2} V,$$

so

$$K_{\widetilde{S}} + \eta_* J \equiv \Big(n + \nu - 1 - \frac{1}{\mu_i}\Big) f + \sum_{i=1}^{n} G_i + \theta^* \tfrac{1}{2} V. \qquad \square$$

Remark 4.8. Because $K_{\bar{S}} + D + \widehat{E}$ and $K_{\bar{S}} + D$ intersect trivially with a general fiber, we can write $K_{\bar{S}} + D + \widehat{E} \equiv \kappa_0 f + G_0$ and $K_{\bar{S}} + D + \widehat{E} \equiv \kappa f + G$, where G_0 and G are some vertical effective and negative definite divisors and $\kappa_0, \kappa \in \mathbb{Q}$. It follows that $\bar{\kappa}(S_0)$ and $\bar{\kappa}(S)$ are determined by the signs of κ_0 and κ. More explicitly, $\bar{\kappa}(S_0)$ equals $-\infty$, 0, or 1 depending on whether $\kappa_0 < 0$, $\kappa_0 = 0$, or $\kappa_0 > 0$, respectively. An analogous statement holds for $\bar{\kappa}(S)$ and κ.

It turns out that κ and κ_0 depend in a quite involved way on the structure of F_0. This dependence can be stated in terms of the properties of $\eta : \bar{S} \to \widetilde{S}$ defined above. Denote the S_0-components of F_0 by C, $\widetilde{C}$ (or just C if there is only one) and their multiplicities by μ, $\tilde{\mu}$ respectively. Note that $\mu \geq 2$ if $\sigma(F_0) = 1$, but if $\sigma(F_0) = 2$, then it can happen that $\mu = 1$ or $\tilde{\mu} = 1$.

Theorem 4.9. *Let $\lambda = n + \nu - 1 - \sum_{i=1}^{n}(1/\mu_i)$. The numbers κ and κ_0 determining the Kodaira dimension of a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane S' and of its smooth locus S_0 defined in Remark 4.8 are as follows*:

(A) *Case $(h, \nu) = (1, 1)$. Denote the component of F_0 intersecting the 2-section contained in D by B.*

   (i) *If $\eta = \mathrm{id}$ and $F_0 = [2, 1, 2]$, then $\kappa = \kappa_0 = \lambda - \frac{1}{2}$.*
   (ii) *If $\eta = \mathrm{id}$, B is not a tip of F_0, and $C \cdot B > 0$, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - 1/2\mu)$.*
   (iii) *If $\eta = \mathrm{id}$, $C \cdot B = 0$, and F_0 is a chain, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda)$.*
   (iv) *If $\eta = \mathrm{id}$ and B is a tip of F_0, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - 1/\mu)$.*
   (v) *If $\eta \neq \mathrm{id}$, then $\kappa = \kappa_0 = \lambda$.*

(B) *Case $(h, \nu) = (2, 1)$.*

   (i) *If $\eta = \mathrm{id}$ and $C^2 = \widetilde{C}^2 = -1$, then $(\kappa, \kappa_0) = (\lambda - 1, \lambda - 1/\min(\mu, \tilde{\mu}))$.*
   (ii) *If $\eta = \mathrm{id}$ and $C^2 \neq -1$ or $\widetilde{C}^2 \neq -1$, then $\kappa = \kappa_0 = \lambda - 1/\min(\mu, \tilde{\mu})$.*
   (iii) *If $\eta \neq \mathrm{id}$, then assuming that C is the S_0-component disjoint from $\widehat{E}$, we have $\kappa = \kappa_0 = \lambda - 1/\mu$.*

(C) *Case $(h, \nu) = (2, 0)$. Then $\kappa = \kappa_0 = \lambda$.*

Proof. (A) The unique S_0-component C of F_0 is a (-1)-curve. Otherwise, the p-minimality of D implies that B is the only (-1)-curve in F_0 and that it intersects two other D-components of F_0, giving $F_0 = [2, 1, 2] \subseteq D$, with no place for C. It is now easy to check that the list of cases in (A) is complete. Because $C^2 = -1$, $F_0 - C$ has at most two connected components. The only case when $\widehat{E}$ is not connected is when F_0 contains no D-components, which is only possible if $C = B$ and $F_0 = [2, 1, 2]$. Because C is the unique (-1)-curve in F_0, we know that $\zeta = \theta \circ \eta$ has at most one center on $\zeta_* F_0$, so by symmetry we can and do assume that it does not belong to U' (see Lemma 4.7). Suppose $\eta \neq \mathrm{id}$. The center of η belongs to a unique component of $\eta_* J$. Because D_h does not intersect components contracted by η, this component is a proper transform of a D-component, so $\eta_*(C + \widehat{E}) = 0$ by the connectedness of $\widehat{E}$. If we now factor η as $\eta = \sigma \circ \eta'$, where σ is a sprouting blow-up for $\eta_* J$, then by Lemma 4.7 and Remark 4.6,

$$K + \sigma^{-1}\eta_* J \equiv \lambda f + G + \sigma^* \theta^* \tfrac{1}{2}(U + U') + \mathrm{Exc}(\sigma),$$

where $\mathrm{Exc}(\sigma)$ is the exceptional (-1)-curve contracted by σ and K is a canonical divisor on a respective surface. Because $\eta_*(C + \widehat{E}) = 0$, each component of $C + \widehat{E}$ appears with positive integer coefficient in $\eta'^* \mathrm{Exc}(\sigma)$, which gives $K_{\overline{S}} + \eta^{-1}\eta_* J \equiv \lambda f + G + G_0$, where G_0 is a vertical effective and negative definite divisor for which $G_0 - \widehat{E} - C$ is still effective. Because $\eta^{-1}\eta_* J = J = D + \widehat{E} + C$, we get $\kappa = \kappa_0 = \lambda$. We can now assume that $\eta = \mathrm{id}$, so

$$K_{\overline{S}} + D + \widehat{E} + C \equiv \lambda f + G + \tfrac{1}{2}(U' + \theta^* U).$$

This can be written as

$$K_{\overline{S}}+D\equiv(\lambda-\tfrac{1}{2})f+G+\tfrac{1}{2}(U'+F_0+\theta^*U-2C-2\widehat{E}).$$

All components of F_0 appear in $U'+F_0+\theta^*U$ with coefficients bigger than 1, so $U'+F_0+\theta^*U-2C-2\widehat{E}$ is effective and negative definite, because its support does not contain the $\widehat{E}$-component that is a proper transform of U. This gives $\kappa=\lambda-\frac{1}{2}$. We now compute κ_0. If $F_0=[2,1,2]$, then $\theta^*U=U$ and $\widehat{E}=U+U'$, so $K_{\overline{S}}+D\equiv(\lambda-\frac{1}{2})f+G$ and we get $\kappa_0=\lambda-\frac{1}{2}$. Suppose B is a tip of F_0. Because $\mu(B)=2$, we know that F_0 is a fork with two (-2)-tips as maximal twigs (see Lemma 2.2(ii)) and that $\theta^*U=U$ (U and U' are components of $\widehat{E}$). The divisor $G_0=\frac{1}{2}(U+U')+(1/\mu)F_0-C$ is vertical effective and its support does not contain C. Writing

$$K_{\overline{S}}+D+\widehat{E}\equiv\Big(\lambda-\frac{1}{\mu}\Big)f+G+G_0,$$

we infer that $\kappa_0=\lambda-1/\mu$, and we obtain (iv). Consider the case (ii). Because B is not a tip of F_0, we know F_0 is a chain. The assumption $B\cdot C>0$ implies that $B^2\neq-1$ and $\theta^*U=C+\widehat{E}$. We obtain

$$K_{\overline{S}}+D+\widehat{E}\equiv\Big(\lambda-\frac{1}{2\mu}\Big)f+G+\tfrac{1}{2}\Big(U'+\widehat{E}+\frac{1}{\mu}F_0-C\Big),$$

and $U'+\widehat{E}+(1/\mu)F_0-C$ is effective with support not containing C. This gives $\kappa_0=\lambda-(1/2\mu)$. We are left with the case (iii). As in (ii), F_0 is a chain, and we have now

$$K_{\overline{S}}+D+\widehat{E}\equiv\lambda f+G+\tfrac{1}{2}(U'+\theta^*U-2C).$$

$U'+\theta^*U-2C$ is effective and does not contain B, because $B\cdot C=0$, so $\kappa_0=\lambda$.

(B) Suppose $\eta\neq\mathrm{id}$. Note that η_*F_0 contains a proper transform of one of C, $\widetilde{C}$, for otherwise, F_0 would contain a D-rivet. It follows that η is a connected modification and that its center lies on a birational transform of a D-component (the S_0-component contracted by η has to intersect D). Thus, η_*F_0 is a chain intersected by D_h in two different tips and containing C. Since $D\cap\widehat{E}=\varnothing$, we get $\eta_*(\widetilde{C}+\widehat{E})=0$. Writing $\eta=\sigma\circ\eta'$, where σ is a sprouting blow-down, we see that $\eta'^*\operatorname{Exc}(\sigma)$ is an effective negative definite divisor that does not contain C in its support and for which $\eta'^*\operatorname{Exc}(\sigma)-\widetilde{C}-\widehat{E}$ is effective. By Lemma 4.7, we have

$$K+\sigma^{-1}\eta_*D+C\equiv\lambda f+G+\operatorname{Exc}(\sigma),$$

where K is a canonical divisor on a respective surface. It follows from Remark 4.6 and from arguments analogous to those in part (A) that $\kappa=\kappa_0=\lambda-(1/\mu)$. We can now assume that $\eta=\mathrm{id}$. By Lemma 4.7,

$$K_{\overline{S}}+D+C+\widehat{E}+\widetilde{C}\equiv\lambda f+G,$$

which implies $\kappa_0 = \lambda - (1/\min(\mu, \tilde{\mu}))$. Writing

$$K_{\bar{S}} + D \equiv \Big(\lambda - \frac{1}{\alpha}\Big) f + G + \frac{1}{\alpha}\big(F_0 - \alpha(C + \widehat{E} + \widetilde{C})\big),$$

we see that $\kappa = \lambda - (1/\alpha)$, where α is the lowest multiplicity of a component of $C + \widehat{E} + \widetilde{C}$ in F_0. Note that $C + \widehat{E} + \widetilde{C}$ is a chain. Now if $C^2 \neq -1$, for instance, then F_0 is columnar, and factoring θ into blow-downs, we see that $\widehat{E}$ is contracted before C, and hence $\alpha = \mu \leq \tilde{\mu}$. Suppose $C^2 = \widetilde{C}^2 = -1$, and let θ' be the composition of successive contractions of (-1)-curves in F_0 different than C. Now either $\theta'_* F_0 = \theta'_* C = [0]$ or $\theta'_* F_0$ is columnar. Both possibilities imply that $C + \widehat{E}$ contains a component of multiplicity one, and hence $\alpha = 1$.

(C) C is a (-1)-curve. Indeed, $D \cap F_0$ contains at most one (-1)-curve, and if it does, then by the p-minimality of D, it meets both components of D_h and has multiplicity one, so there is another (-1)-curve in F_0. We infer that $\underline{F}_0 - C$ has two connected components, one being $\widehat{E}$ and the second containing a rivet. The existence of a rivet in F_0 implies that $\eta \neq \mathrm{id}$, so $\eta_*(C + \widehat{E}) = 0$. Factoring out a sprouting blow-down from η as above, we get

$$K + \sigma^{-1}\eta_* D \equiv \lambda f + G + \mathrm{Exc}(\sigma).$$

The divisor $\eta'^* \,\mathrm{Exc}(\sigma) - C - \widehat{E}$ is effective and does not contain all components of F_0, so by Remark 4.6, $\kappa = \kappa_0 = \lambda$. □

Remark. In case (B)(iii), it is not true in general that $\mu = \min(\mu, \tilde{\mu})$.

4C. ***Smooth locus of Kodaira dimension zero.*** As a corollary, we obtain the following information in case $\bar{\kappa}(S_0) = 0$.

Corollary 4.10. *Let S' be a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane, and let D be a p-minimal boundary for an extension p of this ruling to a normal completion, as above. Let D be the p-minimal boundary, and let n be the number of columnar fibers. Then $\bar{\kappa}(S_0) = 0$ exactly in the following cases*:

(i) *p is twisted, $n = 0$, and F_0 is of type* (A)(iii) *or* (A)(v).

(ii) *p is twisted, $n = 1$, $\mu = \mu_1 = 2$, and F_0 is of type* (A)(i) *or* (A)(iv) *with no D-components.*

(iii) *p is untwisted with base $\mathbb{C}^1$, $n = 1$, $\mu_1 = 2$, $\min(\mu, \tilde{\mu}) = 2$, and some connected component of $F_0 \cap D$ is a (-2)-curve.*

(iv) *p is untwisted with base $\mathbb{C}^1$, $n = 2$, $\mu_1 = \mu_2 = 2$, and some S_0-component of F_0 meets D_h.*

(v) *p is untwisted with base $\mathbb{P}^1$, $n = 2$, and $\mu_1 = \mu_2 = 2$.*

Proof. Note that $n - \sum_{i=1}^{n}(1/\mu_i) \geq n/2$ because $\mu_i \geq 2$ for each i. Suppose p is twisted. Then $\mu \geq 2$, and so by Theorem 4.9,

$$\lambda \geq \kappa_0 \geq \lambda - \tfrac{1}{2} \geq \frac{n-1}{2}.$$

If $n = 0$, then $\lambda = 0$, which gives $\kappa_0 = 0$ exactly in cases (A)(iii) and (A)(v). If $n = 1$, then $\kappa_0 = \lambda - \frac{1}{2} = 0$, which is possible in case (A)(i) if $\mu_1 = 2$ and in case (A)(iv) if $\mu = \mu_1 = 2$. In both cases, D_h meets the S_0-component, so F_0 contains no D-components. If p is untwisted with base $\mathbb{P}^1$, then

$$n - 1 \geq \lambda = \kappa_0 \geq \frac{n}{2} - 1,$$

so $n = 2$ (because $\lambda = -1/\mu_1 < 0$ for $n = 1$) and $\kappa_0 = 1 - 1/\mu_1 - 1/\mu_2$, which vanishes only if $\mu_1 = \mu_2 = 2$. Assume now that p is untwisted with base $\mathbb{C}^1$. Then

$$n > \kappa_0 \geq \lambda - 1 \geq \frac{n}{2} - 1,$$

so $n \in \{1, 2\}$. There are no (-1)-curves in $D \cap F_0$ by the p-minimality of D, so at least one S_0-component, say C, is a (-1)-curve. We can also assume that C is contracted by η in case $\eta \neq \mathrm{id}$ and that $\mu \geq \tilde{\mu}$ in case $\eta = \mathrm{id}$. Then $\kappa_0 = \lambda - 1/\tilde{\mu}$. The composition ξ of successive contractions of all (-1)-curves in $\underline{F}_0 - \widetilde{C}$ and its images is a connected modification. Suppose $n = 2$. The inequalities above give $\lambda = 1$, so $\mu_1 = \mu_2 = 2$ and $\tilde{\mu} = 1$. Then $\xi_* F_0 = [0]$, and because ξ is a connected modification, $\widetilde{C}$ is a tip of F_0. So one of C, $\widetilde{C}$ intersects D_h, because otherwise $F_0 - \widetilde{C} - C - \widehat{E}$ would be connected and would intersect both sections from D_h, and hence F_0 would contain a rivet. This gives (iv). Suppose that $n = 1$. Then $\mu_1 = \tilde{\mu} = 2$. By the choice of C, further contractions of F_0 to a smooth fiber are subdivisional for $\xi_* D \cup \xi_* F_0$, so we have $\xi_* F_0 = [2, 1, 2]$ with the birational transform of $\widetilde{C}$ in the middle, and the image of D_h intersects both (-2)-tips of $\xi_* F_0$. Since ξ is a connected modification, it does not touch one of these tips, so one of the connected components of $D \cap F_0$ is a (-2)-curve. If $\mu = 1$, then $\mu < \tilde{\mu}$, so by our assumption $\eta \neq \mathrm{id}$. But then $\mu > 1$, because $C^2 = -1$ and C intersects $\widehat{E}$ and D. This contradiction ends the proof of (iii). □

4D. *Constructions.* Lemmas 4.5 and 2.12 give a practical method of reconstructing all $\mathbb{C}^*$-ruled $\mathbb{Q}$-homology planes. We summarize it in here. We denote irreducible curves and their proper transforms by the same letters.

Construction 4.11. Case 1 (twisted ruling). Let D_h be a smooth conic on $\mathbb{P}^2$, let L_0, L_∞ be tangents to D_h at distinct points x_0, x_∞, and let L_i, for $i = 1, \ldots, n$ and $n \geq 0$, be distinct lines through $L_0 \cap L_\infty$, other than L_0, L_∞. Blow up once at $L_0 \cap L_\infty$; let $p\colon \mathbb{F}_1 \to \mathbb{P}^1$ be the $\mathbb{P}^1$-ruling of the resulting Hirzebruch surface. Over each of $p(L_0)$, $p(L_\infty)$, blow up on D_h twice, creating singular fibers $\widetilde{F}_0 = [2, 1, 2]$

and $F_\infty = [2, 1, 2]$. For each $i = 1, \dots, n$, by a connected sequence of blow-ups subdivisional for $L_i + D_h$, create a column fiber F_i over $p(L_i)$ and denote its unique (-1)-curve by C_i. By some connected sequence of blow-ups with a center on $\widetilde{F}_0$, create a singular fiber F_0, and denote the newly created (-1)-curve by C (if the sequence is empty, define C as the (-1)-curve of $\widetilde{F}_0$). Denote the resulting surface by $\bar{S}$, put

$$T = D_h + \underline{F_\infty} + (\underline{F_1} - C_1) + \cdots + (\underline{F_n} - C_n) + \underline{F_0} - C,$$

and construct S' as in Lemma 2.12. S' is a $\mathbb{Q}$-homology plane (singular as long as T is not connected) because conditions 2.12(i)–(iii) are satisfied by construction and (iv) by Lemma 4.4. To see that each S' admitting a twisted $\mathbb{C}^*$-ruling can be obtained in this way, note that by the p-minimality of D, even if F_0 contains two (-1)-curves C and $B \subseteq D$, then B is not a tip of F_0 and ζ does not touch it, so in each case the modification $F_0 \to \zeta_* F_0$ induced by ζ is connected, and we are done by Lemma 4.5.

Case 2 (untwisted ruling with base $\mathbb{C}^1$). Let $x_0, x_1 \dots x_n, x_\infty, y \in \mathbb{P}^2$, for $n \geq 0$, be distinct points, such that all but y lie on a common line D_1. Let L_i be a line through x_i and y. Blow up y once and let D_2 be the negative section of the $\mathbb{P}^1$-ruling of the resulting Hirzebruch surface $p\colon \mathbb{F}_1 \to \mathbb{P}^1$. For each $i = 0, 1, \dots, n$, by a connected sequence of blow-ups (which can be empty if $i = 0$), with first center x_i and subdivisional for $D_1 + L_i$, create a column fiber F_i ($\widetilde{F}_0$ if $i = 0$) over $p(x_i)$ and denote its unique (-1)-curve by C_i if $i \neq 0$ and by $\widetilde{C}$ if $i = 0$ (put $\widetilde{C} = L_0$ if the sequence over $p(x_0)$ is empty). Choose a point $z \in F_0$ that lies on $D_1 + \underline{\widetilde{F}_0} - \widetilde{C}$, and by a nonempty connected sequence of blow-ups with first center z, create some singular fiber F_0 over $p(x_0)$. Let C be the new (-1)-curve. Denote the resulting surface by $\bar{S}$, put

$$T = D_1 + D_2 + L_\infty + (\underline{F_1} - C_1) + \cdots + (\underline{F_n} - C_n) + \underline{F_0} - C - \widetilde{C},$$

and construct S' as in Lemma 2.12. The surface S' is a $\mathbb{Q}$-homology plane by Lemma 4.4, because Lemma 4.4(ii) is satisfied by the choice of z. To see that all S' admitting an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ can be obtained in this way, note that by changing the completion of S' by a flow if necessary, we can assume that one of the components of D_h is a (-1)-curve. $D \cap F_0$ contains no (-1)-curves, and as was shown in the proof of Theorem 4.9, η contracts at most one of C, $\widetilde{C}$. Then by Lemma 4.5, we are done.

Case 3 (untwisted ruling with base $\mathbb{P}^1$). Let D_2 be the negative section of the $\mathbb{P}^1$-ruling of a Hirzebruch surface $p\colon \mathbb{F}_N \to \mathbb{P}^1$, with $N > 0$. Let $x_0, x_1, \dots, x_n$, with $n \geq 0$ be points on a section D_1 of p disjoint from D_2. For each $i = 0, 1, \dots, n$, by a connected sequence of blow-ups (which can be empty if $i = 0$), with first center

x_i and subdivisional for $D_1 + p^{-1}(p(x_i))$, create a column fiber F_i ($\widetilde{F}_0$ if $i = 0$) over $p(x_i)$ and denote its unique (-1)-curve by C_i if $i \neq 0$ and by B if $i = 0$ (put $B = p^{-1}(p(x_0))$ if the sequence over $p(x_0)$ is empty). Assume that the intersection matrix of at least one of two connected components of

$$D_1 + D_2 + (\underline{F_1} - C_1) + \cdots + (\underline{F_n} - C_n) + (\underline{\widetilde{F}_0} - B)$$

is nondegenerate. By a connected sequence of blow-ups starting from a sprouting blow-up for $D_1 + \widetilde{F}_0$ with center on B, create some singular fiber F_0 over $p(x_0)$ and let C be the new (-1)-curve. Denote the resulting surface by $\bar{S}$, put

$$T = D_1 + D_2 + (\underline{F_1} - C_1) + \cdots + (\underline{F_n} - C_n) + (\underline{F_0} - C),$$

and construct S' as in Lemma 2.12. D is connected because the modification $F_0 + D_1 \to \widetilde{F}_0 + D_1$ is not subdivisional, so S' is a $\mathbb{Q}$-homology plane by Lemma 4.4. By Lemmas 4.5 and 4.4, each S' with an untwisted $\mathbb{C}^*$-ruling having a base $\mathbb{P}^1$ can be obtained in this way.

5. Corollaries

5A. ***Completions and singularities.*** Recall that $\mathbb{Q}$-homology planes with nonquotient singularities have unique snc-minimal completions (and hence also balanced ones) and unique singular points [Palka 2008, 1.2]. The completions and singularities in case $\bar{\kappa}(S_0) = -\infty$ are described in Section 3. In case $\bar{\kappa}(S_0) = 2$, the singular point is unique and of quotient type [ibid.]. Also, the snc-minimal boundary cannot contain nonbranching b-curves with $b \geq 0$, because these induce $\mathbb{C}^1$- or $\mathbb{C}^*$-rulings of S_0, and hence the snc-minimal completion is unique. Theorem 1.1 summarizes the remaining cases.

Proof of Theorem 1.1. (1) Suppose S' has at least two different balanced completions. These differ by a flow, which implies that the boundary contains a nonbranching rational component F_∞ with zero self-intersection. Then F_∞ is a fiber of a $\mathbb{P}^1$-ruling p of a balanced completion (V, D). We may assume that F_∞ is not contained in any maximal twig of D. Indeed, after moving the 0-curve by a flow to a tip of a new boundary, it gives an affine ruling of S', which is possible only if $\bar{\kappa}(S_0) = -\infty$. Because F_∞ is nonbranching, the induced ruling restricts to an untwisted $\mathbb{C}^*$-ruling of S'. It follows from the connectedness of the modification η (see the proof of Theorem 4.9) that $n > 0$, so this restriction has more than one singular fiber. Both components of D_h are branching in D. Since F_∞ is the only nonbranching 0-curve in D, centers of elementary transformations lie on the intersection of the fiber at infinity with D_h. If D is strongly balanced, then one of the components of D_h is a 0-curve, and hence there are at most two strongly balanced completions. Conversely, suppose that S' has an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and that $n > 0$, and let (V, D, p) be a completion of this ruling. Because S' is not affine-ruled, the horizontal components H, H' of D are branching, so (V, D) is balanced and we can assume $H'^2 = 0$. Because H, H' are proper transforms of two disjoint sections on a Hirzebruch surface, we have $H^2 + H'^2 + n \leq 0$, so $H^2 \neq 0$ and we can obtain a different strongly balanced completion of S' by a flow that makes H into a 0-curve.

(2), (3) By [Palka 2008, 4.5] and [Palka 2011a], we may assume that S' is $\mathbb{C}^*$-ruled. If this ruling is untwisted, it follows from the proof of Theorem 4.9 that S' has a unique singular point, and it is a cyclic singularity. In the twisted case, because $\widehat{E} \subseteq F_0$, if $\widehat{E}$ is not connected then F_0 is of type (A)(i), and if $\widehat{E}$ is not a chain then F_0 is of type (A)(iv). □

Remark. The set of isomorphism classes of strongly balanced boundaries that a given surface admits is an invariant of the surface, which allows us to easily distinguish between many $\mathbb{Q}$-acyclic surfaces.

5B. ***Singular planes of negative Kodaira dimension.*** As another corollary of Theorem 4.9 we give a detailed description of singular $\mathbb{Q}$-homology planes of negative Kodaira dimension. We assume that $\bar{\kappa}(S_0) \neq 2$, but as we show in [Palka and Koras 2010], this assumption is redundant.

Theorem 5.1. *Suppose that S' is a singular $\mathbb{Q}$-homology plane of negative Kodaira dimension and that S_0 is its smooth locus. If $\bar{\kappa}(S_0) \neq 2$, then exactly one of the following holds*:

(i) $\bar{\kappa}(S_0) = -\infty$; *$S'$ is affine-ruled or isomorphic to $\mathbb{C}^2/G$ for a small finite noncyclic subgroup $G <$* GL(2, $\mathbb{C}$).

(ii) $\bar{\kappa}(S_0) \in \{0, 1\}$; *$S'$ is nonlogarithmic and is isomorphic to a quotient of an affine cone over a smooth projective curve by an action of a finite group acting freely off the vertex of the cone and preserving the set of lines through the vertex.*

(iii) $\bar{\kappa}(S_0) \in \{0, 1\}$; *$S'$ has an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and two singular fibers. One of them consists of two $\mathbb{C}^1$'s meeting in a cyclic singular point*; *after taking a resolution and completion*, *the respective completed singular fiber is of type* (B)(i) *with μ, $\tilde{\mu} \geq 2$* (see Figure 3 and Theorem 4.9).

Proof. By [Palka 2011a; Palka 2008, 4.5] and Section 3, we may assume that S' is logarithmic and $\mathbb{C}^*$-ruled and that $\bar{\kappa}(S_0) \geq 0$. We need to show (iii). Let (V, D, p) be a minimal completion of the $\mathbb{C}^*$-ruling. By Theorem 4.9, if p is twisted, then

$$0 > \kappa_0 \geq \lambda - \tfrac{1}{2} \geq \frac{n-1}{2},$$

so $n = \lambda = 0$. The inequalities $\kappa < 0$ and $\kappa_0 \geq 0$ can be satisfied only in case (A)(iii), and then $D_h^2 = 0$ by Lemma 4.5, so D_h induces an untwisted $\mathbb{C}^*$-ruling of

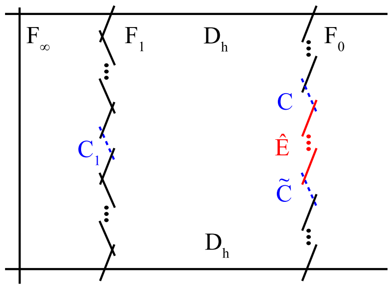

Figure 3. Untwisted $\mathbb{C}^*$-ruling, $\bar{\kappa}(S') = -\infty$.

S'. Suppose p is untwisted. Because $\kappa \neq \kappa_0$, p has base $\mathbb{C}^1$ and is of type (B)(i). Because

$$0 > \kappa = \lambda - 1 \geq \frac{n}{2} - 1,$$

we get $n \leq 1$, but for $n = 0$ we get $\kappa_0 < \lambda < 0$, so in fact $n = 1$. Then $0 \leq \kappa_0 = 1 - 1/\mu_1 - 1/\min(\mu, \tilde{\mu})$, and hence $\min(\mu, \tilde{\mu}) \geq 2$. □

By Proposition 2.11, $H_i(S', \mathbb{Z})$ vanishes for $i > 1$. If S' is of type $\mathbb{C}^2/G$ or of type (ii), then it is contractible. $H_1(S', \mathbb{Z})$ for affine-ruled S' was computed in Remark 3.4. For completeness, we now compute the fundamental group of S' of type (iii), which by Proposition 2.11 is the same as $\pi_1(S)$. Let E_0 be a component of $\widehat{E}$ intersecting C. Contract $\widetilde{C}$ and successive vertical (-1)-curves until C is the only (-1)-curve in the fiber (C cannot became a 0-curve, because it does not intersect D_h), and denote this contraction by θ. Let θ' be the contraction of $\theta_* F_0$ and F_1 to smooth fibers. Put $U = S_0 \setminus (C_1 \cup C \cup \widetilde{C})$ and let $\gamma_1, \gamma, t \in \pi_1(U)$ be the vanishing loops of the images of F_1, F_0 under $\theta' \circ \theta$ and of some component of D_h (see [Fujita 1982, 4.17]). We need to compute the kernel of the epimorphism $\pi_1(U) \to \pi_1(S)$. Because θ does not touch C, $\theta_* F_0$ is columnar and $\theta_* E_0 \neq 0$. Using [ibid., 7.17], one can show by induction on the number of components of a columnar fiber that because $E_0 \cdot C \neq 0$, the vanishing loops of E_0 and C, which are of type $\gamma^a t^b$ and $\gamma^c t^d$, satisfy $ad - bc = \pm 1$. Thus γ and t are in the kernel, and hence

$$\pi_1(S) = \langle \gamma_1 : \gamma^{\mu_1} \rangle \cong \mathbb{Z}_{\mu_1}.$$

In particular, S' is not a $\mathbb{Z}$-homology plane.

6. Uniqueness of $\mathbb{C}^*$-rulings

6A. ***The number of*** $\mathbb{C}^*$***-rulings.*** We consider the question of uniqueness of $\mathbb{C}^*$-rulings of S_0 and S'. Recall that a $\mathbb{C}^*$-ruling of S_0 is *extendable* if it extends to a ruling (morphism) of S'. Two rational rulings of a given surface are considered the same if they differ by an automorphism of the base. When a $\mathbb{C}^*$-ruling of S_0

exists, using the information on snc-minimal boundaries, we are able to compute the number of different $\mathbb{C}^*$-rulings.

Theorem 6.1. *Let S' be a singular $\mathbb{Q}$-homology plane that is not affine-ruled. Let $p_1, \dots, p_r$ for $r \in \mathbb{N} \cup \{\infty\}$ be all different $\mathbb{C}^*$-rulings of the smooth locus S_0 of S'. Let D be an snc-minimal boundary of S'.*

(1) *If $\bar{\kappa}(S_0) = 2$ or if S' is exceptional (so that $\bar{\kappa}(S_0) = 0$), then $r = 0$.*

(2) *If $\bar{\kappa}(S_0) = 1$ or if S' is nonlogarithmic, then $r = 1$.*

(3) *If $\bar{\kappa}(S_0) = -\infty$, then $r \geq 1$ and p_1 is nonextendable. Also, $r \neq 1$ only if the fork that is an exceptional divisor of the snc-minimal resolution of S' is of type $(2, 2, k)$. In this case we have:*

   (i) *If $k \neq 2$, then $r = 2$, p_2 is twisted, and it has a unique singular fiber, which is of type* (A)(iv).

   (ii) *If $k = 2$, then $r = 4$, p_2, p_3, p_4 are twisted, and they have unique singular fibers, which are of type* (A)(iv).

(4) *Assume that $\bar{\kappa}(S_0) = 0$ and that S' is logarithmic and not exceptional. Then all p_i extend to $\mathbb{C}^*$-rulings of S' and the following hold:*

   (i) *If the dual graph of D is*

$$\begin{tikzcd}[arrows=dash] -2 \arrow[r] & -1 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & -2 \\ & -2 & -2 & \end{tikzcd}$$

   with $k \leq -2$, then $r = 1$ and p_1 is twisted.

   (ii) *If the dual graph of D is*

$$\begin{tikzcd}[arrows=dash] -2 \arrow[r] & -1 \arrow[r] \arrow[d] & -1 \arrow[r] \arrow[d] & -2 \\ & -2 & -2 & \end{tikzcd}$$

   then $r = 2$ and p_1, p_2 are twisted.

   (iii) *If the dual graph of D is*

$$\begin{tikzcd}[arrows=dash] -2 \arrow[r] & k \arrow[r] \arrow[d] & 0 \arrow[r] & m \arrow[r] \arrow[d] & -2 \\ & -2 & & -2 & \end{tikzcd}$$

   then $r = 3$, p_1, p_2 are twisted and p_3 is untwisted with base $\mathbb{C}^1$.

   (iv) *In all other cases, $r = 2$, p_1 is twisted and p_2 is untwisted.*

Proof. (1) By definition, exceptional $\mathbb{Q}$-homology planes are not $\mathbb{C}^*$-ruled. If S_0 is of general type, then by Iitaka's easy addition formula [Iitaka 1982, 10.4], S_0 is not $\mathbb{C}^*$-ruled.

(2) If S' is nonlogarithmic, then by [Palka 2008, 4.1], the $\mathbb{C}^*$-ruling of S' is unique. Assume that $\bar{\kappa}(S_0) = 1$. Let $(\bar{S}, D)$ be some normal completion of the snc-minimal resolution $S \to S'$. Denote the exceptional divisor of the resolution by $\widehat{E}$. By [Fujita 1982, 6.11], for some $n > 0$, the base locus of $|n(K_{\bar{S}} + D + \widehat{E})^+|$ is empty and the linear system gives a $\mathbb{P}^1$-ruling of $\bar{S}$ that restricts to a $\mathbb{C}^*$-ruling of S_0; see also [Miyanishi 2001, 2.6.1]. Consider another $\mathbb{C}^*$-ruling of S_0. Modifying $\bar{S}$ if necessary, we can assume that it extends to a $\mathbb{P}^1$-ruling of $\bar{S}$. Let f' be a general fiber of this extension. Then

$$f' \cdot (K_{\bar{S}} + D + \widehat{E}) = f' \cdot K_{\bar{S}} + 2 = 0,$$

and hence

$$f' \cdot (K_{\bar{S}} + D + \widehat{E})^+ + f' \cdot (K_{\bar{S}} + D + \widehat{E})^- = 0.$$

However, $(K_{\bar{S}} + D + \widehat{E})^-$ is effective and $(K_{\bar{S}} + D + \widehat{E})^+$ is numerically effective, so

$$f' \cdot (K_{\bar{S}} + D + \widehat{E})^+ = f' \cdot (K_{\bar{S}} + D + \widehat{E})^- = 0,$$

and we see that the rulings are the same.

(3), (4) We need to understand how to find all twisted $\mathbb{C}^*$-rulings of a given S'. Consider a twisted $\mathbb{C}^*$-ruling of S' and let $(\widetilde{V}, \widetilde{D}, \tilde{p})$ be a minimal completion of this ruling. By the $\tilde{p}$-minimality of $\widetilde{D}$, the only component of $\widetilde{D}$ that can be a nonbranching (-1)-curve is $\widetilde{D}_h$, so there is a connected modification $(\widetilde{V}, \widetilde{D}) \to (V, D)$ with snc-minimal D. Let $\widetilde{D}_0 \subseteq \widetilde{D}$ be the (-1)-curve of the fiber at infinity (see Lemma 4.1). D is not a chain; otherwise S' would be affine-ruled. Let $D_0 \subseteq D$ be the image of $\widetilde{D}_0$, and let T be the connected component of $D - D_0$ containing the image of the horizontal component (which is a point if the modification is nontrivial). In this way, a twisted $\mathbb{C}^*$-ruling of S' determines a pair (D_0, T) (with $D_0 + T$ contained in a boundary of some snc-minimal completion), such that $\beta_D(D_0) = 3$, $D_0^2 \geq -1$, T is a connected component of $D - D_0$ containing the image of the horizontal section, and both connected components of $D - D_0 - T$ are (-2)-curves. Conversely, if we have an snc-minimal normal completion (V, D) and a pair as above, we make a connected modification $(\widetilde{V}, \widetilde{D}) \to (V, D)$ over D by blowing successively on the intersection of the total transform of T with the proper transform of D_0 until D_0 becomes a (-1)-curve. The (-1)-curve together with the transform of $D - T - D_0$ induce a $\mathbb{P}^1$-ruling of V' and constitute the fiber at infinity for this ruling. The restriction to S' is a twisted $\mathbb{C}^*$-ruling.

Suppose $\bar{\kappa}(S_0) = -\infty$. Since S_0 is not affine-ruled, $S' \cong \mathbb{C}^2/G$ for a finite noncyclic small subgroup $G < \mathrm{GL}(2, \mathbb{C})$ (see Section 3). Let (V, D) be an snc-minimal normal completion of S' and let $\bar{S} \to V$ be a minimal resolution with exceptional divisor $\widehat{E}$. We saw in the proof of Proposition 3.1 that S_0 admits a platonic $\mathbb{C}^*$-ruling, which extends to a $\mathbb{P}^1$-ruling of $\bar{S}$. Also, D and $\widehat{E}$ are forks for which D_h and $\widehat{E}_h$ are the unique branching components of D and E respectively. In particular, the $\mathbb{C}^*$-ruling does not extend to a ruling of S', and because nonbranching components of D have negative self-intersections, $(\bar{S}, D + \widehat{E})$ is a unique snc-minimal smooth completion of S_0 (and hence (V, D) is a unique snc-minimal normal completion of S'). It follows from the proof of [Palka 2008, 4.1] that the nonextendable $\mathbb{C}^*$-ruling of S_0 is unique. Suppose there is a $\mathbb{C}^*$-ruling of S_0 that does extend to S'. Since $\widehat{E}$ is not a chain, it follows from the proof of Theorem 4.9 that this ruling is twisted. Since maximal twigs of $\widehat{E}$ and D are adjoint chains of columnar fibers, we see that a maximal twig of $D - D_h$ is a (-2)-curve if and only if the respective maximal twig of $\widehat{E} - \widehat{E}_h$ is a (-2)-curve. Also, $0 < d(\widehat{E})$, so $\widehat{E}_h^2 \le -2$, and because $\widehat{E}_h^2 + D_h^2 = -3$, we have $D_h^2 \ge -1$. Therefore, S' admits a twisted $\mathbb{C}^*$-ruling if and only if $\widehat{E}$ is a fork of type $(2, 2, k)$ for some $k \ge 2$. If $k \ne 2$, then the choice of (D_0, T) as above is unique, and if $k = 2$, then there are three such choices. If (V', D', p) is a minimal completion of such a ruling, then D' is a fork, so because $\kappa_0 < 0$, we have $n = 0$ and F_0 is of type (A)(iv) (see the proof of Theorem 4.9). This gives (3).

We can now assume that $\bar{\kappa}(S_0) = 0$ and that S' is logarithmic and not exceptional. Then S_0 is $\mathbb{C}^*$-ruled and by [Palka 2008, 4.7(iii)], each $\mathbb{C}^*$-ruling of S_0 extends to a $\mathbb{C}^*$-ruling of S'. Let $r \in \{1, 2, \dots\} \cup \{\infty\}$ be the number of different (up to automorphism of the base) $\mathbb{C}^*$-rulings of S' and let (V_i, D_i, p_i), for $i \le r$, be their minimal completions. Minimality implies that nonbranching (-1)-curves in D_i are p_i-horizontal. We add consequently an upper index (i) to objects defined previously for any $\mathbb{C}^*$-ruling when we refer to the ruling p_i. If p_i is untwisted, we denote the horizontal components of $D_h^{(i)}$ by $H^{(i)}$, $H'^{(i)}$.

Suppose p_1 is untwisted with base $\mathbb{P}^1$. Then $F_0^{(1)}$ contains a rivet and by Corollary 4.10, $n^{(1)} = 2$, so D_1 does not contain nonbranching b-curves with $b \ge -1$. Then (V_1, D_1) is balanced and S' does not admit an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$, because it does not contain nonbranching 0-curves (see Lemma 4.1). By Corollary 4.10, each component of $D_h^{(1)}$ has $\beta_{D_1} = 3$ and intersects two (-2)-tips of D_1. Note that $\zeta^{(1)}$ (see Lemma 4.5) touches $D_h^{(1)}$ two times if both components of $D_h^{(1)}$ intersect the same horizontal component of $F_0^{(1)}$ and three times if not. By Lemma 4.5 and the properties of Hirzebruch surfaces, we get $-3 \le (D_h^{(1)})^2 \le -2$. In particular, one of the components of $D_h^{(1)}$, say $H^{(1)}$, has $(H^{(1)})^2 \ge -1$, so by the discussion about twisted $\mathbb{C}^*$-rulings above, $H^{(1)}$ together with two (-2)-tips of D_1 gives rise to a twisted $\mathbb{C}^*$-ruling p_2 of S'. Because $H'^{(1)}$ together with two

(-2)-tips of D_1 intersecting it are contained in a fiber of p_2, $(H'^{(1)})^2 \le -2$. Thus p_2 is the only twisted ruling of S', because $H^{(1)}$ is the only possible choice for a middle component of the fiber at infinity of a twisted ruling. Suppose $r \ge 3$. Then p_3 is untwisted with base $\mathbb{P}^1$. Because D_1 does not contain nonbranching 0-curves, any flow in D_1 is trivial, so $V_3 = V_1$. Because p_3 and p_1 are different after restriction to S', the S_0-components $C^{(1)}$, $C^{(3)}$ contained respectively in $F_0^{(1)}$, $F_0^{(3)}$ are different. Because they both intersect $\widehat{E}$, they are contained in the same fiber of p_2, which contradicts $\Sigma_{S_0}^{(2)} = 0$. Because D contains no nonbranching 0-curves, D is not of type (4)(iii). Since $n^{(1)} = 2$, D contains at least seven components, so D is not of type (4)(i) or (4)(ii).

We can now assume that each untwisted $\mathbb{C}^*$-ruling of S' has base $\mathbb{C}^1$. Suppose p_1 is such a ruling. By Corollary 4.10, both horizontal components of D_1 have $\beta_{D_1} = 3$, and one of them, say $H'^{(1)}$, intersects two (-2)-tips T and T' of D_1. In particular, D_1 is snc-minimal. Because $F_\infty^{(1)} = [0]$, changing V_1 by a flow if necessary, we may assume that $H'^{(1)}$ is a (-1)-curve. Then

$$F_\infty^{(2)} = T + 2H'^{(1)} + T'$$

induces a $\mathbb{P}^1$-ruling $p_2 : V_1 \to \mathbb{P}^1$, which is a twisted $\mathbb{C}^*$-ruling after restricting it to S'. Suppose $r \ge 3$. If p_3 is untwisted, then its base is $\mathbb{C}^1$, and changing V_3 by a flow if necessary, we can assume that $V_3 = V_1$. But then $F_\infty^{(1)} = F_\infty^{(3)}$, because D_1 contains only one nonbranching 0-curve, so p_1 and p_3 have a common fiber and hence cannot be different after restriction to S', which is a contradiction. Thus p_3 is twisted. By the discussion above, p_3 can be recovered from a pair (D_0, T) on some snc-minimal completion of S'. All such completions of S' differ from (V_1, D_1) by a flow, which is an identity on $V_1 - F_\infty^{(1)}$, and hence the birational transform of D_0 on V_1 is either $H^{(1)}$ or $H'^{(1)}$. Because the restrictions of p_1 and p_2 to S' are different, it is $H^{(1)}$. It follows that $r = 3$ and that $D_1 - H'^{(1)}$ has two (-2)-tips as connected components, and hence the dual graph of D_1 is as in (iii). Conversely, if S' has a boundary as in (iii), then besides the untwisted $\mathbb{C}^*$-ruling induced by the 0-curve, it has also two twisted rulings, each with one of the branching components as the middle component of the fiber at infinity.

We can finally assume that all $\mathbb{C}^*$-rulings of S' are twisted. Let (V, D) be a balanced completion of S'. Because S' does not admit untwisted $\mathbb{C}^*$-rulings, D does not contain nonbranching 0-curves, so (V, D) is a unique snc-minimal completion of S'. Thus, to find all twisted $\mathbb{C}^*$-rulings of S', we need to determine all pairs (D_0, T) such that $D_0 + T \subseteq D$, $D_0^2 \ge -1$, $\beta_D(D_0) = 3$, and $D - T - D_0$ consists of two (-2)-tips. Let (D_0, T) and (D_0', T') be two such pairs. Suppose $D_0 \ne D_0'$ and, say, $D_0'^2 \ge D_0^2$. We have $D_0 \cdot D_0' \ne 0$, for otherwise the chain $D - T'$, which is not negative definite, would be contained in (and not equal to, because $\nu \le 1$) a fiber of the twisted ruling associated with (D_0, T), which is impossible. Then D

has six components and we check that

$$d(D) = 16((D_0^2+1)(D_0'^2+1)-1),$$

so $(D_0^2+1)(D_0'^2+1) \leq 0$, because $d(D) < 0$. Then $D_0^2 = -1$ and D_0' is a 2-section of the twisted ruling associated with (D_0, T). Because $\beta_D(D_0') = 3$, by Corollary 4.10 and Lemma 4.5 for this ruling $n = 1$, D_0' is a (-1)-curve and D has dual graph as in (ii). Conversely, it is easy to see that S' with such a boundary has two twisted $\mathbb{C}^*$-rulings. Therefore, we can assume that the choice of D_0 for a pair (D_0, T) as above is unique. Let p_1 be a twisted $\mathbb{C}^*$-ruling associated with some pair (D_0, T). Suppose $n^{(1)} = 0$. By Lemma 4.5, $\zeta_* D_h^{(1)}$ is a 0-curve, so

$$F = \zeta^* \zeta_* D_h^{(1)}$$

induces a $\mathbb{P}^1$-ruling p of V. If ζ touches $D_h^{(1)}$, then F contains the S_0-component of $F_0^{(1)}$, so $F \nsubseteq D$ and p restricts to an untwisted $\mathbb{C}^*$-ruling of S' with base $\mathbb{P}^1$. If ζ does not touch $D_h^{(1)}$, then p restricts to a $\mathbb{C}^*$-ruling of S' with base $\mathbb{C}^1$. This contradicts the assumption. By Corollary 4.10 we get that $n^{(1)} = 1$, $F_0^{(1)}$ contains no D_1-components, and $\mu_1 = 2$. In particular, $D_1 = D$. By Lemma 4.5, $(D_h^{(1)})^2 \leq -1$ because $n^{(1)} = 1$, so D has a dual graph as in (i) or (ii). Conversely, if D is of type (i) or (ii), then $r = 2$ if $k = -1$ and $r = 1$ if $k \leq -2$. □

6B. ***The number of affine lines.*** Theorem 6.1 has interesting consequences. It is known [Zaĭdenberg 1987; Gurjar and Miyanishi 1992] that $\mathbb{Q}$-homology planes with smooth locus of general type (in particular the smooth ones) do not contain topologically contractible curves. In fact, the number $\ell \in \mathbb{N} \cup \{\infty\}$ of contractible curves on a $\mathbb{Q}$-homology plane S' is known except two cases: when S' is non-logarithmic and when S' is singular and $\bar{\kappa}(S_0) = 0$ (see [Palka 2011b, 10.1] and references there). Clearly, in the first case $\ell = \infty$ by the main result of [Palka 2008]. The case when S' is smooth and of Kodaira dimension zero has been considered in [Gurjar and Parameswaran 1995]. Theorem 1.3 is the missing piece of information, and the method can be easily applied to recover the result of Gurjar and Parameswaran.

Proof of Theorem 1.3. We can assume that S' is logarithmic. Suppose S' contains a topologically contractible curve L. We show that L is vertical for some $\mathbb{C}^*$-ruling of S'. The proper transform of L on $\bar{S}$ meets each connected component of $\widehat{E}$ in at most one point. We use the logarithmic Bogomolov–Miyaoka–Yau inequality as in [Koras and Russell 2007, 2.12] to show that $\bar{\kappa}(S_0 - L) \leq 1$. In case $\bar{\kappa}(S_0 - L) = 1$, the surface $S_0 - L$ is $\mathbb{C}^*$-ruled [Fujita 1982, 6.11], so we may assume that $\bar{\kappa}(S_0 - L) = 0$. Let $\mathbb{Z}[D + \widehat{E}]$ be a free abelian group generated by the components of $D + \widehat{E}$. Because

$$\operatorname{Pic} S_0 = \operatorname{Coker}(\mathbb{Z}[D + \widehat{E}] \to \operatorname{Pic} \bar{S})$$

is torsion, the class of L in Pic S_0 is torsion. So there is a surjection $f: S_0 - L \to \mathbb{C}^*$, and taking its Stein factorization, we get a $\mathbb{C}^*$-ruling of $S_0 - L$, which (because $\bar{\kappa}(S_0) \neq -\infty$) extends to a $\mathbb{C}^*$-ruling of S_0. Since S_0 is logarithmic, each $\mathbb{C}^*$-ruling of S_0 extends in turn to a $\mathbb{C}^*$-ruling of S'. Therefore L is vertical for some $\mathbb{C}^*$-ruling of S' and we are done. In particular, exceptional $\mathbb{Q}$-homology planes do not contain contractible curves. It follows from Corollary 4.10 that if the ruling is twisted or untwisted with base $\mathbb{P}^1$, then the vertical contractible curve is unique and is contained in the unique singular noncolumnar fiber. For an untwisted ruling with base $\mathbb{C}^1$, there are at most two such curves. In particular, in cases (4)(i) and (4)(ii) of Theorem 6.1, L needs to intersect the horizontal component of the boundary, so we get respectively $\ell = 1$ and $\ell = 2$. In case (4)(iii), the unique vertical contractible curves for the twisted rulings p_1 and p_3 are distinct and do not intersect the horizontal components of respective rulings, and hence are both vertical for the untwisted ruling p_3, so $\ell = 2$. In the remaining case (4)(iv), $r = 2$, p_1 is twisted and p_2 is untwisted. We can assume that the base of p_2 is $\mathbb{C}^1$ and the unique noncolumnar singular fiber contains two contractible curves, L_1 and L_2, for otherwise $\ell \leq 2$ by the above remarks and we are done. Since the twisted ruling is unique, there is exactly one horizontal component H of $D_h^{(2)}$ that meets two (-2)-tips of $D_h^{(1)}$ (together with these tips it induces the twisted ruling). Clearly, only one L_i can intersect H, so the second one is vertical for p_1 and we get $\ell \leq 2$ is this case too. $\square$

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