CLASSIFICATION OF SINGULAR $\mathbb{Q}$-HOMOLOGY PLANES
II: $C^1$- AND $C^*$-RULINGS

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A \( \mathbb{Q} \)-homology plane is a normal complex algebraic surface having trivial rational homology. We classify singular \( \mathbb{Q} \)-homology planes that are \( \mathbb{C}^1 \)-or \( \mathbb{C}^* \)-ruled. We analyze their completions, the number of different rulings they have, and the number of affine lines on them; and we give constructions. Together with previously known results, this completes the classification of \( \mathbb{Q} \)-homology planes with smooth locus of nongeneral type. We show also that the dimension of a family of homeomorphic but nonisomorphic singular \( \mathbb{Q} \)-homology planes having the same weighted boundary, singularities and Kodaira dimension can be arbitrarily big.

We work with complex algebraic varieties.

1. Main results

A \( \mathbb{Q} \)-homology plane is a normal surface whose rational cohomology is the same as that of \( \mathbb{C}^2 \). This paper is the last piece of the classification of \( \mathbb{Q} \)-homology planes having smooth locus of nongeneral type. The classification is built on the work of many authors; for a summary of what is known about smooth and singular \( \mathbb{Q} \)-homology planes, see [Miyanishi 2001, §3.4] and [Palka 2011b]. In [Palka 2008], we classified singular \( \mathbb{Q} \)-homology planes with nonquotient singularities, showing in particular that they are quotients of affine cones over projective curves by actions of finite groups that respect the set of lines through the vertex. In [Palka 2011a], we classified singular \( \mathbb{Q} \)-homology planes whose smooth locus is of nongeneral type and admits no \( \mathbb{C}^1 \)- or \( \mathbb{C}^* \)-ruling (exceptional planes). Here we classify singular \( \mathbb{Q} \)-homology planes that admit a \( \mathbb{C}^1 \)- or a \( \mathbb{C}^* \)-ruling. We analyze completions and boundaries rather than the open surfaces themselves. To deal with nonuniqueness of these, we use the notions of a balanced and a strongly balanced weighted boundary and completion of an open surface (see Definitions 2.7 and 2.10).

We classify \( \mathbb{C}^1 \)- and \( \mathbb{C}^* \)-ruled \( \mathbb{Q} \)-homology planes by giving necessary and sufficient conditions for a \( \mathbb{C}^1 \)- or \( \mathbb{C}^* \)-ruled open surface to be a \( \mathbb{Q} \)-homology plane.

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(see Lemmas 2.12, 3.2 and 4.4 and the remarks before the latter) and then giving general constructions (see Construction 3.3 and Section 4D). We compute the Kodaira dimension of a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane and of its smooth locus (Theorem 4.9) in terms of properties of singular fibers, and we list the planes with smooth locus of Kodaira dimension zero (Section 4C). As a corollary of the classification, we obtain the following result.

**Theorem 1.1.** Let $S'$ be a singular $\mathbb{Q}$-homology plane, and let $S_0$ be its smooth locus. Assume that $S'$ is not affine-ruled and that $\bar{\kappa}(S_0) \neq 2$.

1. Either $S'$ has a unique balanced completion up to isomorphism, or it admits an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and more than one singular fiber. In the latter case, $S'$ has exactly two strongly balanced completions.

2. If $S'$ has more than one singular point, then it has exactly two singular points, both of Dynkin type $A_1$, and there is a twisted $\mathbb{C}^*$-ruling of $S'$ such that both singular points are contained in a unique fiber isomorphic to $\mathbb{C}^1$.

3. If $S'$ contains a quotient noncyclic singularity, then either $S' \cong \mathbb{C}^2/G$ for a small finite noncyclic subgroup of $\text{GL}(2, \mathbb{C})$, or $S'$ has a twisted $\mathbb{C}^*$-ruling. In the latter case, the unique fiber isomorphic to $\mathbb{C}^1$ is of type $(A)(iv)$ (see Theorem 4.9) and contains a singular point of Dynkin type $D_k$ for some $k \geq 4$.

We now comment on other corollaries of the classification. First, the case can occur when $S'$ has exactly one singular point and it is a cyclic singularity. Second, we show that if $S'$ is affine-ruled, then its strongly balanced weighted boundary is unique unless it is a chain, but that even if it is unique, there still may be infinitely many strongly balanced completions (see Example 3.6). Third, the singularities of affine-ruled $S'$ are necessarily cyclic, but there may be arbitrarily many of them (see [Miyanishi and Sugie 1991] or Section 3). Regarding the remaining case $\bar{\kappa}(S_0) = 2$, which we do not analyze here, let us mention that it follows from the logarithmic Bogomolov–Miyaoka–Yau inequality (see [Palka 2008], for example) that $S'$ has only one singular point and it is of quotient type.

It is known that smooth $\mathbb{Q}$-homology planes can have moduli [Flenner and Zaïdenberg 1994]. The same is true for singular ones. We prove the following result.

**Theorem 1.2.** There exist arbitrarily high-dimensional families of nonisomorphic singular $\mathbb{Q}$-homology planes having smooth locus of negative Kodaira dimension and having the same singularities, same homeomorphism type, and same weighted strongly balanced boundary.

An important property of any $\mathbb{Q}$-homology plane with smooth locus of general type is that it does not contain topologically contractible curves. In fact, the number of contractible curves on a $\mathbb{Q}$-homology plane is known except in the case when the surface is singular and the smooth locus has Kodaira dimension zero (see Section 6).
In Theorem 6.1, we compute the number of different $\mathbb{C}^*$-rulings a $\mathbb{Q}$-homology plane can have. The computation of the number of contractible curves follows from it.

**Theorem 1.3.** If a singular $\mathbb{Q}$-homology plane has smooth locus of Kodaira dimension zero, then it contains one or two irreducible topologically contractible curves if the smooth locus admits a $\mathbb{C}^*$-ruling, and no such curves otherwise.

The notion of a balanced weighted boundary of an open surface (see Definition 2.10) is a more flexible version of the notion of a standard graph from [Flenner et al. 2007], which has its origin in [Daigle 2008]. It follows from above that every $\mathbb{Q}$-homology plane admits up to isomorphism one or two strongly balanced boundaries, but this is not so for the standard ones. The set of such boundaries is a useful invariant of the surface.

Integral homology groups and necessary conditions for singular fibers of $\mathbb{C}^1$- and $\mathbb{C}^*$-ruled $\mathbb{Q}$-homology planes have already been analyzed in [Miyanishi and Sugie 1991]. For $\mathbb{C}^*$-rulings, however, these conditions are not sufficient (see Examples 4.2 and 4.3), and a more detailed analysis is necessary. Also, some formulas for the Kodaira dimension in terms of singular fibers from [Miyanishi and Sugie 1991] require nontrivial corrections (see Section 4B).

**2. Preliminaries**

We follow the notational conventions and terminology of [Miyanishi 2001], [Fujita 1982] and [Palka 2008]. We recall some of them for the convenience of the reader.

**2A. Divisors and normal pairs.** Let $T = \sum t_i T_i$ be an snc-divisor on a smooth complete surface with distinct irreducible components $T_i$. Then $T = \sum T_i$, where the sum runs over $i$ with $t_i \neq 0$, is the reduced divisor with the same support as $T$, and $\beta_T(T_i) = T \cdot (T - T_i)$ is the branching number of $T_i$. A tip has $\beta_T(T_i) \leq 1$. By $Q(T)$ we denote the intersection matrix of $T$; we put $d(0) = 1$ and $d(T) = \det(-Q(T))$ for $T \neq 0$. The symbol “≡” denotes numerical equivalence of divisors.

If $T$ is reduced and its dual graph is linear, it is called a *chain*, and in writing it as a sum of irreducible components $T = T_1 + \cdots + T_n$, we assume that $T_i \cdot T_{i+1} = 1$ for $1 \leq i \leq n - 1$. We put $T' = T_n + \cdots + T_1$. If $T$ is a rational chain, then we write $T = [-T_1^2, \ldots, -T_n^2]$. A rational chain with all $T_i^2 \leq -2$ is called *admissible*. A *fork* is a rational tree for which the branching component is unique and has $\beta = 3$.

Let $D$ be some reduced snc-divisor that is not an admissible chain. A rational chain with support contained in $D$, not containing branching components of $D$ and containing one of its tips, is called a *twig* of $D$. For an admissible (ordered) chain, we put $e(T) = \frac{d(T - T_1)}{d(T)}$ and $\tilde{e}(T) = e(T')$. 
In general, \( e(T) \) and \( \tilde{e}(T) \) are defined as the sums of respective numbers computed for all maximal admissible twigs of \( T \). Here we use the convention that the tip of the twig is the first component.

If \( X \) is a complete surface and \( D \) is a reduced snc-divisor contained in the smooth part of \( X \), then we call \((X, D)\) an snc-pair and we write \( X - D \) for \( X \setminus D \). The pair is normal (resp. smooth) if \( X \) is normal (resp. smooth). If \( X \) is a normal surface, then an embedding \( \iota : X \to \overline{X} \), where \((\overline{X}, \overline{X} \setminus X)\) is a normal pair, is called a normal completion of \( X \). If \( X \) is smooth, then \( \overline{X} \) is smooth and \((\overline{X}, D, \iota)\) is called a smooth completion of \( X \). A morphism of two completions \( \iota_j : X \to \overline{X}_j \), with \( j = 1, 2 \), of a given surface \( X \) is a morphism \( f : \overline{X}_1 \to \overline{X}_2 \) such that \( t_2 = f \circ t_1 \).

Let \( \pi : (X, D) \to (X', D') \) be a birational morphism of normal pairs. We put \( \pi^{-1}D' = \pi^*D' \); that is, \( \pi^{-1}D' \) is the reduced total transform of \( D' \). Assume \( \pi^{-1}D' = D \). If \( \pi \) is a blow-up, then we call it subdivisional (resp. sprouting) for \( D' \) if its center belongs to two (resp. one) components of \( D' \). In general, we say that \( \pi \) is subdivisional for \( D' \) (and for \( D \)) if for any component \( T \) of \( D' \) we have \( \beta(T) = \pi^{-1}(\pi(T)) \). The exceptional locus of a birational morphism between two surfaces \( \eta : X \to X' \), denoted by \( \text{Exc}(\eta) \), is defined as the locus of points in \( X \) for which \( \eta \) is not a local isomorphism.

A \( b \)-curve is a smooth rational curve with self-intersection \( b \). A divisor is snc-minimal if all of its \((-1)\)-curves are branching. We write \( K_X \) for the canonical divisor on a complete surface \( X \).

**Definition 2.1.** A birational morphism of surfaces \( \pi : X \to X' \) is a connected modification if it is proper, \( \pi(\text{Exc}(\pi)) \) is a smooth point on \( X' \), and \( \text{Exc}(\pi) \) contains a unique \((-1)\)-curve. If \( \pi \) is a morphism of pairs \( \pi : (X, D) \to (X', D') \) such that \( \pi^{-1}(D') = D \) and \( \pi(\text{Exc}(\pi)) \subset D' \), we call it a connected modification over \( D' \).

A sequence of blow-downs (and its reversing sequence of blow-ups) whose composition is a connected modification is called a connected sequence of blow-downs (blow-ups).

**2B. Rational rulings.** A surjective morphism \( p_0 : X_0 \to B_0 \) of a normal surface onto a smooth curve is a rational ruling if general fibers are rational curves. By a completion of \( p_0 \), we mean a triple \((X, D, p)\), where \((X, D)\) is a normal completion of \( X_0 \) and \( p : X \to B \) is an extension of \( p_0 \) to a \( \mathbb{P}^1 \)-ruling, with \( B \) being a smooth completion of \( B_0 \). We say that \( p \) is a minimal completion of \( p_0 \) if \( p \) does not dominate any other completion of \( p_0 \). In this case we also say that \( D \) is \( p \)-minimal. It is easy to check that \( D \) is \( p \)-minimal if and only if all of its nonbranching \((-1)\)-curves are horizontal. Let \( F \) be a fiber of \( p \). An irreducible curve \( G \subset X \) is an \( n \)-section of \( p \) if \( G \cdot F = n \). A section is a 1-section. We call \( p_0 \) a \( C^{(n)} \)-ruling if \( F \cdot D = n + 1 \) for \( n \geq 1 \). In the case \( n = 0 \), we call \( p_0 \) a \( C^1 \)-ruling or an affine ruling; the arithmetic genus of \( F \) \( (p_0(F) = \frac{1}{2} F \cdot (K_X + F) + 1) \) vanishes and
$F^2 = 0$. Conversely, it is well-known that an effective divisor with these properties on a complete surface is a fiber of such a ruling [Barth et al. 2004, V.4.3]. If $J$ is a component of $F$, then we denote by $\mu_F(J)$ the multiplicity of $J$; that is, $F = \mu_F(J)J + F'$, where $F'$ is effective and $J \not\subseteq F'$. The structure of fibers of a $\mathbb{P}^1$-ruling is well known [Fujita 1982, §4].

**Lemma 2.2.** Let $F$ be a singular fiber of a $\mathbb{P}^1$-ruling of a smooth complete surface. Then $F$ is a tree of rational curves and it contains a $(-1)$-curve. Each $(-1)$-curve of $F$ meets at most two other components. If $F$ contains a unique $(-1)$-curve $C$, then:

(i) $\mu(C) > 1$. There are exactly two components of $F$ with multiplicity one, and they are tips of the fiber.

(ii) If $\mu(C) = 2$, then either $F = [2, 1, 2]$ or $C$ is a tip of $F$; in the latter case either $F - C = [2, 2, 2]$ or $F - C$ is a $(-2)$-fork of type $(2, 2, n)$.

(iii) If $F$ is not a chain, then the connected component of $F - C$ not containing curves of multiplicity one is a chain (possibly empty).

We define

$$\Sigma_{X-D} = \sum_{F \not\subseteq D} (\sigma(F) - 1),$$

where $\sigma(F)$ is the number of $(X - D)$-components of a fiber $F$ [Fujita 1982, 4.16]. If $p$ is a $\mathbb{P}^1$-ruling as above, then we call an irreducible curve $G$ *vertical* (for $p$) if $p_* G = 0$; otherwise it is *horizontal*. A divisor is vertical (resp. horizontal) if all of its components are vertical (resp. horizontal). We decompose $D$ as $D = D_h + (D - D_h)$, where $D_h$ is horizontal and $D - D_h$ is vertical. The numbers $h$ and $\nu$ are defined respectively as the number of irreducible components of $D_h$ and as the number of fibers contained in $D$. We have [Fujita 1982, §4]

$$\Sigma_{X-D} = h + \nu + b_2(X) - b_2(D) - 2.$$  

We call a connected component of $F \cap D$ a $D$-*rivet* (or *rivet* if this causes no confusion) if it meets $D_h$ at more than one point or if it is a node of $D_h$.

**Definition 2.3.** Suppose $(X, D, p)$ is a completion of a $\mathbb{C}^*$-ruling of a normal surface $X$. We say that the original ruling $p_0 = p|_{X-D}$ is *twisted* if $D_h$ is a 2-section. If $D_h$ consists of two sections, we say that $p_0$ is *untwisted*. Let $F$ be a singular fiber of $p$ that does not contain singular points of $X$. We say that $F$ is *columnar* if $F$ is a chain that can be written as

$$F = A_n + \cdots + A_1 + C + B_1 + \cdots + B_m,$$

where $C$ is a unique $(-1)$-curve and $D_h$ meets $F$ exactly in $A_n$ and $B_m$. The chains $A = A_1 + \cdots + A_n$ and $B = B_1 + \cdots + B_m$ are called *adjoint chains*. 
Remark. By expansion properties of determinants (see [Koras and Russell 2007, 2.1.1], for example) and the fact that $d(A)$ and $d(A - A_1)$ are coprime, we have $e(A) + e(B) = 1$ and $d(A) = d(B) = \mu_F(C)$. In fact, we have also $\tilde{e}(B) + \tilde{e}(A) = 1$ [Fujita 1982, 3.7].

2C. Balanced completions.

Definition 2.4. A pair $(D, w)$ consisting of a complete curve $D$ and a rationally valued function $w$ defined on the set of irreducible components of $D$ is called a weighted curve. If $(X, D)$ is a normal pair, then $(D, w)$ with $w$ defined by $w(D_i) = D_i^2$ is a weighted boundary of $X - D$.

Definition 2.5. Let $(X, D)$ be a normal pair.

(i) Let $L$ be a 0-curve that is a nonbranching component of $D$, and let $c \in L$ be chosen so that if $L$ intersects two other components of $D$, then $c$ is one of the points of intersection. Make a blow-up of $c$ and contract the proper transform of $L$. The resulting pair $(X', D')$, where $D'$ is the reduced direct image of the total transform of $D$, is called an elementary transform of $(X, D)$. The pair $\Phi = (\Phi^o, \Phi^\ast)$ consisting of an assignment $\Phi^o: (X, D) \mapsto (X', D')$ together with the resulting rational mapping $\Phi^\ast: X \dasharrow X'$ is called an elementary transformation over $D$. $\Phi$ is inner (for $D$) if $\beta_D(L) = 2$, and outer (for $D$) if $\beta_D(L) = 1$. The point $c \in L$ is the center of $\Phi$.

(ii) For a sequence of (inner) elementary transformations

$$\Phi^o_i: (X_i, D_i) \mapsto (X_{i+1}, D_{i+1}),$$

with $i = 1, \ldots, n - 1$, we put $\Phi^o = (\Phi^o_1, \ldots, \Phi^o_{n-1})$, $\Phi^\ast = (\Phi^\ast_1, \ldots, \Phi^\ast_{n-1})$ and we call $\Phi = (\Phi^o, \Phi^\ast)$ an (inner) flow in $D_1$. We denote it by $\Phi: (X_1, D_1) \rightsquigarrow (X_n, D_n)$.

$\Phi^\ast = (\Phi^\ast_1, \ldots, \Phi^\ast_{n-1})$ induces a rational mapping $X_1 \dasharrow X_n$, which we also denote by $\Phi^\ast$. There exists the largest open subset of $X_1$ on which $\Phi^\ast$ is a morphism; the complement of this subset is called the support of $\Phi$. Clearly, $\text{Supp} \ \Phi \subseteq D_1$. If $\text{Supp} \ \Phi = \emptyset$, then $\Phi$ is a trivial flow.

A weighted curve $(D, w)$ determines the weighted dual graph of $D$. If $(D, w)$ is a weighted boundary coming from a fixed normal pair $(X, D)$, we omit the weight function $w$ from the notation. For $\Phi$ as above, $D_1$ and $D_n$ are isomorphic as curves. They have the same dual graphs, but usually different weights of components.

Example 2.6. Let $T = [0, 0, a_1, \ldots, a_n]$. Each chain of type $[0, b, a_1, \ldots, a_n]$, $[a_1, \ldots, a_k - 1, a_k - b, 0, b, a_{k+1}, \ldots, a_n]$ or $[a_1, \ldots, a_n, b, 0]$, where $1 \leq k \leq n$ and $b \in \mathbb{Z}$, can be obtained from $T$ by a flow. This follows from the observation that an elementary transformation interchanges the chains $[w, x, 0, y - 1, z]$ and $[w, x - 1, 0, y, z]$. Looking at the dual graph, we see the weights can “flow” from
one side of a 0-curve to another, and possibly vanish \((b = 0 \text{ or } b = a_k)\). If they do, then again the weights can flow through the new zero.

**Definition 2.7.** A rational chain \(D = [a_1, \ldots, a_n]\) is *balanced* if \(a_1, \ldots, a_n \in \{0, 2, 3, \ldots\}\) or if \(D = [1]\). A reduced snc-divider whose dual graph contains no loops (snc-forest) is *balanced* if all rational chains contained in \(D\) that do not contain branching components of the divisor are balanced. A normal pair \((X, D)\) is *balanced* if \(D\) is balanced.

Recall that if \((X_i, D_i)\) for \(i = 1, 2\) are normal pairs such that \(X_1 - D_1 \cong X_2 - D_2\), then \(D_1\) is a forest if and only if \(D_2\) is a forest.

**Proposition 2.8.** A normal surface that admits a normal completion with a forest as a boundary has a balanced completion. Two such completions differ by a flow.

As we discovered after completing the proof, a more general version of this proposition was proved in a graph-theoretic context in [Flenner et al. 2007, Theorem 3.1 and Corollary 3.36]. We therefore leave our more direct arguments to be published elsewhere. In fact, some key observations were made earlier in [Daigle 2008, 4.23.1, 3.2, 5.2]. Let us restate some definitions from [Flenner et al. 2007] on the level of pairs.

**Definition 2.9.** Let \((X, D)\) be a normal pair and assume \(D\) is an snc-forest.

(i) Connected components of the divisor that remains after subtracting all nonrational and all branching components of \(D\) are called the *segments* of \(D\).

(ii) \(D\) is *standard* if for each of its connected components, either the component is equal to \([1]\) or all of its segments are of types \([0], [0, 0, 0]\) or \([0^{2k}, a_1, \ldots, a_n]\), with \(k \in \{0, 1\}\) and \(a_1, \ldots, a_n \geq 2\).

(iii) Let \(D_0 = [0, 0, a_1, \ldots, a_n]\), with \(a_i \geq 2\) for \(i = 1, \ldots, n\), be a segment of \(D\). A *reversion* of \(D_0\) is a nontrivial flow \(\Phi : (X, D) \leadsto (X', D')\) that is supported in \(D_0\), is inner for \(D_0\), and satisfies \(D' - (\Phi^*)_*(D - D_0) = [a_1, a_2, \ldots, a_n, 0, 0]\).

The condition that \(\Phi\) be nontrivial is introduced for the following reason: we want the reversion to transform the two zeros to the other end of the chain, and the condition in necessary to force this in case \(D\) is symmetric, that is, when \([a_1, \ldots, a_n]^t = [a_1, \ldots, a_n]\). Standard chains are called *canonical* in [Daigle 2008]. The Hodge index theorem implies that if \((X, D)\) is a smooth pair and \(D\) is a forest, then it cannot have segments of type \([0^{2k+1}]\) or \([0^{2k}, a_1, \ldots, a_n]\) for \(k > 1\), and can have at most one such segment for \(k = 1\).

Clearly, not every balanced forest is standard, but by a flow one can easily make it so. It follows from Proposition 2.8 that if \(D\) and \(D'\) are two standard boundaries of the same surface and \(D\) is a chain, then either \(D\) and \(D'\) are isomorphic as weighted curves or \(D'\) is the reversion of \(D\). Unfortunately, the notion of a standard
boundary is not as restrictive as one may imagine, and the difference between two standard boundaries can be more than just a reversion of some segments. An additional ambiguity is related to the existence of segments of type $[0^{2k+1}]$. Specifically, if $[0^{2k+1}]$ is a segment of $D$, then one can change by a flow the self-intersections of the components of $D$ intersecting the segment. For example, consider a surface whose standard boundary is a rational fork with a dual graph

$$
\begin{array}{c}
-2 \\
\uparrow \\
0 \\
\downarrow \\
-2 
\end{array}
$$

for some $b \in \mathbb{Z}$. Then for any $b \in \mathbb{Z}$, there is a completion of this surface for which the boundary is standard and has the dual graph as above.\textsuperscript{1} We therefore introduce the following more restrictive conditions.

**Definition 2.10.** A balanced snc-forest $D$ is strongly balanced if it is standard and either $D$ contains no segments of type $[0]$ or $[0, 0, 0]$, or for at least one such segment there is a component $B \subseteq D$ intersecting it such that $B^2 = 0$. A normal pair $(X, D)$ for which $D$ is a forest is strongly balanced if $D$ is strongly balanced.

2D. **Basic properties of $\mathbb{Q}$-homology planes.** We assume that $S'$ is a singular $\mathbb{Q}$-homology plane, that is, a normal nonsmooth complex algebraic surface with $H^*(S', \mathbb{Q}) \cong \mathbb{Q}$. Let $\epsilon : S \to S'$ be a resolution such that the inverse image of the singular locus is an snc-divisor, and let $(\tilde{S}, D)$ be a smooth completion of $S$. Denote the singular points of $S'$ by $p_1, \ldots, p_q$ and the smooth locus by $S_0$. We put $\hat{E}_i = \epsilon^{-1}(p_i)$ and assume that $\hat{E} = \hat{E}_1 + \hat{E}_2 + \cdots + \hat{E}_q$ is snc-minimal. Recall that $S'$ is called logarithmic if and only if every singular point of $S'$ is locally analytically isomorphic to $\mathbb{C}^2/G$ for some finite subgroup $G < \text{GL}(2, \mathbb{C})$ (a quotient singularity). In [Palka 2008], we classified nonlogarithmic $\mathbb{Q}$-homology planes. In particular, it is known that they do not admit $\mathbb{C}^1$- or $\mathbb{C}^*$-rulings. Therefore, from now on we assume that $S'$ is logarithmic. By the argument in [Fujita 1982, 2.4], it is affine.

**Proposition 2.11.** Let the notation be as above.

(i) $D$ is a rational tree with $d(D) = -d(\hat{E}) \cdot |H_1(S', \mathbb{Z})|^2$.

(ii) The embedding $D \cup \hat{E} \to \tilde{S}$ induces an isomorphism on $H_2(\tilde{S}, \mathbb{Q})$.

\textsuperscript{1}This observation was missed in [Flenner et al. 2007], whose Corollary 3.33 is false. See [Flenner et al. 2011] for corrections. In [Daigle 2008, Solution to problem 5] this ambiguity is implicitly taken into account without restricting to balanced divisors.
Assume that

Then the surface $S$ defined as the image of $\bar{S} - D$ after contraction of connected components of $T - D$ to points is a rational $\mathbb{Q}$-homology plane, and $p$ induces a rational ruling of $S'$. Conversely, if $p': S' \to B$ is a rational ruling of a rational $\mathbb{Q}$-homology plane $S'$, then any completion $(\bar{S}, T, p)$ of the restriction of $p'$ to the smooth locus of $S'$ has the above properties.

**Proof.** Since the base of $p$ has some component of $D$ as a branched cover, it is rational, and hence $\tilde{S}$ is rational. We may assume that $T$ is $p$-minimal. Put $\tilde{E} = T - D$. Since $\tilde{E}$ is vertical and since $\tilde{E} \cap D = \emptyset$, $Q(\tilde{E})$ is negative definite and $b_1(\tilde{E}) = 0$. Fujita’s equation

$$\Sigma_{\bar{S} - T} = h + v - 2 + b_2(\bar{S}) - b_2(D + \tilde{E})$$

gives $b_2(\bar{S}) = b_2(T)$, so by (iv), the inclusion $T \to \bar{S}$ induces an isomorphism on $H_2(\cdot, \mathbb{Q})$. By [Palka 2008, 2.6], $S'$ is normal and affine, and in particular $b_4(S') = b_3(S') = 0$. Since $b_1(D) = 0$, the exact sequence of the pair $(\bar{S}, D)$ together with the Lefschetz duality give

$$b_2(S) = b_2(\bar{S}, D) = b_2(\bar{S}) - b_2(D) = b_2(\tilde{E}).$$

Since $b_1(\tilde{E}) = 0$, we get from the exact sequence of the pair $(S, \tilde{E})$ that $b_2(S') = b_2(S, \tilde{E}) = b_2(S) - b_2(\tilde{E}) = 0$. Now

$$\chi(S') = \chi(\bar{S}) - \chi(D \cup \tilde{E}) + b_0(\tilde{E}) = b_0(D) = 1,$$

so we obtain $b_1(S') = b_2(S') = 0$, and hence $S'$ is $\mathbb{Q}$-acyclic.

Conversely, if $p'$ is as above, then let $\tilde{E}$ be an exceptional divisor of a resolution of singularities of $S'$, and let $D = T - \tilde{E}$. Since $\tilde{E}$ is vertical for the $\mathbb{P}^1$-ruling $p$, we have $b_1(\tilde{E}) = 0$. Then the necessity of the above conditions follows from [ibid., 3.1 and 3.2].
3. Smooth locus of negative Kodaira dimension

Here we assume that the smooth locus $S_0$ of the logarithmic $\mathbb{Q}$-homology plane $S'$ has negative Kodaira dimension, implying that the Kodaira dimension of $S'$ is also negative. This case was analyzed and a structure theorem given in [Miyanishi and Sugie 1991, 2.5–2.8]. We recover these results in Lemma 3.2 and Proposition 3.1, but we concentrate on analyzing possible completions and boundaries instead of $S'$ itself. This gives more information, allowing us to give a construction and to answer the question of uniqueness of an affine ruling of $S_0$ (if it exists). The information about completions is also used in the analysis of an example where moduli occur.

**Proposition 3.1.** If a singular $\mathbb{Q}$-homology plane has smooth locus of negative Kodaira dimension, then it is affine-ruled or isomorphic to $\mathbb{C}^2/G$ for some small finite, noncyclic subgroup $G < \text{GL}(2, \mathbb{C})$. The surfaces $\mathbb{C}^2/G$ and $\mathbb{C}^2/G'$ are isomorphic if and only if $G$ and $G'$ are conjugate in $\text{GL}(2, \mathbb{C})$. The minimal normal completion of $\mathbb{C}^2/G$ is unique and the boundary is a nonadmissible rational fork with admissible twigs.

**Proof.** For the first part of the statement, we follow the arguments of [Koras and Russell 2007, §3]. Assume that $S'$ is not affine-ruled. Then $S_0$ is not affine-ruled. Since $S'$ is affine, $D + \hat{E}$ is not negative definite, so by [Miyanishi 2001, 2.5.1], $S_0$ contains a platonically $\mathbb{C}^*$-fibered open subset $U$, which is its almost minimal model. Also, $\chi(U) \leq \chi(S_0)$ (see [Palka 2011a, 2.8]). The algorithm of construction of an almost minimal model [Miyanishi 2001, 2.3.8, 2.3.11] implies that $S_0 - U$ is a disjoint sum of $s$ curves isomorphic to $\mathbb{C}$ and $s'$ curves isomorphic to $\mathbb{C}^*$, for some $s, s' \in \mathbb{N}$. It follows that

$$0 = \chi(U) = \chi(S_0) - s = \chi(S') - q - s = 1 - q - s,$$

so $s = 0$, $q = 1$, and $s' \leq 1$. If $s' \neq 0$, then the boundary divisor of $U$ is connected, and hence $U$ and $S_0$ are affine-ruled. Thus $s' = 0$ and $S_0 = U$, and by [Miyanishi and Tsunoda 1984], $S' \cong \mathbb{C}^2/G$, where $G$ is a small finite noncyclic subgroup of $\text{GL}(2, \mathbb{C})$.

Suppose $G$ and $G'$ are two subgroups of $\text{GL}(2, \mathbb{C})$ such that $\mathbb{C}^2/G \cong \mathbb{C}^2/G'$. Then $\hat{G}_{\mathbb{C}^2/G, (0)} \cong \hat{G}_{\mathbb{C}^2/G', (0)}$, so if $G$ and $G'$ are small then they are conjugate, by [Prill 1967, Theorem 2]. The $\mathbb{C}^*$-ruled of $S_0$ does not extend to a ruling of $S'$, so by [Palka 2008, 4.5], its boundary is a rational fork with admissible maximal twigs and its minimal normal completion is unique up to isomorphism. (For the description of the boundary, one could also use a more general result [Miyanishi 2001, 2.5.2.14].) \qed
3A. Affine-ruled planes. By Proposition 3.1, we may assume that $S'$ is affine-ruled. This gives an affine ruling of $S_0$. We assume that $(\tilde{S}, D + \hat{E}, p)$ is a minimal completion of the latter. This weakens our initial snc-minimality assumption on $D$; that is, $D$ is now $p$-minimal, but the unique section contained in $D$ may be a nonbranching $(-1)$-curve. The base of $p$ is rational because it is isomorphic to a section contained in $D + \hat{E}$.

**Lemma 3.2.** If $S'$ is affine-ruled, then there exists exactly one fiber of $p$ contained in $D$ (see Figure 1). Each other singular fiber has a unique $(-1)$-curve, which is an $S_0$-component. The singularities of $S'$ are cyclic.

**Proof.** We have $\Sigma_{S_0} = v - 1$ and $v \leq 1$ by Proposition 2.11, so $\Sigma_{S_0} = 0$ and there is exactly one fiber $F_{\infty}$ contained in $D$. The fiber is smooth by the $p$-minimality of $D$. Each singular fiber $F$ of $p$ contains exactly one $(-1)$-curve. Indeed, if $D_0 \subseteq D$ is a vertical $(-1)$-curve, then by the $p$-minimality of $D$, it meets $D_h$ and two $D$-components, so $\mu(D_0) > 1$. This is impossible because $D_h \cdot F = 1$. The $(-1)$-curve, say $C$, has $\mu(C) > 1$ and is the unique $S_0$-component of $F$. There are exactly two components of multiplicity one in $F$; they are tips of $F$ and $D_h$ intersects one of them. Thus the connected component of $F - C$ not contained in $D$ is a chain, so $S'$ has only cyclic singularities. \hfill \Box

**Remark.** In Lemma 3.2, it was assumed (as in the whole paper — see Section 2D) that $S'$ is logarithmic, but there is in fact no need for this. In any case $\hat{E}$ is vertical, so it is a rational forest. Then $D$ is a rational tree, and $\tilde{S}$ and the base of $p$ are rational by [Palka 2008, 3.4(i)]. The rest of the argument goes through.

**Construction 3.3.** Let $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ be the first Hirzebruch surface with a (unique) projection $\tilde{p} : F_1 \to \mathbb{P}^1$. Denote the section coming from the inclusion of the first summand by $D'_h$; then $D'_h^2 = -1$. Choose $n + 1$ distinct points $x_{\infty}, x_1, \ldots, x_n \in D'_h$, and let $F_{\infty}$ be the fiber containing $x_{\infty}$. For each $i = 1, \ldots, n$ starting from a blow-up of $x_i$, create a fiber $F_i$ over $\tilde{p}(x_i)$ containing a unique $(-1)$-curve $C_i$. Let $D_i$ be the connected component of $F_i - C_i$ intersecting $D_h$. 

![Figure 1. Affine-ruled $S'$.](image)
the proper transform of \(D_h'\). By renumbering, we may assume there is \(m \leq n\) such that \(C_i\) is a tip of \(F_i\) if and only if \(i > m\). Assume also that \(m \geq 1\) (for \(m = 0\) we would get a smooth surface). For \(i \leq m\), put \(\widehat{E}_i = F_i - D_i - C_i\). Clearly, each \(\widehat{E}_i\) is a chain. Let \(\tilde{S}\) be the resulting surface and let \(p : \tilde{S} \to \mathbb{P}^1\) be the induced \(\mathbb{P}^1\)-ruling. Put \(D = F_\infty + D_h + \sum_{i=1}^{n} D_i\), \(S = \tilde{S} - D\) and \(\widehat{E} = \sum_{i=1}^{m} \widehat{E}_i\). We define \(\epsilon : S \to S'\) as the morphism contracting \(\widehat{E}_i\)'s.

**Remark 3.4.** Let \(p : \tilde{S} \to \mathbb{P}^1\) be as in 3.3, and for a fiber \(F\) denote the greatest common divisor of multiplicities of all \(S\)-components of \(F\) by \(\mu_S(F)\). By Proposition 2.11, we have \(H_1(S', \mathbb{Z}) = H_1(S, \mathbb{Z})\). By [Fujita 1982, 4.19, 5.9],

\[
H_1(S, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}^{\mu_S(F_i)},
\]

so \(H_1(S', \mathbb{Z})\) can be any finite abelian group. It is easy to see that \(\mu_S(F_i) = \mu(C_i)/d(\widehat{E}_i)\), where \(d(\widehat{E}_i) = d(0) = 1\) if \(i > m\). In particular, \(S'\) is a \(\mathbb{Z}\)-homology plane if and only if \(m = n\) and each \(F_i\) is a chain. In fact in the latter case \(\pi_1(S)\) vanishes and so \(S'\) is contractible.

**Theorem 3.5.** The surface \(S'\) in Construction 3.3 is an affine-ruled singular \(\mathbb{Q}\)-homology plane. Conversely, each singular \(\mathbb{Q}\)-homology plane admitting an affine ruling can be obtained by Construction 3.3. Its strongly balanced boundary is unique if it is branched and is unique up to reversion if it is a chain. The affine ruling of \(S'\) is unique if and only if its strongly balanced boundary is not a chain.

**Proof.** By definition, \(\widehat{E}_i\)'s are admissible chains, so \(S'\) is normal and has only cyclic singularities. We have \(d(D) = -\prod_i d(D_i)\) [Koras and Russell 1999, 2.1.1], so \(d(D) \neq 0\), and hence \(S'\) is a singular \(\mathbb{Q}\)-homology plane by Lemma 2.12. The last part of the statement almost follows from Lemma 3.2. It remains to note that by a flow (see Example 2.6), we can freely change the self-intersection of the horizontal boundary component without changing the rest of \(D\), so we can assume that the construction starts with a negative section on \(\mathbb{F}_1\). (We could, for instance, start with \(D_h'\) equal to the negative section on \(\mathbb{F}_n\), so that the resulting boundary would be strongly balanced; see Definition 2.10). The uniqueness of a strongly balanced boundary follows from Proposition 2.8.

We now consider the uniqueness of an affine ruling. Let \((V_i, D_i, p_i)\) be two minimal completions of two affine rulings of \(S'\) (see Section 2B). By Lemma 3.2, both \(D_i\) contain a 0-curve \(F_{\infty,i}\) as a tip. By flows with supports in \(F_{\infty,i}\), we may assume both \(D_i\) are standard (see Definition 2.9).

Assume that \(D_1\) is not a chain. Then \(D_1\) and \(D_2\) are isomorphic as weighted curves (see Proposition 2.8). Let \(T_i\) be the unique maximal twig of \(D_i\) containing a 0-curve. Then either \(T_i = F_{\infty,i} = \{0\}\), or we can write \(T_i = \{0, 0, a_1, \ldots, a_n\}\) with \([a_1, \ldots, a_n]\) admissible. Then there is a flow \(\Phi : (V_1, D_1) \sim (V_2, D_2)\) by
Proposition 2.8. Because $D_1$ is branched, $\text{Supp} \Phi^* \subseteq T_1$. Also, it follows from Proposition 2.8 and Example 2.6 that $\text{Supp} \Phi^* \subseteq F_{\infty,i}$. For $i = 1, 2$, let $f_i$ be some fiber of $p_i$ other than $F_{\infty,i}$. Since $\Phi^*(f_1)$ is disjoint from $F_{\infty,2}$, we get $\Phi^*(f_1) \cdot f_2 = 0$, so $p_1$ and $p_2$ agree on $S'$.

Let $(V_1, D_1)$ be a standard completion of $S'$ with $D_1 = [0, 0, a_1, \ldots, a_n]$. We may assume that $[a_1, \ldots, a_n]$ is admissible and nonempty; if it is empty, then $S' \cong C^2$ is smooth, and if it is nonadmissible, then by the Hodge index theorem we necessarily have $D_1 = [0, 0, 0]$, which disagrees with Proposition 2.11(i). Let $(V_2, D_2)$ be another completion of $S'$, with $D_2$ being a reversion of $D_1$. The 0-tip $T_i$ of each $D_i$ induces an affine ruling on $S'$. Let $(V, D)$ be a minimal normal pair dominating both $(V_i, D_i)$, such that both affine rulings extend to $\mathbb{P}^1$-rulings of $V$. We argue that these affine rulings are different by proving that $\sigma_1^* T_1 \cdot \sigma_2^* T_2 \neq 0$, where $\sigma_1 : (V, D) \to (V_1, D_1)$ are the dominations. Suppose $\sigma_1^* T_1 \cdot \sigma_2^* T_2 = 0$. Let $H$ be an ample divisor on $V$ and let $(\lambda_1, \lambda_2) \neq (0,0)$ be such that $\lambda_1^2 H = 0$ for $\tilde{T} = \lambda_1 \sigma_1^* T_1 + \lambda_2 \sigma_2^* T_2$. We have $(\sigma_i^* T_i)^2 = T_i^2 = 0$, so

$$\tilde{T}^2 = 2\lambda_1 \lambda_2 \sigma_1^* T_1 \cdot \sigma_2^* T_2 = 0,$$

and hence $\tilde{T} \equiv 0$ by the Hodge index theorem. But $D$ has a nondegenerate intersection matrix, because $d(D) = d(D_1) \neq 0$, so $\tilde{T}$ is a zero divisor. Then $\sigma_1^* T_1 = [0]$, for otherwise $\sigma_1^* T_1$ and $\sigma_2^* T_2$ would contain a common $(-1)$-curve, which contradicts the minimality of $(V, D)$. It follows that $\sigma_1$ (and $\sigma_2$) are identities. This contradicts the fact that the reversion for nonempty $[a_1, \ldots, a_n]$ is a nontrivial transformation of the completion (even if $[a_1, \ldots, a_n]^t \equiv [a_1, \ldots, a_n]$).

The following example shows that even if the strongly balanced boundary is unique, there might be infinitely many strongly balanced completions.

**Example 3.6.** Let $(V, D, \iota)$ be an snc-minimal completion ($\iota$ is the embedding; see Section 2A) of an affine-ruled singular $\mathbb{Q}$-homology plane $S'$ as above. Assume that $D_h$ is branched and that $D_h^2 = -1$. The only change of $D$ that can be made by a flow is a change of the weight of $D_h$. If we now make an elementary transformation $(V, D) \mapsto (V_x, D_x)$ with a center $x \in F_{\infty} \setminus D_h$, then $D$ becomes strongly balanced (see Definition 2.10). Denote the resulting completion by $(V_x, D_x, \iota_x)$ and let $F_{\infty,x}$ be the new fiber at infinity. The isomorphism type of the weighted boundary $D_x$ does not depend on $x$, but for different $x$ the completions (triples) are clearly different. In general, even the isomorphism type of the pair $(V_x, D_x)$ depends on $x$. To see this, let $(V_x, D_x) \cong (V_y, D_y)$. Because the isomorphism maps $F_{\infty, x}$ to $F_{\infty, y}$, we get an automorphism of $(V, D)$ mapping $x$ to $y$. Taking a minimal resolution $\tilde{S} \rightarrow V$, contracting all singular fibers to smooth fibers without touching $D_h$, and contracting $D_h$, we see that for $x \neq y$, this automorphism descends to a nontrivial
automorphism of $\mathbb{P}^2$ fixing points that are images of contracted $S_0$-components and of $D_h$. In general such an automorphism does not exist.

**3B. Moduli.** Repeating Construction 3.3 in a special case, we obtain arbitrarily high-dimensional families of nonisomorphic singular $\mathbb{Q}$-homology planes with negative Kodaira dimension of the smooth locus and the same homeomorphism type. Example 3.7 gives a proof of Theorem 1.2. For smooth $\mathbb{Q}$-homology planes, a similar example was considered in [Flenner and Zaïdenberg 1994, 4.16].

**Example 3.7.** Put $m = 2$ and $n = N + 2$ for some $N > 0$, and let $\Sigma$, $D$, $\hat{E}$, etc. be created as in the construction above, so that $D_1 = [3]$, $D_2 = [2]$ and $D_i = [2, 2, 2]$ for $3 \leq i \leq n$. Then $\hat{E}_1 = [2, 2]$ and $\hat{E}_2 = [2]$ (see Figure 2).

![Figure 2. Singular fibers in Example 3.7.](image-url)

Denoting the contraction of $\sum_{i=3}^{n} C_i$ by $\sigma : \tilde{S} \to V$, we can factor the contraction $\tilde{S} \to F_1$ (which reverses the construction) as the composition $\tilde{S} \xrightarrow{\sigma} V \xrightarrow{\sigma'} F_1$. Put $y_i = \sigma(C_i)$ and $y = (y_3, \ldots, y_n)$. While $\sigma'^{-1}$ is determined uniquely by the choice of $(x_1, \ldots, x_n)$, $\sigma^{-1}$ and the resulting surface $\tilde{S}$ (and hence $S'$) can depend on the choice of $y$. Let us write $\tilde{S}_y$ and $S'_y$ to indicate this dependence. For $3 \leq i \leq n$, let $D_0^i$ be the open subset of the middle component of $D_i$ remaining after subtracting two points belonging to other components of $D_i$. Put

$$U = D_4^0 \times \cdots \times D_n^0 \cong \mathbb{C}^{N-1}.$$  

The family

$$\{S'_y\}_{y \in D_3^0 \times U} \to D_3^0 \times U$$

is $N$-dimensional. Since there is a compactly supported autodiffeomorphism of the pair $(\mathbb{C}^2, \mathbb{C}^* \times \{0\})$ mapping $(p, 0)$ to $(q, 0)$ for any $p, q \neq 0$, the choice of $y \in D_3^0 \times U$ is unique up to a diffeomorphism fixing irreducible components of $\sigma_*(D + \hat{E} + C_1 + C_2)$. Thus all $S'_y$ are homeomorphic.

Let $\pi : \mathcal{X} \to U$ be the subfamily over $\{y_3^0\} \times U$. We show that the fibers of $\pi$ are nonisomorphic. Suppose that $S'_y \cong S'_z$ for $y, z \in \{y_3^0\} \times U$. The isomorphism extends to snc-minimal resolutions. There is a flow $\Phi^* : \tilde{S}_y \to \tilde{S}_z$ by Proposition 2.8,
which is an isomorphism outside $F_\infty$. Clearly, $\Phi^*$ fixes $D_h \setminus \{x_\infty\}$, $F_1$ and $F_2$, and hence restricts to an identity on $D_h \setminus \{x_\infty\}$ and respects fibers. Since the $C_i$ are unique $(-1)$-curves of the fibers, they are fixed by $\Phi^*$. Therefore $\Phi^*_V|_{\overline{V}_\infty - D_h}$ descends to an automorphism $\Phi_V$ of $V - F_\infty - D_h$ fixing the fibers, such that

$$\Phi_V(y_i) = z_i.$$  

Also, $\Phi_V$ descends to an automorphism $\Phi_{F_1}$ of $\overline{F}_1 - F_\infty - D'_h$ fixing fibers. If $(x, y)$ are coordinates on $\overline{F}_1 - F_\infty - D'_h \cong \mathbb{C}^2$ such that $x$ is a fiber coordinate, then

$$\Phi_{F_1}(x, y) = (x, \lambda y + P(x))$$

for some $P \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}$. Introducing successive affine maps for the blow-ups, one can check that in some coordinates $\Phi_V$ acts on $D_i^0$ as $t \to \lambda^{\mu(C_i)}t$. Now the requirement $y_3 = y_3^0$ fixes $\lambda^2 = 1$, so because $\mu(C_i) = 2$ for each $3 \leq i \leq n$, we get that $y = z$.

**Remark.** By [Fujita 1982, 4.19 and 5.9], for $S'$ as above, $\pi_1(S')$ is the $N$-fold free product of $\mathbb{Z}_2$. It follows from Remark 3.4 that given a weighted boundary, there exist only finitely many affine-ruled singular $\mathbb{Z}$-homology planes with this boundary. That is why in Example 3.7 we use branched fibers $F_i$ for $3 \leq i \leq n$; so that the resulting surfaces are $\mathbb{Q}$-, but not $\mathbb{Z}$-homology planes.

4. $C^*$-ruled $\mathbb{Q}$-homology planes

By [Palka 2008, 1.1(2) and 1.2] and Section 3A, to finish the classification of singular $\mathbb{Q}$-homology planes with smooth locus of nongeneral type, one needs to classify $\mathbb{Q}$-homology planes that are $C^*$-ruled. Therefore, we assume here that $S'$ is $C^*$-ruled (and logarithmic; see Section 2D). The first homology group of $S'$ and some necessary conditions for singular fibers of such rulings are analyzed in [Miyanishi and Sugie 1991, 2.9 and 2.10]. As before, we concentrate on completions rather than the affine part itself, because this gives more information and allows us to give a general method of construction. It also allows us to compute the number of different $C^*$-rulings, and as a consequence the number of affine lines on $S'$.

**4A. Properties of $C^*$-rulings.** We can lift the $C^*$-ruling of $S'$ to a $C^*$-ruling of the resolution and extend it to a $\mathbb{P}^1$-ruling $p: \widetilde{S} \to \mathbb{P}^1$ of a smooth completion. Assume that $D + \widehat{E}$ is $p$-minimal. By Proposition 2.11(v), $\Sigma_{S_0} = h + v - 2$ and $v \leq 1$, so $(h, v, \Sigma_{S_0}) = (1, 1, 0), (2, 1, 1)$ or $(2, 0, 0)$. The original $C^*$-ruling of $S'$ is twisted with base $\mathbb{C}$ in the first case, untwisted with base $\mathbb{C}$ in the second case, and untwisted with base $\mathbb{P}^1$ in the third case.

**Lemma 4.1.** Denote by $F_1, \ldots, F_n$ all the columnar fibers of $p: \widetilde{S} \to \mathbb{P}^1$ (see Definition 2.3). Let $F_\infty$ be the fiber contained in $D$ if $v = 1$. There is exactly one more singular fiber $F_0$; it contains $\widehat{E}$. Moreover:
If \((h, v) = (1, 1)\), then \(F_{\infty} = [2, 1, 2]\), \(\sigma(F_0) = 1\), and \(F_0\) and \(F_{\infty}\) contain
branching points of \(p|_{D_h}\).

(ii) If \((h, v) = (2, 1)\), then \(F_{\infty}\) is smooth and \(\sigma(F_0) = 2\).

(iii) If \((h, v) = (2, 0)\), then \(\sigma(F_0) = 1\) and \(F_0\) contains a \(D\)-rivet.

(iv) If \(h = 2\), then the components of \(D_h\) are disjoint.

Proof. Let \((h, v) = (1, 1)\). Then \(\Sigma_{S_0} = 0\), so by [Fujita 1982, 7.6], every singular
fiber other than \(F_{\infty}\) is either columnar or contains a branching point of \(p|_{D_h}\). Now
\(D_h\) is rational and \(p|_{D_h}\) has two branching points, one of them contained in \(F_{\infty}\),
because \(D\) is a tree. Thus \(F_0\) is unique. The \(p\)-minimality of \(D\) implies that
\(F_{\infty} = [2, 1, 2]\). Now let \(h = 2\). We have \(\Sigma_{S_0} = v \in \{0, 1\}\), and the \(p\)-minimality of
\(D\) gives (ii), (iii) and the uniqueness of \(F_0\). Suppose the components of \(D_h\) have a
common point. \(D\) is a tree, so in this case \(v = 0\), which gives \(\sigma(F_0) = 1\). Because
\(D\) is a simple normal crossing divisor, the common point belongs to the unique \(S_0\)-
component of \(F_0\), which therefore has multiplicity one. The connectedness of \(D\)
implies that \(F_0\) contains no \(D\)-components. But then \(F_0\) has a unique \((-1)\)-curve,
which is impossible by Lemma 2.2.

Lemma 4.1 is essentially [Miyanishi and Sugie 1991, 2.10]. While the conditions
stated above are necessary, they are not sufficient. In the following examples the \(\mathbb{C}^*\)-ruling satisfies them, but the \(\mathbb{C}^*\)-ruled surface one obtains is not a \(\mathbb{Q}\)-homology plane.

Example 4.2. For \(n \geq 0\), let \(\mathbb{F}_n\) be the \(n\)-th Hirzebruch surface, and let \(D_0, D_{\infty}\) be
sections with \(D_0^2 = n\) and \(D_{\infty}^2 = -n\). Let \(F_{\infty}\) be a fiber and put \(D = D_0 + D_{\infty} + F_{\infty}\).
Pick a point not belonging to \(D\) and make a connected sequence of blow-ups over
it. Let \(C_0\) be the unique \((-1)\)-curve in the inverse image of the point, and let \(F_0\) and
\(C_1\) be the reduced total and the proper transform of the fiber. Denote the resulting
surface by \(\widehat{S}\), put \(S = \widehat{S} - D\) and \(\widehat{E} = F_0 - C_0 - C_1\), and let \(S \to S'\) be the morphism
contracting \(\widehat{E}\). In particular, \(\widehat{E}\) might be any admissible chain, in which case \(S'\) has
a unique cyclic singular point. \(S'\) is not a \(\mathbb{Q}\)-homology plane because \(d(D) = 0\);
see Lemma 2.12(iv).

Example 4.3. Take the pair \((\mathbb{F}_1, D_0 + D_{\infty})\), where \(\mathbb{F}_1\) is the first Hirzebruch surface
and \(D_0\) and \(D_{\infty}\) are sections with \(D_0^2 = 1\) and \(D_{\infty}^2 = -1\). Pick two points on \(D_0\) and
blow up over it to create two singular fibers \(F_1 = [2, 1, 2]\), \(F_2 = [2, 1, 2]\). Denote
their \((-1)\)-curves by \(C_1, C_2\). These \((-1)\)-curves separate two chains \(T_0 = [2, 1, 2]\)
and \(T_{\infty} = [2, 1, 2]\), where the middle \((-1)\)-curves are \(D_0\) and \(D_{\infty}\), respectively.
We have \(d(T_0) = d(T_{\infty}) = 0\). Pick a point on some \(C_i\), say \(C_1\), that does not belong
to \(T_0 + T_{\infty}\), and make a connected sequence of blow-ups over it. Let \(C_0\) be the
unique \((-1)\)-curve in the inverse image of the point, and let \(F_0\) be the total reduced
transform of the fiber. Denote the resulting complete surface by \(\widehat{S}\). If \(C_0\) is not a
tip of $F_0$, then denote the connected component of $F_0 - C_0$ not meeting $D_0 + D_\infty$ by $\hat{E}$. Let $D$ be the reduced divisor with support $T_0 \cup T_\infty \cup (F_0 - C_0 - \hat{E})$. Put $S = \bar{S} - D$ and $\hat{E} = F_0 - C_0 - C_1$, and let $S \to S'$ be the morphism contracting $\hat{E}$ (which is necessarily an admissible chain). Again, $S'$ is not a $\mathbb{Q}$-homology plane because $d(D) = 0$.

Theoretically, if $X$ is a normal surface and $p' : X \to B$ is a $\mathbb{C}^*$-ruling, then by taking a completion of $X$ and an extension of $p'$ to a $\mathbb{P}^1$-ruling, with Lemma 2.12 we can recognize when $X$ is a $\mathbb{Q}$-homology plane ($B$ has to be rational). However, to give constructions we need to reformulate the condition $d(D) \neq 0$ in a way that is easier to verify by looking at the geometry of singular fibers. Recall that for a family of subsets $(A_i)_{i \in I}$ of a topological space $Y$, a subset $X \subseteq Y$ separates the subsets $(A_i)_{i \in I}$ (inside $Y$) if and only if each $A_i$ is contained in a closure of some connected component of $Y \setminus X$ and none of these closures contains more than one $A_i$. Recall also that by convention, a twig of a fixed divisor is ordered so that its tip is the first component.

**Lemma 4.4.** Let $(\bar{S}, T, p)$ be a triple satisfying conditions (i)–(iii) of Lemma 2.12. Assume also that $T$ is $p$-minimal and that $f \cdot T = 2$ for a general fiber $f$ of $p$. When $(h, \nu) = (2, 0)$, let $D_0$ be some horizontal component of $D$, let $F_0$ be a unique fiber containing a $D$-rivet, let $B$ be a unique component of $D$ separating $D_0$, $D_h - D_0$ and $\hat{E}$ inside $D \cup F_0$, and let $\tilde{D}_0$ be a connected component of $D - B$ containing $D_0$. Then $d(D) \neq 0$ if and only if the following conditions hold:

(i) The base of the fibration is $\mathbb{P}^1$ or $\mathbb{C}^1$ (that is, $\nu \leq 1$).

(ii) If $(h, \nu) = (2, 1)$, both $S - T$-components of the fiber with $\sigma = 2$ intersect $D$.

(iii) If $(h, \nu) = (2, 0)$, then $d(\tilde{D}_0) \neq 0$.

The advantage of condition (iii) over $d(D) \neq 0$ is that $\tilde{D}_0$ is simpler than $D$, containing at most one branching component.

**Proof:** Clearly, if $d(D) \neq 0$, then $S'$ is a $\mathbb{Q}$-homology plane by Lemma 2.12, which implies (i) and (ii) ($D$ meets each curve not contained in $D + \hat{E}$ because $S'$ is affine). Suppose now that (i) and (ii) are satisfied. We show that $d(D) \neq 0$ is equivalent to (iii) (which is an empty condition if $(h, \nu) \neq (2, 0)$). Note that $d(D) \neq 0$ is equivalent to $d(T) \neq 0$, because $T - D$ is negative definite.

Consider the case $h = 1$. We have $\Sigma_{\bar{S} - \bar{T}} = \nu - 1$, and hence $\nu = 1$ and $\Sigma = 0$. The horizontal component $D_h$ meets the unique fiber $F_\infty$ contained in $T$ in one point, because $T$ is a forest. Let $T_\infty$ be the component meeting $D_h$. We have $d(F_\infty) = 0$, so by [Koras and Russell 1999, 2.1.1(i)],

$$d(D) = d(F_\infty)d(D - F_\infty) - d(F_\infty - T_\infty)d(D - F_\infty - D_h),$$
and we obtain
\[ d(D) = -d(F_{\infty} - T_\infty) d(D - F_{\infty} - D_h). \]

Since \( F_{\infty} - T_\infty \) and \( D - F_{\infty} - D_h \) are vertical and do not contain whole fibers, they are negative definite, and hence \( d(D) < 0 \).

We may now assume \( h = 2 \). Then \( \Sigma = v \in \{0, 1\} \). Put \( \hat{E} = T - D \). When \( v = 1 \), let \( F_{\infty} \) be the unique fiber contained in \( D \), and let \( F_0 \) be the unique singular fiber with \( \sigma(F_0) = 2 \). When \( v = 0 \), let \( F_0 \) be the unique fiber containing a \( D \)-rivet. All other singular fibers are columnar by [Fujita 1982, 7.6], so they contain no components of \( \hat{E} \). We need to prepare some tools to proceed. Recall that the Neron–Severi group of \( \hat{S} - T \) is defined as the quotient of \( \text{NS}(\hat{S}) \) by the subgroup generated by components of \( T \). We put \( \rho(\hat{S} - T) = \dim \text{NS}(\hat{S} - T) \otimes \mathbb{Q} \).

Let \((X, R)\) be a smooth pair with \( X \) rational. Suppose \( R = R_1 + R_2 \), where \( R_1 \) and \( R_2 \) meet in unique components \( C_1 \subseteq R_1 \), \( C_2 \subseteq R_2 \) respectively. If at least one of \( R_i \) is negative definite for \( i = 1, 2 \), then we call \( R - C_1 \) a swap of \( R - C_2 \) and vice versa. Similarly, \((X, R - C_i)\) are by definition swaps of each other, and so are \( X - (R - C_i) \), for \( i = 1, 2 \). The basic property of this operation that we need is that

\[ \rho(X - (R - C_1)) = \rho(X - (R - C_2)). \]

To see this, it is enough to show that \( C_1 \), \( C_2 \) do not belong to the subspace \( V \) of \( \text{NS}(X) \otimes \mathbb{Q} \) generated by components of \( R_1 - C_1 + R_2 - C_2 \). By symmetry, we can assume that \( R_2 \) is negative definite. Suppose that \( C_1 \in V \) and write

\[ C_1 \equiv U_1 + U_2, \]

where \( U_i \) is in the subspace generated by components of \( R_i - C_i \). Then \( 0 = C_1 \cdot U_2 = U_1 \cdot U_2 + U_2^2 = U_2^2 \), and hence \( U_2 \equiv 0 \) by the negative definiteness of \( R_2 \). Then \( 0 < C_1 \cdot C_2 = U_1 \cdot C_2 = 0 \), a contradiction. Suppose \( C_2 \in V \) and write \( C_2 \equiv U_1 + U_2 \) as above. Then \( (C_2 - U_2)^2 = (C_2 - U_2) \cdot U_1 = 0 \), so \( C_2 \equiv U_2 \) by the negative definiteness of \( R_2 \). Then \( 0 < C_1 \cdot C_2 = C_1 \cdot U_2 = 0 \), a contradiction. Thus, swapping preserves \( \rho \). Though the definition is of general use, we use only a special kind of swapping, when \( C_2 \) is a \((-1)\)-curve and it is absorbed into the boundary (keeping the assumption that \( R_2 \) is negative definite); that is, we do the swap one way, changing \((X, R - C_2)\) to \((X, R - C_1)\).

Take \((\hat{S}, T)\) and interchangeably perform contractions of \((-1)\)-curves in \( F_0 \) (and its images) that are nonbranching components of the boundary and swaps absorbing vertical \((-1)\)-curves in \( F_0 \) (and its images) into the boundary. Denote the resulting smooth pair by \((X, T')\). By the properties of swaps and blow-ups, the rank of the Neron–Severi group of the open part and the difference between \( b_2 \) of the complete surface and the number of components in the boundary remain constant. Also, \( T' \) is a rational forest. Crucially, \( d(T) = 0 \) if and only if \( d(T') = 0 \). To see this, we
may assume that \((X, T')\) is simply a swap of \((\tilde{S}, T)\) as above. Since the number of components of \(T\) equals \(b_2(\tilde{S})\), we know \(d(T) \neq 0\) if and only if \(\rho(\tilde{S} - T) = 0\), which is equivalent to \(\rho(X - T') = 0\) and then to \(d(T') \neq 0\).

Consider the case \(\Sigma = \nu = 0\). At some point, the process of swapping and contracting makes \(B\) into a 0-curve or a \((-1)\)-curve. It is easy to see that the divisor \(\tilde{D}_0 + \tilde{D}_\infty\) is not affected by the process, so we have \(d(D) \neq 0\) if and only if \(d(\tilde{D}_0) \cdot d(\tilde{D}_\infty) \neq 0\). All singular fibers of the induced \(\mathbb{P}^1\)-ruling at this stage are columnar, so they can be written as \(R_{i,0} + C_i + R_{i,\infty}\), where \(i = 1, \ldots, n'\) enumerates these fibers, \(C_i\) equals \(-1\), and \(R_{i,0}\) and \(R_{i,\infty}\) are chains whose last components meet \(D_0\) and \(D_\infty\), respectively. For \(j = 0, \infty\), put \(\tilde{e}_j = \tilde{e}(\tilde{D}_j)\) (see Section 2A). Then \(\tilde{e}_j = \sum_i \tilde{e}(R_{i,j})\). We have \(d(D_j) = (-D_j^2 - \tilde{e}_j) \cdot \prod_i d(R_{i,j})\). By the properties of columnar fibers,

\[
d(\tilde{D}_0) + d(\tilde{D}_\infty) = -(D_0^2 + D_\infty^2 + n') \cdot \prod_i d(R_{i,0}).
\]

When contracting singular fibers to smooth ones, \(D_0 + D_\infty\) is touched \(n'\) times and its image consists of two disjoint sections on a Hirzebruch surface. It follows that \(D_0^2 + D_\infty^2 + n' = 0\), and hence \(d(\tilde{D}_\infty) + d(\tilde{D}_0) = 0\). Thus \(d(D) \neq 0\) if and only if \(d(\tilde{D}_0) \neq 0\).

Consider the case \(\Sigma = \nu = 1\). We show that \(T'\) has at most one horizontal component. Suppose that it has two. Then \(\sigma(\tilde{F}_0) = \sigma(F_0) = 2\), so \(\tilde{F}_0\) contains a \((-1)\)-curve, say \(C_1\). Because \(T'\) is \(p\)-minimal, \(C_1 \not\subset T\). Because we assumed that every \(\tilde{S} - T\)-component meets \(D\), by the properties of swaps, every \(X - T'\)-component meets \(T'\). By the definition of \(X\), absorbing the \((-1)\)-curve by a swap into the boundary is impossible. In particular, if \(\tilde{F}_0\) has no more \((-1)\)-curves, then \(C_1\) is not a tip of \(\tilde{F}_0\), so \(\tilde{F}_0\) is a chain. However, since \(\sigma(\tilde{F}_0) = 2\), a swap absorbing \(C_1\) into the boundary is possible, which is a contradiction. Thus, \(\tilde{F}_0\) has two \((-1)\)-curves, \(C_1\) and \(C_2\). One of them meets some horizontal component of \(T'\); otherwise, either \(C_1\) or \(C_2\) is a tip or \(\tilde{F}_0 \cap T'\) has three connected components, and in either case a swap absorbing one of the \(C_i\)’s into the boundary would be possible. A similar argument shows that the second \((-1)\)-curve also meets a horizontal component of \(T'\). Thus, \(\tilde{F}_0^\prime\) is a chain with \(C_1\) and \(C_2\) as tips, and again a swap is possible, a contradiction. So \(T'\) has at most one horizontal component. But after the first swap where \(\sigma\) of the image of \(F_0\) drops, the fiber has only one \((-1)\)-curve, which therefore has multiplicity greater than one, so no more swaps of this kind are possible. Thus, \(T'\) has a unique horizontal component \(T_h'\). Then

\[
d(T') = d(F_\infty) d(T' - F_\infty) - d(T' - F_\infty - D_\infty) = -d(T' - F_\infty - D_\infty).
\]

Now \(T' - F_\infty - D_\infty\) is vertical and does not contain whole fibers, so it is negative definite and we obtain \(d(T') = d(T' - F_\infty - D_\infty) \neq 0\).
Remark. By Proposition 2.11, for any \( \mathbb{Q} \)-homology plane, we have \( H_i(S', \mathbb{Z}) = 0 \) for \( i > 1 \) and

\[
|H_1(S', \mathbb{Z})|^2 = \frac{d(D)}{d(\hat{E})},
\]

and hence \( S' \) is a \( \mathbb{Z} \)-homology plane if and only if \( d(D) = d(\hat{E}) \). For a \( \mathbb{C}^* \)-ruled \( S' \), more explicit computations are done in [Miyanishi and Sugie 1991], which we do not repeat here. For example, by [ibid., 2.17], if a \( \mathbb{Z} \)-homology plane with \( \kappa(S_0) \neq -\infty \) is \( \mathbb{C}^* \)-ruled, then \( \kappa(S_0) = 1 \) and the ruling is untwisted with base \( \mathbb{P}^1 \).

The conditions for \( S' \) having such a ruling to be contractible are given in [ibid., 2.11] (in particular \( n = 2 \)).

4B. The Kodaira dimension. In [Miyanishi and Sugie 1991, 2.9–2.17] one can find formulas for the Kodaira dimension of the smooth locus \( \kappa(S_0) \) in terms of properties of singular fibers of the \( \mathbb{C}^* \)-ruling (there, \( \kappa(S') \) is by definition equal to \( \kappa(S_0) \)). Unfortunately, their formulas 2.14(4), 2.15(2), and 2.16(2) are incorrect. The corrections require splitting into cases depending on additional properties of singular fibers. We also compute the Kodaira dimension of \( S' \). We keep the notation for singular fibers as in Lemma 4.1. When \( v = 0 \), put \( F_\infty = 0 \). Let \( J \) be the reduced divisor with support equal to \( D \cup F_0 \). For \( i = 1, \ldots, n \), denote the \((-1)\)-curve of the columnar fiber \( F_i \) by \( C_i \) and the multiplicity of \( C_i \) by \( \mu_i \). Put \( J^+ = J + C_1 + \cdots + C_n \).

Lemma 4.5. The divisor \( J^+ \) has simple normal crossings. Contract vertical \((-1)\)-curves in \( J^+ \) and its images as long as the image is an snc-divisor. Let

\[
\zeta : (\overline{S}, J^+) \to (W, \xi_* J^+)
\]

be the composition of these contractions. Then the \( \xi_* F_i \) are smooth for \( i = 1, \ldots, n \); moreover:

(i) If \( h = 1 \), then \( \xi_* F_0 = [2, 1, 2] \), \( (\xi_* D_h)^2 = 0 \), and one can further contract \( \xi_* F_0 \) and \( F_\infty \) to smooth fibers so that \( W \) maps to \( \mathbb{F}_1 \) and \( \xi_* D_h \) maps to a smooth \( 2 \)-section of the \( \mathbb{P}^1 \)-ruling of \( \mathbb{F}_1 \) disjoint from the negative section.

(ii) If \( h = 2 \), then \( \xi_* F_0 \) is smooth, \( W \) is a Hirzebruch surface, and the components of \( \xi_* D_h \) are disjoint. Also, at least one of the components of \( D_h \) has negative self-intersection, and by changing \( \xi \) if necessary, one can assume that it is not affected by \( \xi \).

Proof. Suppose the crossings of \( J^+ \) at \( x \) are not simple normal. By Lemma 4.1, this only happens if \( h = 2 \). Also, \( x \) belongs to \( D_h \cap F_0 \) and is a branching point of \( p_{D_h} \), and two components of \( F_0 \) of multiplicity one meet at \( x \). Because \( D \) has normal crossings, one of them is the unique \( S_0 \)-component of \( F_0 \). By the \( p \)-minimality of \( D \), it has to be a unique \((-1)\)-curve of \( F_0 \) too, which is impossible.
by Lemma 2.2(i). Thus, \(J^+\) is an snc-divisor. Because \(F_i\) for \(i = 1, \ldots, n\) are columnar, \(\zeta_s F_i\) are smooth.

Suppose \(h = 2\). Write \(D_h = H + H'\). By Lemma 4.1, \(H\) and \(H'\) are disjoint. Since \(H\) and \(H'\) meet \(F_0\) only in the components of multiplicity one, it follows from the definition of \(\zeta\) that the images of \(H'\) and \(H\) intersect the same component of \(\zeta_s F_0\). But this is possible only if \(\zeta_s F_0\) is smooth. Since \(\zeta_s J^+\) is snc, these images are disjoint. Say \(H'^2 = H^2 = 0\). Choosing the contracted \((-1)\)-curves correctly, we may assume that \(H'\) is not affected by \(\zeta\). Since \(\zeta_s D_h\) consists of two disjoint sections on a Hirzebruch surface, we have \((\zeta_s D_h)^2 = 0\), so \(D_h^2 \leq 0\). Suppose \(H^2 = H'^2 = 0\). Then \(\zeta\) does not affect \(D_h\), so \(n = 0\) and \(H\) and \(H'\) intersect the same component \(B\) of \(F_\infty\). If \(v = 1\), then \(B\) is an \(S_0\)-component and the second \(S_0\)-component of \(F_0\) does not intersect \(D\), a contradiction with the affineness of \(S'\). Thus \(v = 0\) and Lemma 4.4 is not satisfied (in other words, \(d(D) = 0\), a contradiction.

Suppose \(h = 1\). By the definition of \(\zeta\), the image of \(D_h\) intersects the unique \((-1)\)-curve of \(\zeta_s F_0\). It follows that \(\zeta_s F_0 = [2, 1, 2]\). Now after the contraction of \(F_0\) and \(F_\infty\) to smooth fibers, the image of \(W\) is a Hirzebruch surface \(\mathbb{F}_N\), where \(N \geq 0\), and the image \(D_h'\) of \(D_h\) is a smooth 2-section. Write \(D_h' \equiv \alpha f + 2H\), where \(H\) is a section with \(H^2 = -N\) and \(f\) is a fiber of the induced \(\mathbb{P}^1\)-ruling of \(\mathbb{F}_N\). We compute

\[
p_a(\alpha f + 2H) = \alpha - N - 1,
\]

so because \(D_h'\) is smooth, its arithmetic genus vanishes and \(\alpha = N + 1\). Also, \(D_h' \cdot H = \alpha - 2N\), and hence \(D_h' \cdot H + N = 1\). Now if \(N = 0\), then \(\mathbb{F}_N = \mathbb{P}^1 \times \mathbb{P}^1\), and an elementary transformation with center equal to the point of tangency of \(D_h'\) and the image of \(F_\infty\) (which corresponds to a different choice of components to be contracted in \(F_\infty\)) leads to \(N = 1\) and \(D_h' \cdot H = 0\).

**Remark 4.6.** Let \((X, D)\) be a smooth pair, and let \(L\) be the exceptional divisor of a blow-up \(\sigma : X' \to X\) of a point in \(D\). Then

\[K_{X'} + \sigma^{-1}D = \sigma^*(K_X + D)\]

if \(\sigma\) is subdivisional for \(D\), and

\[K_{X'} + \sigma^{-1}D = \sigma^*(K_X + D) + L\]

if \(\sigma\) is sprouting for \(D\).

Decompose \(\zeta\) into a sequence of blow-downs \(\zeta = \sigma_k \circ \cdots \circ \sigma_1\), and let \(m \leq k\) be the minimal number such that for \(j > m\), the blow-up \(\sigma_j\) is subdivisional for \((\sigma_j \circ \cdots \circ \sigma_1)\_J^+\). Define \(\eta : \widetilde{S} \to \hat{S}\) and \(\theta : \widehat{S} \to W\) as

\[
\eta = \sigma_m \circ \cdots \circ \sigma_1 \quad \text{and} \quad \theta = \sigma_k \circ \cdots \circ \sigma_{m+1}.
\]
Clearly, $\eta$ is the identity outside $F_0$. We denote a general fiber of a $\mathbb{P}^1$-ruling by $f$.

**Lemma 4.7.** Let $\eta : \widetilde{S} \to \widetilde{S}$ and $\theta : \widetilde{S} \to W$ be as above. Then

$$K_{\widetilde{S}} + \eta_* J \equiv \left(n + v - 1 - \sum_{i=1}^{\mu} \frac{1}{\mu_i} \right)f + G + \theta^* \frac{1}{2}(U + U'),$$

where $G$ is a negative definite effective divisor with support contained in the support of $F_\infty + \sum_{i=1}^{n} F_i$ and $U$, $U'$ are the $(-2)$-tips of $\xi_* F_0$ if $p$ is twisted and are zero otherwise.

**Proof.** Let $V \subseteq W$ be defined as the sum of (four) $(-2)$-tips of $F_\infty + \xi_* F_0$ if $p$ is twisted and as zero otherwise. We check easily that

$$K_W + D_h + F_\infty + \xi_* F_0 \equiv (v - 1)f + \frac{1}{2}V.$$ 

Indeed, if $p$ is untwisted, this is just $K_W + D_h + 2f \equiv 0$ on a Hirzebruch surface, and if $p$ is twisted, then it follows from the numerical equivalences $K_W + D_h + f \equiv 0$ and $F_\infty + \xi_* F_0 - \frac{1}{2}V \equiv f$. By Remark 4.6,

$$K_{\widetilde{S}} + \eta_* J^+ \equiv (n + v - 1)f + \theta^* \frac{1}{2}V.$$ 

For every $i = 1, \ldots, n$, the divisor $G_i = (1/\mu_i)F_i - C_i$ is effective and negative definite because $C_i$ is not contained in its support. We get

$$K_{\widetilde{S}} + \eta_* J \equiv (n + v - 1)f + \sum_{i=1}^{n} \left(G_i - \frac{1}{\mu_i} F_i \right) + \theta^* \frac{1}{2}V,$$

so

$$K_{\widetilde{S}} + \eta_* J \equiv (n + v - 1 - \frac{1}{\mu_i})f + \sum_{i=1}^{n} G_i + \theta^* \frac{1}{2}V.$$ 

**Remark 4.8.** Because $K_{\widetilde{S}} + D + \widehat{E}$ and $K_{\widetilde{S}} + D$ intersect trivially with a general fiber, we can write $K_{\widetilde{S}} + D + \widehat{E} \equiv \kappa_0 f + G_0$ and $K_{\widetilde{S}} + D + \widehat{E} \equiv \kappa f + G$, where $G_0$ and $G$ are some vertical effective and negative definite divisors and $\kappa_0, \kappa \in \mathbb{Q}$. It follows that $\kappa(S_0)$ and $\kappa(S)$ are determined by the signs of $\kappa_0$ and $\kappa$. More explicitly, $\kappa(S_0)$ equals $-\infty$, 0, or 1 depending on whether $\kappa_0 < 0$, $\kappa_0 = 0$, or $\kappa_0 > 0$, respectively. An analogous statement holds for $\kappa(S)$ and $\kappa$.

It turns out that $\kappa$ and $\kappa_0$ depend in a quite involved way on the structure of $F_0$. This dependence can be stated in terms of the properties of $\eta : \widetilde{S} \to \widetilde{S}$ defined above. Denote the $S_0$-components of $F_0$ by $C$, $\widetilde{C}$ (or just $C$ if there is only one) and their multiplicities by $\mu$, $\tilde{\mu}$ respectively. Note that $\mu \geq 2$ if $\sigma(F_0) = 1$, but if $\sigma(F_0) = 2$, then it can happen that $\mu = 1$ or $\tilde{\mu} = 1$. 
Theorem 4.9. Let $\lambda = n + v - 1 - \sum_{i=1}^{n}(1/\mu_i)$. The numbers $\kappa$ and $\kappa_0$ determining the Kodaira dimension of a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane $S'$ and of its smooth locus $S_0$ defined in Remark 4.8 are as follows:

(A) Case $(h, v) = (1, 1)$. Denote the component of $F_0$ intersecting the 2-section contained in $D$ by $B$.

(i) If $\eta = \id$ and $F_0 = [2, 1, 2]$, then $\kappa = \kappa_0 = \lambda - \frac{1}{2}$.

(ii) If $\eta = \id$, $B$ is not a tip of $F_0$, and $C \cdot B > 0$, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - 1/2\mu)$.

(iii) If $\eta = \id$, $C \cdot B = 0$, and $F_0$ is a chain, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda)$.

(iv) If $\eta = \id$ and $B$ is a tip of $F_0$, then $(\kappa, \kappa_0) = (\lambda - \frac{1}{2}, \lambda - 1/\mu)$.

(v) If $\eta \neq \id$, then $\kappa = \kappa_0 = \lambda$.

(B) Case $(h, v) = (2, 1)$. 

(i) If $\eta = \id$ and $C^2 = \tilde{C}^2 = -1$, then $(\kappa, \kappa_0) = (\lambda - 1, \lambda - 1/\min(\mu, \tilde{\mu}))$.

(ii) If $\eta = \id$ and $C^2 \neq -1$ or $\tilde{C}^2 \neq -1$, then $\kappa = \kappa_0 = \lambda - 1/\min(\mu, \tilde{\mu})$.

(iii) If $\eta \neq \id$, then assuming that $C$ is the $S_0$-component disjoint from $\tilde{E}$, we have $\kappa = \kappa_0 = \lambda - 1/\mu$.

(C) Case $(h, v) = (2, 0)$. Then $\kappa = \kappa_0 = \lambda$.

Proof. (A) The unique $S_0$-component $C$ of $F_0$ is a $(-1)$-curve. Otherwise, the $p$-minimality of $D$ implies that $B$ is the only $(-1)$-curve in $F_0$ and that it intersects two other $D$-components of $F_0$, giving $F_0 = [2, 1, 2] \subseteq D$, with no place for $C$. It is now easy to check that the list of cases in (A) is complete. Because $C^2 = -1$, $F_0 - C$ has at most two connected components. The only case when $\tilde{E}$ is not connected is when $F_0$ contains no $D$-components, which is only possible if $C = B$ and $F_0 = [2, 1, 2]$. Because $C$ is the unique $(-1)$-curve in $F_0$, we know that $\xi = \theta \circ \eta$ has at most one center on $\xi_* F_0$, so by symmetry we can and do assume that it does not belong to $U'$ (see Lemma 4.7). Suppose $\eta \neq \id$. The center of $\eta$ belongs to a unique component of $\eta_* J$. Because $D_h$ does not intersect components contracted by $\eta$, this component is a proper transform of a $D$-component, so $\eta_* (C + \tilde{E}) = 0$ by the connectedness of $\tilde{E}$. If we now factor $\eta$ as $\eta = \sigma \circ \eta'$, where $\sigma$ is a sprouting blow-up for $\eta_* J$, then by Lemma 4.7 and Remark 4.6,

$$K + \sigma^{-1} \eta_* J \equiv \lambda f + G + \sigma^* \theta^* \frac{1}{2}(U + U') + \Exc(\sigma),$$

where $\Exc(\sigma)$ is the exceptional $(-1)$-curve contracted by $\sigma$ and $K$ is a canonical divisor on a respective surface. Because $\eta_* (C + \tilde{E}) = 0$, each component of $C + \tilde{E}$ appears with positive integer coefficient in $\eta^* \Exc(\sigma)$, which gives $K_{\tilde{E}} + \eta^{-1} \eta_* J \equiv \lambda f + G + G_0$, where $G_0$ is a vertical effective and negative definite divisor for which $G_0 - \tilde{E} - C$ is still effective. Because $\eta^{-1} \eta_* J = J = D + \tilde{E} + C$, we get $\kappa = \kappa_0 = \lambda$.

We can now assume that $\eta = \id$, so

$$K_{\tilde{E}} + D + \tilde{E} + C \equiv \lambda f + G + \frac{1}{2}(U' + \theta^* U).$$
This can be written as

\[ K_S + D \equiv (\lambda - \frac{1}{2}) f + G + \frac{1}{2} (U' + F_0 + \theta^* U - 2C - 2\widehat{E}). \]

All components of \( F_0 \) appear in \( U' + F_0 + \theta^* U \) with coefficients bigger than 1, so \( U' + F_0 + \theta^* U - 2C - 2\widehat{E} \) is effective and negative definite, because its support does not contain the \( \widehat{E} \)-component that is a proper transform of \( U \). This gives \( \kappa = \lambda - \frac{1}{2} \). We now compute \( \kappa_0 \). If \( F_0 = [2, 1, 2] \), then \( \theta^* U = U \) and \( \widehat{E} = U + U' \), so \( K_S + D \equiv (\lambda - \frac{1}{2}) f + G \) and we get \( \kappa_0 = \lambda - \frac{1}{2} \). Suppose \( B \) is a tip of \( F_0 \). Because \( \mu(B) = 2 \), we know that \( F_0 \) is a fork with two \((-2)\)-tips as maximal twigs (see Lemma 2.2(ii)) and that \( \theta^* U = U \) (\( U \) and \( U' \) are components of \( \widehat{E} \)). The divisor \( G_0 = \frac{1}{2} (U + U') + (1/ \mu) F_0 - C \) is vertical effective and its support does not contain \( C \). Writing

\[ K_S + D + \widehat{E} \equiv \left( \lambda - \frac{1}{\mu} \right) f + G + G_0, \]

we infer that \( \kappa_0 = \lambda - 1/ \mu \), and we obtain (iv). Consider the case (ii). Because \( B \) is not a tip of \( F_0 \), we know \( F_0 \) is a chain. The assumption \( B \cdot C > 0 \) implies that \( B^2 \neq -1 \) and \( \theta^* U = C + \widehat{E} \). We obtain

\[ K_S + D + \widehat{E} \equiv \left( \lambda - \frac{1}{2 \mu} \right) f + G + \frac{1}{2} \left( U' + \widehat{E} + \frac{1}{\mu} F_0 - C \right), \]

and \( U' + \widehat{E} + (1/ \mu) F_0 - C \) is effective with support not containing \( C \). This gives \( \kappa_0 = \lambda - (1/2 \mu) \). We are left with the case (iii). As in (ii), \( F_0 \) is a chain, and we have now

\[ K_S + D + \widehat{E} \equiv \lambda f + G + \frac{1}{2} (U' + \theta^* U - 2C). \]

\( U' + \theta^* U - 2C \) is effective and does not contain \( B \), because \( B \cdot C = 0 \), so \( \kappa_0 = \lambda \).

(B) Suppose \( \eta \neq \text{id} \). Note that \( \eta_* F_0 \) contains a proper transform of one of \( C, \widehat{C} \), for otherwise, \( F_0 \) would contain a \( D \)-rivet. It follows that \( \eta \) is a connected modification and that its center lies on a birational transform of a \( D \)-component (the \( S_0 \)-component contracted by \( \eta \) has to intersect \( D \)). Thus, \( \eta_* F_0 \) is a chain intersected by \( D_h \) in two different tips and containing \( C \). Since \( D \cap \widehat{E} = \emptyset \), we get \( \eta_* (\widehat{C} + \widehat{E}) = 0 \). Writing \( \eta = \sigma \circ \eta' \), where \( \sigma \) is a sprouting blow-down, we see that \( \eta^* \text{Exc}(\sigma) \) is an effective negative definite divisor that does not contain \( C \) in its support and for which \( \eta'^* \text{Exc}(\sigma) - \widehat{C} - \widehat{E} \) is effective. By Lemma 4.7, we have

\[ K + \sigma^{-1} \eta_* D + C \equiv \lambda f + G + \text{Exc}(\sigma), \]

where \( K \) is a canonical divisor on a respective surface. It follows from Remark 4.6 and from arguments analogous to those in part (A) that \( \kappa = \kappa_0 = \lambda - (1/ \mu) \). We can now assume that \( \eta = \text{id} \). By Lemma 4.7,

\[ K_S + D + C + \widehat{E} + \widehat{C} \equiv \lambda f + G, \]
which implies $\kappa_0 = \lambda - (1/\min(\mu, \tilde{\mu}))$. Writing

$$K_S + D \equiv \left(\lambda - \frac{1}{\alpha}\right) f + G + \frac{1}{\alpha} (F_0 - \alpha(C + \widehat{E} + \widehat{C})), $$

we see that $\kappa = \lambda - (1/\alpha)$, where $\alpha$ is the lowest multiplicity of a component of $C + \widehat{E} + \widehat{C}$ in $F_0$. Note that $C + \widehat{E} + \widehat{C}$ is a chain. Now if $C^2 \neq -1$, for instance, then $F_0$ is columnar, and factoring $\theta$ into blow-downs, we see that $\widehat{E}$ is contracted before $C$, and hence $\alpha = \mu \leq \tilde{\mu}$. Suppose $C^2 = \widehat{C}^2 = -1$, and let $\theta'_{s_{F_0}}$ be the composition of successive contractions of $(-1)$-curves in $F_0$ different than $C$. Now either $\theta'_{s_{F_0}}C = [0]$ or $\theta'_{s_{F_0}}F_0$ is columnar. Both possibilities imply that $C + \widehat{E}$ contains a component of multiplicity one, and hence $\alpha = 1$.

(C) $C$ is a $(-1)$-curve. Indeed, $D \cap F_0$ contains at most one $(-1)$-curve, and if it does, then by the $p$-minimality of $D$, it meets both components of $D_h$ and has multiplicity one, so there is another $(-1)$-curve in $F_0$. We infer that $F_0 - C$ has two connected components, one being $\widehat{E}$ and the second containing a rivet. The existence of a rivet in $F_0$ implies that $\eta \neq \operatorname{id}$, so $\eta^*(C + \widehat{E}) = 0$. Factoring out a sprouting blow-down from $\eta$ as above, we get

$$K + \sigma^{-1} \eta^* D \equiv \lambda f + G + \operatorname{Exc}(\sigma).$$

The divisor $\eta^* \operatorname{Exc}(\sigma) - C - \widehat{E}$ is effective and does not contain all components of $F_0$, so by Remark 4.6, $\kappa = \kappa_0 = \lambda$. 

\textbf{Remark.} In case (B)(iii), it is not true in general that $\mu = \min(\mu, \tilde{\mu})$.

\textbf{4C. Smooth locus of Kodaira dimension zero.} As a corollary, we obtain the following information in case $\kappa(S_0) = 0$.

\textbf{Corollary 4.10.} Let $S'$ be a $\mathbb{C}^*$-ruled singular $\mathbb{Q}$-homology plane, and let $D$ be a $p$-minimal boundary for an extension $p$ of this ruling to a normal completion, as above. Let $D$ be the $p$-minimal boundary, and let $n$ be the number of columnar fibers. Then $\kappa(S_0) = 0$ exactly in the following cases:

(i) $p$ is twisted, $n = 0$, and $F_0$ is of type (A)(iii) or (A)(v).

(ii) $p$ is twisted, $n = 1$, $\mu = \mu_1 = 2$, and $F_0$ is of type (A)(i) or (A)(iv) with no $D$-components.

(iii) $p$ is untwisted with base $\mathbb{C}^1$, $n = 1$, $\mu = 2$, $\min(\mu, \tilde{\mu}) = 2$, and some connected component of $F_0 \cap D$ is a $(-2)$-curve.

(iv) $p$ is untwisted with base $\mathbb{C}^1$, $n = 2$, $\mu = \mu_1 = 2$, and some $S_0$-component of $F_0$ meets $D_h$.

(v) $p$ is untwisted with base $\mathbb{P}^1$, $n = 2$, and $\mu_1 = \mu_2 = 2$. 
Proof. Note that \( n - \sum_{i=1}^{n} (1/\mu_i) \geq n/2 \) because \( \mu_i \geq 2 \) for each \( i \). Suppose \( p \) is twisted. Then \( \mu \geq 2 \), and so by Theorem 4.9,

\[
\lambda \geq \kappa_0 \geq \lambda - \frac{1}{2} \geq \frac{n-1}{2}.
\]

If \( n = 0 \), then \( \lambda = 0 \), which gives \( \kappa_0 = 0 \) exactly in cases (A)(iii) and (A)(v). If \( n = 1 \), then \( \kappa_0 = \lambda - \frac{1}{2} = 0 \), which is possible in case (A)(i) if \( \mu_1 = 2 \) and in case (A)(iv) if \( \mu = \mu_1 = 2 \). In both cases, \( D_h \) meets the \( S_0 \)-component, so \( F_0 \) contains no \( D \)-components. If \( p \) is untwisted with base \( \mathbb{P}^1 \), then

\[
n - 1 \geq \lambda = \kappa_0 \geq \frac{n}{2} - 1,
\]

so \( n = 2 \) (because \( \lambda = -1/\mu_1 < 0 \) for \( n = 1 \)) and \( \kappa_0 = 1 - 1/\mu_1 - 1/\mu_2 \), which vanishes only if \( \mu_1 = \mu_2 = 2 \). Assume now that \( p \) is untwisted with base \( \mathbb{C}^1 \). Then

\[
n > \kappa_0 \geq \lambda - 1 \geq \frac{n}{2} - 1,
\]

so \( n \in \{1, 2\} \). There are no \((-1)\)-curves in \( D \cap F_0 \) by the \( p \)-minimality of \( D \), so at least one \( S_0 \)-component, say \( C \), is a \((-1)\)-curve. We can also assume that \( C \) is contracted by \( \eta \) in case \( \eta \neq \text{id} \) and that \( \mu \geq \tilde{\mu} \) in case \( \eta = \text{id} \). Then \( \kappa_0 = \lambda - 1/\tilde{\mu} \).

The composition \( \xi \) of successive contractions of all \((-1)\)-curves in \( F_0 - \tilde{C} \) and its images is a connected modification. Suppose \( n = 2 \). The inequalities above give \( \lambda = 1 \), so \( \mu_1 = \mu_2 = 2 \) and \( \tilde{\mu} = 1 \). Then \( \xi_* F_0 = [0] \), and because \( \xi \) is a connected modification, \( \tilde{C} \) is a tip of \( F_0 \). So one of \( C, \tilde{C} \) intersects \( D_h \), because otherwise \( F_0 - \tilde{C} - C - \tilde{E} \) would be connected and would intersect both sections from \( D_h \), and hence \( F_0 \) would contain a rivet. This gives (iv). Suppose that \( n = 1 \). Then \( \mu_1 = \tilde{\mu} = 2 \). By the choice of \( C \), further contractions of \( F_0 \) to a smooth fiber are subdivisional for \( \xi_* D \cup \xi_* F_0 \), so we have \( \xi_* F_0 = [2, 1, 2] \) with the birational transform of \( \tilde{C} \) in the middle, and the image of \( D_h \) intersects both \((-2)\)-tips of \( \xi_* F_0 \). Since \( \xi \) is a connected modification, it does not touch one of these tips, so one of the connected components of \( D \cap F_0 \) is a \((-2)\)-curve. If \( \mu = 1 \), then \( \mu < \tilde{\mu} \), so by our assumption \( \eta \neq \text{id} \). But then \( \mu > 1 \), because \( C^2 = -1 \) and \( C \) intersects \( \tilde{E} \) and \( D \). This contradiction ends the proof of (iii). \( \square \)

4D. Constructions. Lemmas 4.5 and 2.12 give a practical method of reconstructing all \( \mathbb{C}^* \)-ruled \( \mathbb{Q} \)-homology planes. We summarize it in here. We denote irreducible curves and their proper transforms by the same letters.

Construction 4.11. Case 1 (twisted ruling). Let \( D_h \) be a smooth conic on \( \mathbb{P}^2 \), let \( L_0, L_\infty \) be tangents to \( D_h \) at distinct points \( x_0, x_\infty \), and let \( L_i \), for \( i = 1, \ldots, n \) and \( n \geq 0 \), be distinct lines through \( L_0 \cap L_\infty \), other than \( L_0, L_\infty \). Blow up once at \( L_0 \cap L_\infty \); let \( p : \mathbb{F}_1 \to \mathbb{P}^1 \) be the \( \mathbb{P}^1 \)-ruling of the resulting Hirzebruch surface. Over each of \( p(L_0), p(L_\infty) \), blow up on \( D_h \) twice, creating singular fibers \( \tilde{F}_0 = [2, 1, 2] \)
and \( F_\infty = [2, 1, 2] \). For each \( i = 1, \ldots, n \), by a connected sequence of blow-ups subdivisinal for \( L_1 + D_0 \), create a column fiber \( F_i \) over \( p(L_i) \) and denote its unique \((-1)\)-curve by \( C_i \). By some connected sequence of blow-ups with a center on \( \widetilde{F}_0 \), create a singular fiber \( F_0 \), and denote the newly created \((-1)\)-curve by \( C \) (if the sequence is empty, define \( C \) as the \((-1)\)-curve of \( \widetilde{F}_0 \)). Denote the resulting surface by \( \widetilde{S} \), put

\[
T = D_0 + F_\infty + (F_1 - C_1) + \cdots + (F_n - C_n) + F_0 - C,
\]

and construct \( S' \) as in Lemma 2.12. \( S' \) is a \( \mathbb{Q} \)-homology plane (singular as long as \( T \) is not connected) because conditions 2.12(i)–(iii) are satisfied by construction and (iv) by Lemma 4.4. To see that each \( S' \) admitting a twisted \( \mathbb{C}^* \)-ruling can be obtained in this way, note that by the \( p \)-minimality of \( D \), even if \( F_0 \) contains two \((-1)\)-curves \( C \) and \( B \subseteq D \), then \( B \) is not a tip of \( F_0 \) and \( \zeta \) does not touch it, so in each case the modification \( F_0 \to \zeta_*(F_0) \) induced by \( \zeta \) is connected, and we are done by Lemma 4.5.

**Case 2** (untwisted ruling with base \( \mathbb{C}^1 \)). Let \( x_0, x_1, \ldots, x_n, x_\infty, y \in \mathbb{P}^2 \), for \( n \geq 0 \), be distinct points, such that all but \( y \) lie on a common line \( D_1 \). Let \( L_i \) be a line through \( x_i \) and \( y \). Blow up \( y \) once and let \( D_2 \) be the negative section of the \( \mathbb{P}^1 \)-ruling of the resulting Hirzebruch surface \( p: \mathbb{F}_1 \to \mathbb{P}^1 \). For each \( i = 0, 1, \ldots, n \), by a connected sequence of blow-ups (which can be empty if \( i = 0 \)), with first center \( x_i \) and subdivisinal for \( D_1 + L_i \), create a column fiber \( F_i \) (\( \widetilde{F}_0 \) if \( i = 0 \)) over \( p(x_i) \) and denote its unique \((-1)\)-curve by \( C_i \) if \( i \neq 0 \) and by \( \widetilde{C} \) if \( i = 0 \) (put \( \widetilde{C} = L_0 \) if the sequence over \( p(x_0) \) is empty). Choose a point \( z \in F_0 \) that lies on \( D_1 + \widetilde{F}_0 - \widetilde{C} \), and by a nonempty connected sequence of blow-ups with first center \( z \), create some singular fiber \( F_0 \) over \( p(x_0) \). Let \( C \) be the new \((-1)\)-curve. Denote the resulting surface by \( \widetilde{S} \), put

\[
T = D_0 + D_1 + D_2 + L_\infty + (F_1 - C_1) + \cdots + (F_n - C_n) + F_0 - C - \widetilde{C},
\]

and construct \( S' \) as in Lemma 2.12. The surface \( S' \) is a \( \mathbb{Q} \)-homology plane by Lemma 4.4, because Lemma 4.4(ii) is satisfied by the choice of \( z \). To see that all \( S' \) admitting an untwisted \( \mathbb{C}^* \)-ruling with base \( \mathbb{C}^1 \) can be obtained in this way, note that by changing the completion of \( S' \) by a flow if necessary, we can assume that one of the components of \( D_0 \) is a \((-1)\)-curve. \( D \cap F_0 \) contains no \((-1)\)-curves, and as was shown in the proof of Theorem 4.9, \( \eta \) contracts at most one of \( C, \widetilde{C} \). Then by Lemma 4.5, we are done.

**Case 3** (untwisted ruling with base \( \mathbb{P}^1 \)). Let \( D_2 \) be the negative section of the \( \mathbb{P}^1 \)-ruling of a Hirzebruch surface \( p: \mathbb{F}_N \to \mathbb{P}^1 \), with \( N > 0 \). Let \( x_0, x_1, \ldots, x_n, \) with \( n \geq 0 \) be points on a section \( D_1 \) of \( p \) disjoint from \( D_2 \). For each \( i = 0, 1, \ldots, n \), by a connected sequence of blow-ups (which can be empty if \( i = 0 \)), with first center
and subdivisional for $D_1 + p^{-1}(p(x_i))$, create a column fiber $F_i$ ($\tilde{F}_0$ if $i = 0$) over $p(x_i)$ and denote its unique $(-1)$-curve by $C_i$ if $i \neq 0$ and by $B$ if $i = 0$ (put $B = p^{-1}(p(x_0))$ if the sequence over $p(x_0)$ is empty). Assume that the intersection matrix of at least one of two connected components of

$$D_1 + D_2 + (F_1 - C_1) + \cdots + (F_n - C_n) + (\tilde{F}_0 - B)$$

is nondegenerate. By a connected sequence of blow-ups starting from a sprouting blow-up for $D_1 + \tilde{F}_0$ with center on $B$, create some singular fiber $F_0$ over $p(x_0)$ and let $C$ be the new $(-1)$-curve. Denote the resulting surface by $\tilde{S}$, put

$$T = D_1 + D_2 + (F_1 - C_1) + \cdots + (F_n - C_n) + (F_0 - C),$$

and construct $S'$ as in Lemma 2.12. $D$ is connected because the modification $F_0 + D_1 \to \tilde{F}_0 + D_1$ is not subdivisional, so $S'$ is a $\mathbb{Q}$-homology plane by Lemma 4.4. By Lemmas 4.5 and 4.4, each $S'$ with an untwisted $\mathbb{C}^*$-ruling having a base $\mathbb{P}^1$ can be obtained in this way.

5. Corollaries

5A. Completions and singularities. Recall that $\mathbb{Q}$-homology planes with nonquotient singularities have unique snc-minimal completions (and hence also balanced ones) and unique singular points [Palka 2008, 1.2]. The completions and singularities in case $\overline{\kappa}(S_0) = -\infty$ are described in Section 3. In case $\overline{\kappa}(S_0) = 2$, the singular point is unique and of quotient type [ibid.]. Also, the snc-minimal boundary cannot contain nonbranching $b$-curves with $b \geq 0$, because these induce $\mathbb{C}^1$- or $\mathbb{C}^*$-rulings of $S_0$, and hence the snc-minimal completion is unique. Theorem 1.1 summarizes the remaining cases.

Proof of Theorem 1.1. (1) Suppose $S'$ has at least two different balanced completions. These differ by a flow, which implies that the boundary contains a nonbranching rational component $F_\infty$ with zero self-intersection. Then $F_\infty$ is a fiber of a $\mathbb{P}^1$-ruling $p$ of a balanced completion $(V, D)$. We may assume that $F_\infty$ is not contained in any maximal twig of $D$. Indeed, after moving the 0-curve by a flow to a tip of a new boundary, it gives an affine ruling of $S'$, which is possible only if $\overline{\kappa}(S_0) = -\infty$. Because $F_\infty$ is nonbranching, the induced ruling restricts to an untwisted $\mathbb{C}^*$-ruling of $S'$. It follows from the connectedness of the modification $\eta$ (see the proof of Theorem 4.9) that $n > 0$, so this restriction has more than one singular fiber. Both components of $D_h$ are branching in $D$. Since $F_\infty$ is the only nonbranching 0-curve in $D$, centers of elementary transformations lie on the intersection of the fiber at infinity with $D_h$. If $D$ is strongly balanced, then one of the components of $D_h$ is a 0-curve, and hence there are at most two strongly balanced completions. Conversely, suppose that $S'$ has an untwisted $\mathbb{C}^*$-ruling with
base $\mathbb{C}^1$ and that $n > 0$, and let $(V, D, p)$ be a completion of this ruling. Because $S'$ is not affine-ruled, the horizontal components $H, H'$ of $D$ are branching, so $(V, D)$ is balanced and we can assume $H'^2 = 0$. Because $H, H'$ are proper transforms of two disjoint sections on a Hirzebruch surface, we have $H^2 + H'^2 + n \leq 0$, so $H^2 \neq 0$ and we can obtain a different strongly balanced completion of $S'$ by a flow that makes $H$ into a 0-curve.

(2), (3) By [Palka 2008, 4.5] and [Palka 2011a], we may assume that $S'$ is $\mathbb{C}^*$-ruled. If this ruling is untwisted, it follows from the proof of Theorem 4.9 that $S'$ has a unique singular point, and it is a cyclic singularity. In the twisted case, because $\hat{E} \subseteq F_0$, if $\hat{E}$ is not connected then $F_0$ is of type (A)(i), and if $\hat{E}$ is not a chain then $F_0$ is of type (A)(iv).

**Remark.** The set of isomorphism classes of strongly balanced boundaries that a given surface admits is an invariant of the surface, which allows us to easily distinguish between many $\mathbb{Q}$-acyclic surfaces.

**5B. Singuler planes of negative Kodaira dimension.** As another corollary of Theorem 4.9 we give a detailed description of singular $\mathbb{Q}$-homology planes of negative Kodaira dimension. We assume that $\kappa(S_0) \neq 2$, but as we show in [Palka and Koras 2010], this assumption is redundant.

**Theorem 5.1.** Suppose that $S'$ is a singular $\mathbb{Q}$-homology plane of negative Kodaira dimension and that $S_0$ is its smooth locus. If $\kappa(S_0) \neq 2$, then exactly one of the following holds:

(i) $\kappa(S_0) = -\infty$; $S'$ is affine-ruled or isomorphic to $\mathbb{C}^2/G$ for a small finite noncyclic subgroup $G < \text{GL}(2, \mathbb{C})$.

(ii) $\kappa(S_0) \in \{0, 1\}$; $S'$ is nonlogarithmic and is isomorphic to a quotient of an affine cone over a smooth projective curve by an action of a finite group acting freely off the vertex of the cone and preserving the set of lines through the vertex.

(iii) $\kappa(S_0) \in \{0, 1\}$; $S'$ has an untwisted $\mathbb{C}^*$-ruling with base $\mathbb{C}^1$ and two singular fibers. One of them consists of two $\mathbb{C}^1$'s meeting in a cyclic singular point; after taking a resolution and completion, the respective completed singular fiber is of type (B)(i) with $\mu, \tilde{\mu} \geq 2$ (see Figure 3 and Theorem 4.9).

**Proof.** By [Palka 2011a; Palka 2008, 4.5] and Section 3, we may assume that $S'$ is logarithmic and $\mathbb{C}^*$-ruled and that $\kappa(S_0) \geq 0$. We need to show (iii). Let $(V, D, p)$ be a minimal completion of the $\mathbb{C}^*$-ruling. By Theorem 4.9, if $p$ is twisted, then

$$0 > \kappa_0 \geq \lambda - \frac{1}{2} \geq \frac{n-1}{2},$$

so $n = \lambda = 0$. The inequalities $\kappa < 0$ and $\kappa_0 \geq 0$ can be satisfied only in case (A)(iii), and then $D_h^2 = 0$ by Lemma 4.5, so $D_h$ induces an untwisted $\mathbb{C}^*$-ruling of
Figure 3. Untwisted $C^*$-ruling, $\bar{\kappa}(S') = -\infty$.

Suppose $p$ is untwisted. Because $\kappa \neq \kappa_0$, $p$ has base $C_1$ and is of type (B)(i). Because

$$0 > \kappa = \lambda - 1 \geq \frac{n}{2} - 1,$$

we get $n \leq 1$, but for $n = 0$ we get $\kappa_0 < \lambda < 0$, so in fact $n = 1$. Then $0 \leq \kappa_0 = 1 - 1/\mu_1 - 1/\min(\mu, \tilde{\mu})$, and hence $\min(\mu, \tilde{\mu}) \geq 2$. □

By Proposition 2.11, $H_i(S', \mathbb{Z})$ vanishes for $i > 1$. If $S'$ is of type $C^2/G$ or of type (ii), then it is contractible. $H_1(S', \mathbb{Z})$ for affine-ruled $S'$ was computed in Remark 3.4. For completeness, we now compute the fundamental group of $S'$ of type (iii), which by Proposition 2.11 is the same as $\pi_1(S)$. Let $E_0$ be a component of $\hat{E}$ intersecting $C$. Contract $\tilde{C}$ and successive vertical $(-1)$-curves until $C$ is the only $(-1)$-curve in the fiber ($C$ cannot became a 0-curve, because it does not intersect $D_h$), and denote this contraction by $\theta$. Let $\theta'$ be the contraction of $\theta_*F_0$ and $F_1$ to smooth fibers. Put $U = S_0 \setminus (C_1 \cup C \cup \tilde{C})$ and let $\gamma_1, \gamma, t \in \pi_1(U)$ be the vanishing loops of the images of $F_1$, $F_0$ under $\theta' \circ \theta$ and of some component of $D_h$ (see [Fujita 1982, 4.17]). We need to compute the kernel of the epimorphism $\pi_1(U) \to \pi_1(S)$. Because $\theta$ does not touch $C$, $\theta_*F_0$ is columnar and $\theta_*E_0 \neq 0$. Using [ibid., 7.17], one can show by induction on the number of components of a columnar fiber that because $E_0 \cdot C \neq 0$, the vanishing loops of $E_0$ and $C$, which are of type $\gamma^a t^b$ and $\gamma^c t^d$, satisfy $ad - bc = \pm 1$. Thus $\gamma$ and $t$ are in the kernel, and hence

$$\pi_1(S) = \langle \gamma_1 : \gamma^\mu_1 \rangle \cong \mathbb{Z}_{\mu_1}.$$

In particular, $S'$ is not a $\mathbb{Z}$-homology plane.

6. Uniqueness of $C^*$-rulings

6A. The number of $C^*$-rulings. We consider the question of uniqueness of $C^*$-rulings of $S_0$ and $S'$. Recall that a $C^*$-ruling of $S_0$ is extendable if it extends to a ruling (morphism) of $S'$. Two rational rulings of a given surface are considered the same if they differ by an automorphism of the base. When a $C^*$-ruling of $S_0$
exists, using the information on snc-minimal boundaries, we are able to compute
the number of different $\mathbb{C}^*$-rulings.

**Theorem 6.1.** Let $S'$ be a singular $\mathbb{Q}$-homology plane that is not affine-ruled. Let $p_1, \ldots, p_r$ for $r \in \mathbb{N} \cup \{\infty\}$ be all different $\mathbb{C}^*$-rulings of the smooth locus $S_0$ of $S'$. Let $D$ be an snc-minimal boundary of $S'$.

1. If $\overline{\kappa}(S_0) = 2$ or if $S'$ is exceptional (so that $\overline{\kappa}(S_0) = 0$), then $r = 0$.
2. If $\overline{\kappa}(S_0) = 1$ or if $S'$ is nonlogarithmic, then $r = 1$.
3. If $\overline{\kappa}(S_0) = -\infty$, then $r \geq 1$ and $p_1$ is nonextendable. Also, $r \neq 1$ only if the fork that is an exceptional divisor of the snc-minimal resolution of $S'$ is of type $(2, 2, k)$. In this case we have:
   1. If $k \neq 2$, then $r = 2$, $p_2$ is twisted, and it has a unique singular fiber, which is of type (A)(iv).
   2. If $k = 2$, then $r = 4$, $p_2$, $p_3$, $p_4$ are twisted, and they have unique singular fibers, which are of type (A)(iv).
4. Assume that $\overline{\kappa}(S_0) = 0$ and that $S'$ is logarithmic and not exceptional. Then all $p_i$ extend to $\mathbb{C}^*$-rulings of $S'$ and the following hold:
   1. If the dual graph of $D$ is
      
      $\begin{array}{c}
      \begin{array}{c}
      -2 \\
      -2 \end{array} & \begin{array}{c}
      -1 \end{array} & \begin{array}{c}
      k \end{array} & \begin{array}{c}
      -2 \end{array} \\
      \begin{array}{c}
      -2 \end{array} & \begin{array}{c}
      -2 \end{array}
      \end{array}$

      with $k \leq -2$, then $r = 1$ and $p_1$ is twisted.
   2. If the dual graph of $D$ is
      
      $\begin{array}{c}
      \begin{array}{c}
      -2 \\
      -2 \end{array} & \begin{array}{c}
      -1 \end{array} & \begin{array}{c}
      -1 \end{array} & \begin{array}{c}
      -2 \end{array} \\
      \begin{array}{c}
      -2 \end{array} & \begin{array}{c}
      -2 \end{array}
      \end{array}$

      then $r = 2$ and $p_1$, $p_2$ are twisted.
   3. If the dual graph of $D$ is
      
      $\begin{array}{c}
      \begin{array}{c}
      -2 \\
      -2 \end{array} & \begin{array}{c}
      k \end{array} & \begin{array}{c}
      0 \end{array} & \begin{array}{c}
      m \end{array} & \begin{array}{c}
      -2 \end{array} \\
      \begin{array}{c}
      -2 \end{array} & \begin{array}{c}
      -2 \end{array}
      \end{array}$

      then $r = 3$, $p_1$, $p_2$ are twisted and $p_3$ is untwisted with base $\mathbb{C}^1$.
   4. In all other cases, $r = 2$, $p_1$ is twisted and $p_2$ is untwisted.
Proof. (1) By definition, exceptional \(\mathbb{Q}\)-homology planes are not \(\mathbb{C}^*\)-ruled. If \(S_0\) is of general type, then by Iitaka’s easy addition formula [Iitaka 1982, 10.4], \(S_0\) is not \(\mathbb{C}^*\)-ruled.

(2) If \(S'\) is nonlogarithmic, then by [Palka 2008, 4.1], the \(\mathbb{C}^*\)-ruled of \(S'\) is unique. Assume that \(\tilde{r}(S_0) = 1\). Let \((\tilde{S}, D)\) be some normal completion of the snc-minimal resolution \(S \to S'\). Denote the exceptional divisor of the resolution by \(\tilde{E}\). By [Fujita 1982, 6.11], for some \(n > 0\), the base locus of \(|n(K_{\tilde{S}} + D + \tilde{E})^+|\) is empty and the linear system gives a \(\mathbb{P}^1\)-ruling of \(\tilde{S}\) that restricts to a \(\mathbb{C}^*\)-ruling of \(S_0\); see also [Miyanishi 2001, 2.6.1]. Consider another \(\mathbb{C}^*\)-ruled of \(S_0\). Modifying \(\tilde{S}\) if necessary, we can assume that it extends to a \(\mathbb{P}^1\)-ruling of \(\tilde{S}\). Let \(f'\) be a general fiber of this extension. Then

\[
f' \cdot (K_{\tilde{S}} + D + \tilde{E}) = f' \cdot K_{\tilde{S}} + 2 = 0,
\]

and hence

\[
f' \cdot (K_{\tilde{S}} + D + \tilde{E})^+ + f' \cdot (K_{\tilde{S}} + D + \tilde{E})^- = 0.
\]

However, \((K_{\tilde{S}} + D + \tilde{E})^-\) is effective and \((K_{\tilde{S}} + D + \tilde{E})^+\) is numerically effective, so

\[
f' \cdot (K_{\tilde{S}} + D + \tilde{E})^+ = f' \cdot (K_{\tilde{S}} + D + \tilde{E})^- = 0,
\]

and we see that the rulings are the same.

(3), (4) We need to understand how to find all twisted \(\mathbb{C}^*\)-ruleds of a given \(S'\). Consider a twisted \(\mathbb{C}^*\)-ruled of \(S'\) and let \((\tilde{V}, \tilde{D}, \tilde{p})\) be a minimal completion of this ruling. By the \(\tilde{p}\)-minimality of \(\tilde{D}\), the only component of \(\tilde{D}\) that can be a non-branching \((-1)\)-curve is \(\tilde{D}_h\), so there is a connected modification \((\tilde{V}, \tilde{D}) \to (V, D)\) with snc-minimal \(D\). Let \(\tilde{D}_0 \subseteq \tilde{D}\) be the \((-1)\)-curve of the fiber at infinity (see Lemma 4.1). \(D\) is not a chain; otherwise \(S'\) would be affine-ruled. Let \(D_0 \subseteq D\) be the image of \(\tilde{D}_0\), and let \(T\) be the connected component of \(D - D_0 \) containing the image of the horizontal component (which is a point if the modification is nontrivial). In this way, a twisted \(\mathbb{C}^*\)-ruled of \(S'\) determines a pair \((D_0, T)\) (with \(D_0 + T\) contained in a boundary of some snc-minimal completion), such that \(\beta_D(D_0) = 3\), \(D_0^2 \geq -1\). \(T\) is a connected component of \(D - D_0\) containing the image of the horizontal section, and both connected components of \(D - D_0 - T\) are \((-2)\)-curves. Conversely, if we have an snc-minimal normal completion \((V, D)\) and a pair as above, we make a connected modification \((\tilde{V}, \tilde{D}) \to (V, D)\) over \(D\) by blowing successively on the intersection of the total transform of \(T\) with the proper transform of \(D_0\) until \(D_0\) becomes a \((-1)\)-curve. The \((-1)\)-curve together with the transform of \(D - T - D_0\) induce a \(\mathbb{P}^1\)-ruled of \(V'\) and constitute the fiber at infinity for this ruling. The restriction to \(S'\) is a twisted \(\mathbb{C}^*\)-ruled.
Suppose \( \kappa(S_0) = -\infty \). Since \( S_0 \) is not affine-ruled, \( S' \cong \mathbb{C}^2/G \) for a finite noncyclic small subgroup \( G < \text{GL}(2, \mathbb{C}) \) (see Section 3). Let \((V, D)\) be an snc-minimal normal completion of \( S' \) and let \( \tilde{S} \rightarrow V \) be a minimal resolution with exceptional divisor \( \hat{E} \). We saw in the proof of Proposition 3.1 that \( S_0 \) admits a platonic \( \mathbb{C}^* \)-ruling, which extends to a \( \mathbb{P}^1 \)-ruling of \( \tilde{S} \). Also, \( D \) and \( \hat{E} \) are forks for which \( D_h \) and \( \hat{E}_h \) are the unique branching components of \( D \) and \( E \) respectively. In particular, the \( \mathbb{C}^* \)-ruling does not extend to a ruling of \( S' \), and because nonbranching components of \( D \) have negative self-intersections, \( (\tilde{S}, D + \hat{E}) \) is a unique snc-minimal smooth completion of \( S_0 \) (and hence \((V, D)\) is a unique snc-minimal normal completion of \( S' \)). It follows from the proof of [Palka 2008, 4.1] that the nonextendable \( \mathbb{C}^* \)-ruling of \( S_0 \) is unique. Suppose there is a \( \mathbb{C}^* \)-ruling of \( S_0 \) that does extend to \( S' \). Since \( \hat{E} \) is not a chain, it follows from the proof of Theorem 4.9 that this ruling is twisted. Since maximal twigs of \( \hat{E} \) and \( D \) are adjacent chains of columnar fibers, we see that a maximal twig of \( D - D_h \) is a \((-2)\)-curve if and only if the respective maximal twig of \( \hat{E} - \hat{E}_h \) is a \((-2)\)-curve. Also, \( 0 < d(\hat{E}) \), so \( \hat{E}_h^2 \leq -2 \), and because \( \hat{E}_h^2 + D_h^2 = -3 \), we have \( D_h^2 \geq -1 \). Therefore, \( S' \) admits a twisted \( \mathbb{C}^* \)-ruling if and only if \( \hat{E} \) is a fork of type \((2, 2, k)\) for some \( k \geq 2 \). If \( k \neq 2 \), then the choice of \((D_0, T)\) as above is unique, and if \( k = 2 \), then there are three such choices. If \((V', D', p)\) is a minimal completion of such a ruling, then \( D' \) is a fork, so because \( \kappa_0 < 0 \), we have \( n = 0 \) and \( F_0 \) is of type \((A)(iv)\) (see the proof of Theorem 4.9). This gives (3).

We can now assume that \( \kappa(S_0) = 0 \) and that \( S' \) is logarithmic and not exceptional. Then \( S_0 \) is \( \mathbb{C}^* \)-ruled and by [Palka 2008, 4.7(iii)], each \( \mathbb{C}^* \)-ruling of \( S_0 \) extends to a \( \mathbb{C}^* \)-ruling of \( S' \). Let \( r \in \{1, 2, \ldots \} \cup \{\infty\} \) be the number of different (up to automorphism of the base) \( \mathbb{C}^* \)-ruledings of \( S' \) and let \((V_i, D_i, p_i)\), for \( i \leq r \), be their minimal completions. Minimality implies that nonbranching \((-1)\)-curves in \( D_i \) are \( p_i \)-horizontal. We add consequently an upper index \( (i) \) to objects defined previously for any \( \mathbb{C}^* \)-ruleding when we refer to the ruling \( p_i \). If \( p_i \) is untwisted, we denote the horizontal components of \( D_h^{(i)} \) by \( H^{(i)} \) or \( H^{(i)} \).

Suppose \( p_1 \) is untwisted with base \( \mathbb{P}^1 \). Then \( F_0^{(1)} \) contains a rivet and by Corollary 4.10, \( n^{(1)} = 2 \), so \( D_1 \) does not contain nonbranching \( b \)-curves with \( b \geq -1 \). Then \((V_1, D_1)\) is balanced and \( S' \) does not admit an untwisted \( \mathbb{C}^* \)-ruleding with base \( \mathbb{C}^1 \), because it does not contain nonbranching \( 0 \)-curves (see Lemma 4.1). By Corollary 4.10, each component of \( D_h^{(1)} \) has \( \beta_{D_1} = 3 \) and intersects two \((-2)\)-tips of \( D_1 \). Note that \( \zeta^{(1)} \) (see Lemma 4.5) touches \( D_h^{(1)} \) two times if both components of \( D_h^{(1)} \) intersect the same horizontal component of \( F_0^{(1)} \) and three times if not. By Lemma 4.5 and the properties of Hirzebruch surfaces, we get \(-3 \leq (D_h^{(1)})^2 \leq -2 \). In particular, one of the components of \( D_h^{(1)} \), say \( H^{(1)} \), has \((H^{(1)})^2 \geq -1 \), so by the discussion about twisted \( \mathbb{C}^* \)-ruledings above, \( H^{(1)} \) together with two \((-2)\)-tips of \( D_1 \) gives rise to a twisted \( \mathbb{C}^* \)-ruleding \( p_2 \) of \( S' \). Because \( H^{(1)} \) together with two
Then $p$ is the only twisted ruling of $S'$, because $H^{(1)}$ is the only possible choice for a middle component of the fiber at infinity of a twisted ruling. Suppose $r \geq 3$. Then $p_3$ is untwisted with base $\mathbb{P}^1$. Because $D_1$ does not contain nonbranching 0-curves, any flow in $D_1$ is trivial, so $V_3 = V_1$. Because $p_3$ and $p_1$ are different after restriction to $S'$, the $S_0$-components $C^{(1)}, C^{(3)}$ contained respectively in $F^{(1)}_0, F^{(3)}_0$ are different. Because they both intersect $\hat{E}$, they are contained in the same fiber of $p_2$, which contradicts $\Sigma^{(2)} = 0$. Because $D$ contains no nonbranching 0-curves, $D$ is not of type (4)(iii). Since $n^{(1)} = 2$, $D$ contains at least seven components, so $D$ is not of type (4)(i) or (4)(ii).

We can now assume that each untwisted $\mathbb{C}^*$-ruling of $S'$ has base $\mathbb{C}^1$. Suppose $p_1$ is such a ruling. By Corollary 4.10, both horizontal components of $D_1$ have $\beta_D = 3$, and one of them, say $H^{(1)}$, intersects two $(-2)$-tips $T$ and $T'$ of $D_1$. In particular, $D_1$ is snc-minimal. Because $F^{(1)}_\infty = [0]$, changing $V_1$ by a flow if necessary, we may assume that $H^{(1)}$ is a $(-1)$-curve. Then

$$F^{(2)}_\infty = T + 2H^{(1)} + T'$$

induces a $\mathbb{P}^1$-ruling $p_2 : V_1 \to \mathbb{P}^1$, which is a twisted $\mathbb{C}^*$-ruling after restricting it to $S'$. Suppose $r \geq 3$. If $p_3$ is untwisted, then its base is $\mathbb{C}^1$, and changing $V_3$ by a flow if necessary, we can assume that $V_3 = V_1$. But then $F^{(1)}_\infty = F^{(3)}_\infty$, because $D_1$ contains only one nonbranching 0-curve, so $p_1$ and $p_3$ have a common fiber and hence cannot be different after restriction to $S'$, which is a contradiction. Thus $p_3$ is twisted. By the discussion above, $p_3$ can be recovered from a pair $(D_0, T)$ on some snc-minimal completion of $S'$. All such completions of $S'$ differ from $(V_1, D_1)$ by a flow, which is an identity on $V_1 - F^{(1)}_\infty$, and hence the birational transform of $D_0$ on $V_1$ is either $H^{(1)}$ or $H^{(1)}$. Because the restrictions of $p_1$ and $p_2$ to $S'$ are different, it is $H^{(1)}$. It follows that $r = 3$ and that $D_1 - H^{(1)}$ has two $(-2)$-tips as connected components, and hence the dual graph of $D_1$ is as in (iii). Conversely, if $S'$ has a boundary as in (iii), then besides the untwisted $\mathbb{C}^*$-ruling induced by the 0-curve, it has also two twisted rulings, each with one of the branching components as the middle component of the fiber at infinity.

We can finally assume that all $\mathbb{C}^*$-rulings of $S'$ are twisted. Let $(V, D)$ be a balanced completion of $S'$. Because $S'$ does not admit untwisted $\mathbb{C}^*$-rulings, $D$ does not contain nonbranching 0-curves, so $(V, D)$ is a unique snc-minimal completion of $S'$. Thus, to find all twisted $\mathbb{C}^*$-rulings of $S'$, we need to determine all pairs $(D_0, T)$ such that $D_0 + T \subseteq D$, $D_0^2 \geq -1$, $\beta_D(D_0) = 3$, and $D - T - D_0$ consists of two $(-2)$-tips. Let $(D_0, T)$ and $(D'_0, T')$ be two such pairs. Suppose $D_0 \neq D'_0$ and, say, $D_0^2 \geq D'_0^2$. We have $D_0, D'_0 \neq 0$, for otherwise the chain $D - T'$, which is not negative definite, would be contained in (and not equal to, because $v \leq 1$) a fiber of the twisted ruling associated with $(D_0, T)$, which is impossible. Then $D$
has six components and we check that
\[ d(D) = 16((D_0^2 + 1)(D_0^2 + 1) - 1), \]
so \((D_0^2 + 1)(D_0^2 + 1) \leq 0\), because \(d(D) < 0\). Then \(D_0^2 = -1\) and \(D_0\) is a 2-section of the twisted ruling associated with \((D_0, T)\). Because \(\beta_D(D_0') = 3\), by Corollary 4.10 and Lemma 4.5 for this ruling \(n = 1\), \(D_0'\) is a \((-1)\)-curve and \(D\) has dual graph as in (ii). Conversely, it is easy to see that \(S'\) with such a boundary has two twisted \(\mathbb{C}^*\)-rulings. Therefore, we can assume that the choice of \(D_0\) for a pair \((D_0, T)\) as above is unique. Let \(p_1\) be a twisted \(\mathbb{C}^*\)-ruling associated with some pair \((D_0, T)\). Suppose \(n^{(1)} = 0\). By Lemma 4.5, \(\zeta_sD^{(1)}_h\) is a 0-curve, so
\[ F = \zeta\zeta_sD^{(1)}_h \]
induces a \(\mathbb{P}^1\)-ruling \(p\) of \(V\). If \(\zeta\) touches \(D^{(1)}_h\), then \(F\) contains the \(S_0\)-component of \(F_0^{(1)}\), so \(F \not\subset D\) and \(p\) restricts to an untwisted \(\mathbb{C}^*\)-ruling of \(S'\) with base \(\mathbb{P}^1\). If \(\zeta\) does not touch \(D^{(1)}_h\), then \(p\) restricts to a \(\mathbb{C}^*\)-ruling of \(S'\) with base \(\mathbb{C}^1\). This contradicts the assumption. By Corollary 4.10 we get that \(n^{(1)} = 1\), \(F_0^{(1)}\) contains no \(D_1\)-components, and \(\mu_1 = 2\). In particular, \(D_1 = D\). By Lemma 4.5, \((D^{(1)}_h)^2 \leq -1\) because \(n^{(1)} = 1\), so \(D\) has a dual graph as in (i) or (ii). Conversely, if \(D\) is of type (i) or (ii), then \(r = 2\) if \(k = -1\) and \(r = 1\) if \(k \leq -2\).

6B. The number of affine lines. Theorem 6.1 has interesting consequences. It is known [ Zaïdenberg 1987; Gurjar and Miyanishi 1992] that \(\mathbb{Q}\)-homology planes with smooth locus of general type (in particular the smooth ones) do not contain topologically contractible curves. In fact, the number \(\ell \in \mathbb{N} \cup \{\infty\}\) of contractible curves on a \(\mathbb{Q}\)-homology plane \(S'\) is known except two cases: when \(S'\) is non-logarithmic and when \(S'\) is singular and \(\kappa(S_0) = 0\) (see [Palka 2011b, 10.1] and references there). Clearly, in the first case \(\ell = \infty\) by the main result of [Palka 2008]. The case when \(S'\) is smooth and of Kodaira dimension zero has been considered in [Gurjar and Parameswaran 1995]. Theorem 1.3 is the missing piece of information, and the method can be easily applied to recover the result of Gurjar and Parameswaran.

Proof of Theorem 1.3. We can assume that \(S'\) is logarithmic. Suppose \(S'\) contains a topologically contractible curve \(L\). We show that \(L\) is vertical for some \(\mathbb{C}^*\)-ruling of \(S'\). The proper transform of \(L\) on \(\tilde{S}\) meets each connected component of \(\hat{E}\) in at most one point. We use the logarithmic Bogomolov–Miyaoka–Yau inequality as in [Koras and Russell 2007, 2.12] to show that \(\kappa(S_0 - L) \leq 1\). In case \(\kappa(S_0 - L) = 1\), the surface \(S_0 - L\) is \(\mathbb{C}^*\)-ruled [Fujita 1982, 6.11], so we may assume that \(\kappa(S_0 - L) = 0\). Let \(\mathbb{Z}[D + \hat{E}]\) be a free abelian group generated by the components of \(D + \hat{E}\). Because
\[ \text{Pic } S_0 = \text{Coker}(\mathbb{Z}[D + \hat{E}] \to \text{Pic } \tilde{S}) \]
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is torsion, the class of $L$ in $\text{Pic} S_0$ is torsion. So there is a surjection $f : S_0 - L \rightarrow \mathbb{C}^*$, and taking its Stein factorization, we get a $\mathbb{C}^*$-ruling of $S_0 - L$, which (because $\kappa(S_0) \neq -\infty$) extends to a $\mathbb{C}^*$-ruling of $S_0$. Since $S_0$ is logarithmic, each $\mathbb{C}^*$-ruling of $S_0$ extends in turn to a $\mathbb{C}^*$-ruling of $S'$. Therefore $L$ is vertical for some $\mathbb{C}^*$-ruling of $S'$ and we are done. In particular, exceptional $\mathbb{Q}$-homology planes do not contain contractible curves. It follows from Corollary 4.10 that if the ruling is twisted or untwisted with base $\mathbb{P}^1$, then the vertical contractible curve is unique and is contained in the unique singular noncolumnar fiber. For an untwisted ruling with base $\mathbb{C}^1$, there are at most two such curves. In particular, in cases (4)(i) and (4)(ii) of Theorem 6.1, $L$ needs to intersect the horizontal component of the boundary, so we get respectively $\ell = 1$ and $\ell = 2$. In case (4)(iii), the unique vertical contractible curves for the twisted rulings $p_1$ and $p_3$ are distinct and do not intersect the horizontal components of respective rulings, and hence are both vertical for the untwisted ruling $p_3$, so $\ell = 2$. In the remaining case (4)(iv), $r = 2$, $p_1$ is twisted and $p_2$ is untwisted. We can assume that the base of $p_2$ is $\mathbb{C}^1$ and the unique noncolumnar singular fiber contains two contractible curves, $L_1$ and $L_2$, for otherwise $\ell \leq 2$ by the above remarks and we are done. Since the twisted ruling is unique, there is exactly one horizontal component $H$ of $D^{(2)}_h$ that meets two $(-2)$-tips of $D^{(1)}_h$ (together with these tips it induces the twisted ruling). Clearly, only one $L_1$ can intersect $H$, so the second one is vertical for $p_1$ and we get $\ell \leq 2$ is this case too. □

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