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## A DYNAMICAL INTERPRETATION OF THE PROFILE CURVE OF CMC TWIZZLER SURFACES


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# A DYNAMICAL INTERPRETATION OF THE PROFILE CURVE OF CMC TWIZZLER SURFACES

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**Delaunay showed in 1841 that any surface of revolution of constant mean curvature in $\mathbb{R}^3$ has as its profile curve a roulette — specifically, the curve described by the focus of a quadric rolling on a line. Here we introduce a notion similar to the roulette that we call the *treadmill sled*, and we use it to provide a dynamical interpretation for the profile curves of *twizzlers* — helicoidal surfaces of nonzero constant mean curvature.**

**The treadmill sled is connected with a change of variables that allows us to solve the ordinary differential equation that produces twizzlers in a fairly easy way. This allows us to prove that all twizzlers are isometric to Delaunay surfaces; this is similar to work done by do Carmo and Dajczer.**

**We also provide a moduli space for twizzlers and Delaunay surfaces that shows the connection of each surface with its dynamical interpretation, and we explicitly show the foliation of our moduli space by curves of locally isometric CMC "associated surfaces" analogous to the well-known helicoid-to-catenoid deformation. Our dynamical interpretation for twizzlers also allows us to naturally define the notion of a *fundamental piece* of the profile curve of a twizzler, which yields the fact that, whenever a twizzler is not properly immersed, it is dense in the region bounded by two concentric cylinders if the twizzler does not contain the axis of symmetry, or dense in the region bounded by a cylinder otherwise.**

**Using the change of coordinates induced by the notion of the treadmill sled, we also provide a dynamical interpretation for helicoidal surfaces with constant Gauss curvature, and we find an easy way to describe Delaunay surfaces by a relatively simple first integral.**

## 1. Introduction

Delaunay [1841] showed that if one rolls a conic section on a line in a plane and then rotates about that line the trace of a focus, one obtains a surface of revolution of constant mean curvature (CMC). When the conic is a parabola we obtain a catenoid; when the conic is an ellipse, the surface is embedded and it is called an

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*Keywords:* twizzler, constant mean curvature, helicoidal surfaces, Delaunay surfaces.

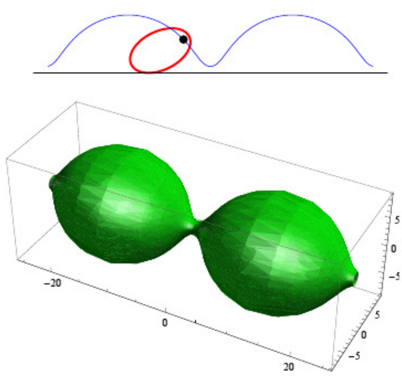


**Figure 1.** Dynamic interpretation of the profile curve of an unduloid.

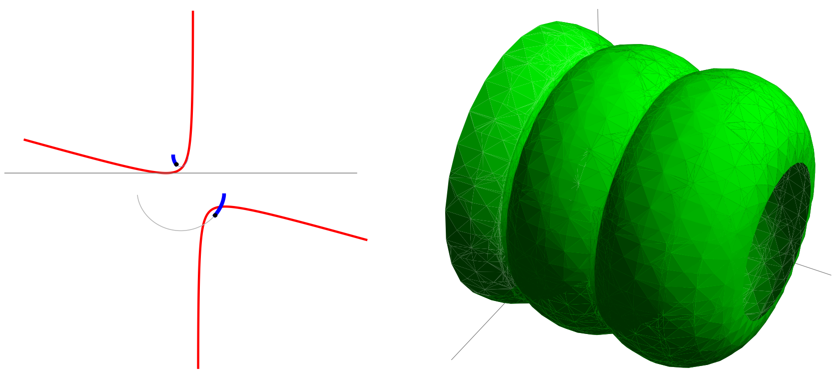

**Figure 2.** A nodoid and the construction of its profile curve.

*unduloid*; and when the conic is a hyperbola the surface is not embedded and it is called a *nodoid*. Unduloids and nodoids are called *Delaunay surfaces*. Figure 1 illustrates the relation between the ellipse and the trace of its focus. Notice that only one focus is used to get the curve that needs to get rotated in order to generate an unduloid. Figure 2 illustrates the relation between the hyperbola and the trace of its foci. Notice that both foci are used to get the curve that needs to get rotated in order to generate a nodoid.

Using the integrability of the Gauss equation and the Mainardi–Codazzi equation, Lawson [1970] showed that for any immersion $f_0 : U \to \mathbb{R}^3$ with constant mean curvature $H$ defined in a simple connected surface $U$, there exists a $2\pi$-periodic 1-parametric family of immersions $\{f_\theta : U \to \mathbb{R}^3 : \theta \in [0, 2\pi]\}$ with constant mean curvature $H$ and with the same induced metric. This family is called the $2\pi$-periodic isometric family *associated to* $f_0$.

**Remark 1.1.** The map $\theta \to f_\theta$ is continuous with respect to the parameter $\theta$.

We can see this family of associated surfaces in the well-known deformation from a helicoid to a catenoid. See Figure 3.

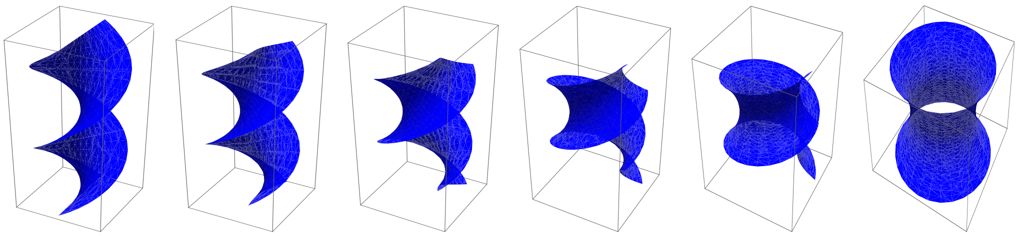

**Figure 3.** All these surfaces are isometric.

In this particular helicoid-to-catenoid deformation, the helicoid corresponds to $\theta = 0$ and the catenoid corresponds to $\theta = \pi/2$. The images in Figure 3 were taken by substituting $\theta = 0, \pi/10, \pi/5, 3\pi/10, 2\pi/5, \pi/2$ in the parametrization

$$\phi_\theta(u, v) = \big(\cos\theta \sinh v \sin u + \cos u \sin\theta \cosh v,\\ \cosh v \sin u \sin\theta - \cos u \cos\theta \sinh v,\ u\cos\theta + v\sin\theta\big).$$

A direct verification shows that the first and second fundamental form of $\phi_\theta$ are given by

$$E = G = \cosh^2 v, \quad F = 0, \quad e = -\sin\theta, \quad f = \cos\theta, \quad g = \sin\theta,$$

from which we can infer that indeed all the elements in this family of surfaces are isometric. It is not difficult to show that the surfaces from $\theta = \pi/2$ to $\pi$ are, up to a rigid motion, in the Euclidean space, the same as the surfaces from $\theta = 0$ to $\pi/2$. In this way, up to a rigid motion, all the surfaces in the $2\pi$-periodic Lawson family of isometric surfaces to a helicoid are contained in those surfaces from $\theta = 0$ to $\theta = \pi/2$. We see in this paper that something similar happens for the isometric associated family to a Delaunay surface.

A surface is called helicoidal with pitch $h \in (-\infty, \infty)$ if it is invariant under the group $g_t : \mathbb{R}^3 \to \mathbb{R}^3$ of rigid motions

$$g_t(x, y, z) = \big(x\cos t + y\sin t, -x\sin t + y\cos t, z + ht\big).$$

When $h = 0$, the group $g_t$ becomes a group of rotations and the helicoidal surfaces become surfaces of revolution. A *twizzler* is an immersion of the form

$$\phi(s, t) = \big(x(s)\cos(wt) + z(s)\sin(wt), t, -x(s)\sin(wt) + z(s)\cos(wt)\big), \tag{1-1}$$

with constant mean curvature. Assume that the curve $(x(s), z(s))$ is parametrized by arc length and call it the *profile curve* of the twizzler. Notice that twizzlers correspond to those helicoidal CMC surfaces with nonzero pitch. Geometrically, up to a rotation about the origin, the profile curve of a twizzler is the intersection of the surface with a plane perpendicular to the axis of symmetry. Here we give an interpretation of the profile curve of twizzlers similar to the interpretation for the profile curves of Delaunay surfaces.

To do this, we introduce an operator taking curves into curves (like the roulette operator), which we call the *treadmill sled*. Given a curve $\alpha$, we imagine a movable plane supporting $\alpha$ rigidly. The trace of the origin of this plane on a stationary plane will be the new curve $\beta$, the treadmill sled of $\alpha$. We now describe the motion of $\alpha$ (and its supporting plane).

First, we choose a point of $\alpha$ and place it at the origin of the fixed plane, so that $\alpha$ has a horizontal tangent there — the $x$-axis of the fixed plane. Then we move the supporting plane of $\alpha$ in such a way that $\alpha$ always remains tangent to the $x$-axis of the fixed plane at the origin. (Another way of thinking of this motion is to imagine a treadmill placed under, and aligned with, the $x$-axis of the fixed plane. The curve $\alpha$ rolls on the treadmill, always keeping one of its points at the origin.)

As already explained, $\beta$ is described by the positions of the origin of $\alpha$'s supporting plane in this process. Obviously, the choice of the moving plane's origin plays an important role in this definition. For example, if $\alpha$ is a circle of radius $R$, its treadmill sled is just a point if the center of $\alpha$ is the origin and it is a circle of radius $r$ with center at $(0, R)$ if the center of the circle is at a distance $r$ from the origin.

Figure 4 shows the treadmill sled of an ellipse with center at the origin. The dot represents the center of the ellipse. (The Electronic Supplement to this article shows this example in motion.)

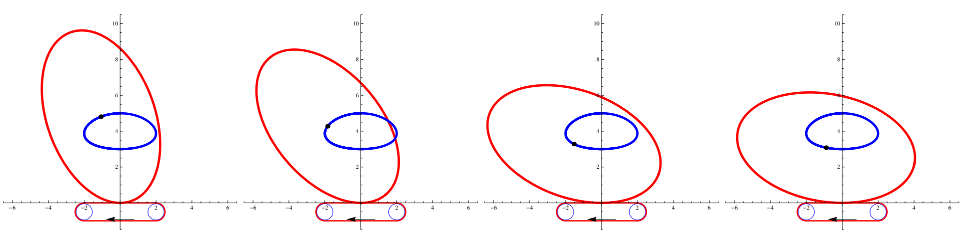

**Figure 4.** Treadmill sled of an ellipse centered at the origin.

We prove that a parametrization of the treadmill sled of an arc-length parametrized curve $(x(s), z(s))$ is given by $(\xi(s), \xi_2(s))$, where $(\xi(s), \xi_2(s))$ are the coordinates of the vector $(x(s), z(s))$ with respect to the orthonormal basis

$$\{(x'(s), z'(s)), (-z'(s), x'(s))\}.$$

It turns out that this treadmill sled notion is linked with the change of variables $x(s)$, $z(s)$ to the variables $\xi_1(s)$, $\xi_2(s)$, which ends up being very convenient for

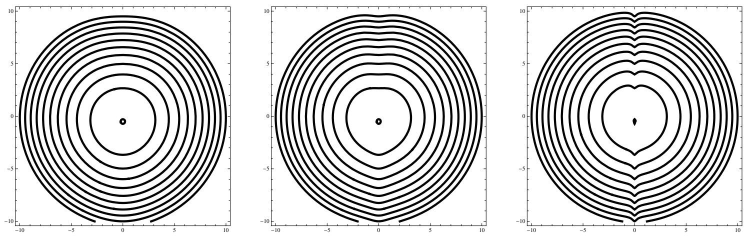

**Figure 5.** Contours, for different values of $w$, of the integral function $h_w(x, y) = x^2 + y^2 + y/\sqrt{1 + w^2x^2}$.

the study of helicoidal surfaces. This paper shows some of the applications of this change of variables. We can relatively easily solve the ODE that generates twizzlers, and as a bonus, we find a dynamical interpretation for their profile curve similar to the dynamical interpretation of the profile curve of Delaunay surfaces using conics. For twizzlers we do not use conics, but rather the level sets of the function

$$h_w(x, y) = x^2 + y^2 + \frac{y}{\sqrt{1 + w^2x^2}},$$

where $w$ is a constant. It is not difficult to check that the range of the function $h_w$ is the interval $[-\frac{1}{4}, \infty)$, that $h_w^{-1}(-\frac{1}{4}) = \{(0, -\frac{1}{2})\}$, that every $M > -\frac{1}{4}$ is a regular value of $h_w$, and that $h_w^{-1}(M)$ is a closed simple curve. We refer to these level sets as *heart-shaped* curves. Figure 5 shows some of them.

Let us denote the origin of the profile curve of a twizzler by $O$; that is, $O$ is the intersection of the plane that contains the profile curve with the axis symmetry of the twizzler. We prove that the level sets of the function $h_w$ are first integrals of the ODE for twizzlers with CMC 1 written in the coordinates $\xi_1$ and $\xi_2$, and therefore geometrically we can say that if we place the profile curve of a twizzler on a treadmill located at the origin and oriented in the positive direction of the $x$-axis, then the trace of the point $O$ is a heart-shaped curve. In other words, the treadmill sled of the profile curve is a heart-shaped curve. It can be shown that the inverse of the treadmill sled of a curve is unique up to a rotation about the origin. Therefore we have a one-to-one correspondence between twizzlers with CMC 1 and the level sets of the function $h_w$. In this way, we can use the two parameters $w$ and $M$ that define the heart-shaped curves to describe twizzlers with CMC 1. Once we have all the twizzlers with CMC 1 described in terms of the treadmill sled of their profile curves, we explicitly describe which twizzlers are in the same associated family of isometric surfaces. Surprisingly for the author, the proof only uses the basic fact that, since Gauss curvatures are invariant under isometries, the quotient between the maximum and minimum of the Gauss curvature is the same for two isometric surfaces. An interesting fact that showed up is that in each one of these families of isometric associated surfaces, there is a twizzler that contains the axis of symmetry. Since such twizzlers are unique in each family and there is an easy formula that relates them with the isometric nodoid and unduloid, we call these twizzlers *special twizzlers*.

Lawson [1970] showed examples of helicoidal surfaces with nonzero pitch and constant mean curvature by proving that the family of CMC surfaces associated to a Delaunay surface is made out of helicoidal surfaces. It was known for a long time [Graustein 1935] that all the isometric surfaces in the associated family of a catenoid are helicoidal surfaces, and also that every helicoidal minimal surface belongs to the associated family of isometric surfaces of a catenoid. This result was generalized by do Carmo and Dajczer [1982] (see also [Haak 1998]), who showed that every helicoidal surface with CMC is in the associated family of a Delaunay surface. Do Carmo and Dajczer provided explicit parametrizations for almost all helicoidal surfaces with CMC. As we pointed out before, we show here that there are as many of these surfaces as unduloids, or as many as nodoids, by proving that there is one in each associated family of a Delaunay surface. The unduloids admit in their isometry group, besides the rotational symmetries, a discrete group of translations. This translational group shows up because the profile curve is periodic, and this periodicity happens because the profile curve is generated by an ellipse, which is a closed curve. The new dynamical interpretation for twizzlers allows us to easily visualize that, besides the helicoidal symmetry, twizzlers are invariant under a group of rotations about the axis of symmetry because the treadmill sled of their profile curve is a closed curve — a heart-shaped curve. If we define the fundamental piece of a twizzler as a connected part of the profile curve with the property that, when placed on a treadmill, the point $O$ traces a heart-shaped curve exactly once, then we have that the whole profile curve is a union of fundamental pieces. Two fundamental pieces differ by a rotation about the origin, and when the angle made by the rays that connect the initial and final point of a fundamental piece is a rational multiple of $2\pi$, then the whole profile curve is the union of only finitely many fundamental pieces, and therefore the twizzler is properly immersed. Otherwise, the twizzler is dense in either the region bounded by two cylinders or the region inside a cylinder. For twizzlers that do not contain the axes of symmetry, this property was shown in [Hitt and Roussos 1991].

## 2. The treadmill sled of a curve

According to the description given in the introduction, we define the treadmill sled of an arc-length parametrized curve $\alpha : [a, b] \to \mathbb{R}^2$ as

$$\begin{aligned}\mathrm{TS}(\alpha) = \bigl\{ T_s(0,0) : T_s \text{ is an oriented isometry of } \mathbb{R}^2,\\ T_s(\alpha(s)) = (0,0), \text{ and } dT_s\alpha'(s)) = (1,0) \bigr\}.\end{aligned}$$

As the following theorem shows, finding a parametrization for the treadmill sled of a curve is not difficult.

**Theorem 2.1.** *If $\alpha(s) = (x(s), z(s))$ is a curve parametrized by arc length and*

$$\xi_1(s) = x(s)\, x'(s) + y(s)\, y'(s) \quad \textit{and} \quad \xi_2(s) = -x(s)\, y'(s) + y(s)\, x'(s),$$

*then the treadmill sled of the curve $\alpha$ is $-(\xi_1(s), \xi_2(s))$.*

*Proof.* Let $\theta(s)$ be such that $\alpha'(s) = \bigl(\cos\theta(s), \sin\theta(s)\bigr)$. A direct computation shows that the transformation

$$\begin{aligned} T_s(X, Y) = {} & \bigl(\cos\theta(s)\, X + \sin\theta(s)\, Y, -\sin\theta(s)\, X + \cos\theta(s)\, Y\bigr) \\ & - \bigl(\cos\theta(s)\, x(s) + \sin\theta(s)\, y(s), -\sin\theta(s) x(s) + \cos\theta(s) y(s)\bigr) \end{aligned}$$

is the only oriented isometry of $\mathbb{R}$ that takes the point $\alpha(s)$ to the origin and for which $dT_{\alpha(s)}(\alpha'(s)) = (1, 0)$. From the definition of $TS(\alpha)$, it follows that

$$T_s(0, 0) = -\bigl(\cos\theta(s)\, x(s) + \sin\theta(s)\, y(s), -\sin\theta(s)\, x(s) + \cos\theta(s)\, y(s)\bigr)$$

must be a point in the treadmill sled of $\alpha$. When we allow $s$ to move through the domain of $\alpha$ we obtain the desired parametrization of $TS(\alpha)$. □

**Remark 2.2.** It easily follows, either from the geometric definition of treadmill sleds or from Theorem 2.1, that the maximum distance from the origin to a curve $\alpha$ equals the maximum distance from the origin to its treadmill sled. Likewise, the minimum distance from the origin to a curve $\alpha$ equals the minimum distance from the origin to its treadmill sled.

## 3. Treadmill sled coordinates on twizzlers: solution of the ODE

The following two results provide a solution for the ODE coming from the problem of finding all twizzlers with CMC 1. As mentioned before, the ODE is greatly simplified when we use treadmill sled coordinates.

**Proposition 3.1.** *The immersions given by* (1-1) *have mean curvature* 1 *if and only if the functions $\xi_1$ and $\xi_2$ defined in Theorem 2.1 satisfy the ordinary differential equations $\xi_1'(s) = f_1(\xi_1(s), \xi_2(s))$ and $\xi_2'(s) = f_2(\xi_1(s), \xi_2(s))$, where*

$$\begin{aligned} f_1(x_1, x_2) &= \frac{-w^2 x_2 + 2(1 + w^2 x_1^2)^{3/2}}{1 + w^2(x_1^2 + x_2^2)}\, x_2 + 1, \\ f_2(x_1, x_2) &= \frac{w^2 x_2 - 2(1 + w^2 x_1^2)^{3/2}}{1 + w^2(x_1^2 + x_2^2)}\, x_1. \end{aligned} \tag{3-1}$$

*Moreover, the function $h_w$ is constant along all solutions $(\xi_1(s), \xi_2(s))$.*

*Proof.* Since the curve $(x(s), z(s))$ is parametrized by arc length, we can consider a function $\theta(s)$ such that

$$x'(s) = \cos\theta(s) \quad \text{and} \quad z'(s) = \sin\theta(s).$$

Let us define the functions $\xi_1(s)$ and $\xi_2(s)$ by

$$\xi_1 = x\cos\theta + z\sin\theta \quad \text{and} \quad \xi_2 = -x\sin\theta + z\cos\theta.$$

A direct verification shows that

$$x = \xi_1\cos\theta - \xi_2\sin\theta, \quad z = \xi_1\sin\theta + \xi_2\cos\theta, \quad \theta' = x'z'' - z'x''. \tag{3-2}$$

Moreover, it is not difficult to check that

$$\xi_1' = \theta'\xi_2 + 1, \quad \xi_2' = -\theta'\xi_1, \quad \xi_1^2 + \xi_2^2 = x^2 + z^2.$$

A direct verification shows that the first fundamental form of $\phi$ is given by

$$E = \langle\phi_s, \phi_s\rangle = 1, \quad F = \langle\phi_s, \phi_t\rangle = w(zx' - xz') = w\xi_2,$$
$$G = \langle\phi_t, \phi_t\rangle = 1 + w^2(x^2 + z^2) = 1 + w^2(\xi_1^2 + \xi_2^2),$$

and therefore,

$$EG - F^2 = 1 + w^2(\xi_1^2 + \xi_2^2) - w^2\xi_2^2 = 1 + w^2\xi_1^2.$$

The Gauss map of the immersion $\phi$ is given by $\nu = \frac{1}{\sqrt{EG-F^2}}\,\phi_s \times \phi_t$. A direct verification shows that

$$\nu(s,t) = \frac{1}{\sqrt{1+w^2\xi_1^2(s)}}\Big(\sin(wt - \theta(s)),\, w\xi_1,\, \cos(wt - \theta(s))\Big).$$

A direct verification shows that the second fundamental form of $\phi$ is given by

$$e = \langle\phi_{ss}, \nu\rangle = \frac{\theta'}{\sqrt{1+w^2\xi_1^2}}, \qquad f = \langle\phi_{st}, \nu\rangle = \frac{-w}{\sqrt{1+w^2\xi_1^2}},$$
$$g = \langle\phi_{tt}, \nu\rangle = \frac{-w^2\xi_2}{\sqrt{1+w^2\xi_1^2}}.$$

Therefore, if we assume that the mean curvature $\frac{eG-2fF+gE}{2(EG-F^2)}$ equals 1, we obtain the ODE

$$\theta' = \frac{-w^2\xi_2 + 2(1+w^2\xi_1^2)^{3/2}}{1+w^2(\xi_1^2+\xi_2^2)}. \tag{3-3}$$

Using this expression for $\theta'$ in the equations $\xi_1' = \theta'\xi_2 + 1$ and $\xi_2' = -\theta'\xi_1$, we obtain that $\xi_1$ and $\xi_2$ satisfy the ODE

$$\xi_1' = f_1(\xi_1, \xi_2), \quad \xi_2' = f_2(x_1, x_2), \tag{3-4}$$

where

$$f_1(x_1, x_2) = \frac{-w^2x_2 + 2(1 + w^2x_1^2)^{3/2}}{1 + w^2(x_1^2 + x_2^2)}\, x_2 + 1,$$

$$f_2(x_1, x_2) = \frac{w^2x_2 - 2(1 + w^2x_1^2)^{3/2}}{1 + w^2(x_1^2 + x_2^2)}\, x_1.$$

A direct verification shows that if we define $h_w : \mathbb{R}^2 \to \mathbb{R}$ as

$$h_w(x_1, x_2) = \frac{x_2}{\sqrt{1 + w^2x_1^2}} + x_1^2 + x_2^2,$$

then $h_w$ is a first integral of the ODE for $\xi_1$ and $\xi_2$; that is, for any solution $\xi_1(s)$ and $\xi_2(s)$ of this system, we have that $h_w(\xi_1(s), \xi_2(s)) = M$, where $M$ is a constant. This completes the proof of the proposition. □

As a consequence of the previous proposition, we have:

**Theorem 3.2.** *The treadmill sled of the profile curve of a twizzler with constant mean curvature* 1 *is a heart-shaped curve* $-h_w^{-1}(M)$ *for some* $M \geq -\frac{1}{4}$. *The value* $M = -\frac{1}{4}$ *is achieved by a cylinder of radius* $\frac{1}{2}$.

Figure 6 shows the profile curve of a twizzler and the heart-shaped curve associated with it. Figure 7 illustrates that, for this twizzler, the treadmill sled of the profile curve is indeed the negative of the heart-shaped curve.

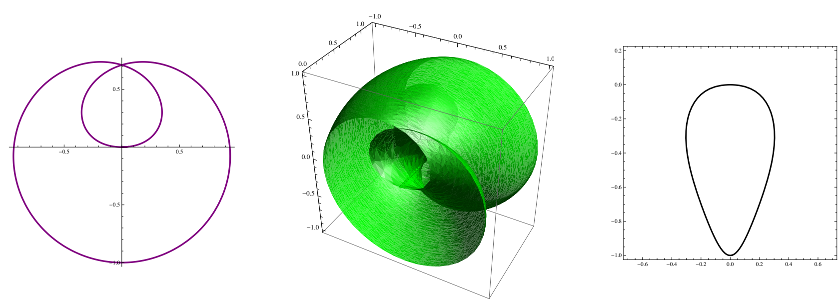

**Figure 6.** Profile curve, surface, and heart-shaped curve associated to a twizzler.

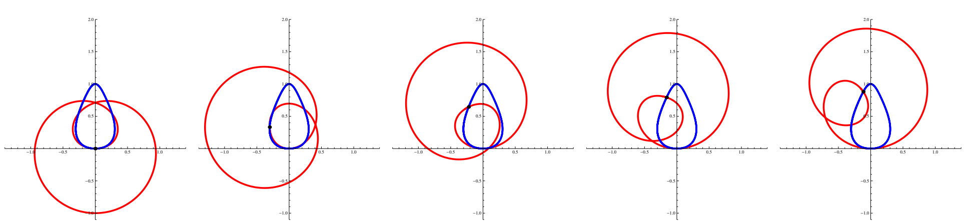

**Figure 7.** Treadmill sled of the profile curve of a twizzler.

## 4. Treadmill sled coordinates on flat surfaces

The following theorem gives us another application of the treadmill sled.

**Theorem 4.1.** *A surface of the form* (1-1) *is flat if and only if either the treadmill sled of the profile curve is a point in the $y$-axis other than the origin* (*in this case the surface is a cylinder*) *or the treadmill sled of the profile curve is contained in a vertical half-line that starts at a point in the $x$-axis other than the origin. The functions $x$ and $z$ can be explicitly computed*:

$$x(s) = \tfrac{1}{2}\cos\frac{2\sqrt{as+b}}{a} + \sqrt{as+b}\,\sin\frac{2\sqrt{as+b}}{a},$$
$$z(s) = \sqrt{as+b}\,\cos\frac{2\sqrt{as+b}}{a} - \tfrac{1}{2}\sin\frac{2\sqrt{as+b}}{a}.$$

*Proof.* If we define the functions $\theta$, $\xi_1$, and $\xi_2$ as in the previous theorem, then the equation for Gauss curvature equal to zero, $eg - f^2 = 0$, reduces to $\theta' = -1/\xi_2$. Substituting this equation in the equations $\xi_1' = \theta'\xi_2 + 1$ and $\xi_2' = -\theta'\xi_1$, we obtain that $\xi_1$ and $\xi_2$ satisfy the ODE

$$\xi_1' = 0, \qquad \xi_2' = \frac{\xi_1}{\xi_2}. \tag{4-1}$$

It follows that $\xi_1(s) = a/2$ for some real number $a$. If $a = 0$, then $\xi_2$ is also a constant other than zero, and the surface $\phi$ is a cylinder. In the case that $a$ is not zero, then $\xi_2 = \pm\sqrt{as+b}$ and $\theta(s) = \mp 2\sqrt{as+b}/a$. This completes the proof. $\square$

Figure 8 illustrates that the treadmill sled of the profile curve of a flat helicoidal surface is a vertical half-line.

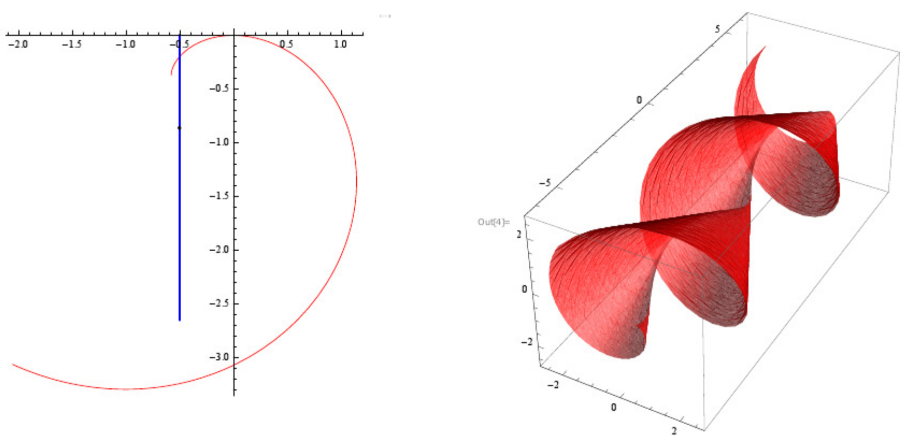


**Figure 8.** A surface with helicoidal symmetry is flat when its treadmill sled is a vertical half-line.

## 5. Treadmill sled coordinates on Delaunay surfaces

Extending the parallel between twizzlers and Delaunay surfaces, we now describe all Delaunay surfaces with CMC 1 using treadmill sled coordinates, and we provide an expression for the quotient of the maximum and minimum values of the Gauss curvature. We use this ratio to find out which unduloid-nodoid pairs are isometric.

**Theorem 5.1.** *For every nonzero real number $M \in (-\frac{1}{4}, \infty)$, the Delaunay surface $\mathbb{D}(M)$ generated by the conic $\{(x, y) : 4x^2 - y^2/M = 1\}$ has constant mean curvature* 1. *The quotient between the maximum value of the Gauss curvature and the minimum value of the Gauss curvature of $\mathbb{D}(M)$ is given by*

$$rs(M) = -\left(\frac{1-\sqrt{1+4M}}{1+\sqrt{1+4M}}\right)^2.$$

*Proof.* Let us assume that $\mathbb{D}(M)$ is parametrized as

$$\phi(s, t) = \big(x(s), z(s)\sin t, z(s)\cos t\big),$$

where the profile curve $(x(s), z(s))$ is parametrized by arc length. A direct verification shows that if $\theta(s)$ is a continuous function such that $x'(s) = \cos\theta(s)$ and $z'(s) = \sin\theta(s)$, then the mean curvature of $\mathbb{D}(M)$ is

$$\tfrac{1}{2}\left(\theta' - \frac{\cos\theta(s)}{z(s)}\right).$$

Since the mean curvature of $\mathbb{D}(M)$ is 1, the functions $\theta(s)$ and $z(s)$ satisfy

$$\theta' = 2 + \frac{\cos\theta}{z} \quad \text{and} \quad z' = \sin\theta.$$

This ODE has as a first integral the function $h(z, \theta) = z(\cos\theta + z)$. Recall that the function $z(s)$ is always positive. Since the minimum of the function $h$ is $-\frac{1}{4}$, it follows that there exists a nonzero constant $k > -\frac{1}{4}$ such that $h(z(s), \theta(s)) = k$. When $k < 0$, the level sets of $h(z, \theta)$ are bounded, and therefore $\mathbb{D}(M)$ represents an unduloid. When $k > 0$, the level sets are not bounded, and $\mathbb{D}(M)$ represents a nodoid. In any case, the $z$-values of the level sets of $h(z, \theta)$ are bounded. A direct computation shows that the maximum and minimum of the $z$-values of the level set $h(z, \theta) = k$ are

$$\frac{1+\sqrt{1+4k}}{2} \quad \text{and} \quad \left|\frac{1-\sqrt{1+4k}}{2}\right|.$$

We can prove that $k$ must be equal to $M$ by comparing these critical values of $z(s)$ with the maximum and the minimum values of the profile curve viewed as the trace of the focus of a conic when it is rolled on a line. A direct computation shows that the Gauss curvature is $-(\theta' \cos\theta)/z$, and since $\theta' = 2 + (\cos\theta)/z$, the Gauss curvature reduces to

$$-\frac{\cos\theta\,(2z+\cos\theta)}{z^2}.$$

Using the Lagrange multiplier method, we see that the maximum and the minimum of the Gauss curvature subject to the constraint $h(z,\theta) = k$ are

$$\frac{4\sqrt{1+4k}}{(1+\sqrt{1+4k})^2} \quad \text{and} \quad -\frac{4\sqrt{1+4k}}{(-1+\sqrt{1+4k})^2},$$

respectively. It follows that the quotient between the maximum of the Gauss curvature and the minimum of the Gauss curvature is

$$-\left(\frac{-1+\sqrt{1+4k}}{1+\sqrt{1+4k}}\right)^2.$$

Since $k = M$, the theorem follows. □

The function $rs$ defines a bijection between the intervals $(-\frac{1}{4}, 0)$ and $(0, 1)$, and it also defines a bijection between the intervals $(0, \infty)$ and $(0, 1)$. On the other hand, each unduloid is isometric to a nodoid [do Carmo and Dajczer 1982]. As a consequence of Theorem 5.1, we have:

**Corollary 5.2.** *Two Delaunay surfaces with CMC* 1 *are isometric if and only if the quotients of the maximum and minimum values of the Gauss curvatures are the same. In particular, for any* $u \in (0, 1)$, *the unduloid* $\mathbb{D}\big(-\sqrt{u}/(1+\sqrt{u})^2\big)$ *is isometric to the nodoid* $\mathbb{D}\big(\sqrt{u}/(1-\sqrt{u})^2\big)$.

## 6. Moduli space for twizzlers

If we exclude the cylinder and the value $M = -\frac{1}{4}$, Theorem 3.2 establishes a 1:1 correspondence between pairs $(M, w)$ with $M > -\frac{1}{4}$ and $w > 0$ and twizzlers with mean curvature 1. Therefore, so far we have that the moduli space of all twizzlers with CMC 1 other than the cylinder is the set $\{(M, w) : M > -\frac{1}{4},\ w > 0\}$. In order to visualize better the boundary of the moduli space of twizzlers, we replace the parameter $w$ with the bounded parameter $v = 1/(1+w^2)$. Therefore, the parameter $v$ moves from 0 to 1 when $w$ moves from $\infty$ to 0. Figure 9 shows pictures from an animation that produces a piece of the twizzler associated with values of $M$ and $v$. We refer to this twizzler as $\mathfrak{T}(M, v)$ when the dependence of $M$ and $v$ is needed.

In [Perdomo 2011], a formula for the inverse of the treadmill sled of a curve is provided. Therefore we can get a parametrization for all twizzlers if we have a parametrization for all heart-shaped curves.

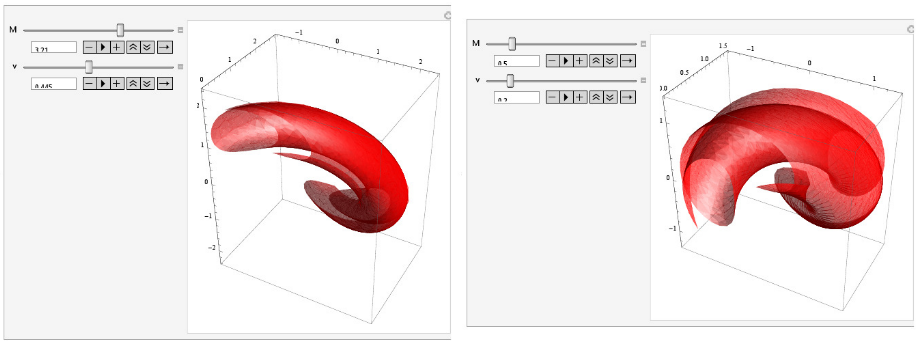


**Figure 9.** Moduli space of twizzler with mean curvature one.

**Lemma 6.1.** *For any $M > -\frac{1}{4}$ and $w > 0$, the curve $\alpha(t) = (\rho_1(t),\ \rho_2(t))$ defined on the interval $[0, 2\pi]$ and given by*

$$\rho_1(u) = A\cos u \quad \textit{and} \quad \rho_2(u) = \frac{-1+\sqrt{1+4M+B\cos^2 u}\,\sin u}{2\sqrt{1+w^2A^2\cos^2 u}},$$

*where*

$$A = \frac{\sqrt{-1+Mw^2+\sqrt{1+(1+2M)w^2+M^2w^4}}}{\sqrt{2}\,w},$$

$$B = \frac{2+2M^2w^4+w^2+2(Mw^2-1)\sqrt{1+(1+2M)w^2+M^2w^4}}{w^2},$$

*is a closed simple regular curve that parametrizes the heart-shaped curve $h_w^{-1}(M)$.*

*Proof.* It is a direct verification. □

Since the maximum and minimum distances from a curve $\alpha$ to the origin agree with the maximum and minimum distances from its treadmill sled to the origin [Perdomo 2011], we have the following proposition.

**Proposition 6.2.** *The maximum distance from a special twizzler with CMC 1 to its axis of symmetry is 1. More generally, the maximum and minimum distances from the twizzler $\mathfrak{T}(M, v)$ to its axis of symmetry are given by*

$$r_1(M) = \left|\frac{\sqrt{1+4M}-1}{2}\right| \quad \textit{and} \quad r_2(M) = \frac{\sqrt{1+4M}+1}{2}.$$

*Proof.* Since the maximum and minimum distances from a twizzler to its axis of symmetry are the same as the maximum and minimum distances from its profile curve to the origin, we only need to show that for any $M > -\frac{1}{4}$, the minimum and maximum distances from the origin to the heart-shaped curve $h_w^{-1}(M)$ are $r_1(M) = \left|(\sqrt{1+4M}-1)/2\right|$ and $r_2(M) = (\sqrt{1+4M}+1)/2$, respectively. We

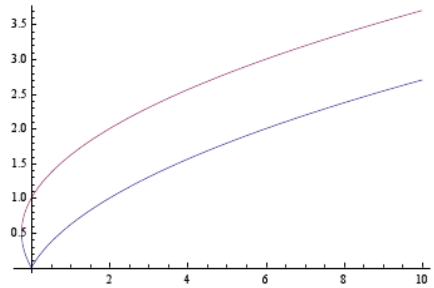


**Figure 10.** The maximum and minimum distances from the origin to the profile curve of $\mathfrak{T}(M, v)$ change with respect to $M$.

prove this by using the method of Lagrange multipliers to find the maximum and minimum values of the function $R(x_1, x_2) = x_1^2 + x_2^2$, subject to the restriction $h_w = M$. A direct verification shows that if $(x_1, x_2)$ and $\lambda_1$ satisfy the Lagrange multiplier equations

$$\frac{\partial R}{\partial x_1} = \lambda_1 \frac{\partial h_w}{\partial x_1} \quad \text{and} \quad \frac{\partial R}{\partial x_2} = \lambda_1 \frac{\partial h_w}{\partial x_2},$$

then $x_1 = 0$. Once we know that $x_1$ must be zero, we obtain from the equation $h_w = M$ that $x_2$ is either $-(\sqrt{1+4M}+1)/2$ or $(\sqrt{1+4M}-1)/2$. Now the result easily follows. □

**Remark 6.3.** From this proposition we can understand the twizzlers in the moduli space that are near the boundary line $M = -\frac{1}{4}$. Since the limit when $M$ goes to $-\frac{1}{4}$ of the functions $r_1(M)$ and $r_2(M)$ is $\frac{1}{2}$ (see Figure 10), then we have that when $M$ is near $-\frac{1}{4}$, the twizzlers $\mathfrak{T}(M, v)$ are near the cylinder of radius $\frac{1}{2}$.

## 7. Fundamental piece of the profile curve of a twizzler and the immersed versus dense property

The fact that the treadmill sled of the profile curve of a twizzler is a closed curve allows us to define a *fundamental piece of the profile curve* as a connected piece of profile curve with the property that the treadmill sled motion of this piece goes exactly once over the heart-shaped curve. It is not difficult to see that the whole profile curve is the union of fundamental pieces. Figure 11 shows the fundamental piece of the profile curve of a properly immersed twizzler, along with the whole profile curve made up of four pieces in this case and the graph of the twizzler.

For the sake of comparison, for an unduloid we could define a fundamental piece as the trace of the focus of the ellipse when this ellipse rolls once. It is clear that the whole profile curve is the union of fundamental pieces, and therefore $\mathbb{Z}$ acts on the group of isometries of the unduloid in the form of translations. Theorem 7.2 shows that the group $\mathbb{Z}$ also acts on the set of isometries of twizzlers.

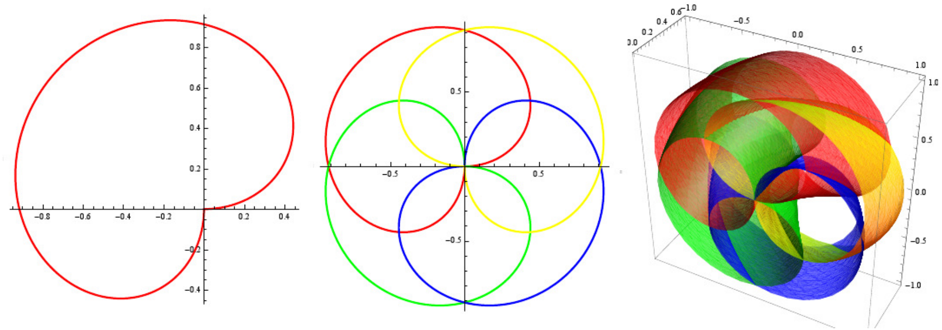

**Figure 11.** Fundamental piece of the profile curve.

Using the parametrization for the heart-shaped curve in Lemma 6.1, we get the following formula for the length of a fundamental piece of a twizzler. (This formula was used in the production of Figure 11.)

**Lemma 7.1.** *The length of the fundamental piece of the twizzler* $\mathfrak{T}(M, v)$ *is*

$$\int_0^{2\pi} \sqrt{\lambda/\mu}\, du,$$

*where* $\lambda(u) = \left(\frac{d\rho_1}{du}\right)^2 + \left(\frac{d\rho_2}{du}\right)^2$ *and* $\mu(u) = f_1^2(\rho_1(u), \rho_2(u)) + f_2^2(\rho_1(u), \rho_2(u))$.

The functions $\rho_1$, $\rho_2$, $f_1$, and $f_2$ are defined in Lemma 6.1 and Proposition 3.1. Recall that $w$ and $v$ are related by the equation $v = 1/(1 + w^2)$.

*Proof.* The proof is straightforward, and is actually included in the proof of the next result, Theorem 7.2. □

In [Perdomo 2011] we showed that two curves with the same treadmill sled differ only by a rotation about the origin. With this in mind, we have that two consecutive fundamental pieces of the same twizzler differ by a rotation about the origin, and therefore the whole profile curve is either a closed curve made out of a finite union of fundamental pieces or the union of infinitely many disjoint fundamental pieces. When the latter happens, it is not difficult to see that the profile curve is either dense in a circle or dense in an annulus depending on whether or not the profile curve passes through the origin. In order to better understand this property, given a twizzler, without loss of generality, let us consider a fundamental piece starting at a point $p_1$ other than the origin and ending in a point $p_2$. We have that $|p_1| = |p_2|$, so in polar coordinates $p_1 = re^{\theta_1}$ and $p_2 = re^{\theta_2}$. We prove that if $\theta_2 - \theta_1$ is a rational multiple of $\pi$, then the profile curve is a closed curve and the twizzler is properly immersed, for otherwise the twizzler is dense in either the region bounded by two concentric cylinders or dense in the region bounded by a cylinder.

The next theorem, along with Theorem 8.2 and Theorem 8.4, gives a precise picture of the moduli space for twizzlers.

**Theorem 7.2.** *The angle between the final and initial points of a fundamental piece of the twizzler $\mathfrak{T}(M, v)$ is given by $\theta_0 = \int_0^{2\pi} \psi\, du$, where*

$$\psi(u) = \frac{-w^2\rho_2(u) + 2(1 + w^2\rho_1^2(u))^{3/2}}{1 + w^2(\rho_1^2(u) + \rho_2^2(u))} \sqrt{\frac{\lambda(u)}{\mu(u)}}.$$

*$\mathfrak{T}(M, v)$ is invariant under a group of rotations of the form $\{R(n\theta_0) : n \in \mathbb{Z}\}$. If $R(m\theta_0) = R(\theta_0)$ for some integer $m$, then the twizzler is properly immersed; otherwise it is dense in the interior of a cylinder of radius* 1 *when $M = 0$, or dense in the region bounded by two concentric cylinders of radii*

$$r_1(M) = \left|\frac{\sqrt{1+4M} - 1}{2}\right| \quad \textit{and} \quad r_2(M) = \frac{\sqrt{1+4M} + 1}{2}$$

*when $M \neq 0$. More precisely, we have that $\mathfrak{T}(M, v)$ is a properly immersed surface with a profile curve consisting of $b$ fundamental pieces if and only if $\theta_0 = 2\pi(a/b)$, with $a$ and $b$ positive relatively prime integers. We also have another type of density: the set of points $(M, v)$ associated with properly immersed twizzlers is uncountable and dense.*

*Proof.* That the twizzler is bounded by a cylinder follows from Proposition 6.2. Let us assume that $(x(s), y(s))$ are such that the surface (1-1) has constant mean curvature 1. Since the curve $(\rho_1, \rho_2)$ defined in Lemma 6.1 is regular, we see that

$$\lambda(u) = \left(\frac{d\rho_1}{du}\right)^2 + \left(\frac{d\rho_2}{du}\right)^2$$

is a periodic positive function. Likewise, since $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ only vanish simultaneously at $(x_1, x_2) = (0, -\frac{1}{2})$, we see that

$$\mu(u) = f_1^2(\rho_1(u), \rho_2(u)) + f_2^2(\rho_1(u), \rho_2(u))$$

is a positive periodic function. Notice that $\xi_1(s) = 0$ and $\xi_2(s) = -\frac{1}{2}$ is the only constant solution of the system (3-4). For any other solution, since $h_w$ is a first integral of the system, there exist $M > -\frac{1}{4}$ and a function $\sigma(s)$ such that

$$\xi_1(s) = \rho_1(\sigma(s)) \quad \text{and} \quad \xi_2(s) = \rho_2(\sigma(s))$$

is a solution of the system (3-4). From the equations above, we have

$$\xi_1'(s)^2 + \xi_2'(s)^2 = \lambda(\sigma(s))\, \sigma'(s)^2. \tag{7-1}$$

On the other hand,

$$\xi_1'(s) = f_1(\xi_1(s), \xi_2(s)) = f_1(\rho_1(\sigma(s)), \rho_2(s)),$$
$$\xi_2'(s) = f_2(\xi_1(s), \xi_2(s)) = f_2(\rho_1(\sigma(s)), \rho_2(s)).$$

It follows that $\sigma$ is either strictly increasing or strictly decreasing; without loss of generality, we can assume that $\sigma$ is strictly increasing. Therefore we get

$$\sigma'(s) = \sqrt{\frac{\mu(\sigma(s))}{\lambda(\sigma(s))}}.$$

If $\kappa(u)$ is the inverse of the function $\sigma(s)$, we have that

$$\kappa'(u) = \frac{1}{\sigma'(\kappa(u))} = \sqrt{\frac{\lambda(u)}{\mu(u)}}. \tag{7-2}$$

If we change from the variable $s$ to the variable $u$, that is, if we consider the functions

$$\tilde{\theta}(u) = \theta(\kappa(u)), \qquad \tilde{\xi}_1(u) = \xi_1(\kappa(u)), \qquad \tilde{\xi}_2(u) = \xi_2(\kappa(u)),$$
$$\tilde{x}(u) = x(\kappa(u)), \qquad \tilde{z}(u) = z(\kappa(u)),$$

it follows from (7-2) and (3-3) that $\tilde{\theta}'(u) = \psi(u)$, where

$$\psi(u) = \frac{-w^2\rho_2(u) + 2(1 + w^2\rho_1^2(u))^{3/2}}{1 + w^2(\rho_1^2(u) + \rho_2^2(u))}\sqrt{\frac{\lambda(u)}{\mu(u)}}.$$

Since the right side of this equation is a periodic function with period $2\pi$, it follows by the existence and uniqueness theorem of ODEs that if $\tilde{\theta}(2\pi) = \theta_0$, then for any integer $j$,

$$\tilde{\theta}(u + 2j\pi) = j\theta_0 + \tilde{\theta}(u). \tag{7-3}$$

Since $\left|(x(s), z(s))\right| = \left|(\xi_1(s), \xi_2(s))\right|$, the piece of profile curve

$$C_{\text{fp}} = C_{\text{fundamental piece}} = \{(\tilde{x}(u), \tilde{z}(u)) : u \in [0, 2\pi]\}$$

also satisfies $r_1(M) = \min\{|q| : q \in C_{\text{fp}}\}$ and $r_2(M) = \min\{|q| : q \in C_{\text{fp}}\}$. Using (3-2) and (7-3), we get

$$\begin{pmatrix} \tilde{x}(u + 2j\pi) \\ \tilde{z}(u + 2j\pi) \end{pmatrix} = R_{\theta_0}^j \begin{pmatrix} \tilde{x}(u) \\ \tilde{z}(u) \end{pmatrix}, \quad \text{where } R_{\theta_0} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}. \tag{7-4}$$

This equality implies that the image of the profile curve can be viewed as the orbit of the group $\{R_{\theta_0}^j\}_{j\in\mathbb{Z}}$ acting on $C_{\text{fp}}$, that is,

$$C = \{(x(t), z(t)) : t \in \mathbf{R}\} = \{R_{\theta_0}^j p : j \in \mathbb{Z} \text{ and } p \in C_{\text{fundamental piece}}\}. \tag{7-5}$$

It follows from this equation that if $\theta_0/2\pi$ is a rational number, then $C$ is a properly immersed curve, and if $\theta_0/2\pi$ is irrational, then $C$ is dense in the annulus

$$\left\{(x_1, x_2) : r_1(M) \le \sqrt{x_1^2 + x_2^2} \le r_2(M)\right\}$$

when $M \neq 0$, or dense in the circle of radius 1 when $M = 0$. Therefore, twizzlers with constant mean curvature 1 have the following property: they are properly immersed, or they are dense in the region contained between two concentric cylinders, or they are dense in the interior of a cylinder of radius 1. We can prove that a surface corresponding to an irrational value $\theta_0/2\pi$ is dense by showing that the profile curve is dense, and we can prove that the profile curve is dense by showing that the intersection of this curve with a circle centered at the origin is either the empty set or dense in the circle. The problem of proving this last statement reduces to that of showing that for any irrational number $\iota$, the set $\{\iota - [n\iota] : n \in \mathbb{Z}\}$ is dense in the interval [0, 1], which is a known fact. To finish, we notice that since the function $(x(s), z(s))$ is parametrized by arc length, the length of the fundamental piece is

$$\kappa(2\pi) = \int_0^{2\pi} \sqrt{\lambda(u)/\mu(u)}\, du.$$

Also, since $\tilde{\theta}'(u) = \psi(u)$, we have that

$$\theta_0 = \int_0^{2\pi} \psi(u) du. \qquad \square$$

For twizzlers that do not contain the axis of symmetry, the "properly immersed versus dense" property established in Theorem 7.2 was proved in [Hitt and Roussos 1991]. By numerically solving the equation $\int_0^{2\pi} \psi\, du = 2\pi(a/b)$ in that theorem, we can graph profile curves of twizzlers with any desired property.

In Figure 12, we solve the numerical equation $\int_0^{2\pi} \psi\, du = 2\pi(a/b)$, fixing $M = 0$ and taking several integer values for $a$ and $b$. Since $M = 0$, these profile curves

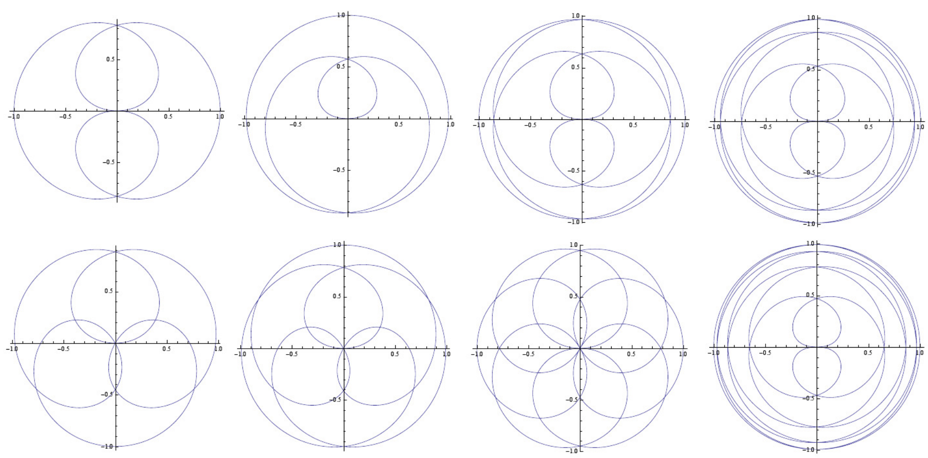

**Figure 12.** Profile curves of properly immersed twizzlers that contain the axis.

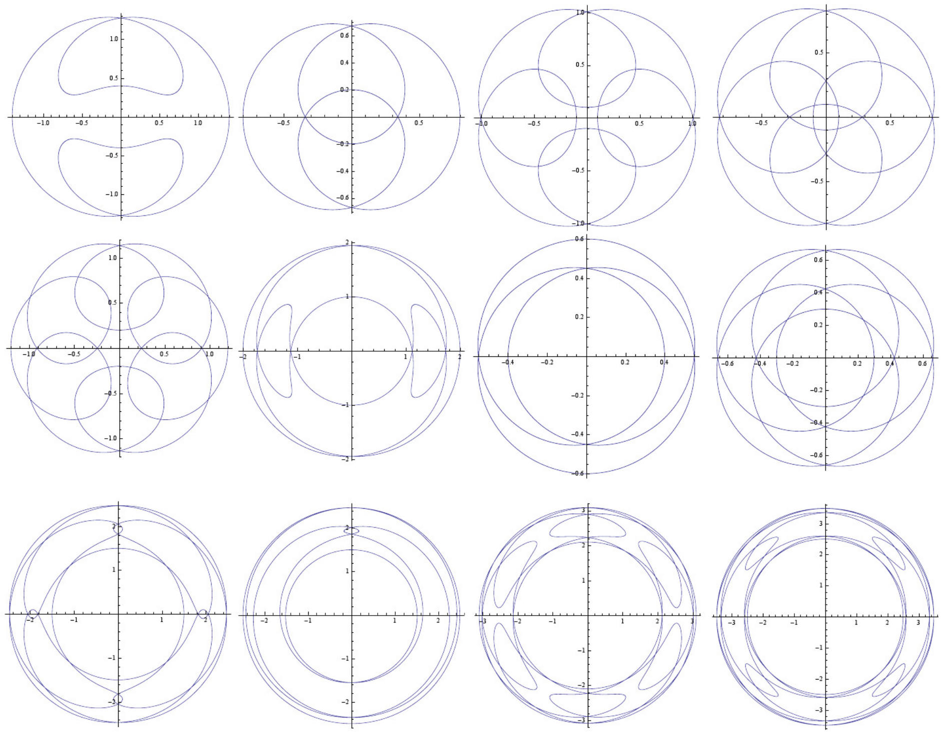

**Figure 13.** Profile curves of properly immersed twizzlers that do not contain their axis.

represent twizzlers that contain the axis of symmetry. In Figure 13, we take several values for $M \neq 0$ and $a$ and $b$ integers to produce properly immersed twizzlers that do not contain the axis of symmetry. In Figure 14, we take $a$ and $b$ such that $a/b$ is not rational, so that the twizzler is not properly immersed. In Figure 15, we take $a = 5$, $b = 4$ and 4 values of $M$ in order to produce properly immersed examples; we also show the points $(M, w)$ associated with these twizzlers.

## 8. Isometric associate family of surfaces

As pointed out before, each nodoid is isometric to an unduloid, and therefore we can replace the word Delaunay by either the word unduloid or nodoid in the result proved in [do Carmo and Dajczer 1982]; that is, we can say that each twizzler is isometric to either a nodoid or an unduloid. Another family of surfaces that holds the same property is the set of twizzlers that contain the axis of symmetry, that is, the set of twizzlers corresponding to $M = 0$ in the moduli space. We call these surfaces *special twizzlers* and we denote them by $\mathbb{S}\mathfrak{T}(v)$; that is, $\mathbb{S}\mathfrak{T}(v) = \mathfrak{T}(0, v)$. Due to a singularity problem on the coordinates used so far to study helicoidal

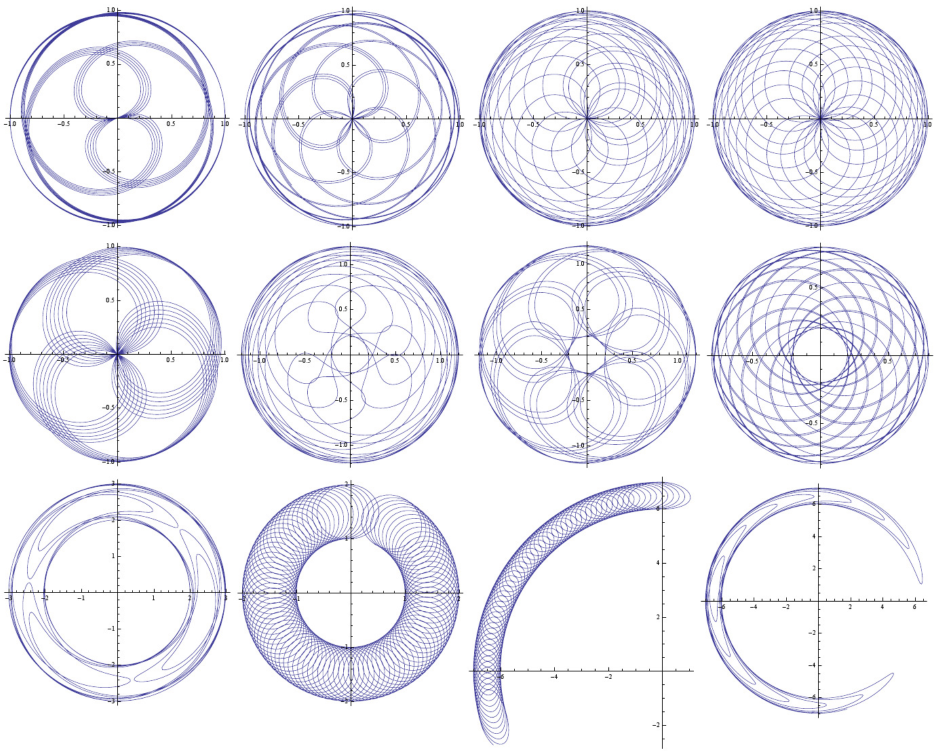

**Figure 14.** Profile curves of nonproperly immersed twizzlers.

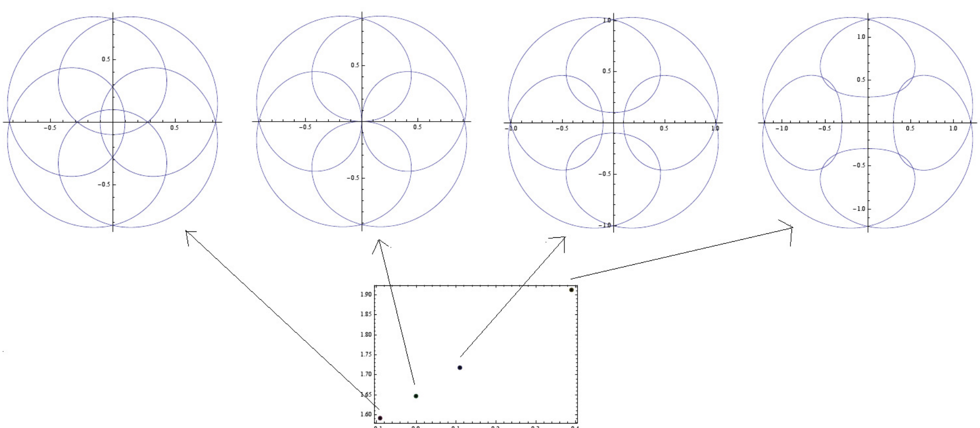

**Figure 15.** Profile curves of twizzlers consisting of four fundamental pieces and their corresponding values $M$ and $w$.

surfaces, twizzlers that contain the axis of symmetry have been overlooked until now. The following theorem gives us the quotient of the maximum value and the minimum value of the Gauss curvature for special twizzlers. Figure 16 shows two sets of isometric nodoid-unduloid-special twizzler surfaces.

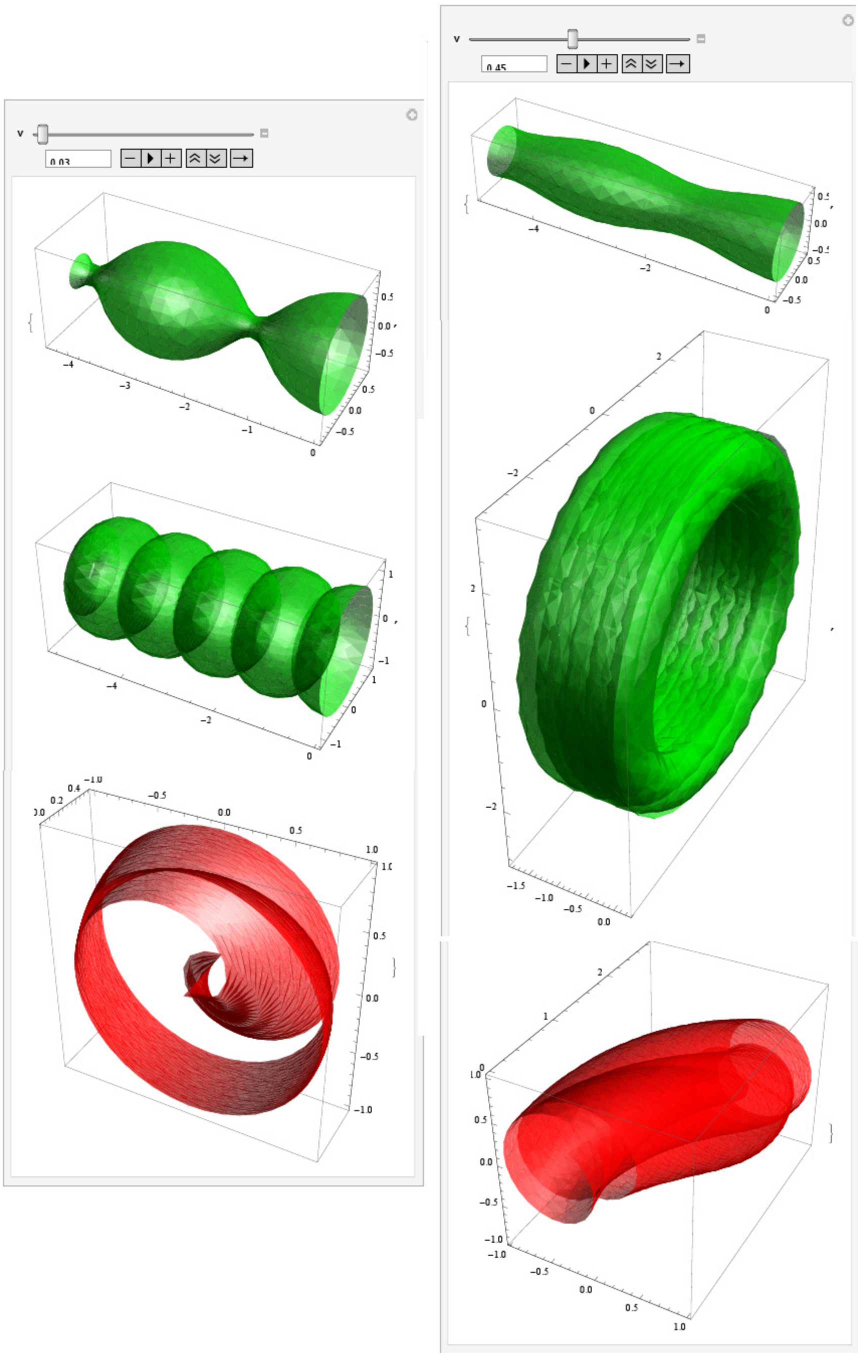

**Figure 16.** Isometric nodoid, unduloid and special twizzler.

**Theorem 8.1.** *For every nonzero real number $v \in (0, 1)$, the quotient between the maximum value of the Gauss curvature and the minimum value of the Gauss curvature of the special twizzler surface $\mathbb{S}\mathfrak{T}(v)$ is $-v$. Moreover, $\mathbb{S}\mathfrak{T}(v)$ is isometric to the unduloid $\mathbb{D}\big(-\sqrt{v}/(1+\sqrt{v})^2\big)$ and the nodoid $\mathbb{D}\big(\sqrt{v}/(1-\sqrt{v})^2\big)$.*

*Proof.* The proof is contained in the proof of Theorem 8.2. □

We can generalize Theorem 8.1 as follows:

**Theorem 8.2.** *If $v = 1/(1 + w^2)$, then the quotient between the maximum and minimum values of the Gauss curvature of the twizzler surface $\mathfrak{T}(M, v)$ is*

$$-\frac{2+(1+2M-\sqrt{1+4M})\,w^2}{2+(1+2M+\sqrt{1+4M})\,w^2}.$$

*Moreover, fixing $c \in (0, 1)$, all the twizzlers in the set*

$$\left\{\mathfrak{T}\left(M, \frac{\sqrt{1+4M}-1-2M+c(\sqrt{1+4M}+1+2M)}{\sqrt{1+4M}+1-2M+c(\sqrt{1+4M}-1+2M)}\right) : M \in \left(-\frac{\sqrt{c}}{(1+\sqrt{c})^2}, \frac{\sqrt{c}}{(1-\sqrt{c})^2}\right)\right\}$$

*are isometric.*

*Proof.* Using the same notation as in the proof of Proposition 3.1, we see that the Gauss curvature $K$ satisfies

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{w^2(1+\theta'\xi_2)}{(1+w^2\xi_1^2)^2} = -\frac{w^2\big(1+2\,\xi_2\sqrt{1+w^2\xi_1^2}\big)}{(1+w^2\xi_1^2)(\xi_1^2+\xi_2^2)}.$$

Taking $\rho_1(u)$ and $\rho_2(u)$ as in Lemma 6.1, we get the following expression for the Gauss curvature in terms of the parameter $u$:

$$\frac{-4w^2\sqrt{1+4M+B\cos^2 u}\,\sin u}{\begin{aligned}\big(4+w^2+4A^4w^4\cos^4 u-2w^2\sqrt{1+4M+B\cos^2 u}\,\sin u\\ +(1+4M)w^2\sin^2 u+w^2(8A^2+B\sin^2 u)\cos^2 u\big)\end{aligned}}$$

A direct computation shows that the derivative of the function $K = K(u)$ is of the form $\cos u\ \mathrm{po}(u)$, where $\mathrm{po}(u)$ is a positive function, and therefore the maximum of the Gauss curvature occurs when $u = 3\pi/2$ and is equal to

$$\frac{2w^2\sqrt{1+4M}}{2+(1+2M+\sqrt{1+4M})w^2},$$

and the minimum of the Gauss curvature occurs when $u = \pi/2$ and is equal to

$$-\frac{2w^2\sqrt{1+4M}}{2+(1+2M-\sqrt{1+4M})w^2}.$$

We conclude that the quotient of the maximum value of the Gauss curvature and the minimum value of the Gauss curvature is

$$-\frac{2+(1+2M-\sqrt{1+4M})\,w^2}{2+(1+2M+\sqrt{1+4M})\,w^2}.$$

This expression in terms of $v$ transforms into

$$-\frac{1+2M-\sqrt{1+4M}+v-2Mv+v\sqrt{1+4M}}{1+2M+\sqrt{1+4M}+v-2Mv-v\sqrt{1+4M}}.$$

A direct verification shows that this expression reduces to $-c$ when we replace $v$ by

$$\frac{\sqrt{1+4M}-1-2M+c(\sqrt{1+4M}+1+2M)}{\sqrt{1+4M}+1-2M+c(\sqrt{1+4M}-1+2M)},$$

and therefore, for any $c \in (0, 1)$, all the twizzlers

$$\left\{\mathfrak{T}\left(M,\ \frac{\sqrt{1+4M}-1-2M+c(\sqrt{1+4M}+1+2M)}{\sqrt{1+4M}+1-2M+c(\sqrt{1+4M}-1+2M)}\right) : M \in \left(-\frac{\sqrt{c}}{(1+\sqrt{c})^2},\ \frac{\sqrt{c}}{(1-\sqrt{c})^2}\right)\right\}$$

must be isometric. This follows because every twizzler with CMC 1 must be in the isometric associated family of a Delaunay surface [Lawson 1970], and it can be shown that the family of curves

$$\Omega_c = \left\{\left(M,\ \frac{\sqrt{1+4M}-1-2M+c(\sqrt{1+4M}+1+2M)}{\sqrt{1+4M}+1-2M+c(\sqrt{1+4M}-1+2M)}\right) : M \in \left(-\frac{\sqrt{c}}{(1+\sqrt{c})^2},\ \frac{\sqrt{c}}{(1-\sqrt{c})^2}\right)\right\}$$

for $c \in (0, 1)$ defines a partition of the set $(-\frac{1}{4}, \infty) \times (0, 1)$. Figure 17 shows these curves $\Omega_c$ for different values of $c$. We know that two twizzlers corresponding to two points in different curves $\Omega_c$ cannot be isometric because their ratios of maximum to minimum Gauss curvatures are different. Using the continuity of the curve $\Omega_c$ and the fact that the $2\pi$-periodic isometric family is continuous (see Remark 1.1), we see that all the isometric surfaces of the $2\pi$-periodic associated family must be contained in a single $\Omega_c$ curve, and therefore all twizzlers there are

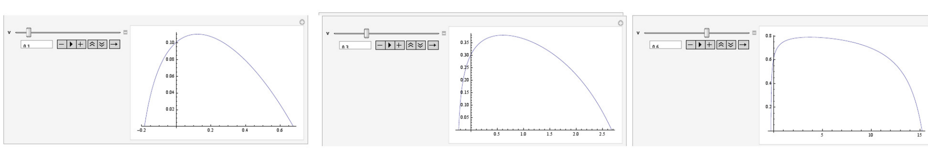

**Figure 17.** Points in the moduli space that represent isometric twizzlers.

isometric. As pointed out in the proof of Theorem 5.1, there are only two isometric Delaunay surfaces whose quotient between maximum and minimum values of the Gauss curvature is $-c$; they are the unduloid $\mathbb{D}(-\sqrt{c}/(1+\sqrt{c})^2)$ and the nodoid $\mathbb{D}(\sqrt{c}/(1-\sqrt{c})^2)$, and they correspond to the limit surfaces of the twizzlers that are in $\Omega_c$. □

**Remark 8.3.** Helicoidal surfaces in the deformation helicoid-catenoid shown in Figure 3 are only a quarter of the whole $2\pi$-periodic isometric family. All other elements in the $2\pi$-periodic family are, up to a rigid motion, contained in the deformation shown in Figure 3. The same situation happens with twizzlers; the family of twizzlers given by points in $\Omega_c$ are only a quarter of the whole $2\pi$-periodic isometric family. All other elements in the $2\pi$-periodic family are, up to a rigid motion, contained in the twizzlers given in $\Omega_c$.

Since the curve

$$\alpha_c(M) = \left(M, \frac{\sqrt{1+4M}-1-2M+c(\sqrt{1+4M}+1+2M)}{\sqrt{1+4M}+1-2M+c(\sqrt{1+4M}-1+2M)}\right)$$

satisfies that $\alpha_c\big(-\sqrt{c}/(1+\sqrt{c})^2\big) = \big(-\sqrt{c}/(1+\sqrt{c})^2, 0\big)$, $\alpha_c(0) = (0, c)$, and $\alpha_c\big(\sqrt{c}/(1-\sqrt{c})^2\big) = \big(\sqrt{c}/(1-\sqrt{c})^2, 0\big)$, as a corollary of Theorems 8.1 and 8.2 and Remark 1.1, we have:

**Theorem 8.4.** *Let $\Omega = \{M+\mathrm{i}v \in \mathbb{C} : M \geq -\frac{1}{4},\ M \neq 0$ and $0 \leq v < 1\}$. The function $\rho$ from $\Omega$ to the set of immersions in $\mathbb{R}^3$ given by*

$$\rho(M+\mathrm{i}v) = \mathfrak{T}(M, v) \text{ for any } v > 0 \text{ and } M \neq -\tfrac{1}{4},$$

$$\rho(M) = \mathbb{D}(M) \text{ for any } M \neq 0 \text{ and } M \neq -\tfrac{1}{4},$$

$$\rho(-\tfrac{1}{4}+\mathrm{i}v) = \{(x, y, z) \in \mathbb{R}^3 : x^2+z^2 = \tfrac{1}{4}\}$$

*is continuous in the sense that for every point $p$ in $\Omega$, there exist a neighborhood $U$ of $p$ in $\Omega$ and a continuous function $f : U \times \mathbb{R}^2 \to \mathbb{R}^3$ such that for any $M+\mathrm{i}v \in U$, the map $(s, t) \to f(M+\mathrm{i}v, s, t)$ defines a parametrization of the surface $\rho(M+\mathrm{i}v)$. Moreover, the function $\rho$ is one-to-one in the interior of $\Omega$.*

The continuity at the points of form $-\frac{1}{4}+\mathrm{i}v$ follows from Theorem 7.2, because each twizzler $\mathfrak{T}(M, v)$ is contained in the region bounded by the two concentric cylinders of radii $r_1(M) = |(\sqrt{1+4M}-1)/2|$ and $r_2(M) = (\sqrt{1+4M}+1)/2$. Figure 17 shows the trace of the curve $\alpha_c$ for several values of $c$.

***Summary.*** We collect some important facts on helicoidal surfaces with constant mean curvature one. Figure 18 shows a picture of the moduli space.

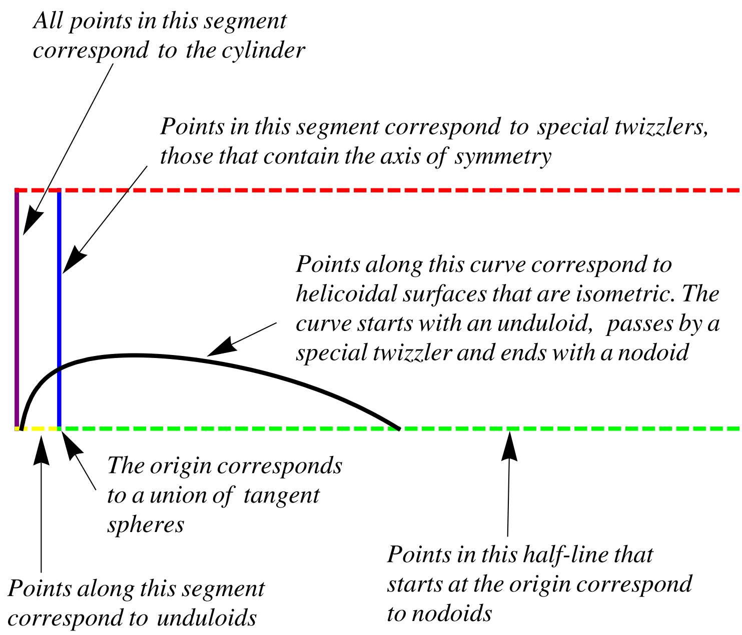


**Figure 18.** Moduli space of twizzler with CMC 1 and its boundary.

*Dynamical interpretation of Delaunay surfaces.* The trace of the focus of each conic $4x^2 - y^2/M = 1$ with $M \in (-\frac{1}{4}, 0) \cup (0, \infty)$, when it is rolled on a line, produces the profile curve of a Delaunay surface with constant mean curvature one. Moreover, every Delaunay surface with CMC 1 but the cylinder corresponds with one of these conics.

*Dynamical interpretation of twizzlers.* The treadmill sled of the profile curve of a twizzler with CMC 1 other than a cylinder is the closed curve

$$x^2 + y^2 - \frac{y}{\sqrt{1 + w^2x^2}} = M$$

for some $M > -\frac{1}{4}$ and $w > 0$.

*Moduli space of twizzlers.* Denote by $\rho(M, v)$ the twizzler whose treadmill sled of its profile curve lies on the heart-shaped curve

$$x^2 + y^2 - \frac{y}{\sqrt{1 + w^2x^2}} = M,$$

where $v = 1/(1 + w^2)$. Then $\rho$ generates a one-to-one correspondence between the half-strip

$$\Omega = \{(M, v) : M > -\tfrac{1}{4} \text{ and } 0 < v < 1\}$$

and all twizzlers with CMC 1 but the cylinder.

*Boundary of the moduli space of twizzlers.* When $M$ goes to $-\frac{1}{4}$, the surfaces $\rho(M, v)$ converge to a cylinder. When $v$ goes to zero, the surfaces $\rho(M, v)$ converge to the Delaunay surface whose profile curve is traced by the focus of the conic $4x^2 - y^2/M = 1$. When $M$ goes to zero, the Delaunay surface whose profile curve is traced by the focus of the conic $4x^2 - y^2/M = 1$ converges to a union of infinitely many tangent spheres [Kapouleas 1990, Appendix A].

*Fundamental piece of the profile curve.* For every twizzler other than a cylinder, we can define the fundamental piece of the profile curve as a connected part of the profile curve whose treadmill sled goes exactly once over the closed curve $x^2+y^2-y/\sqrt{1+w^2x^2}=M$. The function $\theta_0(M, v)$ given in Theorem 7.2 provides a formula for the angle between the initial and final positions of the fundamental piece of the profile curve. The function $\theta_0$ defined on $\Omega$ is given in terms of an integral of an expression involving only sine and cosine functions.

*Properties of twizzlers.* If $M$ is nonzero, then the twizzler $\rho(M, v)$ lies in the region $C_{r_1r_2}$ bounded by two concentric cylinders of radii $r_1(M)=|(\sqrt{1+4M}-1)/2|$ and $r_2(M) = (\sqrt{1+4M}+1)/2$; also, $\rho(M, v)$ is properly immersed if and only if $\theta_0(M, v)/2\pi$ is a rational number, and otherwise it is dense in $C_{r_1r_2}$. If $M = 0$, then, the twizzler $\rho(M, v)$ contains the axis of symmetry and lies inside a cylinder of radius 1; moreover, $\rho(M, v)$ is properly immersed if and only if $\theta_0(M, v)/2\pi$ is a rational number, and otherwise it is dense in the interior of this cylinder.

*Isometric surfaces.* Theorem 8.2 provides an explicit formula for a foliation of the moduli space $\Omega$ by curves with the property that all the twizzlers in each curve are isometric. Each one of these curves starts with an unduloid, passes through a special twizzler (a twizzler that contains the axis of symmetry), and ends with a nodoid. In particular, every twizzler different other than cylinder is isometric to a twizzler that contains the axis of symmetry.

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