Pacific Journal of Mathematics

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Volume 258 No. 2 August 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 258, No. 2, 2012
dx.doi.org/10.2140/pjm.2012.258.487

CLASSIFICATION OF ISING VECTORS IN THE VERTEX OPERATOR ALGEBRA V_L^+

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Let L be an even lattice without roots. In this article, we classify all Ising vectors in the vertex operator algebra V_L^+ associated with L.

Introduction

In vertex operator algebra (VOA) theory, the simple Virasoro VOA $L(\frac{1}{2},0)$ of central charge $\frac{1}{2}$ plays important roles. In fact, for each embedding, an automorphism, called a τ-involution, is defined using the representation theory of $L(\frac{1}{2},0)$ [Miyamoto 1996]. This is useful for the study of the automorphism group of a VOA. For example, this construction gives a one-to-one correspondence between the set of subVOAs of the moonshine VOA isomorphic to $L(\frac{1}{2},0)$ and that of elements in certain conjugacy class of the Monster [Miyamoto 1996; Höhn 2010].

Many properties of τ-involutions are studied using Ising vectors, which are elements of weight 2 generating $L(\frac{1}{2},0)$. For example, the 6-transposition property of τ-involutions was proved in [Sakuma 2007] by classifying the subalgebra generated by two Ising vectors. Hence it is natural to classify Ising vectors in a VOA. For example, this was done in [Lam 1999; Lam et al. 2007] for code VOAs. However, in general, it is hard to even find an Ising vector.

Let L be an even lattice and V_L the lattice VOA associated with L. Then the subspace V_L^+ fixed by a lift of the -1-isometry of L is a subVOA of V_L. There are two constructions of Ising vectors in V_L^+ related to sublattices of L isomorphic to $\sqrt{2}A_1$ [Dong et al. 1994] and $\sqrt{2}E_8$ [Dong et al. 1998; Griess 1998].

The main theorem of this article is this:

Theorem 2.3. *Let L be an even lattice without roots and e an Ising vector in V_L^+. There is a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ and such that $e \in V_U^+$.*

This theorem was conjectured in [Lam et al. 2007], and proved there and in [Lam and Shimakura 2007] in the case that $L/\sqrt{2}$ is even and L is the Leech lattice. We

H. Shimakura was partially supported by Grants-in-Aid for Scientific Research (grant number 23540013), JSPS.
MSC2010: 17B69.
Keywords: vertex operator algebra, lattice vertex operator algebra, Ising vector.

note that if L has roots then the automorphism group of V_L^+ is infinite, and V_L^+ may have infinitely many Ising vectors.

In this article, we prove Theorem 2.3, and hence we classify all Ising vectors in V_L^+. Our result shows that the study of τ-involutions of V_L^+ is essentially equivalent to that of sublattices of L isomorphic to $\sqrt{2}E_8$ (see [Griess and Lam 2011; 2012]).

The key is to describe the action of the τ-involution on the Griess algebra B of V_L^+. Let e be an Ising vector in V_L^+ and $L(4;e)$ the norm 4 vectors in L which appear in the description of e with respect to the standard basis of $(V_L^+)_2$ (see Section 2 for the definition of $L(4;e)$). By [Lam and Shimakura 2007], the τ-involution τ_e associated to e is a lift of an automorphism g of L. We show in Lemma 2.1 that g is trivial on $\{\{\pm v\} \mid v \in L(4;e)\}$. This lemma follows from the decomposition of B with respect to the adjoint action of e [Höhn et al. 2012], the action of τ_e on it [Miyamoto 1996] and the explicit calculations on the Griess algebra [Frenkel et al. 1988]. By this lemma, we can obtain a VOA V containing e on which τ_e acts trivially. By [Lam et al. 2007] e is fixed by the group A generated by τ-involutions associated to elements in $L(4;e)$. Hence e belongs to the subVOA V^A of V fixed by A. Using the explicit action of A, we can find a lattice N satisfying $e \in V_N^+$ and $N/\sqrt{2}$ is even. This case was done in [Lam et al. 2007].

1. Preliminaries

VOAs associated with even lattices. In this subsection, we review the VOAs V_L and V_L^+ associated with even lattice L of rank n and their automorphisms. Our notation for lattice VOAs here is standard (see [Frenkel et al. 1988]).

Let L be a (positive-definite) even lattice with inner product $\langle \cdot , \cdot \rangle$. Let also $H = \mathbb{C} \otimes_{\mathbb{Z}} L$ be an abelian Lie algebra and $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be its affine Lie algebra. Let $\hat{H}^- = H \otimes t^{-1}\mathbb{C}[t^{-1}]$ and let $S(\hat{H}^-)$ be the symmetric algebra of $\hat{H}^-$. Then $M_H(1) = S(\hat{H}^-) \cong \mathbb{C}[h(m) \mid h \in H, m < 0] \cdot \mathbf{1}$ is the unique irreducible $\hat{H}$-module such that $h(m) \cdot \mathbf{1} = 0$ for $h \in H$, $m \geq 0$ and $c = 1$, where $h(m) = h \otimes t^m$. Note that $M_H(1)$ has a VOA structure.

The twisted group algebra $\mathbb{C}\{L\}$ can be described as follows. Let $\langle \kappa \rangle$ be a cyclic group of order 2 and $1 \to \langle \kappa \rangle \to \hat{L} \to L \to 1$ a central extension of L by $\langle \kappa \rangle$ satisfying the commutator relation $[e^\alpha, e^\beta] = \kappa^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let $L \to \hat{L}, \alpha \mapsto e^\alpha$ be a section and $\varepsilon(\, , \,) : L \times L \to \langle \kappa \rangle$ the associated 2-cocycle, that is, $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$. We may assume that $\varepsilon(\alpha, \alpha) = \kappa^{\langle \alpha, \alpha \rangle /2}$ and $\varepsilon(\, , \,)$ is bilinear by [Frenkel et al. 1988, Proposition 5.3.1]. The twisted group algebra is defined by

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}]/(\kappa + 1) \cong \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\},$$

where $\mathbb{C}[\hat{L}]$ is the usual group algebra of the group $\hat{L}$. The lattice VOA V_L associated with L is defined as $M_H(1) \otimes \mathbb{C}\{L\}$ [Borcherds 1986; Frenkel et al. 1988].

For any sublattice E of L, let $\mathbb{C}\{E\} = \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in E\}$ be a subalgebra of $\mathbb{C}\{L\}$ and let $H_E = \mathbb{C} \otimes_{\mathbb{Z}} E$ be a subspace of $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. Then the subspace $S(\hat{H}_E^-) \otimes \mathbb{C}\{E\}$ forms a subVOA of V_L and it is isomorphic to the lattice VOA V_E.

Let $O(\hat{L})$ be the subgroup of $\mathrm{Aut}\,\hat{L}$ induced by $\mathrm{Aut}\,L$. By [Frenkel et al. 1988, Proposition 5.4.1] there is an exact sequence of groups

$$1 \longrightarrow \mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \longrightarrow O(\hat{L}) \xrightarrow{-} \mathrm{Aut}\,L \longrightarrow 1.$$

Note that for $f \in O(\hat{L})$,

$$f(e^\alpha) \in \left\{\pm e^{\bar{f}(\alpha)}\right\}. \tag{1-1}$$

By [Frenkel et al. 1988, Corollary 10.4.8], $f \in O(\hat{L})$ acts on V_L as an automorphism by

$$\begin{aligned} f(h_{i_1}(n_1)h_{i_2}(n_2)\ldots h_{i_k}(n_k) \otimes e^\alpha) \\ = \bar{f}(h_{i_1})(n_1)\bar{f}(h_{i_2})(n_2)\ldots \bar{f}(h_{i_k})(n_k) \otimes f(e^\alpha), \end{aligned} \tag{1-2}$$

where $n_i \in \mathbb{Z}_{<0}$ and $\alpha \in L$. Hence $O(\hat{L})$ is a subgroup of $\mathrm{Aut}\,V_L$.

Let θ be the automorphism of $\hat{L}$ defined by $\theta(e^\alpha) = e^{-\alpha}$ for $\alpha \in L$. Then $\bar{\theta} = -1 \in \mathrm{Aut}\,L$. Using (1-2) we view θ as an automorphism of V_L. Let V_L^+ be the subspace $\{v \in V_L \mid \theta(v) = v\}$ of V_L fixed by θ. Then V_L^+ is a subVOA of V_L. Since θ is a central element of $O(\hat{L})$, the quotient group $O(\hat{L})/\langle\theta\rangle$ is a subgroup of $\mathrm{Aut}\,V_L^+$. Note that V_L^+ is a simple VOA of CFT type.

Later, we will consider the subVOA of V_L^+ generated by the weight 2 subspace.

Lemma 1.1 [Frenkel et al. 1988, Proposition 12.2.6]. *Let L be an even lattice without roots. Let N be the sublattice of L generated by $L(4)$. Then the subVOA of V_L^+ generated by $(V_L^+)_2$ is $(V_N \otimes M_{H'}(1))^+$, where $H' = (\langle N\rangle_{\mathbb{C}})^\perp$ in $\langle L\rangle_{\mathbb{C}}$.*

Ising vectors and τ-involutions. In this subsection, we review Ising vectors and corresponding τ-involutions.

Definition 1.2. A weight 2 element e of a VOA is called an *Ising vector* if the vertex subalgebra generated by e is isomorphic to the simple Virasoro VOA of central charge $\frac{1}{2}$ and e is its conformal vector.

For an Ising vector e, the automorphism τ_e, called the *τ-involution* or *Miyamoto involution*, was defined in [Miyamoto 1996, Theorem 4.2] based on the representation theory of the simple Virasoro VOA of central charge $\frac{1}{2}$ [Dong et al. 1994].

Let V be a VOA of CFT type with $V_1 = 0$. The first product $(a, b) \mapsto a\cdot b = a_{(1)}b$ provides a (nonassociative) commutative algebra structure on V_2. This algebra V_2 is called the *Griess algebra* of V, and τ_e acts on it as follows:

Lemma 1.3 [Höhn et al. 2012, Lemma 2.6]. *Let V be a simple VOA of CFT type with $V_1=0$ and e an Ising vector in V. Then $B=V_2$ has the decomposition*

$$B=\mathbb{C}e\oplus B^e(0)\oplus B^e(\tfrac{1}{2})\oplus B^e(\tfrac{1}{16})$$

with respect to the adjoint action of e, where $B^e(k)=\{v\in B\mid e\cdot v=kv\}$. The automorphism τ_e acts on B as

$$1 \text{ on } \mathbb{C}e\oplus B^e(0)\oplus B^e(\tfrac{1}{2}) \quad \text{and} \quad -1 \text{ on } B^e(\tfrac{1}{16}).$$

In the proof of our main theorem, we need:

Lemma 1.4 [Lam et al. 2007, Lemma 3.7]. *Let V be a VOA of CFT type with $V_1=0$. Suppose that V has two Ising vectors e, f and that $\tau_e=\mathrm{id}$ on V. Then e is fixed by τ_f, namely $e\in V^{\tau_f}$.*

Let L be an even lattice of rank n without roots, that is,

$$L(2)=\{v\in L\mid \langle v,v\rangle=2\}=\varnothing.$$

Then $(V_L^+)_1=0$, and we can consider the Griess algebra $B=(V_L^+)_2$ of V_L^+. Let $\{h_i\mid 1\le i\le n\}$ be an orthonormal basis of the vector space $H=\mathbb{C}\otimes_{\mathbb{Z}}L=\langle L\rangle_{\mathbb{C}}$. Set $L(4)=\{v\in L\mid\langle v,v\rangle=4\}$. For $1\le i\le j\le n$ and $\alpha\in L(4)$, set $h_{ij}=h_i(-1)h_j(-1)\mathbf{1}$ and $x_\alpha=e^\alpha+e^{-\alpha}=e^\alpha+\theta(e^\alpha)$. Note that $x_\alpha=x_{-\alpha}$.

Lemma 1.5 [Frenkel et al. 1988, Section 8.9]. (1) *The set*

$$\{h_{ij},x_\alpha\mid 1\le i\le j\le n,\ \{\pm\alpha\}\subset L(4)\}$$

is a basis of B.

(2) *The products of the basis vectors of B given in* (1) *are*

$$\begin{aligned}
h_{ij}\cdot h_{kl}&=\delta_{ik}h_{jl}+\delta_{il}h_{jk}+\delta_{jk}h_{il}+\delta_{jl}h_{ik},\\
h_{ij}\cdot x_\alpha&=\langle h_i,\alpha\rangle\langle h_j,\alpha\rangle x_\alpha,\\
x_\alpha\cdot x_\beta&=\begin{cases}\varepsilon(\alpha,\beta)x_{\alpha\pm\beta} & \text{if } \langle\alpha,\beta\rangle=\mp2,\\ \alpha(-1)^2\mathbf{1} & \text{if } \alpha=\pm\beta,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $\alpha\in L(4)$. Then the elements $\omega^+(\alpha)$ and $\omega^-(\alpha)$ of V_L^+ defined by

$$\omega^\pm(\alpha)=\tfrac{1}{16}\alpha(-1)^2\cdot\mathbf{1}\pm\tfrac{1}{4}x_\alpha \tag{1-3}$$

are Ising vectors [Dong et al. 1994, Theorem 6.3]. The following lemma is easy:

Lemma 1.6. *The automorphisms $\tau_{\omega^\pm(\alpha)}$ of V_L^+ act by*

$$u\otimes x_\beta\mapsto(-1)^{\langle\alpha,\beta\rangle}u\otimes x_\beta \quad \text{for } u\in M_H(1) \text{ and } \beta\in L.$$

More generally:

Proposition 1.7 [Lam and Shimakura 2007, Lemma 5.5]. *Let L be an even lattice without roots and e an Ising vector in V_L^+. Then $\tau_e \in O(\hat{L})/\langle\theta\rangle$.*

When $L/\sqrt{2}$ is even, our main theorem reduces to something proved earlier:

Proposition 1.8 [Lam et al. 2007, Theorem 4.6]. *Let L be an even lattice and e an Ising vector in V_L^+. Assume that the lattice $L/\sqrt{2}$ is even. There is a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ and such that $e \in V_U^+$.*

2. Classification of Ising vectors in V_L^+

Let L be an even lattice of rank n without roots and e an Ising vector in V_L^+. Then by Lemma 1.5(1),

$$e = \sum_{i \le j} c_{ij}^e h_{ij} + \sum_{\{\pm\alpha\}\subset L(4)} d_{\{\pm\alpha\}}^e x_\alpha, \tag{2-1}$$

where $c_{ij}^e, d_{\{\pm\alpha\}}^e \in \mathbb{C}$. Set $L(4; e) = \{\alpha \in L(4) \mid d_{\{\pm\alpha\}}^e \neq 0\}$, $H_1 = \langle L(4; e)\rangle_{\mathbb{C}}$ and $H_2 = H_1^\perp$ in H. Note that if $\alpha \in L(4; e)$ then $-\alpha \in L(4; e)$. Without loss of generality, we may assume that $h_i \in H_1$ if $1 \le i \le \dim H_1$. Then we have $H_2 = \mathrm{Span}_{\mathbb{C}}\{h_j \mid \dim H_1 + 1 \le j \le n\}$.

By Proposition 1.7, $\tau_e \in O(\hat{L})/\langle\theta\rangle$. Since $e \in V_L$, we regard τ_e as an automorphism of V_L. Then $\tau_e \in O(\hat{L})$, and set $g = \bar{\tau}_e \in \mathrm{Aut}\, L$. Since τ_e is of order 1 or 2, so is g. We now state the key lemma in this article:

Lemma 2.1. *Let $\beta \in L(4; e)$. Then $g(\beta) \in \{\pm\beta\}$.*

Proof. By (1-1) and (1-2),

$$\tau_e(x_\beta) \in \{\pm x_{g(\beta)}\}. \tag{2-2}$$

On the other hand, $\tau_e(e) = e$, (1-2) and (2-1) show that

$$\tau_e(d_{\{\pm\beta\}}^e x_\beta) = d_{\{\pm g(\beta)\}}^e x_{g(\beta)}. \tag{2-3}$$

By (2-2) and (2-3),

$$d_{\{\pm g(\beta)\}}^e / d_{\{\pm\beta\}}^e \in \{\pm 1\}. \tag{2-4}$$

Suppose $g(\beta) \notin \{\pm\beta\}$. Then $x_\beta - \tau_e(x_\beta)$ is nonzero, and it is an eigenvector of τ_e with eigenvalue -1. By Lemma 1.3, we have

$$e \cdot (x_\beta - \tau_e(x_\beta)) = \tfrac{1}{16}(x_\beta - \tau_e(x_\beta)). \tag{2-5}$$

We calculate the image of both sides of (2-5) under the canonical projection $\mu : (V_L^+)_2 \to \mathrm{Span}_{\mathbb{C}}\{h_{ij} \mid 1 \le i \le j \le n\}$ with respect to the basis given in

Lemma 1.5(1). By (2-2) the image of the right side of (2-5) under μ is

$$\mu\left(\tfrac{1}{16}(x_\beta - \tau_e(x_\beta))\right) = 0. \tag{2-6}$$

Let us discuss the left side of (2-5). By Lemma 1.5(2) and (2-4), we have

$$\begin{aligned} e\cdot(x_\beta - \tau_e(x_\beta)) &= \Bigg(\sum_{i\le j} c^e_{ij}h_{ij} + \sum_{\{\pm\alpha\}\subset L(4)} d^e_{\{\pm\alpha\}}x_\alpha\Bigg)\cdot\big(x_\beta - \tau_e(x_\beta)\big) \\ &\in d^e_{\{\pm\beta\}}\big(\beta(-1)^2\mathbf{1} - g(\beta)(-1)^2\mathbf{1}\big) + \mathrm{Span}_{\mathbb{C}}\big\{x_\gamma \;\big|\; \{\pm\gamma\}\subset L(4)\big\}. \end{aligned}$$

Thus

$$\begin{aligned} \mu(e\cdot(x_\beta - \tau_e(x_\beta))) &= d^e_{\{\pm\beta\}}\big(\beta(-1)^2\mathbf{1} - g(\beta)(-1)^2\mathbf{1}\big) \\ &= d^e_{\{\pm\beta\}}\,(\beta - g(\beta))\,(-1)(\beta + g(\beta))(-1)\mathbf{1}. \end{aligned}$$

This is not zero since $g(\beta)\notin\{\pm\beta\}$, which contradicts (2-5) and (2-6). Therefore $g(\beta)\in\{\pm\beta\}$. □

For $\varepsilon\in\{\pm\}$, set

$$L(4;e,\varepsilon) = \{v\in L(4;e)\mid g(v)=\varepsilon v\},\quad L^{e,\varepsilon} = \langle L(4;e,\varepsilon)\rangle_{\mathbb{Z}},\quad H_1^\varepsilon = \langle L^{e,\varepsilon}\rangle_{\mathbb{C}}.$$

Since g preserves the inner product, $H_1 = H_1^+\perp H_1^-$ and g acts on $H_2 = H_1^\perp$. Let $H_2^\pm$ be ±1-eigenspaces of g in H_2. For $\varepsilon\in\{\pm\}$, let W^ε be a lattice of full rank in H_2^ε isomorphic to an orthogonal direct sum of copies of $2A_1$. Then

$$M_{H_2^\varepsilon}(1)\subset V_{W^\varepsilon}. \tag{2-7}$$

Lemma 2.2. *The Ising vector e belongs to the VOA*

$$V^+_{L^{e,+}\oplus W^+}\otimes V^+_{L^{e,-}\oplus W^-},$$

and $\tau_e = \mathrm{id}$ *on this VOA.*

Proof. By Lemma 2.1, $L(4;e) = L(4;e,+)\cup L(4;e,-)$. Hence, by (2-1) and (2-7),

$$e\in (V_{L^{e,+}}\otimes M_{H_2^+}(1)\otimes V_{L^{e,-}}\otimes M_{H_2^-}(1))^+ \subset V^+_{L^{e,+}\oplus W^+\oplus L^{e,-}\oplus W^-}. \tag{2-8}$$

Since g acts by ±1 on $L^{e,\pm}\oplus W^\pm$, the subspace of (2-8) fixed by τ_e is

$$V^+_{L^{e,+}\oplus W^+}\otimes V^+_{L^{e,-}\oplus W^-}.$$

Since e is fixed by τ_e, we have the desired result. □

We now prove the main theorem.

Theorem 2.3. *Let L be an even lattice without roots and e an Ising vector in* V_L^+. *There is a sublattice U of L isomorphic to* $\sqrt{2}A_1$ *or* $\sqrt{2}E_8$ *and such that* $e\in V_U^+$.

Proof. Set $V = V^+_{L^{e,+}\oplus W^+} \otimes V^+_{L^{e,-}\oplus W^-}$. By Lemma 2.2, e belongs to V and $\tau_e = \mathrm{id}$ on V. Let $A = \langle \tau_{\omega^\pm(\beta)} \mid \beta \in L(4; e)\rangle$. By Lemma 1.4, e belongs to the subVOA V^A of V fixed by A. Since e is a weight 2 element, it is contained in the subVOA generated by $(V^A)_2$. By Lemmas 1.1 and 1.6 and (2-7) (see (2-8)),

$$e \in V^+_{N^+\oplus K^+} \otimes V^+_{N^-\oplus K^-} \subset V^+_N,$$

where for $\varepsilon \in \{\pm\}$, $N^\varepsilon = \mathrm{Span}_{\mathbb{Z}}\{v \in L(4; e, \varepsilon) \mid \langle v, L(4; e)\rangle \in 2\mathbb{Z}\}$, K^ε is a lattice of full rank in $(\langle N^\varepsilon\rangle_{\mathbb{C}})^\perp \cap (H_1^\varepsilon \oplus H_2^\varepsilon)$ isomorphic to an orthogonal direct sum of copies of $2A_1$, and $N = N^+ \oplus K^+ \oplus N^- \oplus K^-$. Since N is generated by norm 4 and 8 vectors, and the inner products of the generator belong to $2\mathbb{Z}$, the lattice $N/\sqrt{2}$ is even. By Proposition 1.8, there is a sublattice U of N isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$. It follows from $K^+(4) = K^-(4) = \varnothing$ that $N(4) = N^+(4) \cup N^-(4) \subset L$. Since $\sqrt{2}A_1$ and $\sqrt{2}E_8$ are spanned by norm 4 vectors as lattices, we have $U \subset L$. Hence V_U^+ is a subVOA of V_L^+. □

As an application of the main theorem, we count the total number of Ising vectors in V_L^+ for even lattice L without roots.

Let us describe Ising vectors in V_L^+. The Ising vector $\omega^\pm(\alpha)$ associated to α in $L(4)$ was described in (1-3) as

$$\omega^\pm(\alpha) = \tfrac{1}{16}\alpha(-1)^2 \cdot \mathbf{1} \pm \tfrac{1}{4}x_\alpha.$$

Let E be an even lattice isomorphic to $\sqrt{2}E_8$ and $\{u_i \mid 1 \le i \le 8\}$ an orthonormal basis of $\mathbb{C}\otimes_{\mathbb{Z}} E$. We consider the trivial 2-cocycle of $\mathbb{C}\{E\}$ for V_E. Then for $\varphi \in \mathrm{Hom}(E, \mathbb{Z}/2\mathbb{Z}) (\cong (\mathbb{Z}/2\mathbb{Z})^8)$,

$$\omega(E,\varphi) = \frac{1}{32}\sum_{i=1}^{8} u_i(-1)^2 \cdot \mathbf{1} + \frac{1}{32}\sum_{\{\pm\alpha\}\subset E(4)} (-1)^{\varphi(\alpha)} x_\alpha$$

is an Ising vector in V_E^+ [Dong et al. 1998; Griess 1998]. Since $E(4)$ spans E as a lattice, $\omega(E,\varphi) = \omega(E,\varphi')$ if and only if $\varphi = \varphi'$. Hence V_E^+ has 256 Ising vectors of form $\omega(E,\varphi)$. Thus $V^+_{\sqrt{2}A_1}$ and $V^+_{\sqrt{2}E_8}$ have exactly 2 and 496 Ising vectors, respectively [Lam et al. 2007, Propositions 4.2 and 4.3].

Corollary 2.4. *Let L be an even lattice without roots. Then the number of Ising vectors in V_L^+ is*

$$|L(4)| + 256 \times \left|\{U \subset L \mid U \cong \sqrt{2}E_8\}\right|.$$

Proof. Set $m = |L(4)| + 256 \times |\{E \subset L \mid E \cong \sqrt{2}E_8\}|$. Theorem 2.3 shows that the number of Ising vectors in V_L^+ is less than or equal to m. Let us show that there are exactly m Ising vectors in V_L^+, that is, the Ising vectors $\omega^\pm(\alpha)$ and $\omega(E,\varphi)$ are distinct. By Lemma 1.5(1), $\omega^\varepsilon(\alpha) = \omega^\delta(\beta)$ if and only if $\alpha = \beta$ and $\varepsilon = \delta$. Also $\omega^\varepsilon(\alpha) \ne \omega(E,\varphi)$ for all $\alpha \in L(4)$, $L \supset E \cong \sqrt{2}E_8$ and $\varphi \in \mathrm{Hom}(E, \mathbb{Z}/2\mathbb{Z})$.

Let E_1, E_2 be sublattices of L such that $E_1 \cong E_2 \cong \sqrt{2}E_8$. Let φ_i, $i = 1, 2$, be two elements of $\mathrm{Hom}(E_i, \mathbb{Z}/2\mathbb{Z})$. Then it follows from Lemma 1.5(1) and $\langle E_i(4)\rangle_{\mathbb{Z}} = E_i$ that $\omega(E_1, \varphi_1) = \omega(E_2, \varphi_2)$ if and only if $E_1 = E_2$ and $\varphi_1 = \varphi_2$. Therefore, there are exactly m Ising vectors in V_L^+. □

Acknowledgement

The author thanks the referee for valuable advice.

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Received October 2, 2011. Revised February 23, 2012.

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Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

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