CONVERGENCE OF AXIALLY SYMMETRIC VOLUME-PRESERVING MEAN CURVATURE FLOW

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We study the convergence of axially symmetric hypersurfaces evolving by volume-preserving mean curvature flow. Assuming the surfaces do not develop singularities along the axis of rotation at any time during the flow, and without any additional conditions, as for example on the curvature, we prove that the flow converges to a hemisphere, when the initial hypersurface has a free boundary and satisfies Neumann boundary data, and to a sphere when it is compact without boundary.

1. Introduction

Consider $n$-dimensional hypersurfaces $M_t$, defined by a one-parameter family of smooth immersions $x_t : M^n \to \mathbb{R}^{n+1}$. The hypersurfaces $M_t$ are said to move by mean curvature if $x_t(x(p), t)$ satisfies

(1-1) \[
\frac{d}{dt} x(p, t) = -H(p, t)\nu(p, t), \quad p \in M^n, \quad t > 0,
\]

where $\nu(p, t)$ denotes a smooth choice of unit normal on $M_t$ at $x(p, t)$ (outer normal in case of compact surfaces without boundary), and $H(p, t)$ the mean curvature with respect to this normal.

If in addition the evolving compact surfaces $M_t$ are assumed to enclose a prescribed volume $V$, the corresponding evolution equation is

(1-2) \[
\frac{d}{dt} x(p, t) = -(H(p, t) - h(t))\nu(p, t), \quad p \in M^n, \quad t > 0,
\]

where $h(t)$ is the average of the mean curvature,

$$h(t) = \frac{\int_{M_t} H \, dg_t}{\int_{M_t} dg_t},$$

and $g_t$ denotes the metric on $M_t$. As under the flow (1-1), the surface area $|M_t|$ of the hypersurface is known to decrease under (1-2), while the enclosed volume remains constant in the latter; see [Athanassenas 1997].

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We are interested in an axially symmetric surface, enclosing a given volume $V$, and which has a nonempty boundary contained in a plane $\Pi$ that is perpendicular to the axis of rotation. Motivated by the fact that the stationary solution to the associated variational problem satisfies a Neumann boundary condition, we also assume the surface to meet that plane $\Pi$ at right angles along its boundary. Assuming the surface to be smooth, it will also intersect the axis of rotation orthogonally.

We consider the case where the evolving hypersurfaces do not develop singularities, in particular they do not pinch off along the axis of rotation during the flow, having only one intersection with that axis at the point that is the furthest from the supporting plane $\Pi$, and prove that the flow converges to a half-sphere.

The methods we use apply also in the case of evolving axially symmetric hypersurfaces without boundary having a similar lower height bound, and in that case we prove in Section 8 that the flow converges to a sphere.

The results in this paper make use of the axial symmetry, but no additional conditions on the curvature of the surface are assumed. Convergence to spheres has been previously proved for the volume flow in [Huisken 1987], for compact, uniformly convex initial surfaces, while Li [2009] assumes bounds on the traceless second fundamental form.

Our results can be seen as complementing the work in [Athanassenas 1997; 2003], and in the PhD dissertation [Kandanaarachchi 2011]: in the case of the surface behaving like a “bridge” between two parallel surfaces, if one were able to flow through singularities, the axially symmetric volume-preserving flow would converge to a number of spheres and (possibly) two hemispheres on the parallel planes, like beads strung along the axis of rotation.

2. Notation, definitions and assumptions

In the case of the surface $M_t$ intersecting the obstacle $\Pi$, we will at different stages divide it into two parts as in [Altschuler et al. 1995]: one adjacent to the plane and one containing the (only) intersection with the axis of rotation.

Let $\Pi = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = 0 \}$ and let $M_t$ be contained in the right half-space, $M_t \subset \{ x_1 > 0 \}$. We use $R_t$ as the generic notation for the part of the surface closest to the plane, and $C_t$ for the rest — the cap that intersects the axis of rotation — and we will introduce various superscripts depending on the situation that will be made clear in the text.

We denote by $P(t) = (d(t), 0)$ the pole: the point of intersection of $M_t$ with the axis of rotation. We assume that there are no singularities developing, so that $P(t)$ is the only point of intersection of $M_t$ with the axis of rotation for all time. We are interested in those solutions where the generating curve of the initial hypersurface is smooth and can be written as a graph over the $x_1$ axis except at the pole.
We use the notation

$$\rho_t : [0, d(t)] \to \mathbb{R}$$

for the radius function of the surface of revolution.

Let $$i_1, \ldots, i_{n+1}$$ be the standard basis in $$\mathbb{R}^{n+1}$$ and let $$i_1$$ be the direction of the axis of rotation. We denote the quantities associated with the cap with a tilde $$\tilde{\cdot}$$, and in this context we work with the vertical graph equation [Altschuler et al. 1995].

Furthermore we define the following quantities on $$M_t$$:

Let $$\omega = \hat{x}/|\hat{x}| \in \mathbb{R}^{n+1}, \hat{x} = (0, x_2, \ldots, x_{n+1})$$, denote the outer unit normal to the cylinder intersecting $$M_t$$ at the point $$x(p, t)$$. We call $$u = \langle x, \omega \rangle$$ the height function of $$M_t$$, and set $$v = \langle v, \omega \rangle^{-1}$$. Note that $$v$$ corresponds to $$\sqrt{1 + \dot{\rho}^2}$$; it will be used to obtain gradient estimates.

The corresponding quantities on the cap $$C_t$$ are the height measured from the plane $$5$$, $$\tilde{u} = \langle x, i_1 \rangle$$ and $$\tilde{v} = \langle v, i_1 \rangle^{-1}$$.

We divide the hypersurface into two regions using a plane $$L_\alpha(t)$$, which is parallel to $$5$$ and intersects the surface at points where $$\langle v, i_1 \rangle |_{L_\alpha(t) \cap M_t} = 1/\alpha$$, with $$\alpha > 1$$ being a constant. We define the cap, determined by the inclination angle, as the connected component of $$M_t$$ containing the pole $$P$$,

$$C_t^\alpha = \{x(p, t) \in M_t : 1/\alpha < \langle v, i_1 \rangle \leq 1\},$$

and we call $$R_t^\alpha = M_t \setminus C_t^\alpha$$ the cylindrical part of the surface. Note that $$L_\alpha(t)$$ is chosen such that the specific inclination angle is achieved nowhere else between that plane and the pole $$P(t)$$. As long as the flow is smooth, $$C_t^\alpha$$ is by definition a graph over the $$x_1$$ axis except at the pole. We denote by $$l_\alpha(t)$$ the $$x_1$$ coordinate of $$L_\alpha(t)$$, so $$L_\alpha(t) = \{x_1 = l_\alpha(t)\}$$.

**Assumption 2.1.** We assume that for any $$\alpha > 1$$ there exists a constant $$c(\alpha) > 0$$, depending only on $$\alpha$$, such that $$u|_{R_t^\alpha} > c(\alpha)$$, that is, we assume a lower height bound in $$R_t^\alpha$$, independent of time, dependent on $$\alpha$$.

Thus $$P(t)$$ is assumed to be the only point of intersection of $$M_t$$ with the axis of rotation for all time. The assumption prevents singularities from developing on the axis of rotation.

For an axially symmetric surface the mean curvature is given by

$$H = -\frac{\ddot{\rho}}{(1+\dot{\rho}^2)^{3/2}} + \frac{n-1}{\rho(1+\dot{\rho}^2)^{1/2}},$$

while the principal curvatures are $$k = -\ddot{\rho}/(1+\dot{\rho}^2)^{3/2}$$ and $$p = 1/(\rho \sqrt{1 + \dot{\rho}^2})$$.

We also introduce another quantity, $$q = \langle v, i_1 \rangle u^{-1}$$; thus $$p^2 + q^2 = u^{-2}$. 
3. Height estimates

In this section we prove that $M_t$ satisfies uniform height bounds: both the height function $u$ defined above and the height when measured as distance from the obstacle $\Pi$, denoted by $\tilde{u}$, are bounded. That is then used to show that the length of the generating curve of the surface remains bounded.

**Lemma 3.1.** The evolving surfaces $M_t$ satisfy the uniform height bound

$$ u \leq R = \left( |M_0|/\omega_n \right)^{1/n}. $$

**Proof.** We follow a method used in [Athanassenas 1997] to get bounds for $u$. Given $R > 0$, assume that $u_{M_t} \geq R$ at some given time $t$. Since the surface area is decreasing under the flow, and by comparing to the projection of the surface onto the plane $\Pi$, we have

$$ |M_0| \geq |M_t| > \omega_n R^n, $$

where $\omega_n$ is the volume of the $n$ dimensional unit ball. Therefore $R > (|M_0|/\omega_n)^{1/n}$ would contradict the fact that the evolution decreases the surface area. \hfill \Box

**Lemma 3.2.** There is a constant $l$ such that the evolving surfaces $M_t$ satisfy the height bound $\tilde{u} \leq l$, that is, the distance from the plane $\Pi$ is uniformly bounded.

**Proof.** Here $\alpha = 1/\cos \theta$,

$$ C_\alpha^t = \{ x(p, t) \in M_t : 1/\alpha < \langle v, i_1 \rangle \leq 1 \}, $$

and $R_\alpha^t = M_t \setminus C_\alpha^t$. From Assumption 2.1, we know that $u > c(\alpha)$ in $R_\alpha^t$. As $u_{|\partial C_\alpha^t} \leq R$ by Lemma 3.1 and $|\dot{\rho}| \geq \tan(\pi/2 - \theta) = 1/\sqrt{\alpha^2 - 1}$ in $C_\alpha^t$, we have

$$ d(t) - \tilde{u}_{|\partial C_\alpha^t} \leq R \tan \theta = R \sqrt{\alpha^2 - 1}. $$

![Figure 1. The cylinder of radius $c(\alpha)$.](image-url)
Assume there exists a length $l_1$ such that $\tilde{u}|_{R^\alpha_t} > l_1$. Then
\[ |M_0| \geq |M_t| > n\omega_n c^{n-1}(\alpha)l_1, \]
where now we compared $|M_t|$ to the surface area of an $n$ dimensional cylinder of radius $c(\alpha)$ and length $l_1$. Having
\[ l_1 > \frac{|M_0|}{n\omega_n c^{n-1}(\alpha)} \]
would contradict the fact that the evolution decreases the surface area. Therefore
\[ \tilde{u} < \frac{|M_0|}{n\omega_n c^{n-1}(\alpha)} + R\sqrt{\alpha^2 - 1} =: l. \]
□

Next we show that the length of the generating curve is bounded.

**Lemma 3.3.** Assume $M_t$ to be smooth, axially symmetric hypersurfaces, evolving by (1-2) and with a radius function satisfying $\rho(x_1, t) > 0$ for $x_1 \in [0, d(t))$. Then there exists a constant $c_\ast$, independent of time, such that
\[ \int_0^{d(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq c_\ast. \]

**Proof.** Let us divide $M_t$ into $R^\alpha_t$ and $C^\alpha_t$ for any $\alpha > 1$. As the surface area is decreasing under the flow, $|M_t| \leq |M_0|$, we have
\[ 2\pi \int_0^{d(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq |M_0|, \]
\[ 2\pi \int_0^{l_\alpha(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq 2\pi \int_0^{d(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq |M_0|. \]
From Assumption 2.1,
\[ 2\pi c^{n-1}(\alpha) \int_0^{l_\alpha(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq |M_0| \quad \text{and} \quad \int_0^{l_\alpha(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)}. \]
We can estimate the length of the generating curve of the cap $C^\alpha_t$ by $l + R$. Therefore
\[ \int_0^{d(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)} + l + R =: c_\ast. \]
□

### 4. Estimates on $h$

We now derive a priori estimates for $h(t)$ for solutions of the graphical equation.

**Lemma 4.1.** Assume $M_t$ to be smooth, axially symmetric hypersurfaces, evolving by (1-2) and with a radius function satisfying $\rho(x_1, t) > 0$ for $x_1 \in [0, d(t))$. Then there is a constant $c_1$ such that $0 \leq h(t) \leq c_1$ throughout the flow.
Proof. Following [Athanassenas 2003] we parametrize $M_t$ by its radius function $\rho \in C^\infty([0, d(t)))$, then clearly

$$H = -\frac{\ddot{\rho}}{(1 + \dot{\rho}^2)^{3/2}} + \frac{n-1}{\rho(1 + \dot{\rho}^2)^{1/2}}.$$

From Lemma 3.3, we know that $\int_0^{d(t)} \sqrt{1 + \rho^2} dx_1 \leq c_*$. Our proof follows the ideas of [Athanassenas 1997], the difference being the boundary term when integrating by parts. For the sake of completeness we include it here. For the second term of

$$h(t) = \frac{1}{|M_t|} \int_{M_t} (k + (n - 1)p) dg_t, \quad t \in [0, T),$$

we have

$$0 \leq \frac{n-1}{|M_t|} \int_0^{d(t)} \rho^{n-2}(x_1, t) dx_1 \leq \frac{(n-1)R^{n-2}l}{|M_t|},$$

since $\rho \leq R$ and $d(t) \leq l$ by Lemmas 3.1 and 3.2.

For the first term note that $\dddot{\rho}/(1 + \dot{\rho}^2) = \frac{d}{dx_1}(\arctan \dot{\rho})$. Therefore

$$\text{(4-1)} \quad \int_{M_t} k dg_t = -\int_0^{d(t)} \frac{d}{dx_1}(\arctan \dot{\rho})\rho^{n-1}dx_1$$

$$= (\arctan \dot{\rho})\rho^{n-1}|_{x_1=0} - (\arctan \dot{\rho})\rho^{n-1}|_{x_1=d(t)}$$

$$+ (n-1) \int_0^{d(t)} (\arctan \dot{\rho})\dot{\rho}\rho^{n-2}dx_1$$

$$= (n-1) \int_0^{d(t)} (\arctan \dot{\rho})\dot{\rho}\rho^{n-2}dx_1,$$

because $\arctan \dot{\rho} = 0$ when $x_1 = 0$, and we have $\rho(d(t)) = 0$ at the pole. Since $0 \leq (\arctan \dot{\rho})\dot{\rho} \leq \frac{\pi}{2} |\dot{\rho}| \leq \frac{\pi}{2} \sqrt{1 + \dot{\rho}^2}$, we obtain

$$0 \leq \frac{1}{|M_t|} \int_{M_t} k dg_t \leq \frac{(n-1)\pi}{2|M_t|} \int_0^{d(t)} \sqrt{1 + \dot{\rho}^2}\rho^{n-2}dx_1$$

$$\leq \frac{(n-1)R^{n-2}\pi}{2|M_t|} \int_0^{d(t)} \sqrt{1 + \dot{\rho}^2}dx_1 \leq \frac{(n-1)c_*R^{n-2}}{|M_t|} \frac{\pi}{2},$$

where we have used Lemma 3.3.

From the isoperimetric inequality and the fact that the flow decreases surface area we know that

$$V^{n/(n+1)} < c|M_t| \leq c|M_0|.$$  

Combining these arguments we conclude that $0 \leq \frac{\int H dg_t}{\int dg_t} \leq c_1$.  
\qed
5. Evolution equations and gradient estimates

The maximum principle for noncylindrical or time dependent domains is discussed in [Lumer and Schnaubelt 1999]. We use that version of the maximum principle in this paper.

Lemma 5.1. For the flow (1-2) we have the following evolution equations:

(i) \( \left( \frac{d}{dt} - \Delta \right) u = h/v - (n-1)/u. \)

(ii) \( \left( \frac{d}{dt} - \Delta \right) \tilde{u} = h/\tilde{v}. \)

(iii) \( \left( \frac{d}{dt} - \Delta \right) v = -|A|^2 v + (n-1)v/u^2 - (2/v)|\nabla v|^2. \)

(iv) \( \left( \frac{d}{dt} - \Delta \right) \tilde{v} = -|A|^2 \tilde{v} - (2/\tilde{v})|\nabla \tilde{v}|^2. \)

(v) \( \left( \frac{d}{dt} - \Delta \right) H = (H - h)|A|^2. \)

(vi) \( \left( \frac{d}{dt} - \Delta \right) |A|^2 = -2|\nabla A|^2 + 2|A|^4 - 2hC. \)

(vii) \( \left( \frac{d}{dt} - \Delta \right) p = |A|^2 p + 2q^2(k-p) - hp^2. \)

(viii) \( \left( \frac{d}{dt} - \Delta \right) k = |A|^2 k - 2(n-1)q^2(k-p) - hk^2. \)

where \( C = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}, \) with \( g^{ij} \) denoting the components of the inverse of the first fundamental form, and \( h_{ij} \) those of the second fundamental form.

Proof. (i) and (iii) are proved in [Athanassenas 1997]; (v) and (vi) in [Huisken 1987].

(ii) For \( \tilde{u} = \langle x, i_1 \rangle \) we have

\[
\frac{d}{dt} \tilde{u} = \left( \frac{d}{dt} x, i_1 \right) = -(H - h)\langle v, i_1 \rangle \quad \text{and} \quad \Delta \tilde{u} = \langle \Delta x, i_1 \rangle = -H\langle v, i_1 \rangle,
\]

so that

\[
\left( \frac{d}{dt} - \Delta \right) \tilde{u} = h\langle v, i_1 \rangle.
\]

(iv) For \( \tilde{v} = \langle v, i_1 \rangle^{-1} \) we have

\[
\frac{d}{dt} \tilde{v} = -\tilde{v}^2 \left( \frac{d}{dt} v, i_1 \right) = -\tilde{v}^2 \langle \nabla H, i_1 \rangle.
\]

The evolution equation follows from the well-known identity [Ecker and Huisken 1989]

\[
\Delta \tilde{v} = -\tilde{v}^2 \langle \nabla H, i_1 \rangle + \tilde{v}|A|^2 + 2\tilde{v}^{-1}\nabla \tilde{v}^2.
\]

(vii) Using the same approach as in [Huisken 1990], we start with

\[
\frac{d}{dt} p = \frac{d}{dt} (u^{-2} - q^{-2})^{1/2} = \Delta p + p^{-1}|\nabla p|^2 + p^{-1}|\nabla q|^2 - 3p^{-1}u^{-4}|\nabla u|^2 + p^{-1}u^{-4} - qp^{-1}(|A|^2 q + q(p^2 - q^2 - 2kp)) - hu^{-2} + hq^2.
\]
Equation (vii) follows then from the relations

\begin{align}
\nabla_i u &= \delta_{i1} q u, \quad \nabla_1 (v, i_1) = k p u, \quad \nabla_i q = \delta_{i1} (q^2 + k p), \\
\nabla_i p &= \delta_{i1} q (p - k), \quad |A|^2 = k^2 + (n - 1) p^2, \quad u^{-4} = p^4 + 2 p^2 q^2 + q^4.
\end{align}

(viii) The evolution equation for \( H \) was derived in [Huisken 1987], and (viii) follows from (v), (vii), and the fact that \( H = k + (n - 1) p \).

We proceed to obtain gradient estimates in the different parts of the surface: for the cap by using the vertical graph equation and part (iv) from Lemma 5.1 above, and for the cylindrical part away from the cap by using the evolution equation (iii) in Lemma 5.1.

The quantities \( \tilde{u} \) and \( \tilde{v} \) are used on the cap.

**Lemma 5.2.** Assume \( M_t \) to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then the gradient estimate \( \tilde{v} \leq \alpha \) holds on the cap \( C^\alpha_t \). In addition, there is a constant \( c_2(\alpha) \) such that \( v \leq c_2(\alpha) \) for the cylindrical part \( R^\alpha_t \).

**Proof.** Note that

\[
\left( \frac{d}{dt} - \Delta \right) \tilde{v} \leq 0,
\]

so that by the maximum principle \( \tilde{v} \leq \max(\max_{C^\alpha_t} \tilde{v}, \max_{\partial C^\alpha_t} \tilde{v}) \). By definition in \( C^\alpha_t \) we have \( \tilde{v} \leq \alpha \), and this is supported by the evolution equation!

From the assumption we know that \( u > c(\alpha) \) in \( R^\alpha_t \). As in [Athanassenas 1997, Proposition 4] we calculate

\[
\left( \frac{d}{dt} - \Delta \right) u^2 v = -|A|^2 u^2 v + (n - 1) v + 2 u h - 2(n - 1) v - 2 v |\nabla u|^2 - \frac{2}{v} \nabla v \nabla (u^2 v) \\
\leq 2 h u - (n - 1) v.
\]

If \( v > 2 c_1 R/(n - 1) \) the right side is negative, and proceeding as in [Athanassenas 1997] we conclude \( v \leq c_2(\alpha) \) in \( R^\alpha_t \). It is important to note that on the boundary of \( R^\alpha_t \), either \( v = 1 \) (along the intersection with \( \Pi \)), or \( v = \alpha/\sqrt{\alpha^2 - 1} \). Thereby, we have bounds for \( v \) and \( \tilde{v} \) in \( R^\alpha_t \) and \( C^\alpha_t \) respectively.

**Remark 5.3.** (i) The gradient bounds from Lemma 5.2 guarantee that \( R^\alpha_t \) remains a graph. As \( C^\alpha_t \) remains a graph for all \( \alpha > 1 \), we see that \( M_t \setminus P(t) \) remains a graph throughout the flow.

(ii) As the height of the graph is bounded we find a lower bound for the minimum \( d(t) \) from

\[
V = \int_0^{d(t)} \omega_n \rho^n(x) \, dx_1 \leq \omega_n R^n \int_0^{d(t)} \omega_n \, dx_1 = \omega_n R^n d(t).
\]
Lemma 5.4. Assume the $M_t$ to be axially symmetric surfaces as described in Section 2 and to evolve by (1-2). Let $x_0(t)$ be a boundary point of $C_t^{\sqrt{2}}$, which we can assume without loss of generality to lie on the generating curve and be such that $\langle v(x_0(t)), i_1 \rangle = 1/\sqrt{2}$ (with some abuse of notation for the corresponding normal $v(x_0(t))$). Then $H(x_0(t)) \geq 0$ for $0 \leq t \leq T_{\text{max}} \leq \infty$.

Proof. Suppose $H(x_0(t)) < 0$, then by continuity there is a connected region $C_t^{\sqrt{2}, H^-} \subset C_t^{\sqrt{2}}$, with $x_0(t) \in \partial C_t^{\sqrt{2}, H^-}$, which clearly can be chosen to be axially symmetric, and such that $H|_{C_t^{\sqrt{2}, H^-}} < 0$. Let $x_1(t)$ denote the other boundary point along the generating curve in $C_t^{\sqrt{2}, H^-} \subset C_t^{\sqrt{2}}$, and let $a(t) = \langle x_0(t), i_1 \rangle$ and $b(t) = \langle x_1(t), i_1 \rangle$ denote the $x_1$ coordinate of $x_0(t)$, $x_1(t)$, respectively. Then

$$0 > \int_{C_t^{\sqrt{2}, H^-}} H dg = \int_{a(t)}^{b(t)} \left( -\frac{\ddot{\rho}}{1+\rho^2} \rho^{n-1} + (n-1)\rho^{n-2} \right) dx_1.$$ 

The second term being positive means that the first is negative, and given the bounds on the radius we find

$$\int_{a(t)}^{b(t)} \left( -\frac{\ddot{\rho}}{1+\rho^2} \right) dx_1 = \int_{a(t)}^{b(t)} \left( -\frac{d}{dx_1} (\arctan \dot{\rho}) \right) dx_1 < 0.$$ 

This results in

$$\arctan \dot{\rho}(a(t)) < \arctan \dot{\rho}(b(t)) \quad \text{and} \quad -\frac{\pi}{4} < \arctan \dot{\rho}(b(t)),$$

by the choice of $a(t)$. But this is not possible in $C_t^{\sqrt{2}}$, where $-\frac{\pi}{2} \leq \arctan \dot{\rho} < -\frac{\pi}{4}$, contradicting our assumption and therefore $H(x_0(t)) \geq 0$. □
6. Curvature estimates

**Proposition 6.1.** Assume $M_t$ to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then there is a constant $c_2$ depending only on the initial hypersurface, such that the principal curvatures $k$ and $p$ satisfy $k/p < c_2$, independently of time.

**Proof.** We calculate from Lemma 5.1 that

$$\frac{d}{dt} \left( \frac{k}{p} \right) = \Delta \frac{k}{p} + 2 \nabla_i p \nabla_i \left( \frac{k}{p} \right) + 2 \frac{q^2}{p^2} (p - k) ((n - 1) p + k) + \frac{h k}{p} (p - k).$$

If $k/p \geq 1$ then $(hk/p)(p - k) < 0$. This implies that

$$\frac{k}{p} \leq \max \left( 1, \max_{M_0} \frac{k}{p} \right).$$

Note that for this consideration, the smooth function $k/p$ is defined over the whole surface, and in view of the orthogonality on the boundary, via a reflection argument there are no boundary data involved. □

**Proposition 6.2.** Assume $M_t$ to be axially symmetric surfaces as described in Section 2 that evolve by (1-2) and let $A$ be the second fundamental form. Then there exists a constant $c_3$, independent of time, such that $|A|^2 \leq c_3$.

**Proof.** We proceed as in [Ecker and Huisken 1991] and [Athanassenas 1997] and calculate the evolution equation for the product $g = |A|^2 \varphi(v^2)$ in $R_t^{\sqrt{2}}$, where $\varphi(r) = r / (\lambda - \mu v^2)$, with $v = (v, \omega)^{-1}$ and appropriately chosen constants $\lambda, \mu > 0$. From the evolution equation of $g$ we find the inequality

$$\left( \frac{d}{dt} - \Delta \right) g \leq -2\mu g^2 - 2\lambda \varphi v^{-3} \nabla v \cdot \nabla g - \frac{2\lambda \mu}{(\lambda - \mu v^2)^2} |\nabla v|^2 g - 2hC \varphi(v^2) + \frac{2(n-1)}{u^2} v^2 \varphi' |A|^2.$$

We estimate the second last term as in [Athanassenas 1997] using Young’s inequality and obtain

$$-2hC \varphi(v^2) \leq 2h |A|^3 \varphi(v^2) \leq \frac{3}{2} |A|^4 \varphi(v^2) + \frac{1}{2} h^4 \varphi^{-2}(v^2) = \frac{3}{2} g^2 + \frac{1}{2} h^4 \varphi^{-2}(v^2).$$

We choose $\mu > \frac{3}{4}$ and $\lambda > \mu \max v^2$. As $\varphi' v^2 = \frac{\lambda}{(\lambda - \mu v^2)^2} \varphi$ we have

$$\frac{2(n-1)}{u^2} v^2 \varphi' |A|^2 = \frac{2(n-1)\lambda}{u^2(\lambda - \mu v^2)} g.$$

As $u > c(1/\sqrt{2}) = c_0$ in $R_t^{\sqrt{2}}$ we get

$$\frac{2(n-1)\lambda}{u^2(\lambda - \mu v^2)} g \leq c_4 g.$$
Therefore we have
\[
\left( \frac{d}{dt} - \Delta \right) g \leq -c_5 g^2 + c_6 g - c_7 \nabla v \cdot \nabla g + c_8 (h, \max v)
\leq -c_5 \left( g - \frac{c_6}{2c_5} \right)^2 - c_7 \nabla v \cdot \nabla g + c_9.
\]

The right side of this inequality is negative at a maximum of \(g\), where
\[
g > \frac{c_6}{2c_5} + \sqrt{\frac{c_9}{c_5}}.
\]

On \(\partial R_t^{\sqrt{2}}\) we have \(H = k + (n - 1) p \geq 0\) by Lemma 5.4. Also, as \(k/p < c_2\), we get \(|k|/p < c\) on this boundary and thus we have
\[
|A|^2 = k^2 + (n - 1) p^2 \leq (c^2 + n - 1) p^2 \leq C \rho^{-2} \leq Cc_0^{-2}
\]
on \(\partial R_t^{\sqrt{2}}\). By the maximum principle,
\[
g \leq \max \left( \max_{R_0^{\sqrt{2}}} g, \max_{\partial R_t^{\sqrt{2}}} |A|^2 \varphi(v^2) \right).
\]

Since \(v \leq c_2(\sqrt{2})\) and \(\varphi(v^2)\) is bounded, we have a bound for \(g\) in \(R_t^{\sqrt{2}}\).

The evolution equation for \(\tilde{g} = |A|^2 \varphi(\tilde{v}^2)\) on \(C_t^{\sqrt{2}}\) is the same as the one for \(g\) without the last term on the right side. Thereby we obtain a bound for \(\tilde{g}\) in the same way as above. \(\square\)

**Proposition 6.3.** Assume \(M_t\) to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then for each \(m \geq 1\) there is a \(C_m\) such that
\[
|\nabla^m A|^2 \leq C_m,
\]
uniformly on \(M_t\), for \(0 \leq t \leq T_{\max} \leq \infty\).

**Proof.** Having obtained uniform bounds on \(|A|^2\) and \(h\) the proof is a repetition of that of Theorem 4.1 in [Huisken 1987]. \(\square\)

Thus we have long-time existence for the flow:

**Corollary 6.4.** Let \(M_t\) be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then
\[
T_{\max} = \infty.
\]

### 7. Convergence to surfaces of constant mean curvature

Having long-time existence, Proposition 8 of [Athanassenas 1997] gives convergence to a constant mean curvature surface, which in our case is axially symmetric. By the classification of the Delaunay surfaces [1841] it has to be a half-sphere.
8. Other convergence results

Using the same estimates with very few changes one can show that a compact, axially symmetric surface without boundary, which encloses a volume \( V \) and intersects the axis only at two endpoints throughout the flow by (1-2), will converge to a sphere. We will only explain the parts that are different from the previous result.

We denote the surface again by \( M_t \). We split it into a cylindrical part \( R_\alpha^t \) and two caps \( C_i^{(\alpha,1)} \) for \( i = 1, 2 \), in this case. The left side cap, \( C_1^{(\alpha,1)} \), intersects the axis of rotation at \( x_1 = e(t) \), while the (only other) intersection on the right for \( C_2^{(\alpha,2)} \) is located at \( x_1 = d(t) \). Assumption 2.1 holds on \( M_t \).

**Height estimates.** The height estimates of Lemmas 3.1 and 3.2 change as follows:

**Lemma 8.1.** Assume \( M_t \) to be axially symmetric, compact without boundary and evolving by (1-2). Then the height function \( u \) satisfies

\[
|M_0| \geq |M_t| > 2 \omega_n R^n.
\]

Proof. Given \( R > 0 \) assume that \( u_{\partial M_t} \geq R \) at some given time \( t \). Take a plane perpendicular to the \( x_1 \)-axis and intersecting the surface. This plane divides the surface into two parts, and by projecting both parts onto it we find

\[
|M_0| \geq |M_t| > 2 \omega_n R^n.
\]

Taking \( R > \left( \frac{|M_0|}{2 \omega_n} \right)^{1/n} \) would contradict the fact that the evolution decreases the surface area. \( \square \)

The next lemma gives an estimate for the diameter of \( M_t \) in the \( x_1 \) direction.

**Lemma 8.2.** Assume \( M_t \) to be smooth, axially symmetric, compact without boundary and evolving by (1-2). Then

\[
d(t) - e(t) < l = \frac{|M_0|}{n \omega_n c_0^{n-1}} + 2R.
\]

Proof. As in Lemma 3.2 let \( \alpha = 1/\cos \theta \). From Assumption 2.1 we know that \( u > c(\alpha) \) in \( R_\alpha^t \). As \( u|_{\partial C_i^{\alpha,i}} \leq R \) and \( |\dot{\rho}| \geq \tan \left( \frac{\pi}{2} - \theta \right) \) in \( C_i^{\alpha,i} \) for \( i = 1, 2 \), we have

\[
d(t) - \tilde{u}|_{\partial C_i^{\alpha,1}} \leq R \tan \theta = R \sqrt{\alpha^2 - 1},
\]

\[
\tilde{u}|_{\partial C_i^{\alpha,2}} - e(t) \leq R \tan \theta = R \sqrt{\alpha^2 - 1}.
\]

Assume there exists a length \( l_1 \) such that \( \tilde{u}|_{R_i^\alpha} > l_1 \). Then by the previous argument,

\[
|M_0| \geq |M_t| > n \omega_n c_0^{n-1}(\alpha) l_1,
\]

where now we compared \( |M_t| \) to the surface area of an \( n \) dimensional cylinder of radius \( c(\alpha) \) and length \( l_1 \). If \( l_1 > |M_0|/(n \omega_n c_0^{n-1}(\alpha)) \) this would contradict the fact
that the evolution decreases the surface area. Therefore
\[ \tilde{u} < \frac{|M_0|}{n\omega n c_{n-1}} + 2R\sqrt{\alpha^2 - 1}. \]

Again we can estimate the length of the generating curve throughout the flow:

**Lemma 8.3.** Assume \( M_t \) to be smooth, axially symmetric, compact without boundary, evolving by (1-2) and with a radius function satisfying \( \rho(x_1, t) > 0 \) for \( x_1 \) in \((e(t), d(t))\). Then there exists a constant \( c_* \), independent of time, such that
\[ \int_0^{d(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq c_* \cdot \]

**Proof.** The proof is the same as that of Lemma 3.3 after taking into account the two caps on either side. Here we have
\[ \int_{e(t)}^{d(t)} \sqrt{1 + \dot{\rho}^2} \, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)} + 2l + 2R =: c_* \cdot \]

**Lemma 8.4** (estimates on \( h \)). Assume \( M_t \) to be smooth, axially symmetric, compact without boundary, evolving by (1-2) and with a radius function that satisfies \( \rho(x_1, t) > 0 \) for \( x_1 \in (e(t), d(t)) \). Then there is a constant \( c_1 \) such that \( 0 \leq h(t) \leq c_1 \) throughout the flow.

**Proof.** The only change to the proof of Lemma 4.1 is in the boundary values when integrating by parts in (4-1). Here the new boundary values are
\[ (\arctan \dot{\rho})\rho^{n-1}|_{x_1=a(t)} - (\arctan \dot{\rho})\rho^{n-1}|_{x_1=b(t)}. \]
As \( \rho(a(t)) = \rho(b(t)) = 0 \), the boundary terms disappear and we get the same estimate for \( h \).

**Lemma 8.5** (gradient estimates). Under the above assumptions, the gradient estimate \( |\tilde{v}| \leq \alpha \) holds on the caps \( C_t^{a,i}, i = 1, 2 \). In addition there is a constant \( c \), such that \( v \leq c \) for the cylindrical part \( R_t^a \).

**Proof.** The gradient estimates are as in Lemma 5.2, but in this setting instead of one cap \( C_t^a \) we have two caps on either side, and the same estimate holds for both caps.

Concluding this section, we remark that \( H \geq 0 \) at points where the caps \( C_t^{\sqrt{2},i}, i = 1, 2 \), meet the cylindrical part \( R_t^{\sqrt{2}} \) of the surface. The proof is using the same arguments as the one for Lemma 5.4 after the appropriate adjustments of the sign of \( \arctan \dot{\rho} \) for the cap on the left of the surface. The results on curvature estimates and the convergence to a limiting surface of constant mean curvature follow along the same lines as previously proved. In this case the limit surface is a sphere.
References


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