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We make two additions to recent results of Mendel and Naor on nonlinear versions of Dvoretzky's theorem. We consider the cases of metric spaces with infinite Hausdorff dimension and countably infinite metric spaces.

1. Introduction and the statement of the results

We say that a metric space X is *embedded with distortion* $D \geq 1$ in a metric space Y if there exist a map $f : X \to Y$ and a constant $r > 0$ such that

$$r\, d_X(x, y) \leq d_Y(f(x), f(y)) \leq Dr\, d_X(x, y) \quad \text{for all } x, y \in X.$$

Such a map f is called a *D-embedding*.

Dvoretzky's theorem [1961] states that for every $\varepsilon > 0$, every n-dimensional normed space contains a $k(n, \varepsilon)$-dimensional subspace that embeds into a Hilbert space with distortion $1 + \varepsilon$. This theorem was conjectured by Grothendieck [1953]. See [Milman 1971; 1992; Milman and Schechtman 1999; Schechtman 2006; 2011] for the estimates of $k(n, \varepsilon)$ and the further developments related to this theorem.

Bourgain, Figiel, and Milman proved a natural nonlinear variant of Dvoretzky's theorem:

Theorem 1.1 [Bourgain et al. 1986]. *There exists two universal constants $c_1, c_2 > 0$ such that for every $\varepsilon > 0$, every finite metric space X contains a subset S that embeds into a Hilbert space with distortion $1 + \varepsilon$ and*

$$|S| \geq \frac{c_1 \varepsilon}{\log(c_2/\varepsilon)} \log |X|.$$

See [Bartal et al. 2005; Mendel and Naor 2007; Naor and Tao 2012] for further discussion. It is natural to try to get some versions of the above theorem in the case where $|X| = \infty$. In this paper we prove the following.

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Theorem 1.2. *For every $\varepsilon > 0$, every countable infinite metric space X has an infinite subset which embeds into an ultrametric space with distortion $1+\varepsilon$.*

Recall that a metric space (U, ρ) is called an *ultrametric space* if for every $x, y, z \in X$ we have $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$. Since every separable ultrametric space isometrically embeds into a Hilbert space [Vestfrid and Timan 1979], we verify that Theorem 1.1 holds in the case where $|X| = \infty$.

Recently Mendel and Naor [2012] proved another variant of Dvoretzky's theorem, answering a question by T. Tao. For a metric space X we denote its Hausdorff dimension by $\dim_H X$. A subset of a complete separable metric space is called an *analytic set* if it is an image of a complete separable metric space under a continuous map. Note that analytic sets are not necessarily complete. For example, any Borel subset of a complete separable metric space is an analytic set (refer to [Kechris 1995] for analytic sets).

Theorem 1.3 (compare [Mendel and Naor 2012, Theorem 1.7]). *There exists a universal constant $c \in (0, \infty)$ such that for every $\varepsilon \in (0, \infty)$, every analytic set X whose Hausdorff dimension is finite has a closed subset $S \subseteq X$ that embeds with distortion $2+\varepsilon$ in an ultrametric space, and*

$$\dim_H S \geq \frac{c\varepsilon}{\log(1/\varepsilon)} \dim_H X.$$

Mendel and Naor [2012] stated this theorem only for compact metric spaces. As remarked in their introduction, the theorem is valid for more general metric spaces. For example, the theorem holds for every analytic set X, since the problem can be reduced to the case of a compact subset of X with the same Hausdorff dimension (see [Carleson 1967; Howroyd 1995, Corollary 7]).

In the following theorem we consider the case where $\dim_H X = \infty$.

Theorem 1.4. *For every $\varepsilon \in (0, \infty)$, every analytic set X whose Hausdorff dimension is infinite has a closed subset S that can be embedded into an ultrametric space with distortion $2+\varepsilon$ and has infinite Hausdorff dimension.*

It follows from the proof of Theorem 1.3 in [Mendel and Naor 2012] that if $\dim_H X = \infty$, then X contains an arbitrary large-dimensional closed subset that embeds into an ultrametric space. Combining Theorem 1.3 with Theorem 1.4 we find that a nonlinear Dvoretzky theorem holds for all analytic sets.

The following theorem asserts that the distortion in Theorems 1.3 and 1.4 cannot be strictly less than two.

Theorem 1.5 [Mendel and Naor 2012, Theorem 1.8]. *For every $\alpha > 0$ there exists a compact metric space (X, d) of Hausdorff dimension α, such that if $S \subseteq X$ embeds into a Hilbert space with distortion strictly smaller than 2 then $\dim_H S = 0$.*

Theorem 1.5 immediately implies that the same result holds in the case $\alpha = \infty$.

It is known that ℓ_2 does not embed into ℓ_p with finite distortion for any p in $[1, \infty) \setminus \{2\}$ [Albiac and Kalton 2006, Corollary 2.1.6]. In particular, an infinite-dimensional analogue of Dvoretzky's theorem is no longer true in the linear setting. In contrast to this fact, Theorem 1.4 asserts that an infinite-dimensional Dvoretzky theorem holds in the nonlinear setting.

2. Proof

Lemma 2.1. *Let X be a separable metric space such that $\dim_H X = \infty$. Then there exists a sequence $\{K_i\}_{i=1}^{\infty}$ of mutually disjoint closed subsets of X such that*

$$\lim_{i\to\infty} \operatorname{diam} K_i = 0 \quad \textit{and} \quad \lim_{i\to\infty} \dim_H K_i = \infty.$$

Proof. For every $x \in X$ we take a closed neighborhood K_x of x with $\operatorname{diam} K_x \le 1$. Since X is separable, applying the Lindelöf covering theorem we get a countable subset $F \subseteq X$ such that $X = \bigcup_{x\in F} K_x$. Since

$$\dim_H\left(\bigcup_{x\in F} K_x\right) = \sup_{x\in F} \dim_H K_x,$$

there exists $x_1 \in F$ such that $\dim_H K_{x_1} = \infty$ or there exists a sequence $\{y_i\}_{i=1}^{\infty} \subseteq F$ such that $\{\dim_H K_{y_i}\}_{i=1}^{\infty}$ is strictly increasing and $\lim_{i\to\infty} \dim_H K_{y_i} = \infty$.

We first consider the latter case. We put $K_1 := K_{y_1}$. By the monotonicity of $\dim_H K_{y_i}$ we have $\dim_H(K_{y_i} \setminus \bigcup_{j=1}^{i-1} K_{y_j}) = \dim_H K_{y_i}$ for $i \ge 2$. Covering $K_{y_i} \setminus \bigcup_{j=1}^{i-1} K_{y_j}$ by countably many closed subsets of diameter $\le 1/i$, we thus find a closed subset $K_i \subseteq K_{y_i} \setminus \bigcup_{j=1}^{i-1} K_{y_j}$ such that $\dim_H K_i = \dim_H K_{y_i}$ and $\operatorname{diam} K_i \le 1/i$. This $\{K_i\}_{i=1}^{\infty}$ is the desired sequence.

We now consider the former case. Covering K_{x_1} by countably many closed subsets $\{K_y^1\}_{y\in F_1}$ so that $\operatorname{diam} K_y^1 \le 2^{-1} \operatorname{diam} K_{x_1}$, we have two cases: There exists $x_2 \in F_1$ such that $\dim_H K_{x_2}^1 = \infty$ or there exists a sequence $\{y_i\}_{i=1}^{\infty} \subseteq F_1$ such that $\{\dim_H K_{y_i}^1\}_{i=1}^{\infty}$ is strictly increasing and $\lim_{i\to\infty} \dim_H K_{y_i}^1 = \infty$. Since we have already proved the lemma in the latter case, we consider the former case. Continuing this process we may assume there is a chain $K_{x_2}^1 \supseteq K_{x_3}^2 \supseteq K_{x_4}^3 \supseteq \cdots$ of closed subsets of X such that

$$\dim_H K_{x_i}^{i-1} = \infty \quad \text{and} \quad \operatorname{diam} K_{x_i+1}^{i} \le 2^{-1} \operatorname{diam} K_{x_i}^{i-1}.$$

Since $K_{x_i}^{i-1} \setminus \bigcup_{j=i}^{\infty} (K_{x_j}^{j-1} \setminus K_{x_j+1}^{j})$ consists of at most one point, we get

$$\limsup_{i\to\infty} \dim_H (K_{x_i}^{i-1} \setminus K_{x_{i+1}}^{i}) = \infty.$$

By taking a subsequence we may assume that $\lim_{i\to\infty} \dim_H(K^{i-1}_{x_i} \setminus K^i_{x_{i+1}}) = \infty$. Taking a closed $K_i \subseteq K^{i-1}_{x_i} \setminus K^i_{x_{i+1}}$ such that $\dim_H K_i \geq 2^{-1} \dim_H(K^{i-1}_{x_i} \setminus K^i_{x_{i+1}})$ we easily see that this $\{K_i\}_{i=1}^{\infty}$ is the desired sequence. □

We first prove Theorem 1.4. It turns out that Theorem 1.2 follows from the proof of Theorem 1.4.

Proof of Theorem 1.4. We take a sequence $\{K_i\}_{i=1}^{\infty}$ of closed subsets of X in Lemma 2.1. For each i we fix an element $x_i \in K_i$. Note that closed subsets of analytic sets are also analytic sets. By Theorem 1.3 there exist $A_i \subseteq K_i$ such that $\lim_{i\to\infty} \dim_H A_i = \infty$ and A_i embeds into some ultrametric space (U_i, ρ_i) with distortion $2+\varepsilon$, i.e., there exist $f_i : A_i \to U_i$ satisfying

$$d(x, y) \leq \rho_i(f_i(x), f_i(y)) \leq (2+\varepsilon)d(x, y) \quad \text{for any } x, y \in A_i. \tag{2-1}$$

We divide the proof into three cases.

Case 1. *$\{x_i\}_{i=1}^{\infty}$ is not bounded.*

By taking a subsequence we may assume that $\lim_{n\to\infty} d(x_1, x_i) = \infty$ and that $\operatorname{diam} K_i \leq 1/(2+\varepsilon)$. By taking a further subsequence we may also assume that

$$1 \leq \min\left\{\frac{\sqrt{1+\varepsilon}-1}{\sqrt{1+\varepsilon}\sqrt{1+2^{-1}\varepsilon}}, \frac{\sqrt{1+\varepsilon}-\sqrt{1+2^{-1}\varepsilon}}{\sqrt{1+2^{-1}\varepsilon}}, \frac{\sqrt{1+2^{-1}\varepsilon}-1}{2}\right\} d(A_1, A_2) \tag{2-2}$$

and

$$d(A_1, A_{i-1}) \leq \min\left\{\frac{\sqrt{1+\varepsilon}-1}{\sqrt{1+\varepsilon}\sqrt{1+2^{-1}\varepsilon}}, \frac{\sqrt{1+\varepsilon}-\sqrt{1+2^{-1}\varepsilon}}{\sqrt{1+2^{-1}\varepsilon}}\right\} d(A_1, A_i) \tag{2-3}$$

for any $i \geq 2$. Put $R_i := d(A_i, A_1)$ for $i \geq 2$. Note that $\operatorname{diam} f_i(A_i) \leq 1$ since f_i satisfies (2-1) and $\operatorname{diam} A_i \leq \operatorname{diam} K_i \leq 1/(2+\varepsilon)$.

For each $i \geq 2$ we take a point $u_{i,0}$ not in $f_i(A_i)$ and put $Y_i := f_i(A_i) \cup \{u_{i,0}\}$. Define the distance function $\tilde{\rho}_i$ on Y_i as follows: $\tilde{\rho}_i(u, u_{i,0}) := R_i$ for $u \in f_i(A_i)$ and $\tilde{\rho}_i(u, v) := \rho_i(u, v)$ for $u, v \in f_i(A_i)$. Since $\operatorname{diam} f_i(A_i) \leq 1 \leq R_i$, each $(Y_i, \tilde{\rho}_i)$ is an ultrametric space. Let us consider the space

$$U := \left\{(u_i) \in \prod_{i=2}^{\infty} Y_i \;\middle|\; u_i \neq u_{i,0} \text{ only for finitely many } i\right\} \tag{2-4}$$

and define the distance function ρ on U by

$$\rho((u_i), (v_i)) := \sup_i \tilde{\rho}_i(u_i, v_i). \tag{2-5}$$

It is easy to verify that (U, ρ) is an ultrametric space. For each $x \in A_i$ we put

$$f(x) := (u_{2,0}, u_{3,0}, \ldots, u_{i-1,0}, f_i(x), u_{i+1,0}, u_{i+2,0}, \ldots). \tag{2-6}$$

We shall prove that f is a $(2+\varepsilon)$-embedding from the closed subset $\bigcup_{i=2}^{\infty} A_i \subseteq X$ to the ultrametric space (U, ρ). Note that $\dim_H(\bigcup_{i=2}^{\infty} A_i) = \infty$.

We take two arbitrary points $x \in A_i$ and $y \in A_j$ $(i < j)$ and fix $z \in A_1$. By (2-2) and (2-3), we get

$$d(x, z) \leq R_i + \operatorname{diam} A_1 + \operatorname{diam} A_i \leq R_i + 2 \leq \sqrt{1+2^{-1}\varepsilon}\, R_i.$$

Combining this inequality with (2-2) and (2-3) also implies

$$\begin{aligned} d(x, y) \geq d(y, z) - d(x, z) &\geq R_j - \sqrt{1+2^{-1}\varepsilon}\, R_i \\ &\geq \frac{1}{\sqrt{1+\varepsilon}} R_j = \frac{1}{\sqrt{1+\varepsilon}} \rho(f(x), f(y)), \end{aligned}$$

and

$$\begin{aligned} d(x, y) \leq d(x, z) + d(y, z) &\leq \sqrt{1+2^{-1}\varepsilon}\, R_i + \sqrt{1+2^{-1}\varepsilon}\, R_j \\ &\leq \sqrt{1+\varepsilon}\, R_j = \sqrt{1+\varepsilon}\, \rho(f(x), f(y)). \end{aligned}$$

Hence f is a $(2+\varepsilon)$-embedding.

Case 2. $\{x_i\}_{i=1}^{\infty}$ *is bounded but not totally bounded.*

By taking a subsequence, we may assume that there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \leq d(x_i, x_j) \leq c_2 \text{ for any distinct } i, j.$$

For any $\delta > 0$ we divide $[c_1, c_2] = \bigcup_{j=1}^{m} I_j$ so that $\operatorname{diam} I_j < \delta$ for any j.

Pick $j_1 \in \{1, 2, \dots, m\}$ such that $d(x_i, x_1) \in I_{j_1}$ holds for infinitely many i. Put

$$X_1 := \{x_i \mid d(x_i, x_1) \in I_{j_1}\} = \{x_{k_1(1)}, x_{k_1(2)}, \dots\}.$$

We then choose $j_2 \in \{1, 2, \dots, m\}$ so that $d(x_{k_1(i)}, x_{k_1(1)}) \in I_{j_2}$ holds for infinitely many i and put

$$X_2 := \{x_{k_1(i)} \in X_1 \mid d(x_{k_1(i)}, x_{k_1(1)}) \in I_{j_2}\} = \{x_{k_2(1)}, x_{k_2(2)}, \dots\}.$$

Repeatedly we obtain a sequence $\{j_i\}_{i=1}^{\infty}$ whose terms are elements of the set $\{1, 2, \dots, m\}$ and $X_i = \{x_{k_i(1)}, x_{k_i(2)}, \dots\}$. By a pigeonhole argument we find a subsequence $\{j_{h(i)}\}_{i=1}^{\infty} \subseteq \{j_i\}_{i=1}^{\infty}$ that is monochromatic, i.e., $j_{h(i)} \equiv l$ for some $l \in \{1, 2, \dots, m\}$. We then get $d(x_{k_{h(i)}(i)}, x_{k_{h(j)}(j)}) \in I_l$. Since $\operatorname{diam} I_l < \delta$ and $\lim_{i\to\infty} \operatorname{diam} A_i = 0$, by choosing sufficiently small δ and taking a subsequence, we see that there exists a number $\alpha \geq c_1$ such that

$$\alpha \leq d(u, v) \leq (1+\varepsilon)\alpha \quad \text{for any } u \in A_i \text{ and } v \in A_j \ (i \neq j) \tag{2-7}$$

and $\operatorname{diam} A_i \leq (2+\varepsilon)^{-1}\alpha$. As in Case 1 we take a point $u_{i,0}$ not in $f_i(A_i)$ and put

$Y_i := f_i(A_i) \cup \{u_{i,0}\}$. We define the distance function $\tilde{\rho}_i$ on Y_i by

$$\tilde{\rho}_i(u, u_{i,0}) := \alpha \quad \text{and} \quad \tilde{\rho}_i(u, v) := \rho_i(u, v)$$

for $u, v \in f_i(A_i)$. Since $\operatorname{diam} f_i(A_i) \le (2+\varepsilon) \operatorname{diam} A_i \le \alpha$, each $(Y_i, \tilde{\rho}_i)$ is an ultrametric space. From these $(Y_i, \tilde{\rho}_i)$ we construct an ultrametric space (U, ρ) by (2-4) and (2-5). Then a map $f : \bigcup_{i=2}^{\infty} A_i \to (U, \rho)$ defined by (2-6) is a $(2+\varepsilon)$-embedding.

Case 3. $\{x_i\}_{i=1}^{\infty}$ *is totally bounded.*

The proof is similar to Case 1. From total boundedness, by taking a subsequence, we may assume that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since $\lim_{i\to\infty} \operatorname{diam} A_i = 0$, the sequence $\{A_i\}_{i=1}^{\infty}$ Hausdorff converges to a point x_∞. Let $\delta > 0$ be specified later. Note that $x_\infty \notin A_i$ for any sufficiently large i since A_i are mutually disjoint closed subsets of X. Hence, by taking a subsequence, we may also assume that $d(A_i, x_\infty)/d(A_{i-1}, x_\infty) \le \delta$ for each i. Covering A_i by countably many closed subsets $\{B_{ij}\}_j$ of diameter $\le \delta d(A_i, x_\infty)$ we find a subset B_{ij} such that $\dim_H(B_{ij}) \ge 2^{-1} \dim_H(A_i)$ and

$$\frac{\operatorname{diam} B_{ij}}{d(B_{ij}, x_\infty)} \le \frac{\operatorname{diam} B_{ij}}{d(A_i, x_\infty)} \le \delta.$$

Hence by replacing A_i with B_{ij}, we may assume that $\operatorname{diam} A_i/d(A_i, x_\infty) \le \delta$ for every i.

As in Cases 1 and 2 we add a point $u_{i,0}$ to $f_i(A_i)$ and put $Y_i := f_i(A_i) \cup \{u_{i,0}\}$. Define the distance function $\tilde{\rho}_i$ on Y_i by

$$\tilde{\rho}_i(u, u_{i,0}) := d(A_i, x_\infty) \quad \text{and} \quad \tilde{\rho}_i(u, v) := \rho_i(u, v)$$

for $u, v \in f_i(A_i)$. If $\delta \le (2+\varepsilon)^{-1}$, then we have

$$\operatorname{diam} f_i(A_i) \le (2+\varepsilon) \operatorname{diam} A_i \le d(A_i, x_\infty),$$

which implies that each $(Y_i, \tilde{\rho}_i)$ is an ultrametric space. From these $(Y_i, \tilde{\rho}_i)$ we define an ultrametric space (U, ρ) by (2-4) and (2-5). If we trace the proof of Case 1 by replacing R_i with $d(A_i, x_\infty)$, then we easily see that a map $f : \bigcup_{i=2}^{\infty} A_i \to (U, \rho)$ defined by (2-6) is a $(2+\varepsilon)$-embedding, provided that $\delta > 0$ is small enough. $\square$

Proof of Theorem 1.2. Let $X := \{x_1, x_2, \dots\}$. Apply the proof of Theorem 1.4 by identifying each x_i with K_i. Note that the loss of the distortion in the proof only comes from (2-1), which we can ignore in the case where $A_i = x_i$. Hence the space X can be embedded into an ultrametric space with distortion $1+\varepsilon$. $\square$

Remark 2.2. After this work was completed, the author proved in [Funano 2012] that every proper ultrametric space isometrically embeds into ℓ_p for any $p \ge 1$. In particular the subset S in Theorem 1.3 also embeds into ℓ_p. Theorems 1.2 and 1.4

also hold in the case where the target metric space is ℓ_p instead of an ultrametric space. In fact, in the proof of Theorem 1.4, observe that we may assume that A_i is compact [Carleson 1967; Howroyd 1995, Corollary 7]. Since $\bigcup_{i=2}^{\infty} A_i$ is a proper subset which embeds into an ultrametric space in the case of Cases 1 and 3, we consider only Case 2. Since we have (2-7) in Case 2 we easily see that $\bigcup_{i=2}^{\infty} A_i$ embeds into ℓ_p. It was mentioned in [Funano 2012, Proposition 3.4] that an ℓ_p analogue of Theorem 1.5 also holds.

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