Pacific Journal of Mathematics

MAHLO CARDINALS AND THE TORSION PRODUCT OF PRIMARY ABELIAN GROUPS

PATRICK W. KEEF

Volume 259 No. 1 September 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 259, No. 1, 2012
dx.doi.org/10.2140/pjm.2012.259.117

MAHLO CARDINALS AND THE TORSION PRODUCT OF PRIMARY ABELIAN GROUPS

PATRICK W. KEEF

Nunke's problem asks when the torsion product of two abelian p-groups is isomorphic to a direct sum of cyclic groups. A complete solution to the problem is given using a new invariant, denoted by L_G, whose values are certain collections of finite sets of uncountable regular cardinals. This is a refinement of a previous approach to the problem that only worked up to the first cardinal that is weakly Mahlo. The multiplicative properties of L_G are then related to the generalized continuum hypothesis.

Introduction and terminology

A fundamental question of Nunke asks when the torsion product of two abelian p-groups is isomorphic to a direct sum of cyclic groups. Early work on the problem included [Hill 1983b; Keef 1988; 1990; 1991; 1993; Nunke 1964; 1967a; 1967b]. More recently, the paper [Keef 2008] presented a new approach that unfortunately had two drawbacks: it was rather complicated, and it only worked for groups whose cardinality did not exceed the first regular limit cardinal. Still more recently, in [Balof and Keef 2009] a second approach was presented. This refinement was more straightforward and also had the advantage of working up to the first weakly Mahlo cardinal, which is substantially larger than the first regular limit cardinal. (The definitions of these terms will be reviewed later.) The purpose of this paper is to give a complete solution to Nunke's problem. In so doing, we will show that the limitations of the techniques of [Balof and Keef 2009] are unavoidable.

We stress that, except where explicitly stated, the results in this paper are valid in ZFC; that is, they do not depend upon special set-theoretic assumptions such as the axiom of constructibility (V = L) or the generalized continuum hypothesis (GCH).

To begin, by the term "group" we will mean an abelian p-group, where p is a fixed prime. Our terminology and notation will generally follow [Fuchs 1970; 1973], and we will on occasion refer the reader to [Eklof and Mekler 2002] or [Jech 2003] for set-theoretic material. A group will be said to be Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. We will denote the torsion product of the groups G

MSC2010: 20K10, 20K40.
Keywords: primary abelian group, Mahlo cardinals, torsion product.

and H by the (admittedly nonstandard) $G \triangledown H$. This notation is considerably more compact than the usual $\mathrm{Tor}(G, H)$ and will make our computations significantly clearer.

To answer Nunke's problem we need to find an invariant that gives positive answers to the following two questions:

(N1) Does this invariant tell us when a group is Σ-cyclic?

(N2) Does this invariant behave well with respect to the torsion product?

Before describing two such possible invariants, which we designate by K_G and L_G, we first introduce some set-theoretic terminology.

Suppose $\mathcal{R}$ is the class of uncountable regular cardinals, and let $\mathcal{R}_f$ denote the class of finite subsets of $\mathcal{R}$. By an *$\mathcal{R}_f$-antichain* we mean a set M of finite subsets of $\mathcal{R}$ (that is, a subset of $\mathcal{R}_f$) such that whenever $S, T \in M$ and $S \subseteq T$, then $S = T$. Given an $\mathcal{R}_f$-antichain M, let $\mathcal{A}$ be the class of all $T \in \mathcal{R}_f$ such that $S \subseteq T$ for some $S \in M$. We call such a class an *$\mathcal{R}_f$-invariant* and we say M *generates* $\mathcal{A}$. Note that if $\mathcal{A}$ is an $\mathcal{R}_f$-invariant, then the set of minimal elements of $\mathcal{A}$ under the inclusion ordering is precisely M, and if $S \in \mathcal{A}$ and $S \subseteq T \in \mathcal{R}_f$, then $T \in \mathcal{A}$. In other words, $\mathcal{R}_f$-antichains and $\mathcal{R}_f$-invariants are two terms for essentially the same phenomenon.

What we have just defined as an $\mathcal{R}_f$-invariant was called an $\mathcal{R}_K$-invariant in [Balof and Keef 2009]. The reason for the difference is that we define below another $\mathcal{R}_f$-invariant, L_G, in addition to the $\mathcal{R}_f$-invariant K_G from the earlier work.

We now point out two special, and extreme, cases of the above notions. Clearly, $0_{\mathcal{R}} := \varnothing$ is the $\mathcal{R}_f$-invariant generated by $M = \varnothing$, and $1_{\mathcal{R}} := \mathcal{R}_f$ is the $\mathcal{R}_f$-invariant generated by $M = \{\varnothing\}$. Inclusion, $\subseteq$, is clearly reflexive, antisymmetric and transitive on $\mathcal{R}_f$-invariants; and under this ordering, $0_{\mathcal{R}}$ and $1_{\mathcal{R}}$ are the least and greatest $\mathcal{R}_f$-invariants, respectively.

If κ is a regular uncountable cardinal, then $C \subseteq \kappa$ is a *CUB* if it is closed and unbounded in the order topology; $W \subseteq \kappa$ is *stationary* if $W \cap C \neq \varnothing$ for all CUB subsets $C \subseteq \kappa$. Further, κ is *weakly Mahlo* if $\kappa \cap \mathcal{R} = \{\tau < \kappa : \tau \in \mathcal{R}\}$ is stationary in κ. Let $\mathcal{M}$ denote the class of all weakly Mahlo cardinals; if $\mathcal{M}$ is nonempty, let δ_m be its smallest element, and otherwise, let $\delta_m = \infty$. A regular cardinal κ is *Mahlo* if it is weakly Mahlo and strongly inaccessible (that is, $\gamma < \kappa$ implies $2^{|\gamma|} < \kappa$). Clearly, in the context of the generalized continuum hypothesis (GCH), every weakly Mahlo cardinal is, in fact, Mahlo.

Consider first the degree to which K_G answered question (N1).

Theorem 0.1 [Balof and Keef 2009, Theorems 3(b) and 6]. *Suppose G is a group.*

(a) *If G is Σ-cyclic, then $K_G = 0_{\mathcal{R}}$.*

(b) *If $K_G = 0_{\mathcal{R}}$ and the final rank of G is strictly less than δ_m, then G is Σ-cyclic.*

So, at least for groups of cardinality less than δ_m, K_G can be used to characterize when they are direct sums of cyclics. One of our goals is to show that the cardinality restriction in Theorem 0.1 is truly necessary. In our first main result, Theorem 1.2, we show that in the context of the constructible universe (V = L), if there is a (weakly) Mahlo cardinal, then there is, in fact, a group G such that $K_G = 0_{\mathcal{R}}$ that is *not* Σ-cyclic.

On the other hand, our new $\mathcal{R}_f$-invariant L_G will be much better at answering question (N1). In Theorem 1.6, we show that for *any* group of *whatever* cardinality, G is Σ-cyclic if and only if $L_G = 0_{\mathcal{R}}$. So, when it comes to the first of the two questions, L_G is a definite improvement over K_G.

We now consider the second question, (N2). If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{R}_f$-invariants, define

$$\mathcal{A} \cdot \mathcal{B} = \{\, S \cup T : S \in \mathcal{A},\, T \in \mathcal{B},\, S \cap T = \varnothing \,\}.$$

Clearly, this very natural product is associative and commutative, $0_{\mathcal{R}} \cdot \mathcal{A} = 0_{\mathcal{R}}$ and $1_{\mathcal{R}} \cdot \mathcal{A} = \mathcal{A}$. The following result states that K_G answered (N2) perfectly.

Theorem 0.2 [Balof and Keef 2009, Theorem 4]. *If G and H are any two groups, then*

$$K_{G \triangledown H} = K_G \cdot K_H.$$

The obvious question is whether $L_{G \triangledown H}$ also always agrees with $L_G \cdot L_H$. We *do* verify that this is the case in the situation of relevance to Nunke's problem. In Theorem 2.2 we show that $G \triangledown H$ is Σ-cyclic if and only if $L_{G \triangledown H} = 0_{\mathcal{R}}$ if and only if $L_G \cdot L_H = 0_{\mathcal{R}}$. Again, this is a theorem in ZFC; it means that $G \triangledown H$ is Σ-cyclic if and only if for every $S \in L_G$ and $T \in L_H$, the intersection $S \cap T$ is nonempty. In particular, in some sense, the invariant L_G is sufficient to "solve" Nunke's problem. However, as will be seen, calculating L_G, even for some very familiar groups, quickly involves undecidable questions of cardinal arithmetic.

Next, we show that the full statement that $L_{G \triangledown H} = L_G \cdot L_H$ is true for all G and H cannot be decided in ZFC. This statement does follow from GCH (Corollary 2.4). However, if it is consistent with ZFC that there is a weakly Mahlo cardinal, then it is consistent with ZFC that there are groups G and H for which it fails (Corollary 3.5). Surprisingly, this example is simply where G and H are copies of the standard torsion-complete group $\bar{B}$, where $B = \bigoplus_{n<\omega} \mathbb{Z}_{p^n}$. This dramatically illustrates the point that computing K_G or L_G, even for well-known groups such as $\bar{B}$, can depend upon questions of set theory and cardinal arithmetic.

Although the solution to Nunke's problem does necessitate going from the invariant K_G to the new invariant L_G, it is not true that L_G is in all ways superior to K_G. We have already noted that it is undecidable if Theorem 0.2 can be generalized to L_G. In addition, by [Balof and Keef 2009, Theorem 3(a)], K_G behaves very nicely with respect to direct sums. In Theorem 3.6 we show that the same holds for L_G if and only if there are no weakly Mahlo cardinals.

We close the paper by noting a few of the results from [Balof and Keef 2009] that we can generalize to groups of arbitrary cardinality using L_G. For example, in Theorem 3.8 we show that a group G is a "$\triangledown$-zero divisor" (that is, there is a group H which is not Σ-cyclic such that $G \triangledown H$ is Σ-cyclic) if and only if it is "$\triangledown$-nilpotent" (that is, for some positive integer n, $G^n = G \triangledown G \triangledown \cdots \triangledown G$ is Σ-cyclic); in [Balof and Keef 2009] we were only able to verify this for groups of cardinality less than δ_m.

1. The first question: K_G, L_G and Σ-cyclic groups

We begin with a few elementary definitions regarding elements of $\mathcal{R}_f$. If $T \in \mathcal{R}_f$, let $\nu(T)$ be the number of elements in T, so $\nu(T)$ is a nonnegative integer. In addition, let $\mu(\varnothing) = \aleph_0$, and if T is nonempty, let $\mu(T)$ be its largest element; further, let $T' = T - \{\mu(T)\}$.

We now review the definition of K_G given in [Balof and Keef 2009]. The following formulation is clearly equivalent to that given in the earlier work, but it will be more convenient for our purposes, especially when we want to introduce L_G. Given $T \in \mathcal{R}_f$, we answer the question of whether to place it in K_G by a traditional induction on $\nu(T)$. First, the base case:

(K0) If $\nu(T) = 0$ (that is, $T = \varnothing$), then $T \in K_G$ if and only if $p^\omega G \neq \{0\}$.

Next, suppose n is a positive integer and for all groups H and all $S \in \mathcal{R}_f$ with $\nu(S) < n$ we have defined when $S \in K_H$. Let $T \in \mathcal{R}_f$ have n elements; note that since $\nu(T') = n - 1$, for all groups H we have already answered the question of when $T' \in K_H$. We now let $T \in K_G$ if and only if either

(K1) $T' \in K_G$, or

(K2) G has a subgroup A of cardinality $\kappa := \mu(T) \in \mathcal{R}$ with a filtration $\mathcal{A} = \{A_i\}_{i<\kappa}$ such that
$$\Gamma_{T'}(A) := \{\, i < \kappa : T' \in K_{A/A_i} \,\}$$
is stationary in κ.

Recall that to say $\mathcal{A} = \{A_i\}_{i<\kappa}$ is a filtration of A means that it is a smoothly ascending chain of subgroups, its union is all of A and each A_i has cardinality less than κ. In (K2) we are only concerned with whether $\Gamma_{T'}(A)$ is stationary in κ. If $\mathcal{A}'$ is another filtration of A, it follows that $\mathcal{A}$ and $\mathcal{A}'$ will agree on a CUB subset of κ so that the property that $\Gamma_{T'}(A)$ is stationary does not depend upon which filtration is chosen. As a result, we will often, without extensive comment, replace one filtration by another, e.g., one composed of pure subgroups.

We now review a useful realization theorem from [Balof and Keef 2009]. If κ is a cardinal, then a group G is said to be *κ-Σ-cyclic* if all its subgroups of cardinality strictly less than κ are Σ-cyclic. If $T \in \mathcal{R}_f$, then the $\mathcal{R}_f$-invariant generated by the antichain $\{T\}$ is called *T-principal*.

Lemma 1.1 [Balof and Keef 2009, Lemma 9]. (V = L) *Suppose that $T \in \mathcal{R}_f$, $T \cap \mathcal{M} = \varnothing$ and $\kappa = \mu(T)$. Assuming the axiom of constructibility, there is a κ-Σ-cyclic group G of cardinality κ such that K_G is T-principal.*

The following shows that K_G is inherently limited with respect to answering question (N1):

Theorem 1.2. *Assuming the axiom of constructibility, there is a group G which is not Σ-cyclic such that $K_G = 0_{\mathcal{R}}$ if and only if there exists a weakly Mahlo cardinal.*

Proof. Certainly, if there are no weakly Mahlo cardinals, then $\delta_m = \infty$ and the result follows from Theorem 0.1. (Note that this direction does not use V = L.) Suppose, then, that $\delta_m < \infty$. Let $\mathcal{R}_m = \mathcal{R} \cap \delta_m$, so $\mathcal{R}_m$ is stationary in δ_m. If $\kappa \in \mathcal{R}_m$, let $T \in \mathcal{R}_f$ be chosen so that $T \subseteq \mathcal{R} - \mathcal{M}$ and $\mu(T) = \kappa$ (e.g., $T = \{\kappa\}$) and G_κ be a group defined as in Lemma 1.1 so that K_{G_κ} is T-principal. In particular, since $K_{G_\kappa} \neq 0_{\mathcal{R}}$, G_κ will not be Σ-cyclic.

We define a chain of Σ-cyclic groups $\{A_i\}_{i<\delta_m}$ satisfying the following:

(0) For all $n < \omega$, $f_{A_i}(n)$, the n-th Ulm–Kaplansky invariant of A_i, is $|i| \cdot \aleph_0$.

(1) If $i < j$, then A_i is a pure subgroup of A_j.

(2) If j is a limit, then $\bigcup_{i<j} A_i = A_j$ (that is, the chain is smoothly ascending).

(3) If $i < j$ and $i \notin \mathcal{R}_m$, then A_i is a summand of A_j.

(4) If $i \in \mathcal{R}_m$, then $A_{i+1}/A_i \cong G_i$.

Let $A_0 = \{0\}$. Suppose A_i has been defined for all $i < j$; we then describe how to construct A_j. If j is a limit, then (2) forces the definition of A_j; note that (0), (1) and (4) follow easily. To verify that (3) continues to hold when $i \notin \mathcal{R}_m$, suppose first that $j \in \mathcal{R}_m$. Since $j < \delta_m$, it is not weakly Mahlo, so there is a CUB subset $C \subseteq j$ such that $C \cap \mathcal{R}_m = \varnothing$. Let $\{\alpha_l\}_{l<j}$ be an increasing enumeration of C; there is clearly no loss of generality in assuming that $\alpha_0 = i$. Since by induction on j, for all $l < j$, A_{α_l} is a summand of $A_{\alpha_{l+1}}$, it follows that

$$A_j \cong A_i \oplus \Big(\bigoplus_{l<j} A_{\alpha_{l+1}}/A_{\alpha_l}\Big).$$

On the other hand, if $j \notin \mathcal{R}_m$, then let λ be the cofinality of j. Define a closed and unbounded subset $C' = \{\beta_l\}_{l<\lambda} \subseteq j$ starting with $\beta_0 = i$, $\beta_1 > \lambda$, such that if $l > 0$ is isolated, then so is β_l. We again claim that $C' \cap \mathcal{R}_m = \varnothing$. Clearly, if l is isolated, then so is β_l, so that $\beta_l \notin \mathcal{R}_m$. And if l is a limit, then $\mathrm{cf}(\beta_l) \le l < \lambda < \beta_l$

so that again, $\beta_l \notin \mathcal{R}_m$. It again follows that $A_j \cong A_i \oplus \left(\bigoplus_{l<\lambda} A_{\beta_{l+1}}/A_{\beta_l}\right)$. (The last two paragraphs are usually summarized by saying $\mathcal{R}_m$ is a *nonreflecting* stationary subset of κ.)

Suppose next that $j = l+1$ is isolated. If $l \notin \mathcal{R}_m$, we merely let $A_j = A_l \oplus \left(\bigoplus_{m<\omega} \mathbb{Z}_{p^m}\right)$. Clearly, (0) and (1) hold, (2) and (4) say nothing new and (3) holds for j because it holds for l.

Finally, suppose $j = l+1$ and $l \in \mathcal{R}_m$. In this case, we let A_j be a Σ-cyclic group containing A_l as a pure subgroup such that $A_j/A_l \cong G_l$; such a group can easily be constructed from a pure-projective resolution of G_l. Clearly (4) holds for l, (1) follows from the transitivity of purity and (2) does not involve any new conditions. Since G_l has cardinality l and for all $n < \omega$ we have $f_{A_j}(n) = f_{A_l}(n) + f_{G_l}(n)$, it follows that (0) holds for A_j.

With regards to (3), assume $i \notin \mathcal{R}_m$ and $i < j$; in particular, $i < l$. Note that $\{A_i\}_{i<l}$ is a filtration of A_l. Similarly, there is a filtration $\{S_i\}_{i<l}$ of A_j consisting of summands. By a standard "back-and-forth" argument, there is an ordinal i' such that $i < i' < l$ and $A_{i'} = S_{i'} \cap A_l$. Note that $A_{i'}$ will be pure in A_j so that it is also pure in $S_{i'}$. In addition, $S_{i'}/A_{i'}$ has cardinality less than l, and it maps injectively into $A_j/A_l \cong G_l$. Since G_l is l-Σ-cyclic, it follows that $S_{i'}/A_{i'}$ is also Σ-cyclic. Therefore, $A_{i'}$ will be a summand of $S_{i'}$. Since A_i is a summand of $A_{i'}$ by induction on (3) and $S_{i'}$ is a summand of A_j by construction, A_i will be a summand of A_j, as required.

Let $G = \bigcup_{i<\delta_m} A_i$. We first claim that G is not Σ-cyclic. To verify this, recall that $\{A_i\}_{i<\delta_m}$ is a filtration of G and $\mathcal{R}_m$ is a stationary subset of δ_m. For every $i \in \mathcal{R}_m$, the quotient G/A_i contains a subgroup, $A_{i+1}/A_i \cong G_i$, which is not Σ-cyclic. It follows that G/A_i also fails to be Σ-cyclic. This means that G cannot be Σ-cyclic since that would imply that it has a filtration consisting of subgroups (that is, summands) such that these quotients are all Σ-cyclic.

We now need to verify that $K_G = 0_{\mathcal{R}}$. If this fails, then let T be a minimal subset of K_G. Because G clearly has no nonzero element of infinite height, we can conclude that T is nonempty. Let $\kappa = \mu(T)$; by the minimality of T, (K2) and not (K1) must pertain. So there is a subgroup of A of G of cardinality κ such that $\Gamma_{T'}(A)$ is stationary; it follows that $\kappa \le \delta_m$.

Suppose first that $\kappa < \delta_m$. Since $\{A_i\}_{i<\delta_m}$ is a filtration of G and δ_m is regular, we can conclude that $A \subseteq A_i$ for some $i < \delta_m$. However, A_i is Σ-cyclic, so $K_{A_i} = 0_{\mathcal{R}} = \varnothing$. This contradicts the fact that $T \in K_A \subseteq K_{A_i}$.

We may therefore assume that $\kappa = \delta_m$. It follows that there is a stationary subset $W \subseteq \delta_m$ such that $T' \in K_{G/A_i}$ for all $i \in W$ (see [Balof and Keef 2009, Lemma 2(b)], for example). Let $\lambda = \mu(T') < \delta_m$, and let $i \in W$ be chosen so that $\lambda < i < \delta_m$. It follows that $T' \in K_X$ for some subgroup $X \subseteq G/A_i$ such that $|X| = \lambda$. We will show, however, that this X must be Σ-cyclic so that $K_X = \varnothing$.

Since δ_m is regular, we can conclude that $X \subseteq A_{i'}/A_i$ for some $i < i' < \delta_m$. Since $i+1 \notin \mathcal{R}$, by (3) we have that A_{i+1} is a summand of $A_{i'}$ so that

$$A_{i'}/A_i \cong (A_{i'}/A_{i+1}) \oplus (A_{i+1}/A_i).$$

Clearly, $X \subseteq X_1 \oplus X_2$, where $X_1 \subseteq A_{i'}/A_{i+1}$, $X_2 \subseteq A_{i+1}/A_i$ and $|X_1|, |X_2| \leq \lambda$. Since $A_{i'}/A_{i+1}$ is Σ-cyclic, it follows that X_1 is as well. In addition, A_{i+1}/A_i is either Σ-cyclic (if $i \notin \mathcal{R}_m$) or i-Σ-cyclic (if $i \in \mathcal{R}_m$). Since $|X_2| \leq \lambda < i$, we can conclude that X_2 is also Σ-cyclic. This shows that X must be Σ-cyclic and completes the argument. □

Again, the last result illustrates that in the presence of Mahlo cardinals, the invariant K_G is insufficiently robust to answer question (N1) for arbitrarily large groups. To address this, we amend it significantly, concentrating on how it behaves at weakly Mahlo cardinals. We begin with some notation. If $T \in \mathcal{R}_f$ and $i < \mu(T)$ is an ordinal, let T_i be $T_i = T' \cup \{i\}$ whenever i is an uncountable regular cardinal, and otherwise, let $T_i = T'$, so $\mu(T_i) = (i \text{ or } \mu(T')) < \mu(T)$.

Given a group G, we define $L_G \subseteq \mathcal{R}_f$ not, as in the case of K_G, by traditional induction on $\nu(T)$ but rather by *transfinite* induction on $\mu(T)$. Suppose $T \in \mathcal{R}_f$ and we want to decide if $T \in L_G$. We begin with the same base case.

(L0) If $\mu(T) = \aleph_0$ (that is, $T = \varnothing$), we again let $T \in L_G$ if and only if $p^\omega G \neq \{0\}$.

Next, suppose for all groups H we have defined all the elements $S \in L_H$ such that $\mu(S) < \kappa := \mu(T)$. In order to define when $T \in L_G$, we first observe that for any group H we have defined when $T_i \in L_H$ for any $i < \kappa$. Replacing condition (K1), we say

(L1) $T \in L_G$ when

$$\Upsilon_T(G) := \{\, i < \kappa : T_i \in L_G \,\}$$

is stationary in κ.

Replacing condition (K2), we also say

(L2) $T \in L_G$ when G has a subgroup A of cardinality κ with a filtration $\mathcal{A} = \{A_i\}_{i<\kappa}$ such that

$$\Lambda_T(A) := \{\, i < \kappa : T_i \in L_{A/A_i} \,\}$$

is stationary in κ.

As in the definition of K_G, in this definition we are only concerned with whether $\Upsilon_T(G)$ and $\Lambda_T(A)$ are stationary in κ, and this property does not depend upon which particular filtration is chosen. Similarly, when we write something like $\Lambda_T(A) \subseteq \Lambda_T(G)$, we mean that $\Lambda_T(G) - \Lambda_T(A)$ is *not* stationary in κ, which again does not depend upon exactly which filtrations are used for A and G.

Since $T' \subseteq T_i$ for all $T \in \mathcal{R}_f$ and ordinals $i < \mu(T)$, an easy induction shows that $K_G \subseteq L_G$ for all groups G. On the other hand, suppose that $\kappa \in \mathcal{R}$ is not weakly Mahlo. It follows that there is a CUB subset $C \subseteq \kappa$ such that $C \cap \mathcal{R} = \varnothing$. If $\kappa = \mu(T)$, then for all $i \in C$ we have $T_i = T'$. This means that, away from weakly Mahlo cardinals, the two definitions agree. We state this more formally, as follows:

Proposition 1.3. *If G is a group, $T \in \mathcal{R}_f$ and $T \cap \mathcal{M} = \varnothing$, then $T \in L_G$ if and only if $T \in K_G$.*

By [Balof and Keef 2009, Theorem 10], whose proof uses the above Lemma 1.1, and the last result, assuming the axiom of constructibility (V = L), if $\mathcal{A}$ is an $\mathcal{R}_f$-invariant whose minimal sets contain no weakly Mahlo cardinals, then there is a group G such that $L_G = K_G = \mathcal{A}$. The proof of this uses stationary sets that are nonreflecting. In particular, such sets are plentiful in the constructible universe if one steers clear of weakly Mahlo cardinals (see [Eklof and Mekler 2002, Theorem VI, 3.13], for example). It is not clear what such a realization result would look like outside of V = L and in the presence of weakly Mahlo cardinals.

It is consistent with ZFC that there are no weakly Mahlo cardinals (in fact, it is consistent that there are no regular limit cardinals at all). In any such model, Proposition 1.3 says that $K_G = L_G$ for all groups G. We pause for some straightforward observations that parallel facts about K_G from [Balof and Keef 2009].

Lemma 1.4. *Suppose G and H are groups and $S, T \in \mathcal{R}_f$.*

(a) *If $S \in L_G$ and $S \subseteq T$, then $T \in L_G$.*

(b) *$L_G = 1_{\mathcal{R}}$ if and only if G has elements of infinite height.*

(c) *If G is a subgroup of H, then $L_G \subseteq L_H$.*

(d) *If G is a subgroup of H and $|G| = |H| = \mu(T)$, then $\Lambda_T(G) \subseteq \Lambda_T(H)$.*

(e) *$L_{G\oplus H} = L_G \cup L_H$.*

(f) *If $T \in L_G$, then there is subgroup $A \subseteq G$ such that $|A| \leq \mu(T)$ and $T \in L_A$.*

(g) *If $T \in L_G$ is minimal under inclusion, then $\mu(T) \leq |G|$.*

Proof. Consider (a); we prove this by induction on $\kappa := \mu(T)$.

If $\kappa = \aleph_0$, then $T = \varnothing$, and so therefore $T = S \in L_G$. Suppose now that $\kappa > \aleph_0$, $S \subseteq T$ and $S \in L_G$, so $T \neq \varnothing$. Suppose first that $\kappa \notin S$. If $i < \kappa$, then $S \subseteq T_i$, so by induction, for all such i, we have $T_i \in L_G$. In particular, $\Upsilon_T(G)$ is stationary, and (L1) is satisfied.

Suppose next that $\kappa \in S$. It follows that for all $i < \kappa$, $S_i \subseteq T_i$. Therefore, by induction on κ, $\Upsilon_S(G) \subseteq \Upsilon_T(G)$. Also, if A is a subgroup of G of cardinality κ, then $\Lambda_S(A) \subseteq \Lambda_T(A)$. For S, one of these two sets must be stationary in κ, so the same must hold for T.

Now (b) follows from (a) since $L_G = 1_{\mathcal{R}}$ if and only if $\varnothing \in L_G$ if and only if $p^\omega G \neq \{0\}$.

Turning to (c), we again show that if $T \in L_G$, then $T \in L_H$ by induction on $\mu(T)$. If $\mu(T) = \aleph_0$, then $T = \varnothing$. This means G has elements of infinite height, so the same holds for H, and $T \in L_H$, as required. Suppose now that this holds for all sets $S \in \mathcal{R}_f$ with $\mu(S) < \kappa := \mu(T)$. By induction on κ, $\Upsilon_T(G) \subseteq \Upsilon_T(H)$, so if (L1) holds for G, it holds for H. On the other hand, if A is a subgroup of G of cardinality κ such that $\Lambda_T(A)$ is stationary in κ, then A is equally a subgroup of H, so again $T \in L_H$.

Next, in (d), if $\{B_i\}_{i<\kappa}$ is a filtration of H, then letting $A_i = G \cap B_i$ gives a filtration of G. For all $i < \kappa$, G/A_i embeds in H/B_i, and the result follows from (c).

Turning to (e), the containment $\supseteq$ follows from (c). We therefore need to show that every $T \in L_{G\oplus H}$ is in $L_G \cup L_H$, which we again do by our usual induction on $\kappa := \mu(T)$. If $\kappa = \aleph_0$, then $T = \varnothing$, and $G \oplus H$ has elements of infinite height. So either G or H has elements of infinite height; that is, $T \in L_G$ or $T \in L_H$.

Suppose now that this holds for all sets $S \in \mathcal{R}_f$ with $\mu(S) < \kappa := \mu(T)$. By induction on κ, $\Upsilon_T(G\oplus H) = \Upsilon_T(G) \cup \Upsilon_T(H)$. Therefore, if (L1) holds for $G \oplus H$, one of the latter two sets is stationary, and appealing again to (L1) gives the result.

If (L2) pertains, then there is a subgroup $C \subseteq G \oplus H$ of cardinality κ such that $\Lambda_T(C)$ is stationary in κ. Note that C is contained in a subgroup of the form $C' = A \oplus B$ for some subgroups $A \subseteq G$ and $B \subseteq H$, where $|C'| = \kappa$. By (d), $\Lambda_T(C')$ will also be stationary. Replacing C by C', there is no loss of generality in assuming that $C = A \oplus B$.

If $|A| = \kappa$, let $\{A_i\}_{i<\kappa}$ be a filtration of A, and otherwise, let each $A_i = A$; define $\{B_i\}_{i<\kappa}$ similarly. Clearly, $\{A_i \oplus B_i\}_{i<\kappa}$ is a filtration of C. If $|A| < \kappa$, then eventually $C/C_i \cong B/B_i$ so that $\Lambda_T(B)$ is stationary. This shows $T \in L_H$, which completes the proof. Similarly reasoning applies if $|B| < \kappa$, so assume $|A| = |B| = \kappa$.

By induction on κ, we can conclude that $\Lambda_T(C) = \Lambda_T(A) \cup \Lambda_T(B)$. It follows that if (L2) holds for C, then it holds for either A or B, proving (e).

For (f), if $\mu(T) = \aleph_0$, then G has elements of infinite height. So G has a countable subgroup with elements of infinite height, which is what is being asserted.

Next, if $W := \Upsilon_T(G)$ is stationary in κ, then by induction, for all $i \in W$ there is a subgroup A_i of cardinality at most $|i|$ such that $T_i \in L_{A_i}$. We need only let $A = \langle A_i : i \in W \rangle$. On the other hand, if (L2) holds, then we need only let A be the subgroup mentioned there.

Finally, (g) is equivalent to the statement that if $T \in L_G$ and $|G| < \mu(T)$, then T is not minimal in L_G. Observe first that if $\mu(T) = \aleph_0$, then G is finite. In particular, G has no elements of infinite height so that $T \notin L_G$. Next, for the induction step, (L2) is clearly prohibited by $|G| < \mu(T)$. Therefore, (L1) must hold. Choose some $i \in \Upsilon_T(G)$ such that $|G| < i < \mu(T)$. By induction, there is a $T_0 \subseteq T_i$ such that $T_0 \in L_G$ and $\mu(T_0) \le |G|$. But this second condition implies that $T_0 \subseteq T' \subseteq T$, contradicting the minimality of T. □

Part (e) of Lemma 1.4 can be generalized to much larger direct sums. However, we will show later that it does not generalize to arbitrary direct sums, showing again that, in some respects at least, the invariant K_G is better behaved than the invariant L_G.

The following minor variation on Fodor's lemma (see [Jech 2003, Theorem 8.7], for example) will be crucial:

Lemma 1.5. *Suppose $\kappa \in \mathfrak{R}$ and $W \subseteq \kappa$ is a stationary subset. If $f : W \to \mathfrak{R}_f$ is a function such that $f(i) \subseteq i$ for all $i \in W$, then there is a stationary subset $W' \subseteq W$ such that $f(i) = f(j)$ for all $i, j \in W'$.*

Proof. If $U_0 = \{ i \in W : f(i) = \varnothing \}$ is stationary in κ, then we are clearly done. If not, let $V_0 = W - U_0$, so V_0 is also stationary. For every $i \in V_0$, let $\phi_1(i) = \mu(f(i)) \in \mathfrak{R}$. It follows that ϕ_1 is a regressive function, so by Fodor's lemma, there is an $\alpha_1 \in \kappa$ and a stationary subset $W_1 \subseteq V_0 \subseteq W$ such that $\phi_1(i) = \alpha_1$ for all $i \in W_1$.

We start the process over again. If $U_1 = \{ i \in W_1 : f(i) = \{\alpha_1\} \}$ is stationary in κ, then we are clearly done. Otherwise, let $V_1 = W_1 - U_1$, so V_1 is also stationary. For every $i \in V_1$, let $\phi_2(i) = \mu(f(i) - \{\alpha_1\}) \in \mathfrak{R}$. It follows that ϕ_2 is a regressive function, so by Fodor's lemma, there is an $\alpha_2 \in \kappa$ and a stationary subset $W_2 \subseteq V_1 \subseteq W_1$ such that $\phi_2(i) = \alpha_2$ for all $i \in W_2$. Clearly, $\alpha_1 > \alpha_2$.

Once again, if $U_2 = \{ i \in W_2 : f(i) = \{\alpha_1, \alpha_2\} \}$ is stationary in κ, then we are clearly done. Continuing in this way, we keep constructing stationary subsets $W \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq W_k$ and ordinals $\alpha_1 > \alpha_2 > \cdots > \alpha_k$. Since this sequence of ordinals cannot continue indefinitely, at some point we must have constructed the desired W'. □

Using an obvious extension of the usual terminology, we will call a function f as in Lemma 1.5 regressive. We have now arrived at one of our main results. Again, observe that it is valid for groups of arbitrarily large cardinality.

Theorem 1.6. *A group G is Σ-cyclic if and only if $L_G = 0_{\mathfrak{R}}$.*

Proof. Suppose first that G is Σ-cyclic. We again show by induction on $\kappa := \mu(T)$ that $T \notin L_G$. If $T = \varnothing$, then $T \notin L_G$ since G has no elements of infinite height. Suppose $S \notin L_G$ whenever $\mu(S) < \mu(T)$. Since $\mu(T_i) < \kappa$ for every $i < \kappa$, by induction we can conclude $\Upsilon_T(G) = \varnothing$ so that (L1) does not hold and $T \notin L_G$.

Next considering (L2), let A be any subgroup of G of cardinality κ and let $\{A_i\}_{i<\kappa}$ be a filtration of A. Since A will also be Σ-cyclic, we may assume that each A_i is a summand of A. It follows by induction that for all $i < \kappa$, $T_i \notin L_{A/A_i}$; that is, $\Lambda_T(A) = \varnothing$, and this implies that $T \notin L_G$, as required.

For the converse, we show by induction on $\kappa := |G|$ that if $L_G = 0_{\mathfrak{R}}$, then G is Σ-cyclic. If $\kappa = \aleph_0$, then G will be a countable group without elements of infinite height (since $\varnothing \notin L_G$). However, a countable group is Σ-cyclic if and only if it has no nonzero elements of infinite height, so the result holds in this case (see [Fuchs 1970, Theorem 17.3], for example). Next, assume the result holds for all groups of cardinality less than $\kappa > \aleph_0$ and that $L_G = 0_{\mathcal{R}}$. If A is a subgroup of G with $|A| < \kappa$, then by Lemma 1.4(c) we can conclude that $L_A = 0_{\mathcal{R}}$. By induction, then, G is κ-Σ-cyclic.

Suppose first that κ is singular. In this case, a variation of Shelah's "Singular Compactness Theorem" (see [Keef 1990, Lemma 3.1], for example) implies that G is Σ-cyclic, as required.

Suppose now that κ is regular. Let $\{A_i\}_{i<\kappa}$ be a filtration of G consisting of pure subgroups; there is clearly no loss of generality in assuming that $|A_i| = \aleph_0 \cdot |i|$ for all $i < \kappa$.

We claim that

$$C := \{\, i < \kappa : G/A_i \text{ is } \kappa\text{-}\Sigma\text{-cyclic} \,\}$$

contains a CUB subset of κ. If we can establish this, and we enumerate this subset by $\{\beta_j\}_{j<\kappa}$, then it follows that

$$G \cong A_0 \oplus \left(\bigoplus_{j<\kappa} A_{\beta_{j+1}}/A_{\beta_j} \right),$$

and as all the terms in this decomposition are Σ-cyclic, the result follows.

Observe that if C does not contain a CUB subset of κ, then

$$W := \kappa - C = \{\, i < \kappa : G/A_i \text{ is not } \kappa\text{-}\Sigma\text{-cyclic} \,\}$$

must be stationary in κ. We assume this holds and derive a contradiction. There is clearly no loss of generality in assuming that every element of W is infinite so that for all $i \in W$, $|A_i| = |i|$.

If $i \in W$, then since G/A_i is not κ-Σ-cyclic, we can find a subgroup $X \subseteq G/A_i$ with $|X| < \kappa$ that is not Σ-cyclic. Let Y be the subgroup of G defined by the equation $Y/A_i = X$. Since G is κ-Σ-cyclic and $|Y| < \kappa$, it follows that Y is Σ-cyclic. Note that A_i is contained in a summand Z of Y such that $|Z| = |A_i| = |i|$. This means that $X = Y/A_i \cong (Y/Z) \oplus (Z/A_i)$. Observe that Y/Z is Σ-cyclic so that Z/A_i fails to be Σ-cyclic. Since $|Z/A_i| \le i < \kappa$, by induction, $L_{Z/A_i} \ne 0_{\mathcal{R}}$. Therefore, if T^i is some minimal element in L_{Z/A_i}, then by Lemma 1.4(g), we have $\mu(T^i) \le |i|$.

Define $f : W \to \mathcal{R}_f$ by $f(i) = T^i - \{i\}$, so $f(i) = T^i$ unless i is actually a regular cardinal that is the largest element of T^i, in which case $f(i) = (T^i)'$. By Lemma 1.5, there is a stationary subset $W' \subseteq W$ and $U \in \mathcal{R}_f$ such that $f(i) = U$ for all $i \in W'$. The result then follows from the next statement, which contradicts the assumption that $L_G = 0_{\mathcal{R}}$.

Claim. $V := U \cup \{\kappa\} \in L_G$. Indeed, let $i \in W'$. If $i \notin \mathcal{R}$, then $V_i = U = T^i \in L_{G/A_i}$. And if $i \in \mathcal{R}$, then $V_i = U \cup \{i\} = T^i \cup \{i\} \in L_{G/A_i}$. Therefore, $W' \subseteq \Lambda_V(G)$ so that $V \in L_G$, completing the proof. □

The example constructed in the proof of Theorem 1.2 (assuming V = L and the existence of Mahlo cardinals) has $K_G = 0_{\mathcal{R}}$, but since it is not Σ-cyclic, we have $L_G \neq 0_{\mathcal{R}}$. In particular, it is not the case that in all conceivable models of set theory, $K_G = L_G$ for all groups G.

2. The second question: Does $L_{G \triangledown H} = L_G \cdot L_H$?

In this section we investigate the question of whether $L_{G \triangledown H}$ always agrees with $L_G \cdot L_H$. One containment is straightforward.

Theorem 2.1. *If G and H are groups, then $L_G \cdot L_H \subseteq L_{G \triangledown H}$.*

Proof. As usual, if $T \in L_G \cdot L_H$, we show $T \in L_{G \triangledown H}$ by induction on $\kappa = \mu(T)$. Clearly, if $\kappa = \aleph_0$, then $T = \varnothing$; however, this implies that $\varnothing = \varnothing \cup \varnothing$ is in both L_G and L_H. It then follows that both G and H have elements of infinite height. This, in turn, implies that $G \triangledown H$ also has elements of infinite height (see [Fuchs 1970, 62.4]) and $T = \varnothing \in L_{G \triangledown H}$. So assume the result holds for all groups G and H and all $S \in \mathcal{R}_f$ such that $\mu(S) < \kappa = \mu(T)$.

By definition, T is the disjoint union of $U \in L_G$ and $V \in L_H$. Without loss of generality, assume $\kappa \in U$; let $\gamma = \mu(V) < \kappa$.

Consider the reason why $U \in L_G$. Since $\kappa \in U$, it is nonempty, so (L0) does not apply. Next, suppose (L1) holds so that $\Upsilon_U(G)$ is stationary in κ. Since whenever $\gamma < i < \kappa$, T_i is the disjoint union $U_i \cup V$, it follows by transfinite induction that

$$\Upsilon_U(G) \cap (\kappa - \gamma) \subseteq \Upsilon_T(G \triangledown H).$$

This shows that $T \in L_{G \triangledown H}$, as required.

Now suppose that B is a subgroup of G of cardinality κ such that $\Lambda_U(B)$ is stationary; let $\mathcal{B} = \{B_i\}_{i<\kappa}$ be a pure filtration of B. By Lemma 1.4(f) there is a subgroup C of H such that $|C| \leq \gamma < \kappa$ and $V \in L_C$. Note that $\{B_i \triangledown C\}_{i<\kappa}$ is a filtration of $B \triangledown C$ and for all $i < \kappa$ we have a pure exact sequence

$$0 \to B_i \triangledown C \to B \triangledown C \to (B/B_i) \triangledown C \to 0$$

(see [Fuchs 1970, 63.2], for example). By transfinite induction, for all $\gamma < i < \kappa$, if $i \in \Lambda_U(B)$, then $T_i = U_i \cup V \in L_{B/B_i} \cdot L_C$ implies

$$T_i \in L_{[(B/B_i) \triangledown C]} = L_{[(B \triangledown C)/(B_i \triangledown C)]}.$$

However, this means that $T \in L_{B \triangledown C} \subseteq L_{G \triangledown H}$, as required. □

This brings us to our next major result.

Theorem 2.2. *If G and H are groups, then $G \triangledown H$ is Σ-cyclic if and only if $L_{G \triangledown H} = 0_{\mathfrak{R}}$ if and only if $L_G \cdot L_H = 0_{\mathfrak{R}}$.*

Proof. The first equivalence is an immediate consequence of Theorem 1.6. For the second, note first that if $G \triangledown H$ is Σ-cyclic, then $L_G \cdot L_H \subseteq L_{G \triangledown H} = 0_{\mathfrak{R}}$ by Theorem 2.1 so that $L_G \cdot L_H = 0_{\mathfrak{R}}$. For the converse, unlike the case of K_G, we do not actually show that $L_{G \triangledown H} \subseteq L_G \cdot L_H$ for all groups G and H. We do, however, prove the following, which shows that if $L_G \cdot L_H$ is empty, then so is $L_{G \triangledown H}$:

Claim. If $T \in L_{G \triangledown H}$, then there is an $S \in L_G \cdot L_H$ such that $\mu(S) \leq \mu(T)$.

Once again, we prove this by a transfinite induction on $\mu(T) = \kappa$. Note first that if $\kappa = \aleph_0$, then $G \triangledown H$ has nonzero elements of infinite height. Therefore, both G and H also have such elements. We can then let $S = \varnothing = \varnothing \cup \varnothing \in L_G \cdot L_H$.

So assume the result holds for all groups G and H and all finite sets of regular cardinals R with $\mu(R) < \kappa = \mu(T)$.

Suppose first that $\Upsilon_T(G \triangledown H)$ is stationary in κ. If i is in this set, then $\mu(T_i) < \kappa$ so that by induction there is an $S \in L_G \cdot L_H$ such that $\mu(S) \leq \mu(T_i) < \kappa$.

Next, suppose there is a subgroup A of $G \triangledown H$ of cardinality κ such that $\Lambda_T(A)$ is stationary. After possibly expanding A a bit, we may assume $A = B \triangledown C$, where B and C are subgroups of G and H, respectively, and $\max(|B|, |C|) = \kappa$. We will be done if we can find an $S \in L_B \cdot L_C \subseteq L_G \cdot L_H$ with $\mu(S) \leq \kappa$.

Define $\{B_i\}_{i<\kappa}$ as follows: if $|B| < \kappa$, let each $B_i = B$; otherwise, let it be a filtration of B consisting of pure subgroups. Define $\{C_i\}_{i<\kappa}$ in C in an analogous fashion. It follows that $\{B_i \triangledown C_i\}_{i<\kappa}$ is a pure filtration of $B \triangledown C$. For each $i < \kappa$, the kernel of the obvious map

$$B \triangledown C \to [(B/B_i) \triangledown C] \oplus [B \triangledown (C/C_i)]$$

is

$$(B_i \triangledown C) \cap (B \triangledown C_i) = B_i \triangledown C_i$$

(see [Nunke 1967b, Lemma 7], for example) so that there is an embedding

$$(B \triangledown C)/(B_i \triangledown C_i) \to [(B/B_i) \triangledown C] \oplus [B \triangledown (C/C_i)].$$

Let $W = \Lambda_T(B \triangledown C)$, so W is a stationary subset of κ. If $i \in W$, then T_i is in $L_{(B \triangledown C)/(B_i \triangledown C_i)}$, and it follows from Lemma 1.4(c) that T_i is either in $L_{(B/B_i) \triangledown C}$ or $L_{B \triangledown (C/C_i)}$. Without loss of generality, then, we may suppose that T_i is in $L_{(B/B_i) \triangledown C}$ for all i in a stationary subset $Y \subseteq W$; replacing Y by $Y \cap (\mu(T'), \kappa)$, we may assume $\mu(T') < i$ for all $i \in Y$. For each $i \in Y$, by transfinite induction there are disjoint sets $U^i \in L_{B/B_i}$ and $V^i \in C$ such that $\mu(U^i \cup V^i) \leq \mu(T_i) \leq i$.

Observe that if $|B| < \kappa$, then $B/B_i = \{0\}$ for all $i \in Y$. But this would mean that $L_{B/B_i} = 0_{\mathfrak{R}}$, which contradicts that $U^i \in L_{B/B_i}$. We can conclude that $|B| = \kappa$ and that $\{B_i\}_{i<\kappa}$ actually is a filtration of B.

Consider the function $Y \to \mathcal{R}_f$ given by $f(i) = U^i \cap i = \{\, x \in U^i : x < i \,\}$. Since f is clearly regressive, by Lemma 1.5 there is a stationary subset $Z \subseteq Y$ such that for all $i \in Z$ we have $f(i) = P'$ for some fixed set $P' \in \mathcal{R}_f$.

If we let $P = P' \cup \{\kappa\}$, then $Z \subseteq \Lambda_P(B)$ so that $P \in L_B$. Choose any $i \in Z$, and set $Q := V^i \in L_C$; then $\kappa \notin V_i$, $P \subseteq \{\kappa\} \cup U^i$ and $U_i \cap V_i = \varnothing$ implies that $P \cap Q = \varnothing$. Therefore, $S := P \cup Q \in LG \cdot LH$, and $\mu(S) = \kappa$, as required. $\square$

Let $\mathcal{M}_A$ be the class of weakly Mahlo cardinals that are accessible. So a weakly Mahlo cardinal is, in fact, Mahlo if and only if it is not in $\mathcal{M}_A$. Note that in the presence of the generalized continuum hypotheses, $\mathcal{M}_A$ is empty. Let $\mathcal{R}' = \mathcal{R} - \mathcal{M}_A$ and $\mathcal{R}'_f$ be the class of finite subsets of $\mathcal{R}'$. The following states that our product works fine as long as we stay clear of $\mathcal{M}_A$:

Theorem 2.3. *If G and H are groups, then*

$$L_{G \triangledown H} \cap \mathcal{R}'_f = L_G \cdot L_H \cap \mathcal{R}'_f.$$

Proof. Basically, we preserve the notation of the proof of Theorem 2.2 and show how to amend the argument to fit the present circumstances. By Theorem 2.1 we have the containment $\supseteq$; for the reverse, we perform our usual induction. If $\mu(T) = \aleph_0$, then $T = \varnothing$. So if $T \in L_{G \triangledown H}$, then $G \triangledown H$ has elements of infinite height, so the same is true of G and H so that $T \in L_G \cdot L_H$. Again, suppose the result holds for all $S \in \mathcal{R}'_f$ such that $\mu(S) < \kappa := \mu(T)$ and $T \in L_{G \triangledown H} \cap \mathcal{R}'_f$.

Suppose first that (L1) holds so that $\Upsilon_T(G \triangledown H)$ is stationary in κ. If there is an $i \in \Upsilon_T(G \triangledown H) - \mathcal{R}$, then $T' = T_i \in L_{G \triangledown H}$. By induction, $T' \in L_G \cdot L_H$, and this gives $T \in L_G \cdot L_H$, as required.

We may therefore assume that $\Upsilon_T(G \triangledown H) \subseteq \mathcal{R}$; in particular, this means that κ is a weakly Mahlo cardinal. Since $T \cap \mathcal{M}_A = \varnothing$, κ must actually be Mahlo, that is, strongly inaccessible. If we consider the function $f : \kappa \to \kappa$ given $f(\alpha) = 2^{|\alpha|}$, then there is a CUB subset $E \subseteq \kappa$ such that $E \cap \mu(T') = \varnothing$ and for all $\beta \in E$, if $\alpha < \beta$, then $2^{|\alpha|} < \beta$. Clearly any element of E is strongly inaccessible. So any element of E that is weakly Mahlo is actually Mahlo; that is, $E \cap \mathcal{M}_A = \varnothing$.

Note that $F := \Upsilon_T(G \triangledown H) \cap E$ will also be stationary in κ. If $i \in F$, then $T_i \in \mathcal{R}'_f$, so by transfinite induction, $T_i = U^i \cup V^i$, where $U^i \in L_G$ and $V^i \in L_H$ are disjoint. For every $i \in F$, either $i \in U^i$ or $i \in V^i$. Without loss of generality, assume $i \in U^i$ for a stationary subset $F' \subseteq F$. By Lemma 1.5, there is a stationary subset $F'' \subseteq F'$ such that $U' := U^i - \{i\}$ and $V := V^i$ are constant for all $i \in F''$. Note that if we let $U = U' \cup \{\kappa\}$, then $F'' \subseteq \Upsilon_U(G)$ so that $U \in L_G$. Since $V = V^i \in L_H$ and T is the disjoint union of U and V, we can conclude that $T \in L_G \cdot L_H$.

On the other hand, if (L2) holds, then there is a subgroup A of $G \triangledown H$ of cardinality κ such that $W := \Lambda_T(A)$ is stationary. Again, we may assume $A = B \triangledown C$, where B and C are subgroups of G and H, respectively, and $\max(|B|, |C|) = \kappa$.

We define $\{B_i\}_{i<\kappa}$ and $\{C_i\}_{i<\kappa}$ as before, so in particular, they are always pure subgroups of B and C, respectively.

Continuing the argument as in Theorem 2.1 by using exact sequences for the torsion product, by transfinite induction we may assume there is a stationary subset $Y \subseteq W$ such that $Y \cap \mu(T') = \varnothing$ and for all $i \in Y$, $T_i = U^i \cup V^i$ for disjoint subsets $U^i \in L_{B/B_i}$ and $V^i \in L_C$. And further, that $|B| = \kappa$ and $\{B_i\}_{i<\kappa}$ actually is a filtration of B.

Appealing again to Lemma 1.5, we can find a stationary subset $Z \subseteq Y$ such that for all $i \in Z$, $U^i \cap i$ always agrees with some fixed set P and $V^i \cap i$ always agrees with some fixed set Q. In particular, this means that T is the disjoint union $P \cup Q \cup \{\kappa\}$.

We now divide the argument into two possibilities.

Case 1: $Z_1 := \{\, i \in Z : V^i = Q \,\}$ is stationary in κ.

If $i \in Z_1$, then since $i \notin V^i$, it follows that $i \in U^i$ exactly when $i \in \mathcal{R}$. Therefore, if $S := P \cup \{\kappa\}$ and $i \in Z_1$, it follows that S_i will coincide with $U^i \in L_{B/B_i}$. This means $Z_1 \subseteq \Lambda_S(B)$ so that $S \in L_B \subseteq L_G$. On the other hand, if $i \in Z_1$, then $Q = V_i \in L_C \subseteq L_H$, and T is the disjoint union of Q and S, completing this case.

This brings us to the more interesting possibility.

Case 2: $Z_2 := \{\, i \in Z : V^i \neq Q \,\}$ is stationary in κ.

Observe that if $i \in Z_2$, then $i \in V^i \subseteq \mathcal{R}$; that is, $Z_2 \subseteq \mathcal{R}$. This means that κ must be a (weakly Mahlo and hence) Mahlo cardinal. If $R := Q \cup \{\kappa\}$, then $R_i = Q \cup \{i\} = V^i \in L_H$ for all $i \in Z_2$, so it follows from (L1) that $R \in L_H$. Since T is the disjoint union $R \cup P$, the result follows from the next statement.

Claim. $P \in L_G$.

Let $\mu = \mu(P) < \kappa$. Again, $\{B_i\}_{i<\kappa}$ is a pure filtration of B, and we may clearly assume that $|B_i| = |i| \cdot \aleph_0$. For every $i \in Z_2$, $i \notin U^i$, so $P = U^i \in L_{B/B_i}$. This means that B/B_i has a subgroup F_i of cardinality μ such that $P \in L_{F_i}$. Now for each $i \in Z_2$, let D_i be a subgroup of B of cardinality μ such that $F_i = (B_i + D_i)/B_i$. If $i \in Z_2 \subseteq \mathcal{R}$, then i is a regular cardinal, and since $i \in Z$, we have $P = i \cap U^i$ so that $\mu < i$. Therefore, whenever $i \in Z_2$ there is an $f(i) < i$ such that $B_i \cap D_i \subseteq B_{f(i)}$. By Fodor's lemma, there is a stationary subset $Z_3 \subseteq Z_2$ such that f is constant on Z_3. Let $\alpha = f(i)$ for all $i \in Z_3$. For $i \in Z_3$, let $E_i = B_\alpha + D_i$. Clearly B_α is a pure subgroup of E_i since it is a pure subgroup of B. Note that

$$E_i/B_\alpha = (B_\alpha + D_i)/B_\alpha \cong D_i/(B_\alpha \cap D_i) = D_i/(B_i \cap D_i) \cong (B_i + D_i)/B_i = F_i$$

and so $|E_i/B_\alpha| = |F_i| = \mu$.

Suppose M_1 is a divisible hull of B_α and M_2 is a divisible group of rank μ, and set $N := M_1 \oplus M_2$. For every $i \in Z_3$, the inclusion $B_\alpha \subseteq M_1$ extends to an injective homomorphism $\phi_i : E_i \to N$. Note that $|N| = |B_\alpha| \cdot \mu < \kappa$, so N has only $2^{|N|} < \kappa$

subsets (since κ is strongly inaccessible). On the other hand, $|Z_3| = \kappa$. It follows that there are distinct $i, j \in Z_3$ such that $\phi_i(E_i) = \phi_j(E_j)$; assume $j < i$. Consider the homomorphism $\rho : E_i \to B$ given by $1_{E_i} - \phi_j^{-1} \circ \phi_i$. Since ϕ_i and ϕ_j are both the identity on B_α, it follows that B_α is contained in the kernel of ρ.

But if $\pi : B \to B/B_i$ is the canonical epimorphism, then since $B_j \subseteq B_i$, we have $\pi \circ \rho = \pi - \pi \circ \phi_j^{-1} \circ \phi_i = \pi$. It follows that the kernel of ρ is contained in the kernel of π, that is, B_i. However, $B_i \cap E_i = B_i \cap (B_\alpha + D_i) = B_\alpha + (B_i \cap D_i) = B_\alpha$ so that B_α must, in fact, be the kernel of ρ.

Therefore, ρ induces an isomorphism between $E_i/B_\alpha \cong F_i$ and a subgroup F of B. And since $i \in Z_3 \subseteq Z_2$, we know that $P \in L_{F_i} = L_F \subseteq L_G$. This established the claim and so completes the proof of the theorem. □

Corollary 2.4. *Assuming the generalized continuum hypothesis, if G and H are any groups, then*

$$L_{G \triangledown H} = L_G \cdot L_H.$$

Proof. This follows directly from Theorem 2.3 since GCH implies that every regular limit cardinal is strongly inaccessible so that $\mathcal{M}_A = \varnothing$. □

3. Applications and an independence result

The following is essentially [Keef 2008, Theorem 16] stated in terms of the invariants K_G and L_G. It shows that, though in the presence of weakly Mahlo cardinals K_G and L_G may differ at times, they will always agree for torsion-complete groups.

Proposition 3.1. *If G is a torsion-complete group of final rank γ, then $K_G = L_G$ is generated by the $\mathcal{R}_f$-antichain $\{\,\{\kappa\} : \kappa \in \mathcal{R}, \kappa \le \gamma\,\}$. In other words, it is the $\mathcal{R}_f$-invariant consisting of those $T \in \mathcal{R}_f$ such that $T \cap \gamma^+ \neq \varnothing$.*

Proof. Let B denote a basic subgroup of G. Consider a decomposition $G \cong X \oplus G'$, where X is bounded. Since $L_X = K_X = 0_{\mathcal{R}}$, we can replace G by G'; that is, we may assume that G has the same cardinality and final rank. Similarly, we may assume that B has the same cardinality and final rank.

Claim. If $\kappa \in \mathcal{R}$ and $\kappa \le \gamma$, then $\{\kappa\} \in K_G$.

We split the argument into two cases. Suppose first that $|B| < \kappa$. It follows that $G/B \cong \bigoplus_J \mathbb{Z}_{p^\infty}$, where $|J| = \gamma$. If $I \subseteq J$ with $|I| = \kappa$, then define A by the equation $A/B \cong \bigoplus_I \mathbb{Z}_{p^\infty}$. Let $\{A_i\}_{i<\kappa}$ be a filtration of A; we may certainly assume that $A_0 = B$. It follows that A/A_i will always be an epimorphic image of A/B, so A/A_i will always be divisible. This shows that $\Gamma_\varnothing(A) = \kappa$, and in particular, it is stationary. This implies that $\{\kappa\} \in K_G$.

Next, suppose $|B| \ge \kappa$. Let $B' = \bigoplus_{j<\kappa} C_j$ be a summand of B such that each C_j is a countable, unbounded Σ-cyclic group. Let E be the set of all limit ordinals in κ of countable cofinality. If $i \in E$, let X_i be a countable subgroup of G such that there are proper containments

$$\bigcup_{l<i} \overline{\bigoplus_{j<l} C_j} \subset X_i + \bigcup_{l<i} \overline{\bigoplus_{j<l} C_j} \subset \overline{\bigoplus_{j<i} C_j}$$

and such that $D_i = \big(X_i + \bigoplus_{j<i} C_j\big)/\big(\bigoplus_{j<i} C_j\big)$ is divisible. (In other words, X_i is in the p-adic closure of $\bigoplus_{j<i} C_j$ but not in the p-adic closure of $\bigoplus_{j<l} C_j$ for any $l < i$; the existence of such a subgroup follows from the countable cofinality of $i \in E$.)

For all $i < \kappa$, let $E_i = \{k \in E : k < i\}$ and $A_i = \big(\bigoplus_{j<i} C_j\big) + (\sum_{k \in E_i} X_k)$. Clearly, $\{A_i\}_{i<\kappa}$ is a filtration of $A := \bigcup_{i<\kappa} A_i$ and $|A| = \kappa$. Now, if $i \in E$, then we assert that A/A_i has elements of infinite height: to see this, note that $\bigoplus_{j<i} C_j \subseteq A_i$ and $X_i \subseteq A$, so there is a homomorphism $D_i \to A/A_i$, and since X_i is not a subgroup of $A_i \subseteq \bigcup_{l<i} \overline{\bigoplus_{j<l} C_j}$, the image of this map is nonzero, establishing the assertion.

We have shown that $E \subseteq \Gamma_{\varnothing}(A)$. Since E is a stationary subset of κ, we can conclude that $\{\kappa\} \in K_G$. This proves the claim.

Note that if $T \in \mathcal{R}_f$ is minimal in L_G, then by Lemma 1.4(g), we can conclude that $\mu(T) \leq \gamma$. So if we choose any $\kappa \in T$, then by our claim $\{\kappa\} \in K_G \subseteq L_G$. The minimality of T then implies that $T = \{\kappa\}$. This means that K_G and L_G correspond to the same $\mathcal{R}_f$-antichain, namely this collection of singletons, and so they really are the same $\mathcal{R}_f$-invariant. □

If G is a group of final rank γ, then every minimal element of K_G or L_G must be contained in γ^+. Proposition 3.1 states that if G is torsion-complete, then $L_G = K_G$ is the largest possible $\mathcal{R}_f$-invariant generated by nonempty subsets of γ^+.

We now consider a particularly simple case, that is, the torsion product of two unbounded torsion-complete groups with countable basic subgroups. We show that, though K_G and L_G agree on all torsion-complete groups, if we take torsion products, this may no longer be the case. Of course, $c = 2^{\aleph_0}$ denotes the continuum.

Theorem 3.2. *Suppose G is an unbounded torsion-complete group with a countable basic subgroup B. Then $K_{G \triangledown G}$ is generated by the $\mathcal{R}_f$-antichain consisting of all two-element subsets of $\mathcal{R} \cap c^+$, whereas $L_{G \triangledown G}$ is generated by the $\mathcal{R}_f$-antichain consisting of all two-element subsets of $(\mathcal{R} - \mathcal{M}) \cap c^+$, together with all one-element subsets of $\mathcal{M} \cap c^+$.*

Proof. Because $K_{G \triangledown G} = K_G \cdot K_G$, the first statement follows directly from Proposition 3.1. Next, note that if T is any two-element subset of $\mathcal{R} - \mathcal{M}$ with $\mu(T) \leq c$, then by Proposition 1.3, $T \in K_{G \triangledown G}$ will also be minimal in $L_{G \triangledown G}$. Finally, we need to show that the following holds:

Claim. If $\delta \leq c$ is weakly Mahlo, then $\{\delta\} \in L_{G \triangledown G}$.

Let A be a pure subgroup of G containing B with $|A| = \delta$, and let $\{A_i\}_{i<\delta}$ be a pure filtration of A starting with $A_0 = B$. We may clearly assume that for all $0 < i < \delta$ we have $|A_i| = |i| \cdot \aleph_0$. Therefore, if $i \in \mathcal{R} \cap \delta$, then $\{A_j\}_{j<i}$ will be a pure filtration of A_i and A_i/A_j will always be divisible. This implies that $\Lambda_{\{i\}}(A_i) = i$ so that $\{i\} \in L_{A_i}$ whenever $i \in \mathcal{R} \cap \delta$.

Now $\{A_i \triangledown A_i\}_{i<\delta}$ will be a filtration of $A \triangledown A$. If $i \in \mathcal{R} \cap \delta$, then

$$(A_i \triangledown A)/(A_i \triangledown A_i) \subseteq (A \triangledown A)/(A_i \triangledown A_i).$$

The pure exact sequence $0 \to A_i \to A \to A/A_i \to 0$ leads to a pure exact sequence

$$0 \to A_i \triangledown A_i \to A_i \triangledown A \to A_i \triangledown (A/A_i) \to 0.$$

Since A/A_i is divisible, it is isomorphic to a direct sum of δ copies of $\mathbb{Z}_{p^\infty}$. So

$$(A_i \triangledown A)/(A_i \triangledown A_i) \cong A_i \triangledown (A/A_i)$$

will be isomorphic to a direct sum of copies of A_i. In particular, this shows that $(A \triangledown A)/(A_i \triangledown A_i)$ has a subgroup isomorphic to A_i. So if $i \in \mathcal{R} \cap \delta$, then $\{i\} \in L_{A_i} \subseteq L_{(A \triangledown A)/(A_i \triangledown A_i)}$.

The above computation shows that

$$\mathcal{R} \cap \delta \subseteq \Lambda_{\{\delta\}}(A \triangledown A).$$

But by (L2), this implies that $\{\delta\} \in L_{G \triangledown G}$, as required. □

Theorem 3.2 makes it clear why the continuum hypothesis (CH) is equivalent to $G \triangledown G$ being Σ-cyclic. (This was [Keef 1991, Proposition 5] though the result was known before that paper.) If CH holds, then $(\mathcal{R} - \mathcal{M}) \cap c^+ = \{\aleph_1\}$ has no two-element subsets, and $\mathcal{M} \cap c^+ = \varnothing$ has no one-element subsets; this means that $L_{G \triangledown G}$ is empty. And on the other hand, if CH fails, then $\{\aleph_1, \aleph_2\}$ will be a two-element subset of $(\mathcal{R} - \mathcal{M}) \cap c^+$, so $L_{G \triangledown G}$ is nonempty. The next result is a striking parallel with regards to L_G, c and δ_m.

Corollary 3.3. *If G is an unbounded torsion-complete group with a countable basic subgroup, then $L_{G \triangledown G} = L_G \cdot L_G$ if and only if $c = 2^{\aleph_0} < \delta_m$.*

Proof. If $c < \delta_m$, then $(\mathcal{R} - \mathcal{M}) \cap c^+ = \mathcal{R} \cap c^+$ and $\mathcal{M} \cap c^+ = \varnothing$. It follows then from Proposition 3.1 and Theorem 3.2 that $L_{G \triangledown G} = K_{G \triangledown G} = K_G \cdot K_G = L_G \cdot L_G$.

Conversely, if $\delta_m \leq c$, then by Theorem 3.2, $\{\delta_m\} \in L_{G \triangledown G}$. However, since every set in $L_G \cdot L_G$ has at least two-elements, $\{\delta_m\} \notin L_G \cdot L_G$. □

It is tempting to think that in any model of ZFC we must have $c < \delta_m$. After all, a Mahlo cardinal, even a weakly Mahlo cardinal, ought to be extremely large; certainly much larger than the continuum. The following, however, shows that this need not be the case. (I am thankful to Prof. Joan Bagaria for this argument.)

Theorem 3.4. *If there is a model of ZFC that contains a weakly Mahlo cardinal, then there is a model of ZFC in which there is a weakly Mahlo cardinal smaller than the continuum.*

Proof. Start with a model V of ZFC in which there is a weakly Mahlo cardinal δ. Let γ be any regular cardinal greater than δ. The usual way to construct a model of ZFC with at least γ Cohen reals is to define the notion of forcing P to be the set of all functions from a finite subset of $\gamma \times \omega$ to $\{0, 1\}$ (see [Jech 2003, 15.1], for example). If $p, q \in P$, then let $p \leq q$ (that is, p is stronger than q) if and only if $p \supseteq q$. By [Jech 2003, Lemma 14.25], this P satisfies the c.c.c. (countable chain condition). Therefore, if we use P to construct a generic extension $V[G]$ of V, then in $V[G]$ we will have $\delta < \gamma \leq c$ since $G \in V[G]$ can be thought of as a collection of γ distinct functions $\omega \to \{0, 1\}$. (In fact, $c^{V[G]} = (2^{\aleph}_0)^{V[G]} = (\gamma^{\aleph_0})^V$, but we do not need to be precise.)

By [Jech 2003, Theorem 14.34], V and $V[G]$ have the same cardinals and cofinalities. This implies that the class $\mathcal{R}$ does not change when we go from V to $V[G]$. And in particular, this means that δ remains a regular cardinal.

Next, note that for all $\kappa \in \mathcal{R}$, since P is c.c.c., it is κ-c.c. (every antichain in P has cardinality less than κ). It follows from [Jech 2003, Lemma 22.25] that if $S \in V$ is a stationary subset of κ in V, then S remains stationary in $V[G]$. In particular, this means that $\mathcal{R} \cap \delta$ will remain stationary in δ so that δ remains weakly Mahlo in $V[G]$. □

Corollary 3.5. *If ZFC is consistent, then there exists a model of ZFC in which $L_{G \triangledown H} = L_G \cdot L_H$ holds for all groups G and H. On the other hand, if there is a model of ZFC in which there is a weakly Mahlo cardinal, then there is a model of ZFC in which $L_{G \triangledown H} \neq L_G \triangledown L_H$ for some pair of groups G and H.*

Proof. For the first statement, just take any model in which GCH holds (e.g., a model of V = L). For the second, consider a model in which $\delta_m < c$, and let $G = H$ be an unbounded torsion-complete group with a countable basic subgroup. □

We have seen that the $\mathcal{R}_f$-invariants K_G have some advantages over the corresponding $\mathcal{R}_f$-invariants L_G. That is, for all groups G and H, we know that $K_{G \triangledown H} = K_G \cdot K_H$, but this equation is much more complicated for L_G. In addition, if $\{G_i\}_{i \in I}$ is a collection of groups, by [Balof and Keef 2009, Theorem 3(a)] we have $K_{\bigoplus_{i \in I} G_i} = \bigcup_{i \in I} K_{G_i}$. On the other hand, we have the following result:

Theorem 3.6. *A weakly Mahlo cardinal exists if and only if there is a collection of groups $\{G_i\}_{i \in I}$ such that*

$$L_{\bigoplus_{i \in I} G_i} \neq \bigcup_{i \in I} L_{G_i}.$$

Proof. If no weakly Mahlo cardinal exists, then $\delta_m = \infty$, and for all groups G we have $L_G = K_G$. So the result follows from [Balof and Keef 2009, Theorem 3(a)].

On the other hand, if $\delta_m < \infty$, then let $I = \mathfrak{R} \cap \delta_m$, and let H be a torsion-complete group of final rank at least δ_m. According to Proposition 3.1 and Lemma 1.4(f), if $i \in I$, then H has a subgroup G_i of cardinality i such that $\{i\} \in L_{G_i}$. We let G be the external direct sum $\bigoplus_{i \in I} G_i$. If for all $j < \delta_m$ we let $A_j = \bigoplus_{i<j} G_i$, then $\{A_j\}_{j<\delta_m}$ is a filtration of G. Since for all $j \in I$, G_j embeds in G/A_j, it follows that $\Lambda_\varnothing G \supseteq I$ is stationary. By (L2), this implies that $\{\delta_m\} \in L_G$. However, if $\{\delta_m\}$ was an element of some L_{G_i}, since $|G_i| = i < \delta_m$ we could conclude from Lemma 1.4(f) that $\{\delta_m\}$ is not minimal. This, in turn, would imply that $\varnothing \in L_{G_i}$ so that G_i has nonzero elements of infinite height. Therefore, $\{\delta_m\} \notin \bigcup_{i \in I} L_{G_i}$, so the two sets disagree. □

We will need the following extension of Theorem 2.2:

Corollary 3.7. *If $G_1, \dots, G_n$ are groups, then $G_1 \triangledown \cdots \triangledown G_n$ is Σ-cyclic if and only if $L_{G_1} \cdots L_{G_n} = 0_{\mathfrak{R}}$.*

It would be tempting to say that Corollary 3.7 follows directly from Theorem 2.2 by simply inducting on n. This does not quite work, however. For example, if $n = 3$, then $G_1 \triangledown G_2 \triangledown G_3$ will be Σ-cyclic if and only if $L_{G_1 \triangledown G_2} \cdot L_{G_3} = 0_{\mathfrak{R}}$, but at this stage we are stuck since we do not necessarily have $L_{G_1 \triangledown G_2} = L_{G_1} \cdot L_{G_2}$.

The way to verify Corollary 3.7 is to go back to the proof of Theorem 2.2. That proof was done with only two terms, but essentially the same argument can be made with any finite number of terms. The notation is much more cumbersome, but the ideas are identical, and so we omit them. This brings us to our final result, an extension of [Balof and Keef 2009, Theorem 55]. Since that proof was embedded in a much more involved discussion of the structure of $\mathfrak{R}_f$-invariants, we provide this self-contained argument. Observe that it is a complete characterization of the groups that positively answer Nunke's question.

Theorem 3.8. *If G is a group (of arbitrary cardinality), then the following are equivalent:*

(a) *There is a group H that is not Σ-cyclic such that $G \triangledown H$ is Σ-cyclic.*

(b) *There is no infinite pairwise disjoint subset of L_G.*

(c) *For some positive integer n, $G^n = G \triangledown G \triangledown \cdots \triangledown G$ is Σ-cyclic.*

Proof. Suppose (a) holds so that $L_H \neq 0_{\mathfrak{R}}$ and $L_G \cdot L_H = 0_{\mathfrak{R}}$. Let $S \in L_H$, and suppose $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$. We assume $\{T_k\}_{k<\omega}$ is an infinite pairwise disjoint collection from L_G and derive a contradiction. Note that each T_k must intersect S nontrivially since $L_G \cdot L_H = 0_{\mathfrak{R}}$. On the other hand, since the T_ks are disjoint, every element of S can be in at most one of them. So the fact that there are only a finite number of elements of S contradicts the fact that there are an infinite number of T_ks.

We suppose now that (b) holds and verify (c); actually, we will assume that (c) fails, and verify that (b) must also fail. To that end, suppose we have constructed a pairwise disjoint collection in L_G and $T_1, \dots, T_k$; we will verify that there must be a $T_{k+1} \in L_G$ that is disjoint from them all. Let n be one more that the total number of elements in $T_1 \cup \cdots \cup T_k$. Since by Corollary 3.7, $(L_G)^n \neq 0_{\mathscr{R}}$, there is a pairwise disjoint collection $S_1, \dots, S_n$ in L_G. Since the S_js are disjoint, no element of $T_1 \cup \cdots \cup T_k$ is in more than one S_j. Therefore, there must be an S_j disjoint from all the T_is, and we just let T_{k+1} be this S_j.

We suppose now that (c) holds and verify (a). If G is Σ-cyclic, then we can let H be any group that is not Σ-cyclic. Otherwise, choose n to be the least positive integer such that G^n is Σ-cyclic. So $n > 1$ (since G is not Σ-cyclic) and $H := G^{n-1}$ is not Σ-cyclic, but $G \triangledown H = G^n$ is. □

The following shows that torsion-complete groups can be used as "test groups" for the Σ-cyclic groups:

Corollary 3.9. *Suppose G is a torsion-complete group of final rank at least $\aleph_\omega$. If H is any group, then H is Σ-cyclic if and only if $G \triangledown H$ is Σ-cyclic.*

Proof. By Proposition 3.1, for every $m < \omega$, $\{\aleph_m\} \in L_G$, so G satisfies the denial of Theorem 3.8(b), and the result follows from the denial of Theorem 3.8(a). □

In summary, the new invariant L_G "solves" Nunke's problem in the sense that it reduces it from a question involving the torsion product to a question of being able to compute L_G. As we have seen, however, even for a standard torsion-complete group, calculating L_G involves significant questions of cardinal arithmetic.

Nunke's problem can be generalized to asking when $G \triangledown H$ is a direct sum of countable groups (which we abbreviate to d.s.c. group) of length $\lambda \leq \omega_1$. If $\lambda = \omega_1$, we showed in [Keef 1989] that the answer to this question depends upon a set-theoretic statement known as *Kurepa's Hypothesis* (see [Jech 2003, Definition 9.24]). On the other hand, for all countable $\lambda < \omega_1$ the above techniques can be generalized to answer the question of when $G \triangledown H$ is a d.s.c. group of length λ.

Define an invariant L_G^λ inductively as follows:

($L^\lambda 0$) If $\mu(T) = \aleph_0$ (that is, $T = \varnothing$), $T \in L_G^\lambda$ if and only if $p^\lambda G \neq \{0\}$.

Next, suppose for all groups H we have defined all the elements $S \in L_H^\lambda$ such that $\mu(S) < \kappa := \mu(T)$. We then say $T \in L_G^\lambda$ if and only if one of two things occurs:

($L^\lambda 1$) $\Upsilon_{T'}^\lambda(G) := \{\, i < \kappa : T_i \in L_G^\lambda \,\}$ is stationary in κ, or

($L^\lambda 2$) G has a subgroup A of cardinality κ with a filtration $\{A_i\}_{i<\kappa}$ such that

$$\Lambda_T^\lambda(A) := \{\, i < \kappa : T_i \in L_{A/A_i}^\lambda \,\}$$

is stationary in κ.

So all that really has to be changed is the base case, ($L^\lambda 0$) versus (L0).

Next, recall from [Keef 1991] that the group G is a C_λ-group if for every $\alpha < \lambda$, G has an α-high subgroup that is a d.s.c. group. In particular, a d.s.c. group will always be a C_λ group, and every group is a C_ω-group. The following can be proven in an essentially identical manner to the results of this paper:

Theorem 3.10. *If G and H are groups and $\lambda < \omega_1$ is a countable ordinal, then $G \triangledown H$ is a d.s.c. group of length λ if and only if G and H are C_λ groups of length at least λ and $L^\lambda_G \cdot L^\lambda_H = 0_{\mathscr{R}}$.*

Similarly, virtually all of the results of this paper can be recast to statements involving d.s.c. groups of countable length.

There are difficulties in translating these results to groups of uncountable length and, in particular, to simply presented groups of arbitrary length. The base case in the definition L^λ_G really depends upon the countability of λ. In particular, if $\varnothing \in L^\lambda_G$, we would like to conclude that there is a *countable* subgroup $A \subseteq G$ such that $\varnothing \in L^\lambda_A$ (cf., Lemma 1.4(f)), but this is not true if $\lambda \geq \omega_1$. Correspondingly, as was shown in [Hill 1983a], if G and H are reduced groups, then $G \triangledown H$ will *never* be a simply presented group of uncountable length. Some of these issues will be discussed in later work.

Acknowledgements

This research was completed while I was a visiting scholar at the Institut de Matemática, Universitat de Barcelona. I am grateful to the Institut, and to its director, Carles Casacuberta, for their hospitality during my stay.

The argument in the proof of Theorem 3.4 was provided by Professor Joan Bagaria of the Universitat de Barcelona. I am very grateful to him for his assistance in clarifying this part of my investigations.

References

[Balof and Keef 2009] B. A. Balof and P. W. Keef, "Invariants on primary abelian groups and a problem of Nunke", *Note Mat.* **29**:2 (2009), 83–114. MR 2012d:20113 Zbl 1215.20051

[Eklof and Mekler 2002] P. C. Eklof and A. H. Mekler, *Almost free modules: set-theoretic methods*, North-Holland Mathematical Library **65**, North-Holland, Amsterdam, 2002. MR 2003e:20002 Zbl 1054.20037

[Fuchs 1970] L. Fuchs, *Infinite abelian groups*, vol. 1, Pure and Applied Mathematics **36**, Academic Press, New York, 1970. MR 41 #333 Zbl 0209.05503

[Fuchs 1973] L. Fuchs, *Infinite abelian groups*, vol. 2, Pure and Applied Mathematics **36**, Academic Press, New York, 1973. MR 50 #2362 Zbl 0257.20035

[Hill 1983a] P. Hill, "The balance of Tor", *Math. Z.* **182**:2 (1983), 179–188. MR 84f:20059 Zbl 0488.20042

[Hill 1983b] P. Hill, "When Tor(A, B) is a direct sum of cyclic groups", *Pacific J. Math.* **107**:2 (1983), 383–392. MR 84h:20052 Zbl 0538.20028

[Jech 2003] T. Jech, *Set theory*, 3rd ed., Springer, Berlin, 2003. MR 2004g:03071 Zbl 1007.03002

[Keef 1988] P. W. Keef, "On the Tor functor and some classes of abelian groups", *Pacific J. Math.* **132**:1 (1988), 63–84. MR 89b:20115 Zbl 0617.20033

[Keef 1989] P. W. Keef, "On set theory and the balanced-projective dimension of C_Ω groups", pp. 31–41 in *Abelian group theory* (Perth, 1987), edited by L. Fuchs et al., Contemp. Math. **87**, American Mathematical Society, Providence, RI, 1989. MR 90d:20100 Zbl 0673.20031

[Keef 1990] P. W. Keef, "Some isomorphisms of abelian groups involving the Tor functor", *Proc. Amer. Math. Soc.* **110**:1 (1990), 27–37. MR 91b:20072 Zbl 0713.20048

[Keef 1991] P. W. Keef, "On iterated torsion products of abelian p-groups", *Rocky Mountain J. Math.* **21**:3 (1991), 1035–1055. MR 92k:20102 Zbl 0773.20021

[Keef 1993] P. W. Keef, "An application of set theory to the torsion product of abelian groups", *Proc. Amer. Math. Soc.* **118**:3 (1993), 735–738. MR 93i:20063 Zbl 0798.20048

[Keef 2008] P. W. Keef, "On the torsion product and direct sums of primary cyclic groups", *J. Algebra* **319**:7 (2008), 2868–2888. MR 2009b:20098 Zbl 1144.20035

[Nunke 1964] R. J. Nunke, "On the structure of Tor", pp. 115–124 in *Proceedings on the Colloquium on Abelian Groups* (Tihany, 1963), edited by L. Fuchs and E. Schmidt, Akadémiai Kiadó, Budapest, 1964. MR 30 #155 Zbl 0132.27303

[Nunke 1967a] R. J. Nunke, "Homology and direct sums of countable abelian groups", *Math. Z.* **101** (1967), 182–212. MR 36 #1538 Zbl 0173.02401

[Nunke 1967b] R. J. Nunke, "On the structure of Tor, II", *Pacific J. Math.* **22** (1967), 453–464. MR 35 #5508 Zbl 0158.02805

Received December 13, 2011.

PATRICK W. KEEF
DEPARTMENT OF MATHEMATICS
WHITMAN COLLEGE
345 BOYER AVENUE
WALLA WALLA, WA 99362
UNITED STATES

keef@whitman.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 259 No. 1 September 2012
