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**RELATIVELY MAXIMUM VOLUME RIGIDITY IN
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Given compact metric spaces X and Z with Hausdorff dimension n , if there is a distance-nonincreasing onto map $f : Z \rightarrow X$, then the Hausdorff n -volumes satisfy $\text{vol}(X) \leq \text{vol}(Z)$. The *relatively maximum volume conjecture* says that if X and Z are both Alexandrov spaces and $\text{vol}(X) = \text{vol}(Z)$, X is isometric to a gluing space produced from Z along its boundary ∂Z and f is length-preserving. We partially verify this conjecture and give a further classification for compact Alexandrov n -spaces with relatively maximum volume in terms of a fixed radius and space of directions. We also give an elementary proof for a pointed version of the Bishop–Gromov relative volume comparison with rigidity in Alexandrov geometry.

Introduction

Let Z be a compact metric space with Hausdorff dimension α . Consider all compact metric spaces X with Hausdorff dimension α such that there is a distance-nonincreasing onto map $f : Z \rightarrow X$. We let “vol” denote the Hausdorff measure (or volume) in the top dimension. Then $\text{vol } X \leq \text{vol } Z$. A natural question is to determine X (in terms of Z) when $\text{vol } X = \text{vol } Z$. We refer to this as a *relatively maximum volume rigidity problem*.

A possible answer to the relatively maximum volume rigidity problem is closely related to the regularity of underlying geometric and topological structures. For instance, if Z and X are closed Riemannian n -manifolds, f is an isometry (see [Corollary 0.2](#)). On the other hand, taking any measure-zero subset S in Z (a Riemannian manifold) and identifying S with a point $p \in S$, the projection map, $Z \rightarrow X = Z/(S \sim p)$, is a distance-nonincreasing onto map, and it is hopeless to have some rigidity on Y in terms of X .

In this paper, we will study the relatively maximum volume rigidity problem in Alexandrov geometry, partly because an Alexandrov space X has a “right” geometric structure for this problem (see [Conjecture 0.1](#) below). For instance, for

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$p \in X$, the gradient-exponential map, $g \exp_p : T_p X \rightarrow X$, becomes a distance-nonincreasing map, when $T_p X$ is equipped with the κ -cone metric via the cosine law on the space form S_κ^2 ; see [Burago et al. 1992]. When taking Z to be a closed r -ball at the vertex (for $\kappa > 0$, $r \leq \pi/(2\sqrt{\kappa})$ or $r = \pi/\sqrt{\kappa}$), the relatively volume rigidity problem (see Theorem B) indeed extends the (absolutely) maximum radius-volume rigidity theorem proved in [Grove and Petersen 1992]; see Theorem 0.3.

The recent study of Alexandrov spaces was initiated by Burago, Gromov, and Perelman [Burago et al. 1992] and has gotten a lot of attention lately. An Alexandrov space with curvature $\text{curv} \geq \kappa$ is a length metric space such that each point has a neighborhood in which the Toponogov triangle comparison holds with respect to the space form of constant curvature κ . In the rest of the paper, we will freely use basic notions on an Alexandrov space from [Burago et al. 1992] and [Petrunin 2007]; for example, the space of directions, the gradient-exponential maps, and (n, δ) -strained points, among others. Let $\text{Alex}^n(\kappa)$ denote the collection of compact Alexandrov n -spaces with $\text{curv} \geq \kappa$.

Note that the boundary gluing will automatically yield a distance-nonincreasing onto (projection) map, which also preserves the volume (see Examples 2.14 and 2.15). We propose the following relatively maximum volume rigidity conjecture for Alexandrov spaces.

Conjecture 0.1. *Consider $Z, X \in \text{Alex}^n(\kappa)$, and let $f : Z \rightarrow X$ be a distance-nonincreasing onto map. If $\text{vol } Z = \text{vol } X$, X is isometric to a gluing space produced from Z along its boundary ∂Z and f is length-preserving. In particular, Z is isometric to X if $\partial Z = \emptyset$ or if f is injective.*

Our goal in this paper is to partially verify Conjecture 0.1 and give a classification for the boundary gluing maps in a special case (see Theorem A, Corollary 0.2, and Theorem B).

We now begin to state the main results. Throughout this paper, $\tau(\delta)$ denotes a function in δ such that $\tau(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Our first result verifies Conjecture 0.1 for the case where f preserves non- (n, δ) -strained points up to an error $\tau(\delta)$. For $X \in \text{Alex}^n(\kappa)$ and $\delta > 0$, let $X^\delta \subseteq X$ denote the set of all (n, δ) -strained points. Then a small ball centered at an (n, δ) -strained point is almost isometric to an open subset in \mathbb{R}^n [Burago et al. 1992].

Theorem A. *Let Z, X be Alexandrov n -spaces (not necessarily complete) with curvature $\text{curv} \geq \kappa$ and $\text{vol } Z = \text{vol } X$. Suppose that $f : Z \rightarrow X$ is a distance-nonincreasing onto map such that for any $\delta > 0$, $f^{-1}(X^\delta) \subseteq Z^{\tau(\delta)}$. Then f is an isometry.*

A point z in Z is called *regular* if the space of directions Σ_x is isometric to a unit sphere. Clearly, the space Z with all points regular is a topological manifold, but

Z may not be isometric to any Riemannian manifold (for example, the doubling of two flat disks). [Theorem A](#) includes the following case:

Corollary 0.2. *Let $Z, X \in \text{Alex}^n(\kappa)$ with $\text{vol } Z = \text{vol } X$ and all points in Z regular (for example, Z is a Riemannian manifold). If $f : Z \rightarrow X$ is a distance-nonincreasing onto map, f is an isometry.*

In Alexandrov geometry, perhaps the most natural distance-nonincreasing onto map is the gradient-exponential map $g \exp_p : C_\kappa(\Sigma_p) \rightarrow X$, $p \in X \in \text{Alex}^n(\kappa)$, where $C_\kappa(\Sigma_p)$ denotes the tangent cone $T_p X$ equipped with a κ -cone metric via the cosine law in S_κ^2 [[Burago et al. 1992](#)]. Since $g \exp_p$ is distance-nonincreasing and preserves any r -ball, we immediately get the pointed version of the Bishop type volume comparison:

$$\text{vol } B_R(p) \leq \text{vol } C_\kappa^R(\Sigma_p),$$

where $C_\kappa^R(\Sigma_p)$ denotes the open R -ball in $C_\kappa(\Sigma_p)$ at the vertex \tilde{o} . We show that when the equality holds, $g \exp_p$ will satisfy the conditions in [Theorem A](#) ([Lemmas 2.4](#) and [2.5](#)) and thus open ball $C_\kappa^R(\Sigma_p)$ is isometric to $B_R(p)$ with respect to intrinsic metrics (see [Theorem 2.1](#)).

We prove an important case of [Conjecture 0.1](#), which gives a classification of Alexandrov spaces with relatively maximum volume: given any $\kappa, R > 0$ and $\Sigma \in \text{Alex}^{n-1}(1)$, let $\mathcal{A}_\kappa^R(\Sigma)$ be the collection of Alexandrov n -spaces $X \ni p$ satisfying

$$\text{curv} \geq \kappa, \quad X = \bar{B}_R(p), \quad \Sigma_p = \Sigma.$$

Then $\text{vol } X \leq \text{vol } C_\kappa^R(\Sigma) = v(\Sigma, \kappa, R)$. When $\text{vol } X = v(\Sigma, \kappa, R)$, we say that X has the relatively maximum volume.

Theorem B (relatively maximum volume rigidity). *Let $X \in \mathcal{A}_\kappa^R(\Sigma)$ such that $\text{vol } X = v(\Sigma, \kappa, R)$. Then X is isometric to $\bar{C}_\kappa^R(\Sigma)/x \sim \phi(x)$ and $R \leq \pi/(2\sqrt{\kappa})$ or $R = \pi/\sqrt{\kappa}$ for $\kappa > 0$, where $\phi : \Sigma \times \{R\} \rightarrow \Sigma \times \{R\}$ is an isometric involution (which can be trivial). Conversely, given any isometric involution ϕ on Σ , $\bar{C}_\kappa^R(\Sigma)/x \sim \phi(x) \in \mathcal{A}_\kappa^R(\Sigma)$ and has the relatively maximum volume.*

[Theorem B](#) verifies [Conjecture 0.1](#) for the case $f = g \exp_p : Z = \bar{C}_\kappa^R(\Sigma_p) \rightarrow X$, together with a further classification for the boundary identification. Note that [Theorem B](#) implies that if $k > 0$ and $\pi/(2\sqrt{\kappa}) < R < \pi/\sqrt{\kappa}$,

$$\max\{\text{vol } X, X \in \mathcal{A}_\kappa^R(\Sigma)\} < v(\Sigma, \kappa, R).$$

For the case where X is a limit of Riemannian manifolds, a classification was given in [[Grove and Petersen 1992](#)]. A general classification is more complicated, and we wish to discuss it elsewhere.

As mentioned earlier, [Theorem B](#) extends this radius-volume rigidity theorem:

Theorem 0.3 [[Grove and Petersen 1992](#)]. *Let $M_i \xrightarrow{d_{GH}} X$ be a Gromov–Hausdorff convergent sequence of Riemannian n -manifolds such that*

$$\text{sec}_{M_i} \geq \kappa, \quad \text{rad}(M_i) = R, \quad \text{vol } M_i \rightarrow \text{vol } C_\kappa^R(S_1^{n-1}),$$

where $\text{rad}(M_i) = \min\{r, \bar{B}_r(p) \cap M_i, p \in M_i\}$. Then $R \leq \pi/(2\sqrt{\kappa})$ or $R = \pi/\sqrt{\kappa}$ for $\kappa > 0$, and X is isometric to the quotient of $\bar{C}_\kappa^R(S_1^{n-1})$ by the equivalence relation $x \sim \phi(x)$, where $\phi : \partial \bar{C}_\kappa^R(S_1^{n-1}) \rightarrow \partial \bar{C}_\kappa^R(S_1^{n-1})$ is either the antipodal map or a reflection in a totally geodesic hypersurface. Moreover, each M_i is homeomorphic to an n -sphere or a real projective n -space.

Note that $\text{vol } X = \text{vol } C_\kappa^R(S_1^{n-1})$. Choosing $p_i \in M_i$ such that $M_i = \bar{B}_R(p_i)$, $p_i \rightarrow p \in X$ and $\Sigma_p = S_1^{n-1}$. By now [Theorem B](#) implies the rigidity part of [Theorem 0.3](#) (a generalization of the homeomorphic rigidity in [Theorem 0.3](#) will be given in [Theorem C](#)). [Theorem B](#) also implies the following extension of [Theorem 0.3](#).

Theorem 0.4 [[Shteingold 1994](#)]. *Let $X \in \mathcal{A}_\kappa^r(S_1^{n-1})$ with $\text{vol } X = v(S_1^{n-1}, \kappa, r)$. Then $X = \bar{C}_\kappa^r(S_1^{n-1})/x \sim \phi(x)$, $x \in S_1^{n-1} \times \{r\}$, where ϕ is the reflection on an ℓ -dimensional totally geodesic subsphere, $1 \leq \ell \leq n$ (ϕ is trivial for $\ell = n$).*

A further problem concerning [Theorem B](#) is to determine the homeomorphic type of X . We have solved this problem for X being a topological manifold (see [Theorem 0.3](#)).

Theorem C. *Given $\Sigma \in \text{Alex}^{n-1}(1)$, κ and $R > 0$, there exists a constant $\epsilon = \epsilon(\Sigma, \kappa, R) > 0$ such that if $X \in \mathcal{A}_\kappa^R(\Sigma)$ with $\text{vol } X > v(\Sigma, \kappa, R) - \epsilon$ and X is a closed topological manifold, X is homeomorphic to S_1^n or a real projective space $\mathbb{R}P^n$.*

Note that Σ in [Theorem C](#) is not necessarily a topological manifold; for instance, $X = C_1(C_1(N))$, the twice spherical suspensions over a Poincaré sphere N , satisfies [Theorem C](#), but $\Sigma = C_1(N)$ is not a topological manifold. However, X is homeomorphic to a 5-sphere, by [[Kapovitch 2002](#)].

In the proof of [Theorem B](#), we establish a pointed version of the Bishop volume comparison with rigidity ([Theorem 2.1](#)). In general, we will prove the following pointed version of the Bishop–Gromov relative volume comparison with rigidity.

For $p \in X \in \text{Alex}^n(\kappa)$, let $A_R^r(p)$ denote the annulus $\{x \in X : r < |px| < R\}$, $0 \leq r < R$, and let $A_R^r(\Sigma_p)$ denote the corresponding annulus in $C_\kappa(\Sigma_p)$.

Theorem D (pointed Bishop–Gromov relative volume comparison). *Let*

$$X \in \text{Alex}^n(\kappa).$$

Then, for any $p \in X$ and $R_3 > R_2 > R_1 \geq 0$,

$$\frac{\text{vol } A_{R_3}^{R_1}(p)}{\text{vol } A_{R_3}^{R_2}(p)} \geq \frac{\text{vol } A_{R_3}^{R_1}(\Sigma_p)}{\text{vol } A_{R_3}^{R_2}(\Sigma_p)}, \quad \text{or equivalently,} \quad \frac{\text{vol } A_{R_2}^{R_1}(p)}{\text{vol } A_{R_3}^{R_2}(p)} \geq \frac{\text{vol } A_{R_2}^{R_1}(\Sigma_p)}{\text{vol } A_{R_3}^{R_2}(\Sigma_p)}.$$

In particular,

$$\frac{\text{vol } B_{R_1}(p)}{\text{vol } B_{R_3}(p)} \geq \frac{\text{vol } C_\kappa^{R_1}(\Sigma_p)}{\text{vol } C_\kappa^{R_3}(\Sigma_p)}.$$

If any of these inequalities becomes an equality, the open ball $B_{R_3}(p)$ is isometric to $C_\kappa^{R_3}(\Sigma_p)$ with respect to the intrinsic metrics.

Remark 0.5. The Riemannian version of the Bishop–Gromov relative comparison for Alexandrov spaces (that is, the model space is S_κ^n) was stated in [Burago et al. 1992]; compare [Burago et al. 2001]. A notable difference between Theorem D and the Riemannian version is in the rigidity part: the latter is the *absolute maximum* volume rigidity and its model space is *unique*, while the former may be viewed as the *relatively maximum* volume rigidity (relatively to Σ_p), whose model spaces are of *infinitely many* possibilities. Moreover, the proof of Theorem D is considerably difficult; for instance, a dimension-inductive argument (which works in the Riemannian version) does not work.

Remark 0.6. By Lemma 2.1 in [Li 2010], we see that

$$\frac{\text{vol } C_\kappa^R(\Sigma_p)}{\text{vol } C_\kappa^r(\Sigma_p)} = \frac{\text{vol } B_R(S_\kappa^n)}{\text{vol } B_r(S_\kappa^n)},$$

and thus the monotonicity part of Theorem D coincides with that in the Riemannian version. We point out that our proof of the volume ratio monotonicity in Theorem D is different from one suggested in [Burago et al. 1992]; we take an elementary (calculus) approach via finding an (unconventional) partition suitable for triangle comparison arguments, while a proof in [Burago et al. 2001] relies on a coarea formula for Alexandrov spaces. We point out that in the case where $\kappa \leq 0$, a weak form of the above monotonicity was previously obtained in [Liu and Shen 1994, Proposition 1].

We now give some indication on our approach to Theorem A and Theorem B. In the proof of Theorem A, we show that f is a homeomorphism and f preserves the length of curves. Based on basic properties of an Alexandrov space (not necessarily complete), any curve c in X can be approximated by piecewise geodesics c_i in X^{δ_i} ($\delta_i \rightarrow 0$) such that lengths $L(c_i) \rightarrow L(c)$. Thus, it suffices to show that when restricting to $f^{-1}(X^\delta)$ and X^δ , respectively, f is injective and f^{-1} preserves the length of any geodesic up to an error $\tau(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, respectively. We derive this with a volume formula for tube-like ϵ -balls in X^δ , which can be treated as a

replacement of the volume formula of a thin tube around a curve. The proof of the volume formula is based on the fact that a small ball at an (n, δ) -strained point can be almost isometrically embedded into \mathbb{R}^n ; see [Burago et al. 1992].

Our approach to [Theorem B](#) consists of two steps: first, establishing the open ball rigidity: the gradient-exponential map $g \exp_p : C_\kappa^R(\Sigma_p) \rightarrow B_R(p) \subset X$ is an isometry with respect to the intrinsic distance. We achieve this by showing that $g \exp_p$ satisfies the condition in [Theorem A](#); see [Lemmas 2.4](#) and [2.5](#). Consequently, $X = \bar{C}_\kappa^R(\Sigma_p)/\sim$, where \sim is a relation on $\Sigma_p \times \{R\}$: $\tilde{x} \sim \tilde{y}$ if and only if $g \exp_p(\tilde{x}) = g \exp_p(\tilde{y})$. Observe that if $\tilde{x} \neq \tilde{y} \in \Sigma_p \times \{R\}$ with $\tilde{x} \sim \tilde{y}$, then the $g \exp_p$ -images of the two geodesics $[\tilde{o}\tilde{x}]$ and $[\tilde{o}\tilde{y}]$ together form a local geodesic at $g \exp_p \tilde{x} = g \exp_p \tilde{y}$. Because a geodesic does not bifurcate, any equivalent class contains at most two points and thus we obtain an involution $\phi : \Sigma_p \times \{R\} \rightarrow \Sigma_p \times \{R\}$ such that $X = \bar{C}_\kappa^R(\Sigma)/\tilde{x} \sim \phi(\tilde{x}), \tilde{x} \in \Sigma_p \times \{R\}$. The main difficulty is to show that ϕ is an isometry. Our main technical lemma says that ϕ is almost 1-bi-Lipschitz up to a uniform error:

$$\left| \frac{|\phi(\tilde{x})\phi(\tilde{y})|}{|\tilde{x}\tilde{y}|} - 1 \right| \leq 20\tilde{x}\tilde{y}|$$

for $|\tilde{x}\tilde{y}|$ small (see [Lemma 2.12](#)). This implies that ϕ is continuous and preserves the length of a path, and thus ϕ is distance-nonincreasing. Consequently, ϕ is an isometry since ϕ is an involution. Note that without the curvature lower bound, this does not, in general, imply that the metric on $X = \bar{C}_\kappa^R(\Sigma)/\tilde{x} \sim \phi(\tilde{x})$ coincides with the induced metric. For example, $X = \bar{C}_0^1(\mathbb{S}_1^1)/(\tilde{x} \sim \tilde{x}) = \bar{B}_1(\mathbb{R}^2)$ is equipped with the length metric and coincides with the Euclidean metric when restricted to the interior, and $L(\gamma)$ is half of the Euclidean arc length for any $\gamma \subset \partial X$. Our proof relies on the curvature lower bound as well as the cone metric.

Let $L_p(X) = g \exp_p(\Sigma \times \{R\})$, which locally divides a tubular neighborhood of $L_p(X)$ into two components U_1, U_2 . The main difficulty in proving the above inequality is that a geodesic in X connecting two points $a, b \in L_p(X)$ may intersect with $L_p(X)$ at many points other than a, b (called *crossing points*). We show that if a geodesic is not contained in $L_p(X)$, the crossing points are discrete ([Corollary 2.9](#)). Thus we can reduce the proof to the case where $c_1 = [ab] \subset U_1$ has no crossing point. It's sufficient to construct a noncrossing piecewise intrinsic geodesic $c_2 \subset U_2$ connecting a, b , and show that $\text{length}(c_2)$ is close to $\text{length}(c_1) = |ab|$ up to a second order error ([Lemma 2.12](#)).

We remark that the present proof, in an essential way, relies on the κ -cone metric structure; and we believe that establishing a similar inequality in general will be the main obstacle in [Conjecture 0.1](#).

[Theorems A, B, C](#) and [D](#) are proved in [Sections 1, 2, 3](#) and [4](#), respectively.

1. Proof of Theorem A: (n, δ) -strained isometries

Let $f : Z \rightarrow X$ be as in [Theorem A](#). We will establish that f is an isometry through the following properties:

- (i) If a distance-nonincreasing onto map f preserves the volume of the total spaces, then f and f^{-1} preserve volumes of any subsets (see [Lemma 1.1](#)).
- (ii) Based on a local bi-Lipschitz embedding property (see [Lemma 1.2](#)), we show that for δ suitably small, f is injective on $f^{-1}(X^\delta) \subseteq Z^{\tau(\delta)}$. In particular, for any curve $c \subset X^\delta$, $f^{-1}(c) \subseteq Z^{\tau(\delta)}$ is a curve (see [Lemma 1.3](#)).
- (iii) Our main technical lemma is a volume formula for a tube of ϵ -balls (which can be treated as a replacement for an ϵ -tube around a curve, see [Lemma 1.4](#)). Together with (i) and (ii), this formula implies that f^{-1} preserves the length of any geodesic in X^δ up to an error $\tau(\delta)$. Because for any small $\delta < 1/(8n)$, the set X^δ is dense in X (see [Lemma 1.6](#)), we are able to show that f is also distance nondecreasing and thus f is an isometry.

Lemma 1.1. *Let $f : Z \rightarrow X$ be a distance-nonincreasing onto map of two metric spaces of equal Hausdorff dimension. If $\text{vol } X = \text{vol } Z$, then, for any subset $A \subseteq Z$ and $B \subseteq X$,*

$$\text{vol } A = \text{vol } f(A), \quad \text{vol } B = \text{vol } f^{-1}(B).$$

Proof. We argue by contradiction. If $\text{vol } A > \text{vol } f(A)$, then

$$\text{vol } Z = \text{vol } A + \text{vol}(Z - A) > \text{vol } f(A) + \text{vol } f(Z - A) \geq \text{vol } f(Z) = \text{vol } X,$$

a contradiction. Similarly, one can check that $\text{vol } f^{-1}(B) = \text{vol } B$. \square

Let $X^\delta(\rho)$ denote the union of points with an (n, δ) -strainer $\{(a_i, b_i)\}$ of radius $\rho > 0$, where $\rho = \min_{1 \leq i \leq n} \{|pa_i|, |pb_i|\} > 0$.

Lemma 1.2 [[Burago et al. 1992](#), Theorem 9.4]. *Let $X \in \text{Alex}^n(\kappa)$. If $p \in X^\delta(\rho)$, the map $\psi : X \rightarrow \mathbb{R}^n$ defined by $\psi(x) = (|a_1x|, \dots, |a_nx|)$ maps a small neighborhood U of p $\tau(\delta, \delta_1)$ -almost isometrically onto a domain in \mathbb{R}^n , that is,*

$$\left| |\psi(x)\psi(y)| - |xy| \right| < \tau(\delta, \delta_1)|xy|$$

for any $x, y \in U$, where $\delta_1 = \rho^{-1} \text{diam}(U)$. In particular, ψ is a $\tau(\delta)$ -almost isometric embedding when restricting to $B_{\delta\rho}(p)$.

A consequence of [Lemma 1.2](#) is that

$$1 - \tau(\delta) \leq \frac{\text{vol } B_\epsilon(p)}{\text{vol } B_\epsilon(\mathbb{R}^n)} \leq 1 + \tau(\delta)$$

for any $p \in X^\delta(\rho)$ and $\epsilon \leq \delta\rho$.

Lemma 1.3. *Let the assumptions be as in Theorem A. Then $f : f^{-1}(X^\delta) \rightarrow X^\delta$ is injective. Consequently, if $\gamma \subset X^\delta$ is a continuous curve, $f^{-1}(\gamma)$ is also a continuous curve.*

Proof. We argue by contradiction, assuming $z_1 \neq z_2 \in f^{-1}(X^\delta)$ such that $f(z_1) = f(z_2) = x$. We may assume that z_1 and z_2 have $\tau(\delta)$ -strainers of radius $\rho > 0$. Choose $4\epsilon < |z_1 z_2|$ and $\epsilon < \delta\rho$. By Lemma 1.1 and the above consequence of Lemma 1.2, we get

$$1 = \frac{\text{vol } f^{-1}(B_\epsilon(x))}{\text{vol } B_\epsilon(x)} \geq \frac{\text{vol } B_\epsilon(z_1) + \text{vol } B_\epsilon(z_2)}{\text{vol } B_\epsilon(x)} \geq 2(1 - \tau(\delta)),$$

a contradiction. □

We now develop a formula which estimates the volume of an ϵ -ball tube with a higher order error. Let x_1, x_2, \dots, x_{N+1} be $N + 1$ points in $X^\delta(\rho)$. We first give an estimate of the volume of the ϵ -ball tube $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ in terms of $\sum_{i=1}^N |x_i x_{i+1}|$ and ϵ, δ with errors.

Lemma 1.4 (volume of an ϵ -ball tube). *Let $X \in \text{Alex}^n(\kappa)$ and $x_i \in X^\delta(\rho)$, $i = 1, 2, \dots, N + 1$ satisfy that $0 < |x_i x_{i+1}| < 2\epsilon \ll \delta\rho$ and $B_\epsilon(x_i) \cap B_\epsilon(x_j) \cap B_\epsilon(x_k) = \emptyset$ for $i \neq j \neq k$. Then the volume of the ϵ -ball tube $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ (see Figure 1) satisfies*

$$(1-1) \quad (1 + \tau(\delta)) \text{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i) = \text{vol } B_\epsilon(\mathbb{R}^n) + 2\epsilon \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \sum_{i=1}^N \int_{\theta_i}^{\pi/2} \sin^n(t) dt,$$

where $\theta_i \in [0, \pi/2]$ such that $\cos \theta_i = |x_i x_{i+1}|/(2\epsilon)$. If, in addition, $|x_i x_{i+1}| \leq \epsilon^2$ for all $1 \leq i \leq N$,

$$(1-2) \quad (1 + \tau(\delta)) \text{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i) = \text{vol } B_\epsilon(\mathbb{R}^n) + \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^N |x_i x_{i+1}|.$$

Because $B_\epsilon(x_{i-1}) \cup B_\epsilon(x_i) \cup B_\epsilon(x_{i+1}) \subset B_{\delta\rho}(x_i)$, which is $\tau(\delta)$ -almost isometrically embedded into \mathbb{R}^n , one can divide $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ into small pieces $\Gamma^\pm(x_i)$, whose volumes are $(1 + \tau(\delta))$ -proportional to the volumes of the following ‘‘trapezoidal balls’’

$$\Gamma_\epsilon^{h_i^\pm}(\mathbb{R}^n)$$

in \mathbb{R}^n . This allows us to reduce the calculation to Euclidean space.

We define the trapezoidal ball $\Gamma_r^h(\mathbb{R}^n)$ in $\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) : x_n \geq 0\}$ in the following way. Let $u \in \mathbb{R}_+^n$ be a point with $|ou| = h \leq r$. Then the hyper plane H

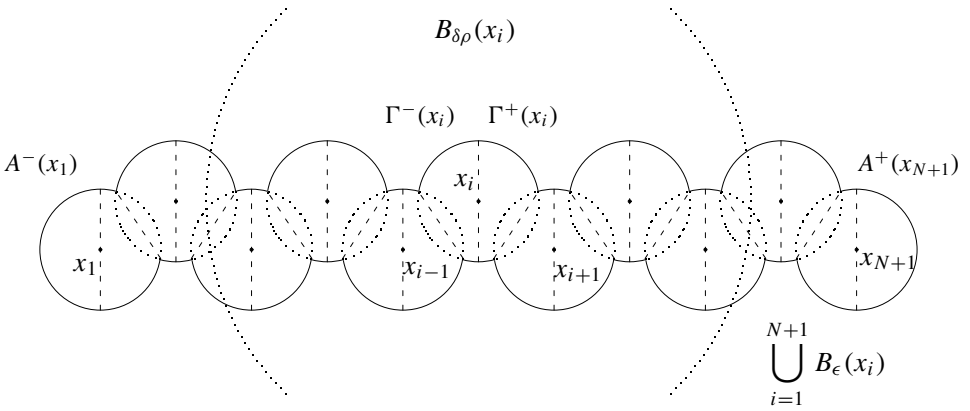


Figure 1

passing through u and perpendicular to $\vec{\partial u}$ divides the half ball $B_r(\mathbb{R}^n) \cap \mathbb{R}_+^n$ into two subsets. Let $\Gamma_r^h(\mathbb{R}^n)$ be the subset which contains the origin (see Figure 3). It's easy to see that $\text{vol } \Gamma_r^h(\mathbb{R}^n)$ depends only on h and r , and not on the direction $\vec{\partial u}$, as long as $H \cap B_r(\mathbb{R}^n) \subset \mathbb{R}_+^n$.

Lemma 1.5. *Let $\Gamma_r^h(\mathbb{R}^n)$ be a trapezoidal ball defined as above. Then*

$$\text{vol } \Gamma_r^h(\mathbb{R}^n) = r \text{vol } B_r(\mathbb{R}^{n-1}) \int_{\theta}^{\pi/2} \sin^n(t) dt,$$

where $\theta \in [0, \pi/2]$ such that $r \cos \theta = h$.

Proof. Let $s = r \cos t \in [0, h]$ be the parameter for the height with the corresponding angle $t \in [\theta, \pi/2]$. Then

$$\begin{aligned} \text{vol } \Gamma_r^h(\mathbb{R}^n) &= \int_0^h \text{vol } B_{r \sin t}(\mathbb{R}^{n-1}) ds = \int_{\theta}^{\pi/2} \text{vol } B_{r \sin t}(\mathbb{R}^{n-1}) r \sin(t) dt \\ &= r \text{vol } B_r(\mathbb{R}^{n-1}) \int_{\theta}^{\pi/2} \sin^n(t) dt. \end{aligned} \quad \square$$

Proof of the volume formula, Lemma 1.4. Because $B_{\epsilon}(x_i) \cap B_{\epsilon}(x_{i+1}) \neq \emptyset$ and $B_{\epsilon}(x_i) \cap B_{\epsilon}(x_j) \cap B_{\epsilon}(x_k) = \emptyset$ for any $i \neq j \neq k$, we can decompose $\bigcup_{i=1}^{N+1} B_{\epsilon}(x_i)$ as the following (see Figure 2): let

$$A^+(x_i) = \{q \in B_{\epsilon}(x_i) : |qx_i| \leq |qx_{i+1}|\}, \quad A^-(x_i) = \{q \in B_{\epsilon}(x_i) : |qx_i| \leq |qx_{i-1}|\}.$$

For $i = 2, 3, \dots, N$, let

$$\begin{aligned} H^+(x_i) &= A^+(x_i) \cap A^-(x_{i+1}) = \{q \in B_{\epsilon}(x_i) \cap B_{\epsilon}(x_{i+1}) : |qx_i| = |qx_{i+1}|\}, \\ H^-(x_i) &= A^-(x_i) \cap A^+(x_{i-1}) = \{q \in B_{\epsilon}(x_i) \cap B_{\epsilon}(x_{i-1}) : |qx_i| = |qx_{i-1}|\}, \end{aligned}$$

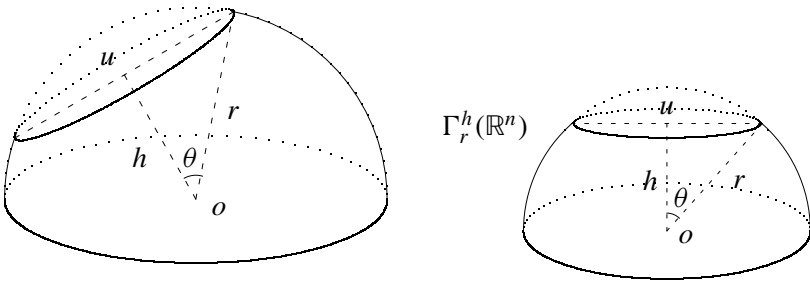


Figure 2

and

$$\Gamma^+(x_i) = \{q \in A^+(x_i) \cap A^-(x_i) : d(q, H^+(x_i)) \leq d(q, H^-(x_i))\},$$

$$\Gamma^-(x_i) = \{q \in A^+(x_i) \cap A^-(x_i) : d(q, H^+(x_i)) \geq d(q, H^-(x_i))\}.$$

By the construction,

$$\bigcup_{i=1}^{N+1} B_\epsilon(x_i) = A^-(x_1) \cup \left(\bigcup_{i=2}^N \Gamma^\pm(x_i) \right) \cup A^+(x_{N+1}).$$

Note that $H^\pm(x_i), i = 2, \dots, N$ consist of all the possible intersections of any two of $A^-(x_1), \Gamma^\pm(x_i), i = 2, \dots, N,$ and $A^+(x_{N+1})$ and $\text{vol } H^\pm(x_i) = 0$. We have

$$(1-3) \quad \text{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i) = \text{vol } A^-(x_1) + \text{vol } A^+(x_{N+1}) + \sum_{i=2}^N \text{vol } \Gamma^+(x_i) + \sum_{i=2}^N \text{vol } \Gamma^-(x_i).$$

Because $B_\epsilon(x_{i-1}) \cup B_\epsilon(x_i) \cup B_\epsilon(x_{i+1}) \subset B_{\delta\rho}(x_i)$, which is homeomorphically and $\tau(\delta)$ -almost isometrically embedded into \mathbb{R}^n , we have that

$$(1 + \tau(\delta)) \text{vol } \Gamma^\pm(x_i) = \text{vol } \Gamma_\epsilon^{h_i^\pm}(\mathbb{R}^n),$$

$$(1 + \tau(\delta)) \text{vol } A^+(x_1) = \frac{1}{2} \text{vol } B_\epsilon(\mathbb{R}^n) + \text{vol } \Gamma_\epsilon^{h_1^+}(\mathbb{R}^n),$$

$$(1 + \tau(\delta)) \text{vol } A^-(x_{N+1}) = \frac{1}{2} \text{vol } B_\epsilon(\mathbb{R}^n) + \text{vol } \Gamma_\epsilon^{h_{N+1}^-}(\mathbb{R}^n),$$

where $h_i^+ = \frac{1}{2}|x_i x_{i+1}|, h_i^- = \frac{1}{2}|x_i x_{i-1}|$. Note that it's our convention that the same symbol $\tau(\delta)$ may represent different functions of δ , as long as $\tau(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Together with (1-3) and the fact that $h_i^+ = h_{i+1}^-$, we get

$$(1-4) \quad (1 + \tau(\delta)) \text{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i) = \text{vol } B_\epsilon(\mathbb{R}^n) + 2 \sum_{i=1}^N \text{vol } \Gamma_\epsilon^{h_i^+}(\mathbb{R}^n).$$

Let $\theta_i \in [0, \pi/2]$ such that $\cos \theta_i = h_i^+/\epsilon = |x_i x_{i+1}|/(2\epsilon)$. By [Lemma 1.5](#), we have

$$\text{vol } \Gamma_\epsilon^{h_i^+}(\mathbb{R}^n) = \epsilon \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \int_{\theta_i}^{\pi/2} \sin^n(t) dt.$$

Plugging this into [\(1-4\)](#), we get [\(1-1\)](#).

To get [\(1-2\)](#), we need to write $\int_{\theta_i}^{\pi/2} \sin^n(t) dt$ in terms of $|x_i x_{i+1}|$. Let

$$g(s) = \int_\theta^{\pi/2} \sin^n(t) dt,$$

where $\theta \in [0, \pi/2]$ with $\cos \theta = s/(2\epsilon)$. Noting that $\theta = \pi/2$ if and only if $s = 0$, we have $g(0) = 0$. Furthermore,

$$\begin{aligned} g'(s) &= -\sin^n \theta \cdot \frac{d\theta}{ds} = -\sin^n \theta \cdot \frac{1}{-2\epsilon \sin \theta} = \frac{\sin^{n-1} \theta}{2\epsilon}; \\ g''(s) &= \frac{1}{2\epsilon} (n-1) \sin^{n-2} \theta \cos \theta \cdot \frac{1}{-2\epsilon \sin \theta} = \frac{n-1}{-4\epsilon^2} \sin^{n-3} \theta \cos \theta; \end{aligned}$$

and thus $g'(0) = 1/(2\epsilon)$, $g''(0) = 0$, and $g'''(0) = c_n/\epsilon^3$. The Taylor expansion of g at $s = 0$ is

$$g(s) = \int_\theta^{\pi/2} \sin^n(t) dt = 0 + \frac{s}{2\epsilon} + \frac{1}{\epsilon^3} \cdot O(s^3).$$

Letting $s = |x_i x_{i+1}| \leq \epsilon^2$, we get

$$\int_{\theta_i}^{\pi/2} \sin^n(t) dt = \frac{1}{2\epsilon} |x_i x_{i+1}| + O(\epsilon) |x_i x_{i+1}|.$$

Plugging this into [\(1-1\)](#), we get [\(1-2\)](#). □

In the rest of this section we assume that $f : Z \rightarrow X$ is a distance-nonincreasing onto map such that $f^{-1}(X^\delta) \subset Z^{\tau(\delta)}$. By [Lemma 1.3](#), f is homeomorphic on $f^{-1}(X^\delta)$.

Lemma 1.6. *Let the assumptions be as in [Theorem A](#). Let $x, y \in X^\delta$. For $\delta > 0$ sufficiently small, there exists a small constant $c = c(\rho, \delta) > 0$ such that if $|xy| \leq c$, $|f^{-1}(x)f^{-1}(y)| \leq 2|xy|$.*

Proof. Assume that $|xy| = \epsilon \ll \delta\rho$ and $|f^{-1}(x)f^{-1}(y)| > 2\epsilon$. Consider the metric balls $B_\epsilon(x)$ and $B_\epsilon(y)$. By [Lemma 1.4](#),

$$\begin{aligned} (1 + \tau(\delta)) \text{vol}(B_\epsilon(x) \cup B_\epsilon(y)) \\ = \text{vol } B_\epsilon(\mathbb{R}^n) + 2\epsilon \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \int_{\pi/3}^{\pi/2} \sin^n(t) dt + O(\epsilon^{n+1}). \end{aligned}$$

Since $B_\epsilon(f^{-1}(x)) \cap B_\epsilon(f^{-1}(y)) = \emptyset$, we have

$$(1 + \tau(\delta)) \text{vol}(B_\epsilon(f^{-1}(x)) \cup B_\epsilon(f^{-1}(y))) = 2 \text{vol } B_\epsilon(\mathbb{R}^n).$$

Because f is distance-nonincreasing,

$$B_\epsilon(f^{-1}(x)) \cup B_\epsilon(f^{-1}(y)) \subset f^{-1}(B_\epsilon(x) \cup B_\epsilon(y)).$$

Together with the fact that f^{-1} is volume-preserving, we get

$$\begin{aligned} 1 &= \frac{\text{vol } f^{-1}(B_\epsilon(x) \cup B_\epsilon(y))}{\text{vol}(B_\epsilon(x) \cup B_\epsilon(y))} \\ &\geq \frac{(1 - \tau(\delta)) \cdot 2 \text{vol } B_\epsilon(\mathbb{R}^n)}{\text{vol } B_\epsilon(\mathbb{R}^n) + 2\epsilon \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \int_{\pi/3}^{\pi/2} \sin^n(t) dt + O(\epsilon^{n+1})} \\ &= \frac{(1 - \tau(\delta)) \cdot 2 \int_0^{\pi/2} \sin^n(t) dt}{\int_0^{\pi/2} \sin^n(t) dt + \int_{\pi/3}^{\pi/2} \sin^n(t) dt + O(\epsilon)}. \end{aligned}$$

(See [Lemma 1.5](#), $\theta = 0$.) This leads to a contradiction for sufficiently small ϵ and δ . □

In the proof of [Theorem A](#), we will need the following result.

Lemma 1.7 [[Burago et al. 1992](#), 10.6.1]. *Let $X \in \text{Alex}^n(\kappa)$. For a fixed sufficiently small $\delta > 0$, the union of interior points which do not admit any (n, δ) -strainer has Hausdorff dimension $\leq n - 2$. In particular, X^δ is dense.*

Proof of Theorem A. Since f is distance-nonincreasing, it suffices to show that f is distance nondecreasing, that is, for any $\tilde{a}, \tilde{b} \in Z$, $|ab| \geq |\tilde{a}\tilde{b}|$, where $a = f(\tilde{a})$ and $b = f(\tilde{b})$.

For any small ϵ_1 , by [Lemma 1.7](#), there are $\tilde{a}_{\epsilon_1}, \tilde{b}_{\epsilon_1} \in Z^{\tau(\delta)}$, $a_{\epsilon_1} = f(\tilde{a}_{\epsilon_1})$, $b_{\epsilon_1} = f(\tilde{b}_{\epsilon_1}) \in X^\delta$, such that $|aa_{\epsilon_1}| \leq |\tilde{a}\tilde{a}_{\epsilon_1}| < \epsilon_1$, $|bb_{\epsilon_1}| \leq |\tilde{b}\tilde{b}_{\epsilon_1}| < \epsilon_1$.

Case 1. Assume that there exists a minimal geodesic $[a_{\epsilon_1}b_{\epsilon_1}] \subset X$. Then, because the spaces of directions are isometric along the interior of a geodesic, $[a_{\epsilon_1}b_{\epsilon_1}] \subset X^{2\delta}$ [[Petrunin 1998](#)]. By [Lemma 1.3](#) (which will be frequently used without mention), $f^{-1}([a_{\epsilon_1}b_{\epsilon_1}])$ is also a continuous curve. Because $[a_{\epsilon_1}b_{\epsilon_1}]$ is compact, we may let $\rho > 0$ such that $[a_{\epsilon_1}b_{\epsilon_1}] \subset X^{2\delta}(\rho)$ and $f^{-1}([a_{\epsilon_1}b_{\epsilon_1}]) \subset Z^{\tau(\delta)}(\rho)$. Let $\{x_i\}_{i=1}^{N+1}$ be an ϵ -partition of $[a_{\epsilon_1}b_{\epsilon_1}]$, where $x_1 = a_{\epsilon_1}$, $x_{N+1} = b_{\epsilon_1}$ for $\epsilon \ll \delta\rho$. Because $[a_{\epsilon_1}b_{\epsilon_1}]$ is a geodesic, [Lemma 1.4](#) can be applied on the partition $\{x_i\}_{i=1}^{N+1}$. Thus we get

$$\begin{aligned} (1 + \tau(\delta)) \text{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i) &= \text{vol } B_\epsilon(\mathbb{R}^n) + \text{vol } B_\epsilon(\mathbb{R}^{n-1}) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^N |x_i x_{i+1}| \\ &= \text{vol } B_\epsilon(\mathbb{R}^{n-1}) |a_{\epsilon_1}b_{\epsilon_1}| + O(\epsilon^n). \end{aligned}$$

Let $z_i = f^{-1}(x_i)$. By [Lemma 1.6](#), $|z_i z_{i+1}| \leq 2|x_i x_{i+1}| = 2\epsilon$. Together with the fact that f is distance-nonincreasing, one can easily check that $\bigcup_{i=1}^{N+1} B_\epsilon(z_i)$ satisfies

the condition of [Lemma 1.4](#). Then we have

$$(1 + \tau(\delta)) \operatorname{vol} \bigcup_{i=1}^{N+1} B_\epsilon(z_i) = \operatorname{vol} B_\epsilon(\mathbb{R}^{n-1}) \sum_{i=1}^N |z_i z_{i+1}| + O(\epsilon^n).$$

Because f is distance-nonincreasing and volume-preserving,

$$\begin{aligned} 1 &= \frac{\operatorname{vol} f^{-1}(\bigcup_{i=1}^{N+1} B_\epsilon(x_i))}{\operatorname{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i)} \geq \frac{\operatorname{vol} \bigcup_{i=1}^{N+1} B_\epsilon(z_i)}{\operatorname{vol} \bigcup_{i=1}^{N+1} B_\epsilon(x_i)} \\ &= (1 - \tau(\delta)) \frac{\operatorname{vol} B_\epsilon(\mathbb{R}^{n-1}) \sum_{i=1}^N |z_i z_{i+1}| + O(\epsilon^n)}{\operatorname{vol} B_\epsilon(\mathbb{R}^{n-1}) |a_{\epsilon_1} b_{\epsilon_1}| + O(\epsilon^n)}, \\ &= (1 - \tau(\delta)) \frac{\sum_{i=1}^N |z_i z_{i+1}| + O(\epsilon)}{|a_{\epsilon_1} b_{\epsilon_1}| + O(\epsilon)} \\ &\geq (1 - \tau(\delta)) \frac{|\tilde{a}_{\epsilon_1} \tilde{b}_{\epsilon_1}| + O(\epsilon)}{|a_{\epsilon_1} b_{\epsilon_1}| + O(\epsilon)}. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get

$$|a_{\epsilon_1} b_{\epsilon_1}| \geq (1 - \tau(\delta)) |\tilde{a}_{\epsilon_1} \tilde{b}_{\epsilon_1}|.$$

Case 2. Assume that there is no minimal geodesic in X^δ from a_{ϵ_1} to b_{ϵ_1} (since X may not be complete). Because spaces of directions along the interior of a geodesic are isometric to each other [[Petrinin 1997](#)], we may assume a curve c_1 in X^δ from a_{ϵ_1} to b_{ϵ_1} such that $L(c_1) < |a_{\epsilon_1} b_{\epsilon_1}| + \epsilon_1$. Since $c_1(t)$ is a compact subset in the open set X^δ , we may assume $\eta > 0$ such that an η -tube of c_1 is also contained in X^δ . Consequently, we may assume a piecewise geodesic c in X^δ such that $L(c) \leq L(c_1) \leq |a_{\epsilon_1} b_{\epsilon_1}| + \epsilon_1$. Applying Case 1 to each geodesic segment of c , we conclude that

$$|a_{\epsilon_1} b_{\epsilon_1}| \geq L(c) - \epsilon_1 \geq (1 - \tau(\delta)) |\tilde{a}_{\epsilon_1} \tilde{b}_{\epsilon_1}| - \epsilon_1.$$

In either Case 1 or Case 2, we have

$$\begin{aligned} |ab| &\geq |a_{\epsilon_1} b_{\epsilon_1}| - 2\epsilon_1 \geq (1 - \tau(\delta)) |\tilde{a}_{\epsilon_1} \tilde{b}_{\epsilon_1}| - 3\epsilon_1 \\ &\geq (1 - \tau(\delta)) (|\tilde{a}\tilde{b}| - 2\epsilon_1) - 3\epsilon_1. \end{aligned}$$

Letting $\delta \rightarrow 0$, $\epsilon_1 \rightarrow 0$, we get $|ab| \geq |\tilde{a}\tilde{b}|$. □

2. Proof of Theorem B: Relatively maximum volume

Our proof of the classification part in [Theorem B](#) is divided into the following two theorems: open ball rigidity ([Theorem 2.1](#)) and isometric involution ([Theorem 2.2](#)). Recall that \tilde{o} denotes the vertex of the cone $\bar{C}_k^R(\Sigma_p)$ and thus $g \exp_p(\tilde{o}) = p$.

Theorem 2.1. *Under the assumptions of [Theorem B](#),*

$$g \exp_p : C_\kappa^R(\Sigma) \rightarrow B_R(p)$$

is an isometry with respect to the intrinsic metrics. In particular, $g \exp_p = \exp_p$.

By [Theorem 2.1](#), $X = \bar{C}_\kappa^R(\Sigma_p)/x \sim x'$, where the equivalent relation $x \sim x'$ if and only if $\exp_p x = \exp_p x'$ and $x, x' \in \Sigma_p \times \{R\}$.

Theorem 2.2. *Let $X = \bar{C}_\kappa^R(\Sigma_p)/x \sim x' \in \text{Alex}^n(\kappa)$ be defined as above. Then each equivalent class contains at most two points. Moreover, the induced involution $\phi : \Sigma_p \times \{R\} \rightarrow \Sigma_p \times \{R\}$, $\phi(x) = x'$ (where $x \sim x'$) is an isometry.*

Recall that the induced gradient-exponential map $g \exp_p : \bar{C}_\kappa^R(\Sigma) \rightarrow \bar{B}_R(p) = X$ is distance-nonincreasing and onto. Indeed, the open ball rigidity is essentially a consequence of [Theorem A](#) and the general property that $\exp_p^{-1} : X \rightarrow T_p X : \exp_p^{-1}$ preserves (n, δ) -strained points up to a constant depending on δ (see [Lemma 2.4](#)). In the proof, let's recall the following property from [[Burago et al. 1992](#)]:

Lemma 2.3 [[Burago et al. 1992](#), Lemmas 7.5 and 11.2]. *Let $p \in X \in \text{Alex}^n(\kappa)$. Then, for any $\delta > 0$, there is a small neighborhood U_p of p such that, for any triangle $\triangle pab$ with $a, b \in U_p$, each angle of $\triangle pab \subset X$ differs from the comparison angle of $\tilde{\triangle pab} \subset \mathbb{S}_\kappa^2$ by less than δ .*

Lemma 2.4. *Let $q \in X^\delta$. Then for any $p \in X$, $\uparrow_p^q \in \Sigma_p^{\tau(\delta)}$. Consequently,*

$$\exp_p^{-1}(q) \in \bar{C}_\kappa^R(\Sigma_p)^{\tau(\delta)}.$$

Proof. Since $q \in X^\delta$, by [Lemma 1.2](#), we may assume an $(n, 2\delta)$ -strainer $\{(a_i, b_i)\}$ for $q_1 \in [pq]$ and near q , such that $b_n = q$, $a_n \in [pq_1]$. Because the spaces of directions are isometric along the interior of a geodesic [[Petrunin 1998](#)], there is $q' \in [pq] \cap U_p$ which has an $(n, \tau(\delta))$ -strainer $\{(a'_i, b'_i)\}$. By the same reason as above, we can assume that $a'_n \in [pq']$ and $b'_n \in [q'q]$.

In addition, we can assume that $|q'a'_i|, |q'b'_i|$ are short so that $a'_i, b'_i \in U_p$ and $\angle a'_i p q', \angle b'_i p q' < 5\delta$. We claim that

$$\left\{ \left(\uparrow_p^{a'_i}, \uparrow_p^{b'_i} \right) \right\}_{i=1}^{n-1}$$

forms an $(n-1, \tau(\delta))$ -strainer at $\uparrow_p^q \in \Sigma_p$. It's easy to see that

$$\angle a'_i p q' = \tilde{\angle} a'_i p q' + \tau(\delta) = \frac{|a'_i q'|}{|p q'|} + \tau(\delta).$$

Thus

$$\cos \tilde{\angle} \uparrow_p^{a'_i} \uparrow_p^{q'} \uparrow_p^{x_j} = \frac{|a'_i q'|^2 + |x_j q'|^2 - |a'_i x_j|}{2|a'_i q'| |x_j q'|} + \tau(\delta) = \cos \tilde{\angle} a'_i q' x_j + \tau(\delta),$$

where $i, j = 1, 2, \dots, n-1$, $x_j = a'_j$ or b'_j . □

To conclude the open ball rigidity by applying [Theorem A](#), we need to check that $g \exp_p^{-1}(X^\delta) \subseteq \bar{C}_\kappa^R(\Sigma_p)^\tau(\delta)$. We do this by showing that $g \exp_p = \exp_p$ when $\text{vol } X = v(\Sigma_p, \kappa, R)$.

Lemma 2.5. *If $\text{vol } B_R(p) = \text{vol } C_\kappa^R(\Sigma_p)$, the gradient exponential map is actually an exponential map $\exp_p : \bar{C}_\kappa^R(\Sigma_p) \rightarrow \bar{B}_R(p)$ which preserves the distance along the radial direction.*

Proof. Clearly, the map $\exp_p^{-1} : \bar{B}_R(p) \rightarrow \bar{C}_\kappa^R(\Sigma_p)$ (If there is more than one image, we will pick one) is distance nondecreasing. Because

$$\text{vol } C_\kappa^R(\Sigma_p) = \text{vol } X \leq \text{vol } \exp_p^{-1}(X) \leq \text{vol } C_\kappa^R(\Sigma_p),$$

$\exp_p^{-1}(X)$ is dense in $C_\kappa^R(\Sigma_p)$. For any $z \in C_\kappa^R(\Sigma_p)$, there is a sequence $x_i \in X$, such that $\exp_p^{-1}(x_i) = z_i \rightarrow z$. Let $\exp_p : C_\kappa^R(\Sigma_p) \rightarrow X$; $\exp_p(z) = \lim_{i \rightarrow \infty} x_i$. Such an \exp_p is well defined, since if there is another sequence $\exp_p^{-1}(x'_i) = z'_i \rightarrow z$,

$$d(\lim_{i \rightarrow \infty} x_i, \lim_{i \rightarrow \infty} x'_i) = \lim_{i \rightarrow \infty} d(x_i, x'_i) \leq \lim_{i \rightarrow \infty} d(z_i, z'_i) = 0.$$

It's clear that \exp_p , defined as an extension of \exp_p^{-1} , is distance-nonincreasing. Moreover, it preserves the distance along the radial direction.

We now show that any geodesic from $p = \exp_p(\tilde{\delta})$ to $q = \exp_p(\tilde{q}) \in B_R(p)$ can be extended. Therefore \exp_p is a bijection, since geodesics do not bifurcate. Let $[\tilde{\delta}\tilde{q}]$ be the geodesic in $C_\kappa^R(\Sigma_p)$ such that $\exp_p([\tilde{\delta}\tilde{q}]) = [pq]$, and $\tilde{q}' \in C_\kappa^R(\Sigma_p)$ the extended point of $[\tilde{\delta}\tilde{q}]$. Then

$$|pq| + |qq'| \leq |\tilde{\delta}\tilde{q}| + |\tilde{q}\tilde{q}'| = |\tilde{\delta}\tilde{q}'| = |pq'|,$$

which forces $[pq] \cup [qq']$ to be a geodesic. \square

Proof of [Theorem 2.1](#). For $X \in \mathcal{A}_\kappa^R(\Sigma)$ with $\text{vol } X = v(\Sigma, \kappa, R)$, by [Lemmas 2.4](#) and [2.5](#), we see that $\exp_p : C_\kappa^R(\Sigma) \rightarrow B_R(p)$ is a distance-nonincreasing onto map that satisfies the assumptions in [Theorem A](#) (note that $\exp_p : \bar{C}_\kappa^R(\Sigma_p) \rightarrow \bar{B}_R(p) = X$ may not satisfy the assumptions of [Theorem A](#)). \square

In the proof of [Theorem 2.2](#), our main technical lemma is [Lemma 2.12](#). Let $\phi : \Sigma \times \{R\} \rightarrow \Sigma \times \{R\}$ be defined as in [Theorem 2.2](#). We first observe that ϕ is an involution. Let $L_p(X) = \exp_p(\Sigma \times \{R\}) = \{x \in X : |px| = R\}$.

Lemma 2.6. *Let $X = \bar{C}_\kappa^R(\Sigma)/x \sim x' \in \text{Alex}^n(\kappa)$ be defined as in [Theorem 2.2](#). For any $q \in L_p(X)$, if $\tilde{q}_1 \neq \tilde{q}_2$ with $\exp_p(\tilde{q}_1) = \exp_p(\tilde{q}_2) = q$, then the loop $\exp_p([\tilde{\delta}\tilde{q}_1]) \cup \exp_p([\tilde{\delta}\tilde{q}_2])$ forms a local geodesic at q . Consequently, $\exp_p^{-1}(q)$ contains at most two points.*

Proof. It's clear that $\exp_p([\tilde{\delta}\tilde{q}_i])$ are minimal geodesics, $i = 1, 2$. Let $x_i \in X$ be a point on $\exp_p([\tilde{\delta}\tilde{q}_i])$ and $\tilde{x}_i = \exp_p^{-1}(x_i)$, $i = 1, 2$. We claim that if x_1, x_2

are both close enough to q , the geodesic $[x_1x_2]$ intersects with $L_p(X)$. If not, $[x_1x_2] \subset B_R(p)$. By the assumption, $|x_1x_2|_X = |\tilde{x}_1\tilde{x}_2|_{\bar{C}_\kappa^R(\Sigma)}$. Let $x_1, x_2 \rightarrow q$. We get that $|x_1x_2|_X \rightarrow 0$ and $|\tilde{x}_1\tilde{x}_2|_{\bar{C}_\kappa^R(\Sigma)} \rightarrow |\tilde{q}_1\tilde{q}_2|_{\bar{C}_\kappa^R(\Sigma)} > 0$, a contradiction.

Let $a \in [x_1x_2] \cap L_p(X)$. It remains to show that $a = q$. For $i = 1, 2$,

$$|x_i a| \geq |pa| - |px_i| = |pq| - |px_i| = |x_i q|.$$

Thus

$$|x_1 q| + |x_2 q| \leq |x_1 a| + |x_2 a| = |x_1 x_2|,$$

which forces both of the above inequalities to be equalities, and thus $a = q$. \square

As a corollary of [Lemma 2.6](#), we conclude that for $X \in \mathcal{A}_\kappa^R(\Sigma)$, $\kappa > 0$, and $\pi/(2\sqrt{\kappa}) < R < \pi/\sqrt{\kappa}$, $\text{vol } C_\kappa^R(\Sigma)$ is not the optimal upper bound for $\text{vol } X$; see [\[Grove and Petersen 1992\]](#). Equivalently, we have:

Corollary 2.7. *Assume $X \in \mathcal{A}_\kappa^R(\Sigma)$ with $\text{vol}(X) = \text{vol } \bar{C}_\kappa^R(\Sigma)$ and $\kappa > 0$. Then $R \leq \pi/(2\sqrt{\kappa})$ or $R = \pi/\sqrt{\kappa}$. In the second case, $X = C_\kappa(\Sigma)$ which is the k -suspension of Σ .*

Proof. Assume $\pi/(2\sqrt{\kappa}) < R < \pi/\sqrt{\kappa}$. Let $p \in X$ such that $\Sigma_p = \Sigma$. It's clear that $\text{rad}_p(X) = R$. We claim that $L_p(X) = \{q\}$ has only one point. Then by [Lemma 2.6](#), $\Sigma_p \times \{R\} = \exp_p^{-1}(q)$ contains at most two points, a contradiction. Let $a \neq b \in L_p(X)$. Consider the triangle $\triangle pab$ and the compared triangle $\tilde{\triangle} pab \in S_\kappa^2$. Take $c \in [ab]$ and the corresponding $\tilde{c} \in [\tilde{a}\tilde{b}]$ with $|ac| = |\tilde{a}\tilde{c}|$. By the triangle comparison, $|pc| \geq |\tilde{p}\tilde{c}| > R$, a contradiction. Note that the case where $R = \pi/\sqrt{\kappa}$ follows from [Theorem 2.1](#). \square

It remains to show that ϕ is an isometry. The following lemma plays an important role in the study of the angles in the gluing space X .

Lemma 2.8. *Let $a, b \in C_\kappa(\Sigma)$. Then $\angle apb = \tilde{\angle} apb$ and $\angle pab = \tilde{\angle} pab$.*

Proof. The proofs are essentially the same for different κ . For simplicity, we only give a proof for $\kappa = 0$. Note that $\angle apb = \tilde{\angle} apb$ by the definition of $C_\kappa(\Sigma)$.

To see $\angle pab = \tilde{\angle} pab$, shortly extend the geodesic $[pa]$ to a' and apply the cosine law to the triangles $\triangle aa'b$, $\triangle pa'b$, and $\triangle pab$. We get

$$(2-1) \quad |a'b|^2 = |aa'|^2 + |ab|^2 - 2|aa'| |ab| \cos \tilde{\angle} a'ab,$$

$$(2-2) \quad |a'b|^2 = |pa'|^2 + |pb|^2 - 2|pa'| |pb| \cos \angle apb \\ = (|pa| + |aa'|)^2 + |pb|^2 - 2(|pa| + |aa'|) |pb| \cos \angle apb,$$

$$(2-3) \quad |ab|^2 = |pa|^2 + |pb|^2 - 2|pa| |pb| \cos \angle apb.$$

Calculating (2-1) + (2-3) - (2-2), we get

$$\begin{aligned} 0 &= |ab| \cos \tilde{\angle} a'ab + |pa| - |pb| \cos \angle apb \\ &\geq |ab| \cos \angle a'ab + |pa| - |pb| \cos \angle apb \\ &= -|ab| \cos \angle pab + |pa| - |pb| \cos \angle apb. \end{aligned}$$

Since $\angle pab \geq \tilde{\angle} pab$ and $\angle apb = \tilde{\angle} apb$, the above inequality implies

$$\begin{aligned} |pa| &\leq |ab| \cos \angle pab + |pb| \cos \angle apb \\ &\leq |ab| \cos \tilde{\angle} pab + |pb| \cos \tilde{\angle} apb = |pa|, \end{aligned}$$

which forces $\angle pab = \tilde{\angle} pab$. \square

Corollary 2.9. *Let $x, y \in X$ be two points. If $[xy] \cap L_p(X) \neq \emptyset$, then either $[xy] \subset L_p(X)$ or $[xy] \cap L_p(X)$ is finite.*

Proof. Let $x \notin L_p(X)$. We show that $[xy] \cap L_p(X)$ is finite. Let $a \in [xy] \cap L_p(X)$ be the accumulation point which is closest to x . Clearly $a \neq x$ since $x \notin L_p(X)$. Thus there is a geodesic segment $[ba]$ of $[xy]$ with $[ba] - \{a\} \subset B_R(p)$. Since $|pb| < |pa| = R$, by Lemma 2.8,

$$\angle pab = \tilde{\angle} pab < \frac{\pi}{2}.$$

On the other hand, because there are $a_i \in [xy] \cap L_p(X)$ with $a_i \rightarrow a$ as $i \rightarrow \infty$ and $|pa| = |pa_i| = R$, by the first variation formula, we get

$$\angle pay = \frac{\pi}{2}.$$

Therefore $\pi = \angle pab + \angle pay < \pi$, a contradiction. \square

As another corollary, we prove Theorem 2.2 for the special case $\kappa > 0$ and $R = \pi/(2\sqrt{\kappa})$.

Corollary 2.10. *Theorem 2.2 holds for the case $\kappa > 0$ and $R = \pi/(2\sqrt{\kappa})$.*

Proof. Let $x, y \in L_p(X)$, $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2 \in \Sigma \times \{R\}$ with $\exp_p(\tilde{x}_1) = \exp_p(\tilde{x}_2) = x$, $\exp_p(\tilde{y}_1) = \exp_p(\tilde{y}_2) = y$. We will show that $|\tilde{x}_1\tilde{y}_1|_{\bar{C}_\kappa^R(\Sigma)} = |\tilde{x}_2\tilde{y}_2|_{\bar{C}_\kappa^R(\Sigma)}$. Assume $|\tilde{x}_1\tilde{y}_1|_{\bar{C}_\kappa^R(\Sigma)} > |\tilde{x}_2\tilde{y}_2|_{\bar{C}_\kappa^R(\Sigma)}$. Then there is a point $a \notin L_p(X)$ (take $\exp_p^{-1}(a)$ close to x_1) such that $[ay] \cap L_p(X)$ contains a point $b \neq y$. Because \exp_p is distance-nonincreasing and $\Sigma \times \{\pi/(2\sqrt{\kappa})\}$ is totally geodesic, $[by] \subset L_p(X)$, which contradicts Corollary 2.9. \square

Let $\text{Fix}(\phi) = \{\tilde{x} \in \Sigma \times \{R\} : \phi(\tilde{x}) = \tilde{x}\}$ be the fixed points set. Let $L_p^1(X) = \exp_p(\text{Fix}(\phi))$ denote the image. Due to Lemma 2.6, let $L_p^2(X) = L_p(X) - L_p^1(X)$ denote the points that are identified from exactly two points, that is, for any

$$x \in L_p^2(X),$$

$\exp^{-1}(x) = \{\tilde{x}^+, \tilde{x}^-\}$ contains exactly two points.

In the rest of the proof of [Theorem 2.2](#), by [Corollaries 2.9](#) and [2.10](#) and their proofs, we can always assume $R < \pi/(2\sqrt{\kappa})$ for $\kappa > 0$ and that for any $x, y \in X$, $[xy] \cap L_p(X)$ is finite if it is not empty. Moreover, the following corollary shows that $]xy[\cap L_p(X) \subset L_p^2(X)$, where $]xy[$ denotes the geodesic connecting x, y without the end points.

Corollary 2.11. *Let the assumption be as in [Theorem 2.2](#). Assume $R < \pi/(2\sqrt{\kappa})$ when $\kappa > 0$. For any $x, y \in X$, if $q \in]xy[\cap L_p(X)$, $q \in L_p^2(X)$.*

Proof. Without losing generality, we assume $x, y \notin L_p(X)$ and $]xy[\cap L_p(X) = \{q\}$. If $q \in L_p^1(X)$, by [Lemma 2.8](#), $\angle xqp = \tilde{\angle}xqp < \pi/2$ and $\angle yqp = \tilde{\angle}yqp < \pi/2$. Thus $\angle xqp + \angle yqp < \pi$, which contradicts the fact that $]xy[$ is a geodesic. \square

Now we are ready to prove our main technical lemma. Let $x \in L_p^2(X)$ and $\{\tilde{x}^+, \tilde{x}^-\} = \exp_p^{-1}(x)$ denote the preimage. Then there are exactly two geodesics $\exp_p([\tilde{\alpha}\tilde{x}^+])$, $\exp_p([\tilde{\alpha}\tilde{x}^-])$ connecting x to p . To distinguish geodesics and angles, we use the following notation.

- Let $[px^+]$ and $[px^-]$ denote $\exp_p([\tilde{\alpha}\tilde{x}^+])$ and $\exp_p([\tilde{\alpha}\tilde{x}^-])$ respectively.

In addition, for $y \in L_p^2(X)$ and $\exp_p^{-1}(y) = \{\tilde{y}^+, \tilde{y}^-\}$:

- let $[x^\pm y^\pm]$ denote $\exp_p([\tilde{x}^\pm \tilde{y}^\pm])$;
- let $|x^\pm y^\pm|$ denote the length of the geodesics $[x^\pm y^\pm]$;
- let $\angle x^\pm p y^\pm$ denote the angle between $[px^\pm]$ and $[py^\pm]$ at p ;
- let $\angle px^\pm y^\pm$ denote the angle between $[px^\pm]$ and $[x^\pm y^\pm]$ at x .

Lemma 2.12. *Let the assumptions be as in [Theorem 2.2](#). Assume $R < \pi/(2\sqrt{\kappa})$ when $\kappa > 0$. Then, for any $\tilde{x} \neq \tilde{y} \in \Sigma \times \{R\}$ with $|\tilde{x}\tilde{y}|$ sufficiently small,*

$$\left| \frac{|\phi(\tilde{x})\phi(\tilde{y})|}{|\tilde{x}\tilde{y}|} - 1 \right| \leq 20 |\tilde{x}\tilde{y}|.$$

Proof. For simplicity, we give a proof for the case $\kappa = 0$. The other cases can be carried out similarly. Throughout the proof, we will frequently use [lemmas 2.6](#) and [2.8](#) and [Corollary 2.11](#) without mentioning it. We will also assume that for any $a, b \in X$, $[ab] \cap L_p(X)$ is finite if it is not empty.

Clearly, ϕ preserves the distance when x and y are both in $L_p^1(X)$. Let

$$x \in L_p^2(X), \quad y \in L_p(X)$$

(if $y \in L_p^1(X)$, $\tilde{y}^+ = \tilde{y}^-$ will denote the same point and the argument will still go through). Because $[xy] \cap L_p(X)$ is finite, not losing generality, assume $[xy] =$

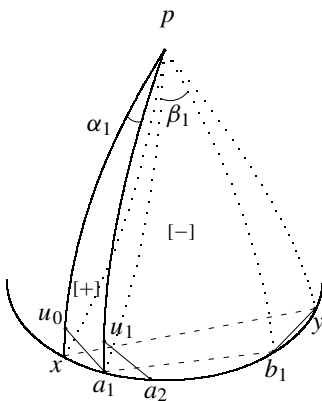


Figure 3

$[x^-y^-]$. Thus $\angle x^-py^- \leq \angle x^+py^+$. Let $\beta_0 = \angle x^-py^-$. Since $|x^-y^-| = 2R \sin \frac{\beta_0}{2}$ and $|x^+y^+| = 2R \sin(\angle x^+py^+/2)$, it's sufficient to show that

$$(2-4) \quad 10\beta_0^2 + \beta_0 \geq \angle x^+py^+.$$

Take $u_0 \in [px^+]$ with $|u_0x^+| = \epsilon$. Let $[u_0y]$ be a geodesic. If

$$[(u_0y)] \cap L_p(X) \neq \emptyset,$$

let $a_1 (\neq y)$ and b_1 (b_1 can be y) be the first and second intersection points in $[u_0y] \cap L_p(X)$ along the direction $\uparrow_{u_0}^y$ (see Figure 3). Assign \pm to $\exp_p^{-1}(a_1)$, $\exp_p^{-1}(b_1)$ such that $\angle pa_1^+u_0 < \pi/2$. Let $\alpha_1 = \angle x^+pa_1^+$ and $\beta_1 = \angle a_1^-pb_1^-$. In the case of $[(u_0y)] \cap L_p(X) = \emptyset$, we take $b_1 = a_1 = y$ and $\beta_1 = 0$.

Because $[u_0a_1^+] * [a_1^-b_1^-] * [b_1^-y]$ is a minimal geodesic, by triangle inequality,

$$|u_0x| + |xy| \geq |u_0a_1^+| + |a_1^-b_1^-| + |b_1y|.$$

This implies

$$(2-5) \quad \epsilon + 2R \sin \frac{\beta_0}{2} \geq |u_0a_1^+| + 2R \sin \frac{\beta_1}{2}.$$

Applying the cosine law (the form in Lemma 4.7(5)) in Δpu_0a_1 with the angle $\angle u_0pa_1^+ = \alpha_1$, we get that

$$|u_0a_1^+| = \sqrt{\epsilon^2 + 4R(R - \epsilon) \sin^2 \frac{\alpha_1}{2}} \geq 2(R - \epsilon) \sin \frac{\alpha_1}{2}.$$

Thus

$$(2-6) \quad \epsilon + 2R \sin \frac{\beta_0}{2} \geq 2(R - \epsilon) \sin \frac{\alpha_1}{2} + 2R \sin \frac{\beta_1}{2}.$$

If $[(u_0y)] \cap L_p(X) = \emptyset$, we stop here. If $[(u_0y)] \cap L_p(X) \neq \emptyset$, we proceed with $u_1 \in [pa_1^+]$ and $|u_1a_1| = \epsilon$. Let $[u_1b_1]$ be a geodesic. Again, if $[(u_1b_1)] \cap L_p(X) \neq \emptyset$, let $a_2 (\neq y)$ and b_2 (can be b_1) be the first and second intersection points in $[u_1b_1] \cap L_p(X)$ along the direction $\uparrow_{u_1}^{b_1}$. Assign \pm to $\exp_p^{-1}(a_2)$, $\exp_p^{-1}(b_2)$ such that $\angle pa_2^+u_1 < \pi/2$. Let $\alpha_2 = \angle a_1^+pa_2^+$ and $\beta_2 = \angle a_2^-pb_2^-$. If $[(u_1b_1)] \cap L_p(X) = \emptyset$, $a_2 = b_2 = b_1$, $\beta_2 = 0$, and we stop the process. Proceed inductively until $[(u_Nb_N)] \cap L_p(X) = \emptyset$, which yields that $a_{N+1} = b_{N+1} = b_N$ and $\beta_{N+1} = 0$. We claim that N is finite, and, moreover,

$$(2-7) \quad (N+1)\epsilon < 5R\beta_0^2.$$

For each $0 \leq i \leq N$, we have

$$(2-8) \quad \epsilon + 2R \sin \frac{\beta_i}{2} \geq |u_i a_{i+1}^+| + 2R \sin \frac{\beta_{i+1}}{2},$$

$$(2-9) \quad \epsilon + 2R \sin \frac{\beta_i}{2} \geq 2(R - \epsilon) \sin \frac{\alpha_{i+1}}{2} + 2R \sin \frac{\beta_{i+1}}{2},$$

where $\alpha_i = \angle a_i^+pa_{i+1}^+$, $\beta_i = \angle a_i^-pb_i^-$. Summing up (2-9) for $i = 0, 1, \dots, N$ and applying (2-7), we get

$$\begin{aligned} 5R\beta_0^2 + 2R \sin \frac{\beta_0}{2} &\geq (N+1)\epsilon + 2R \sin \frac{\beta_0}{2} \\ &\geq 2(R - \epsilon) \sum_{i=1}^N \sin \frac{\alpha_i}{2} \geq 2(R - \epsilon) \sin \frac{\sum_{i=1}^N \alpha_i}{2} \\ &\geq 2(R - \epsilon) \sin \frac{\angle x^+pb_N}{2}. \end{aligned}$$

Since $b_N \rightarrow b_1 \rightarrow y^+$ when taking $\epsilon \rightarrow 0$, (2-4) follows.

It remains to show (2-7). A sum of (2-8) for $i = 0, 1, \dots, N$ indicates that the upper bound of N relies on an estimate of $|u_i a_{i+1}^+|$ in terms of ϵ and β_{i+1} . Noting that $a_{i+1} = [u_i b_{i+1}] \cap ([pa_{i+1}^+] * [pa_{i+1}^-])$ and $[pa_{i+1}^+] * [pa_{i+1}^-]$ is a local geodesic at a_{i+1} , we have $\angle pa_{i+1}^+u_i = \angle pa_{i+1}^-b_{i+1} = \pi/2 - \beta_{i+1}/2$. Applying the cosine law in triangle $\triangle pu_i a_{i+1}^+$, we get

$$(R - \epsilon)^2 = R^2 + |u_i a_{i+1}^+|^2 - 2R|u_i a_{i+1}^+| \sin \frac{\beta_{i+1}}{2},$$

that is,

$$|u_i a_{i+1}^+|^2 - 2R \sin \frac{\beta_i}{2} |u_i a_{i+1}^+| + R\epsilon - \epsilon^2 = 0.$$

Solving for $|u_i a_{i+1}^+|$ and taking into account that $\epsilon > 0$ is small, we have

$$|u_i a_{i+1}^+| \geq R \sin \frac{\beta_{i+1}}{2} - \sqrt{\left(R \sin \frac{\beta_{i+1}}{2}\right)^2 - (R\epsilon - \epsilon^2)} > \frac{\epsilon}{4 \sin(\beta_{i+1}/2)}.$$

Note that β_i is decreasing, which is implied by (2-8) and $|u_i a_{i+1}^+| > |u_i a_i^+| = \epsilon$. We get

$$(2-10) \quad |u_i a_{i+1}^+| > \frac{\epsilon}{4 \sin(\beta_0/2)}.$$

Plugging (2-10) into (2-8), we get

$$(2-11) \quad \epsilon + 2R \sin \frac{\beta_i}{2} > \frac{\epsilon}{4 \sin(\beta_0/2)} + 2R \sin \frac{\beta_{i+1}}{2}.$$

Summing up (2-11) for $i = 0, 1, \dots, N$, we get

$$(N+1)\epsilon + 2R \sin \frac{\beta_0}{2} > (N+1) \frac{\epsilon}{4 \sin(\beta_0/2)}.$$

Therefore

$$(N+1)\epsilon < \frac{8R \sin^2(\beta_0/2)}{1 - 4 \sin(\beta_0/2)} < 5R \beta_0^2. \quad \square$$

Proof of Theorem 2.2, assuming $R < \pi/(2\sqrt{\kappa})$ when $\kappa > 0$. By Lemma 2.12, ϕ is a continuous involution and thus a homeomorphism. It reduces to show that $\phi : \Sigma \times \{R\} \rightarrow \Sigma \times \{R\}$ preserves the length of any curve $c : [0, 1] \rightarrow \Sigma \times \{R\}$. Given $\delta, \epsilon > 0$, we may assume a partition $P : 0 = t_0 < t_1 < \dots < t_N = 1$ with $|c(t_i)c(t_{i+1})| \leq \delta$ such that the length of the curve satisfies

$$L(c) < \sum_{i=0}^{N-1} |c(t_i)c(t_{i+1})| + \frac{\epsilon}{2}, \quad L(\phi(c)) < \sum_{i=0}^{N-1} |\phi(c(t_i))\phi(c(t_{i+1}))| + \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} |L(c) - L(\phi(c))| &\leq \sum_{i=0}^{N-1} \left| |c(t_i)c(t_{i+1})| - |\phi(c(t_i))\phi(c(t_{i+1}))| \right| + \epsilon \\ &\leq \sum_{i=0}^{N-1} 20 |c(t_i)c(t_{i+1})|^2 + \epsilon \\ &\leq 20 \delta \sum_{i=0}^{N-1} |c(t_i)c(t_{i+1})| + \epsilon \\ &\leq 20 \delta L(c) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ and $\delta > 0$ can be chosen arbitrarily small, we get the desired result. \square

Completion of Proof of Theorem B. By Theorems 2.1 and 2.2, we identify X with $\bar{C}_\kappa^R(\Sigma_p)/x \sim \phi(x)$. We show that the metric on X coincides with the metric induced from the identification $x \sim \phi(x)$. It's equivalent to show that

$$\exp_p : \bar{C}_\kappa^R(\Sigma_p) \rightarrow X$$

preserves lengths of geodesics. Let $\gamma \subset \bar{C}_\kappa^R(\Sigma_p)$ be a geodesic and $\sigma = f(\gamma)$. Since $L(\gamma) \geq L(\sigma)$, it remains to show that $L(\sigma) \geq L(\gamma)$. Because either $\gamma \subset \Sigma \times \{R\}$ or $\gamma \cap (\Sigma \times \{R\})$ has at most two points, we need only check for the case $\gamma \subset \Sigma \times \{R\}$, that is, $\sigma \subset L_p(X)$. For any $\epsilon > 0$, let $\{x_i\}_{i=0}^{2N+1} \subset \sigma$ be an ϵ -partition and

$$L(\sigma) = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N} |x_i x_{i+1}|.$$

Let $a_i \in \gamma$ so that $\exp_p(a_i) = x_i$. Choose $b_{2k} \in \bar{C}_\kappa^R(\Sigma)$, $k = 0, 1, \dots, N$, with $|a_{2k} - b_{2k}| < \epsilon^4$. Let $b_{2k+1} = a_{2k+1}$ for $k = 0, 1, \dots, N$ and $y_i = \exp_p(b_i)$ for $i = 0, 1, \dots, 2N + 1$. Then $|y_i - x_i| \leq |b_i - a_i| < \epsilon^4$ and thus

$$L(\sigma) = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N} |y_i y_{i+1}|.$$

We claim that $[y_i y_{i+1}] \cap L_p(X)$ is either y_i or y_{i+1} . By [Corollary 2.9](#), let

$$u, v \in [y_i y_{i+1}] \cap L_p(X)$$

and there is no crossing point in between. Without losing generality, we assume $y_i \notin L_p(X)$ and $|y_i u| < |y_i v|$. Let $[u^- v^-] \subset [y_i y_{i+1}]$. Because the involution ϕ is an isometry ([Theorem 2.2](#)), $L([u^+ v^+]) = L([u^- v^-])$. Thus $[y_i u] \cup [u^+ v^+] \neq [y_i u] \cup [u^- v^-]$ is also a geodesic, which yields a bifurcation of geodesics.

By the claimed property, we have that $|y_i y_{i+1}| = |b_i b_{i+1}|$. Since $\sum_{i=0}^{2N} |b_i b_{i+1}| \geq L(\gamma)$, we have

$$L(\sigma) = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N} |y_i y_{i+1}| = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N} |b_i b_{i+1}| \geq L(\gamma).$$

It remains to show that for $\Sigma \in \text{Alex}^{n-1}(1)$, if $\phi : \Sigma \times \{R\} \rightarrow \Sigma \times \{R\}$ is an isometric involution, $X = \bar{C}_\kappa^R(\Sigma)/(x \sim \phi(x)) \in \text{Alex}^n(\kappa)$.

Case 1. Assume $\partial\Sigma = \emptyset$. Take two copies of $\bar{C}_\kappa^R(\Sigma)$, marked as $\bar{C}_\kappa^R(\Sigma)_1$ and $\bar{C}_\kappa^R(\Sigma)_2$, whose vertices are p_1 and p_2 , respectively. Gluing along their boundaries by ϕ , we obtain a double space $\hat{X} = \bar{C}_\kappa^R(\Sigma)_1 \cup_\phi \bar{C}_\kappa^R(\Sigma)_2$. By the gluing theorem [[Petrunin 1997](#)], $\hat{X} \in \text{Alex}^n(\kappa)$.

Now we extend the isometric \mathbb{Z}_2 -action by ϕ on Σ to an isometric $\hat{\mathbb{Z}}_2$ -action on \hat{X} such that $X = \hat{X}/\hat{\mathbb{Z}}_2$, and thus $X \in \text{Alex}^n(\kappa)$. For any $u \in \bar{C}_\kappa^R(\Sigma)_1$, extend the geodesic $[p_1 u]_{\bar{C}_\kappa^R(\Sigma)_1}$ to $u_1 \in (\Sigma \times \{R\})_1$. Let $\hat{\phi}(u)$ be the point on the geodesic $[p_2 \phi(u_1)]_{\bar{C}_\kappa^R(\Sigma)_2}$ such that $|p_2 \hat{\phi}(u)| = |p_1 u|$ (so $\hat{\phi} : \bar{C}_\kappa^R(\Sigma)_1 \rightarrow \bar{C}_\kappa^R(\Sigma)_2$). Switching the roles of $\bar{C}_\kappa^R(\Sigma)_1$ and $\bar{C}_\kappa^R(\Sigma)_2$, we extend ϕ to an isometric involution $\hat{\phi} : \bar{C}_\kappa^R(\Sigma)_2 \rightarrow \bar{C}_\kappa^R(\Sigma)_1$. Clearly, $\hat{\phi} : \hat{X} \rightarrow \hat{X}$ is an isometric involution such that $X = \hat{X}/\hat{\phi}$.

Case 2. Assume $\partial\Sigma \neq \emptyset$. Let $\hat{\Sigma} = \Sigma^+ \cup \Sigma^-$ denote the double of Σ . We first extend the isometric involution ϕ on Σ to $\hat{\phi} : \hat{\Sigma} \rightarrow \hat{\Sigma}$ by $\hat{\phi}(x_{\pm}) = \phi(x)_{\mp}$, where $x_{+} = x_{-} \in \Sigma$. We then define another isometric involution $\psi : \hat{\Sigma} \rightarrow \hat{\Sigma}$ by the reflection on $\partial\Sigma$, $\psi(x_{\pm}) = x_{\mp}$. Then $\hat{\psi}(\hat{\phi}(x_{\pm})) = \hat{\psi}(\phi(x)_{\pm}) = \phi(x)_{\mp} = \hat{\phi}(x_{\mp}) = \hat{\phi}(\hat{\psi}(x_{\pm}))$. This implies that $\hat{\Sigma}$ admits a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -action. Clearly, the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -action extends uniquely to an isometric $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -action on $\bar{C}_{\kappa}^r(\hat{\Sigma})$. By Case 1, we extend only the $\hat{\phi}$ -action to \hat{X} such that $\bar{C}_{\kappa}^r(\hat{\Sigma})/x \sim \hat{\phi}(x) \in \text{Alex}^n(\kappa)$. Then $X = [\bar{C}_{\kappa}^r(\hat{\Sigma})/x \sim \hat{\phi}(x)]/\hat{\psi} \in \text{Alex}^n(\kappa)$. \square

By [Theorem B](#), the isometric classification of $X \in \mathcal{A}_{\kappa}^r(\Sigma)$ with relatively maximum volume reduces to the isometric classification of all $(n - 1)$ -dimensional Alexandrov spaces Σ with $\text{curv} \geq 1$ and the equivariant isometric \mathbb{Z}_2 -actions on Σ . For $n = 2$, one easily gets a complete list:

Corollary 2.13. *Any 2-dimensional compact Alexandrov space with $\text{curv} \geq \kappa$ and relatively maximum volume is isometric to one of the following:*

$$\bar{C}_{\kappa}^r(S_{\theta}^1)/\phi_i \quad (i = 1, 2, 3) \quad \text{or} \quad \bar{C}_{\kappa}^r([0, \theta])/\psi_i \quad (i = 1, 2),$$

where S_{θ}^1 denotes a circle of length 2θ with $0 < \theta \leq \pi$ and $\phi_i : S_{\theta}^1 \rightarrow S_{\theta}^1$ (respectively $\psi_i : [0, \theta] \rightarrow [0, \theta]$) is trivial, a reflection or the antipodal map respectively for $i = 1, 2$ and 3 (respectively $i = 1$ and 2).

Example 2.14 (cf. [\[Grove and Petersen 1992\]](#)). Let $Z = \mathbb{D}^2$ be a 2-dimensional flat unit disk. Then $\partial Z = \mathbb{S}^1(1)$ is a unit circle. Let $\phi : \partial Z \rightarrow \partial Z$ be a one-to-one map and $X = \mathbb{D}^2/x \sim \phi(x)$ the glued space via identification $z \sim \phi(z)$. By [Theorem B](#), X is an Alexandrov space if and only if ϕ is an isometric involution, that is, ϕ is a reflection, antipodal map, or identity, where X is homeomorphic to \mathbb{S}^2 , $\mathbb{R}P^2$, and \mathbb{D}^2 , respectively.

Example 2.15. Consider a 2-dimensional simplex. We identify points on each side via a reflection about the mid point. Then we get a tetrahedron, in which one vertex is glued from the three vertices of the simplex.

3. Proof of Theorem C: Relatively almost maximum volume

In the proof of [Theorem C](#), we need the following result.

Theorem 3.1 [\[Bredon 1972, Theorem 5.5\]](#). *Let M be a G -manifold. G is a finite group. Assume that, for a given prime p and all p -subgroups, $P \subseteq G$ satisfies*

$$H_i(M^P; \mathbb{Z}_p) = 0, \quad i \leq q \text{ (including } P = \{e\}\text{)}.$$

Then $H_i(M/G; \mathbb{Z}_p) = 0$ for all $i \leq q$. Moreover, if this holds for all prime p and $H_i(M; \mathbb{Z}) = 0$ for $i \leq q$, then $H_i(M/G; \mathbb{Z}) = 0$ for $i \leq q$.

Proof of Theorem C. We first show that if $X \in \mathcal{A}_\kappa^r(\Sigma)$ with $\text{vol } X = v(\Sigma, \kappa, r)$, X is homeomorphic to S^n or $\mathbb{C}P^n$.

By Theorem B, X is isometric to $\overline{C}_\kappa^R(\Sigma)/x \sim \phi(x)$ and $\phi : \Sigma \rightarrow \Sigma$ is an isometric involution. To determine the homeomorphism type of X , we consider the double space $\widehat{X} = \overline{C}_\kappa^R(\Sigma)^+ \cup_\phi \overline{C}_\kappa^R(\Sigma)^-$. As seen in the proof of Theorem B, $\widehat{X} \in \text{Alex}^n(\kappa)$ and ϕ extends an isometric \mathbb{Z}_2 -action on \widehat{X} such that \widehat{X}/\mathbb{Z}_2 .

We claim that \widehat{X} is a homeomorphism sphere. First, \widehat{X} is a topological manifold if every point $\hat{q} \in \partial \overline{C}_\kappa^R(\Sigma) \hookrightarrow \widehat{X}$ is a manifold point. According to [Wu 1997], a point x in an Alexandrov space is a manifold point if and only if Σ_x is simply connected. Because $\Sigma_{\hat{q}}$ is a suspension of $\Sigma_{\hat{q}}(\Sigma)$, \hat{q} is a manifold point. By the Poincaré conjecture (in all dimensions), our claim reduces to the following: \widehat{X} is an integral homotopy sphere. Because \widehat{X} is a suspension, \widehat{X} is simply connected, and thus it suffices to show that \widehat{X} is a homology sphere. Because $\overline{C}_\kappa^R(\Sigma)$ is contractible, from the Mayer–Vietoris exact sequence of $(\overline{C}_\kappa^R(\Sigma)^+, \overline{C}_\kappa^R(\Sigma)^-)$, it is easy to see that \widehat{X} is an integral homology sphere.

If the \mathbb{Z}_2 -action is free, $X = \widehat{X}/\mathbb{Z}_2$ is homeomorphic to $\mathbb{R}P^n$. Otherwise, X is a simply connected topological manifold (the induced map, $\pi_1(\widehat{X}) \rightarrow \pi_1(X)$ is an onto map). Again, it suffices to show that X is an integral homology sphere. By the Smith theorem, the \mathbb{Z}_2 -fixed point set $\widehat{X}^{\mathbb{Z}_2}$ is a \mathbb{Z}_2 -homology sphere. By now we can apply Theorem 3.1 to conclude the claim.

We prove Theorem C by contradiction; assume a sequence $X_i \in \mathcal{A}_\kappa^r(\Sigma)$ such that $\text{vol } X_i > \text{vol } C_\kappa^R(\Sigma) - \epsilon_i$ ($\epsilon_i = i^{-1}$) and none of X_i is homeomorphic to S^n or $\mathbb{R}P^n$. Without loss of generality, we may assume that

$$(X_i, p_i) \xrightarrow{d_{GH}} (X, p) \in \text{Alex}^n(\kappa),$$

where $X_i = \overline{B}_r(p_i)$. By Perelman’s stability theorem [Kapovitch 2007; Perelman 1991], X_i is homeomorphic to X for i large. In particular, X is a topological manifold. We claim that $X \in \mathcal{A}_\kappa^r(\Sigma_p)$ satisfies $\text{vol } X = v(\Sigma_p, \kappa, r)$. By the above, we then conclude that X is homeomorphic to S^n or $\mathbb{R}P^n$, and thus X_i is homeomorphic to X for i large, a contradiction.

To see the claim,

$$\text{vol } X = \lim_{i \rightarrow \infty} \text{vol } X_i = \lim_{i \rightarrow \infty} (\text{vol } C_\kappa^R(\Sigma) - \epsilon_i) = \text{vol } C_\kappa^R(\Sigma).$$

On the other hand, we shall construct a distance-nonincreasing map, $\phi : \Sigma \rightarrow \Sigma_p$. Consequently, $\text{vol } \Sigma_p \leq \text{vol } \Sigma$ and thus

$$\text{vol } X \leq \text{vol } C_\kappa^R(\Sigma_p) \leq \text{vol } C_\kappa^R(\Sigma) \leq \text{vol } X.$$

Let $A = \{v_i\} \subset \Sigma$ be a countable dense subset and $f_i : (X_i, p_i) \rightarrow (X, p)$ a sequence of ϵ_i -Gromov–Hausdorff approximations, $\epsilon_i \rightarrow 0$. For v_1 , the sequence

$\{f_i(g \exp_{p_i} v)\} \subset X$ contains a converging subsequence

$$f_{i_1}(g \exp_{p_{i_1}} q(v)) \rightarrow x_1 \in X.$$

Then $[px_1] = w_1 \in \Sigma_p$ (which may not be unique). We define $\phi(v_1) = w_1$. For v_2 and $\{f_{i_1}\}$, repeating the above, we obtain $w_2 \in \Sigma_p$ and define $\phi(v_2) = w_2$. Iterating this process, we define a map $\phi : A \rightarrow \Sigma_p$, $\phi(v_i) = w_i$. It is easy to check that ϕ is distance-nonincreasing and thus ϕ extends uniquely to a distance-nonincreasing map from Σ to Σ_p . □

4. Proof of Theorem D: Pointed Bishop–Gromov relative volume comparison

Assuming the monotonicity in [Theorem D](#), the rigidity part follows by [Lemma 4.3](#) and [Theorem 2.1](#). For $p \in X \in \text{Alex}^n(\kappa)$, let $A_R^r(p)$ (or briefly A_R^r) denote the annulus $\{x \in X : r < |px| \leq R\}$, $0 \leq r < R$, and let $A_R^r(\Sigma_p)$ (or briefly \tilde{A}_R^r) denote the corresponding annulus in $C_\kappa(\Sigma_p)$. Let B_r denote A_r^0 and let \tilde{B}_r denote \tilde{A}_r^0 . Let's recall the following two lemmas.

Lemma 4.1 [[Li 2010](#), Lemma 2.1]. *Let $\Sigma \in \text{Alex}^{n-1}(1)$ and $0 < r \leq \pi/\sqrt{\kappa}$. Then*

$$\text{vol } C_\kappa^r(\Sigma) = \text{vol } \Sigma \int_0^r \text{sn}_\kappa^{n-1}(t) dt.$$

Lemma 4.2 [[Li 2010](#), Theorem B]. *Let U be an open subset in $X \in \text{Alex}^n(\kappa)$. Then there is a constant $c(n)$ depending only on n such that*

$$V_{r_n}(\bar{U}) = V_{r_n}(U) = c(n) \text{Haus}_n(U) = c(n) \text{Haus}_n(\bar{U}),$$

where V_{r_n} and Haus_n represent the n -dimensional rough volume and Hausdorff measure, respectively.

Lemma 4.3. *If the monotonicity in [Theorem B](#) holds,*

$$\frac{\text{vol } B_r}{\text{vol } \tilde{B}_r} = \frac{\text{vol } B_R}{\text{vol } \tilde{B}_R}$$

for some $0 < r < R$ ($R \leq \pi/\sqrt{\kappa}$ for $\kappa > 0$) if and only if $\text{vol } B_R = \text{vol } \tilde{B}_R$.

Proof. Assume $\text{vol } B_R = \text{vol } \tilde{B}_R$. The desired equation follows by the monotonicity:

$$1 = \frac{\text{vol } B_R}{\text{vol } \tilde{B}_R} \leq \frac{\text{vol } B_r}{\text{vol } \tilde{B}_r} \leq \lim_{r \geq t \rightarrow 0} \frac{\text{vol } B_t}{\text{vol } \tilde{B}_t} = 1.$$

Assume

$$\frac{\text{vol } B_r}{\text{vol } \tilde{B}_r} = \frac{\text{vol } B_R}{\text{vol } \tilde{B}_R}$$

for some $0 < r < R$. Then for any $t < r$,

$$\frac{\text{vol } B_t}{\text{vol } A_R^r} + \frac{\text{vol } A_R^t}{\text{vol } A_R^r} = \frac{\text{vol } B_R}{\text{vol } A_R^r} = \frac{\text{vol } \tilde{B}_R}{\text{vol } \tilde{A}_R^r} = \frac{\text{vol } \tilde{B}_t}{\text{vol } \tilde{A}_R^r} + \frac{\text{vol } \tilde{A}_R^t}{\text{vol } \tilde{A}_R^r}.$$

By the monotonicity, we have

$$\frac{\text{vol } A_R^t}{\text{vol } A_R^r} \geq \frac{\text{vol } \tilde{A}_R^t}{\text{vol } \tilde{A}_R^r}.$$

Also,

$$\frac{\text{vol } B_t}{\text{vol } A_R^r} = \frac{\text{vol } B_t}{\text{vol } A_r^t} \cdot \frac{\text{vol } A_r^t}{\text{vol } A_R^r} \geq \frac{\text{vol } \tilde{B}_t}{\text{vol } \tilde{A}_r^t} \cdot \frac{\text{vol } \tilde{A}_r^t}{\text{vol } \tilde{A}_R^r} = \frac{\text{vol } \tilde{B}_t}{\text{vol } \tilde{A}_R^r}.$$

Consequently

$$\frac{\text{vol } B_t}{\text{vol } A_R^r} = \frac{\text{vol } \tilde{B}_t}{\text{vol } \tilde{A}_R^r}, \quad \text{or equivalently,} \quad \frac{\text{vol } B_t}{\text{vol } \tilde{B}_t} = \frac{\text{vol } A_R^r}{\text{vol } \tilde{A}_R^r}.$$

Letting $t \rightarrow 0$, we get $\text{vol } A_R^r = \text{vol } \tilde{A}_R^r$. Thus

$$1 \geq \frac{\text{vol } B_R}{\text{vol } \tilde{B}_R} \geq \frac{\text{vol } A_R^r}{\text{vol } \tilde{A}_R^r} = 1. \quad \square$$

Now it remains to show the monotonicity in [Theorem D](#). We take an elementary approach by expressing the monotonicity as a form of ‘‘Riemann sum’’ (see [\(4-5\)](#)) and using the Toponogov triangle comparison to bound each term in terms of the desired form (see [Corollary 4.6](#)). To achieve this goal, we choose a special infinite partition (see [\(4-5\)](#) and [\(4-6\)](#)).

We start the proof of [Theorem D](#) by deriving an equivalent form of the monotonicity. For $0 \leq R_1 < R_2 < R_3$ ($< \pi/\sqrt{\kappa}$ when $\kappa > 0$), and $p \in X$, by [Lemma 4.1](#), the monotonicity has the following integral form:

$$\frac{\text{vol } A_{R_3}^{R_1}}{\text{vol } A_{R_2}^{R_1}} \leq \frac{\int_{R_1}^{R_3} \text{sn}_\kappa^{n-1}(t) dt}{\int_{R_1}^{R_2} \text{sn}_\kappa^{n-1}(t) dt},$$

which is equivalent to

$$(4-1) \quad I_1 = \log \left[\frac{\text{vol } A_{R_3}^{R_1}}{\text{vol } A_{R_2}^{R_1}} \right] \leq \log \left[\frac{\int_{R_1}^{R_3} \text{sn}_\kappa^{n-1}(t) dt}{\int_{R_1}^{R_2} \text{sn}_\kappa^{n-1}(t) dt} \right] = I_2.$$

Fixing a small $\delta > 0$, let $m = [(R_3 - R_2)/\delta] + 1$, $\Delta = (R_3 - R_2)/m \approx \delta$, and $r_j = R_2 + j\Delta$, $0 \leq j \leq m$. Then

$$A_{R_2}^{R_1} = A_{r_0}^{R_1} \subset A_{r_1}^{R_1} \subset \dots \subset A_{r_m}^{R_1} = A_{R_3}^{R_1}.$$

Using the Taylor expansion $\log(1/x) = 1 - x + O((1-x)^2)$, we may rewrite the left hand side of (4-1) as

$$(4-2) \quad \begin{aligned} I_1 &= \sum_{j=1}^m \log \frac{\text{vol } A_{R_1}^{r_j}}{\text{vol } A_{R_1}^{r_{j-1}}} = \sum_{j=1}^m \left[\left(1 - \frac{\text{vol } A_{R_1}^{r_{j-1}}}{\text{vol } A_{R_1}^{r_j}} \right) + O(\delta^2) \right] \\ &= \sum_{j=1}^m \frac{\text{vol } A_{R_1}^{r_j}}{\text{vol } A_{R_1}^{r_{j-1}}} + O(\delta). \end{aligned}$$

Let $\phi(r) = \int_{R_1}^r \text{sn}_\kappa^{n-1}(t) dt$. Then the right hand side of (4-1) can be written as

$$(4-3) \quad \begin{aligned} I_2 &= \log \frac{\phi(R_3)}{\phi(R_2)} = \int_{R_2}^{R_3} \frac{\phi'(t)}{\phi(t)} dt \\ &= \sum_{j=1}^m \frac{\phi'(r_j)}{\phi(r_j)} \delta + \tau(\delta) \\ &= \sum_{j=1}^m \frac{\delta \text{sn}_\kappa^{n-1}(r_j)}{\int_{R_1}^{r_j} \text{sn}_\kappa^{n-1}(t) dt} + \tau(\delta). \end{aligned}$$

Comparing (4-1) to (4-2) and (4-3), it's sufficient to show

$$(4-4) \quad \frac{\text{vol } A_{R_1}^{r_j}}{\text{vol } A_{R_1}^{r_{j-1}}} \leq \frac{\delta \text{sn}_\kappa^{n-1}(r_j)}{\int_{R_1}^{r_j} \text{sn}_\kappa^{n-1}(t) dt}.$$

We further divide $A_{R_1}^{r_j}$ into thinner annuli: given a monotonic sequence

$$\{a_i\}_{i=1}^\infty \subset [0, 1]$$

such that $a_j \rightarrow 0$, $\{a_i r_j\}_{i=1}^\infty$ is an infinite partition for $[0, r_j]$, and (4-4) is equivalent to

$$(4-5) \quad \frac{\text{vol } A_{R_1}^{r_j}}{\text{vol } A_{R_1}^{r_{j-1}}} = \sum_{i=1}^\infty \frac{\text{vol } A_{a_i r_j}^{a_{i+1} r_j}}{\text{vol } A_{R_1}^{r_j}} \geq \frac{\int_{R_1}^{r_j} \text{sn}_\kappa^{n-1}(t) dt}{\delta \text{sn}_\kappa^{n-1}(r_j)}.$$

To show (4-5), we need an estimate for $\text{vol } A_{a_i r_j}^{a_{i+1} r_j} / \text{vol } A_{R_1}^{r_j}$ from below (see Corollary 4.6). Assume δ is so small that $R - \delta > 0$ and $r - \lambda\delta > 0$. Let $x \in A_{R-\delta}^R$. We define a map, $\phi: A_{R-\delta}^R \rightarrow A_{r-\lambda\delta}^r$, where $f(x)$ is the point on a minimal geodesic $[px]$ (if not unique, we pick one of them) such that

$$|pf(x)| = r - \lambda(R - |px|).$$

Because a geodesic in X does not branch, ϕ is well-defined and is injective.

In the proof of Theorem D, the following is a main technical lemma, which asserts that ϕ behaves like a bi-Lipschitz function.

Lemma 4.4. *Let $\delta > 0$ be sufficiently small, $\lambda = (\text{sn}_\kappa r / (\text{sn}_\kappa R))$, and*

$$\phi : \tilde{A}_{R-\delta}^R \rightarrow \tilde{A}_{r-\lambda\delta}^r$$

be defined as above. Then

$$c(\kappa, \delta) \lambda \leq \frac{\text{sn}_\kappa(|\phi(x)\phi(y)|/2)}{\text{sn}_\kappa(|xy|/2)} \leq c(\kappa, \delta)^{-1}\lambda,$$

where

$$c(\kappa, \delta) = \begin{cases} 1 & \kappa = 0, \\ 1 - \frac{2\delta}{\text{sn}_\kappa R + \delta} & \kappa > 0, \\ 1 - \delta \frac{\cosh_\kappa R}{R} & \kappa < 0 \end{cases}.$$

Because the proof of [Lemma 4.4](#) is technical and somewhat tedious, we will delay it to the end of this section.

Lemma 4.5. *Let U and V be two open subsets of $X \in \text{Alex}^n(\kappa)$, and let $\phi : V \rightarrow U$ be an injection. If ϕ satisfies $\text{sn}_\kappa(|\phi(x)\phi(y)|/2) \geq c \text{sn}_\kappa(|xy|/2)$ for any $x, y \in V$, $\text{vol } U \geq c^n \text{vol } V$, where c is a constant.*

Proof. By [Lemma 4.2](#), it suffices to prove for rough volume. Recall that the n -dimensional rough volume of a subset V is

$$V_{r_n}(V) = \lim_{\epsilon \rightarrow 0} \epsilon^n \beta_V(\epsilon),$$

where $\beta_V(\epsilon)$ denotes the number of points in an ϵ -net $\{x_i\}$ on V .

By the assumption, $\{\phi(x_i)\}$ is a $2 \text{sn}_\kappa^{-1}(c \text{sn}_\kappa(\epsilon/2))$ -net in U . We get

$$\beta_U\left(2 \text{sn}_\kappa^{-1}\left(c \text{sn}_\kappa \frac{\epsilon}{2}\right)\right) \geq \beta_V(\epsilon),$$

or in another form,

$$\frac{\epsilon^n}{(2 \text{sn}_\kappa^{-1}(c \text{sn}_\kappa(\epsilon/2)))^n} \left(2 \text{sn}_\kappa^{-1}\left(c \text{sn}_\kappa \frac{\epsilon}{2}\right)\right)^n \beta_U\left(2 \text{sn}_\kappa^{-1}\left(\text{sn}_\kappa \frac{\epsilon}{2}\right)\right) \geq \epsilon^n \beta_V(\epsilon).$$

Letting $\epsilon \rightarrow 0$, we get $(1/c^n)V_{r_n}(U) \geq V_{r_n}(V)$. □

Corollary 4.6. *Let $p \in X \in \text{Alex}^n(\kappa)$, $\delta > 0$ small. Then*

$$\frac{\text{vol } A_{r-\lambda\delta}^r}{\text{vol } A_{R-\delta}^R} \geq (1 - \tau(\delta)) \left(\frac{\text{sn}_\kappa r}{\text{sn}_\kappa R}\right)^n.$$

Proof. Consider the map $\phi : A_{R-\delta}^R \rightarrow A_{r-\lambda\delta}^r$ and $\tilde{\phi} : \tilde{A}_{R-\delta}^R \rightarrow \tilde{A}_{r-\lambda\delta}^r$ defined as above. For any $x, y \in A_{R-\delta}^R$, take two points $\tilde{x}, \tilde{y} \in C_\kappa(\Sigma_p)$ such that $|\tilde{o}\tilde{x}| = |px|$,

$|\tilde{\partial}\tilde{y}| = |py|$, and $|\tilde{x}\tilde{y}| = |xy|$. By condition B (see [Burago et al. 1992]), it's easy to see that $|f(x)f(y)| \geq |\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{y})|$. Thus, by Lemma 4.4, we have

$$\operatorname{sn}_\kappa \frac{|f(x)f(y)|}{2} \geq \operatorname{sn}_\kappa \frac{|\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{y})|}{2} \geq (1 - \tau(\delta)) \operatorname{sn}_\kappa \frac{|\tilde{x}\tilde{y}|}{2} = (1 - \tau(\delta)) \operatorname{sn}_\kappa \frac{|xy|}{2}.$$

Then we get the desired estimate by Lemma 4.5. \square

Proof of the monotonicity in Theorem D. Continuing from the earlier discussion, the proof reduces to verifying (4-5). We now take $\delta > 0$ sufficiently small, and choose the sequence $\{a_i\}_{i=0}^\infty$ as

$$(4-6) \quad a_0 = 1, \quad a_{i+1} = a_i - \frac{\operatorname{sn}_\kappa(a_i r_j)}{r_j \operatorname{sn}_\kappa r_j} \delta, \quad i = 0, 1, \dots$$

Then

$$0 < a_{i+1} \leq \begin{cases} \left(1 - \frac{\delta}{r_j}\right) a_i, & \text{if } \kappa \geq 0, \\ \left(1 - \frac{\delta}{\operatorname{sn}_\kappa r_j}\right) a_i, & \text{if } \kappa < 0, \end{cases}$$

and thus $a_i \rightarrow 0$ and is monotonically decreasing. For each $0 \leq i < \infty$ and $0 \leq j \leq m$, consider the map, $\phi : A_{r_j - \delta}^{r_j} \rightarrow A_{a_i r_j - \lambda_i \delta}^{a_i r_j} = A_{a_{i+1} r_j}^{a_i r_j}$, with $\lambda_i = \operatorname{sn}_\kappa(a_i r_j) / \operatorname{sn}_\kappa(r_j)$. By Corollary 4.6, we obtain that

$$\frac{\operatorname{vol} A_{a_{i+1} r_j}^{a_i r_j}}{\operatorname{vol} A_{r_j - \delta}^{r_j}} \geq (1 - \tau(\delta)) \left(\frac{\operatorname{sn}_\kappa(a_i r_j)}{\operatorname{sn}_\kappa r_j} \right)^n.$$

Observe that for $\delta \rightarrow 0$, $\{a_i\}$ will become more dense, and thus we can take $N_\delta > 0$ such that $a_{N_\delta} r_j \geq R_1$ and $a_{N_\delta} r_j \rightarrow R_1$ as $\delta \rightarrow 0$. Summing up for $i = 0, 1, \dots, N_\delta$, we get

$$\begin{aligned} \frac{\operatorname{vol} A_{r_j}^{R_1}}{\operatorname{vol} A_{r_j - \delta}^{r_j}} &\geq \frac{\sum_{i=0}^{N_\delta} \operatorname{vol} A_{a_{i+1} r_j}^{a_i r_j}}{\operatorname{vol} A_{r_j - \delta}^{r_j}} \\ &\geq \sum_{i=0}^{N_\delta} (1 - \tau(\delta)) \left(\frac{\operatorname{sn}_\kappa(a_i r_j)}{\operatorname{sn}_\kappa r_j} \right)^n \\ &\geq (1 - \tau(\delta)) \frac{1}{\delta \operatorname{sn}_\kappa^{n-1}(r_j)} \sum_{i=0}^{N_\delta} \operatorname{sn}_\kappa^{n-1}(a_i r_j) \frac{\delta \operatorname{sn}_\kappa(a_i r_j)}{\operatorname{sn}_\kappa r_j} \\ &= (1 - \tau(\delta)) \frac{1}{\delta \operatorname{sn}_\kappa^{n-1}(r_j)} \left(\int_{R_1}^{r_j} \operatorname{sn}_\kappa^{n-1}(t) dt + \tau(\delta) \right) \\ &= (1 - \tau(\delta)) \frac{\int_{R_1}^{r_j} \operatorname{sn}_\kappa^{n-1}(t) dt}{\delta \operatorname{sn}_\kappa^{n-1}(r_j)}, \end{aligned}$$

or the following equivalent form:

$$\frac{\text{vol } A_{r_j-\delta}^{r_j}}{\text{vol } A_{r_j}^{R_1}} \leq (1 + \tau(\delta)) \frac{\delta \text{sn}_\kappa^{n-1}(r_j)}{\int_{R_1}^{r_j} \text{sn}_\kappa^{n-1}(t) dt}.$$

Summing up for all j and together with (4-2) and (4-3), we get

$$I_1 + O(\delta) \leq (1 + \tau(\delta))I_2 + \tau(\delta).$$

Letting $\delta \rightarrow 0$, we get the desired inequality. □

The rest of this section is devoted to a proof of Lemma 4.4. The following are some properties used in the proof.

Lemma 4.7. (1) For $\lambda \in [0, 1]$ and $x \in [0, \pi]$, $\sin \lambda x \geq \lambda \sin x$.

(2) For $\lambda \in [0, 1]$ and $x \geq 0$, $\sinh \lambda x \leq \lambda \sinh x$.

(3) For $\lambda \geq 0$ and $x \geq 0$, $\sin \lambda x / (\lambda \sin x) \geq 1 - (\lambda x)^2 / 6$.

(4) For $\lambda \geq 0$ and $x \geq 0$, $\sinh \lambda x / (\lambda \sinh x) \geq x / \sinh x \geq 1 - x$.

(5) Let $\triangle pab$ be a triangle in S_κ^2 . The cosine law can be written as

$$\text{sn}_\kappa^2 \frac{|ab|}{2} = \text{sn}_\kappa^2 \frac{|pa| - |pb|}{2} + \sin^2 \frac{\angle apb}{2} \text{sn}_\kappa |pa| \text{sn}_\kappa |pb|.$$

Proof.

(1) Let $h(x) = \sin \lambda x - \lambda \sin x$. Then

$$h'(x) = \lambda \cos \lambda x - \lambda \cos x = \lambda(\cos \lambda x - \cos x) \geq 0,$$

since $0 \leq \lambda x \leq x \leq \pi$.

(2) Let $h(x) = \sinh \lambda x - \lambda \sinh x$. Then

$$h'(x) = \lambda \cosh \lambda x - \lambda \cosh x = \lambda(\cosh \lambda x - \cosh x) \leq 0,$$

since $0 \leq \lambda x \leq x$.

(3) For $x > 0$, one can show that $x \geq \sin x \geq x - x^3/6$. Then

$$\frac{\sin \lambda x}{\lambda \sin x} \geq \frac{\lambda x - (\lambda x)^3/6}{\lambda x} = 1 - (\lambda x)^2/6.$$

(4) The first equality is easy to see through $\sinh \lambda x \geq \lambda x$. Obviously, the second equality is true for $x \geq 1$. For $0 < x < 1$,

$$\sinh x = x + \frac{x^3}{6} + \dots \leq x(1 + x + x^2 + \dots) = \frac{x}{1-x}.$$

(5) Follows by trigonometric metric identities. □

Proof of Lemma 4.4. By scaling, we only need to check for $\kappa = 1, -1$ and $\kappa = 0$.

Case 1 ($\kappa = 1$). Noting that

$$\frac{|px'| - |py'|}{|px| - |py|} = \frac{\lambda(|px| - |py|)}{|px| - |py|} = \lambda,$$

by Lemma 4.7(3) and $0 \leq ||px| - |py|| \leq \delta < \frac{1}{2} \sin R$, we have

$$\begin{aligned} \sin \frac{||px'| - |py' ||}{2} &= \sin \left(\lambda \frac{||px| - |py||}{2} \right) \\ &\geq \left(1 - \frac{(\lambda\delta)^2}{6} \right) \lambda \sin \frac{||px| - |py||}{2} \\ &\geq \left(1 - \frac{\delta^2}{6 \sin^2 R} \right) \lambda \sin \frac{||px| - |py||}{2} \\ &\geq \left(1 - \frac{2\delta}{\sin R + \delta} \right) \lambda \sin \frac{||px| - |py||}{2} \\ &= \tau_1 \lambda \sin \frac{||px| - |py||}{2}. \end{aligned}$$

Thus

$$(4-7) \quad \tau_1 \lambda \leq \frac{\sin(||px'| - |py' ||/2)}{\sin(||px| - |py||/2)} \leq \frac{\lambda ||px| - |py||/2}{\sin(||px| - |py||/2)} \leq \lambda \frac{\delta}{\sin \delta} \leq \tau_1^{-1} \lambda.$$

For any $x \in \tilde{A}_{R-\delta}^R$, by Lemma 4.7(1), we have

$$\sin |px'| \geq \frac{|px'|}{r} \sin r \geq \frac{r - \lambda\delta}{r} \sin r = \frac{r - (\sin r / \sin R) \delta}{r} \sin r \geq \left(1 - \frac{\delta}{\sin R} \right) \sin r,$$

which, together with

$$\sin |px'| - \sin r = 2 \sin \frac{|px'| - r}{2} \cos \frac{|px'| + r}{2} \leq r - |px'| \leq \lambda\delta,$$

gives us

$$\left(1 - \frac{\delta}{\sin R} \right) \sin r \leq \sin |px'| \leq \sin r + \lambda\delta = \left(1 + \frac{\delta}{\sin R} \right) \sin r.$$

Similarly,

$$\sin |px| \geq \frac{|px|}{R} \sin R \geq \frac{R - \delta}{R} \sin R \geq \left(1 - \frac{\delta}{\sin R} \right) \sin R$$

and

$$\sin |px| - \sin R = 2 \sin \frac{|px| - R}{2} \cos \frac{|px| + R}{2} \leq R - |px| \leq \delta,$$

hence

$$\left(1 - \frac{\delta}{\sin R} \right) \sin R \leq \sin |px| \leq \sin R + \delta = \left(1 + \frac{\delta}{\sin R} \right) \sin R.$$

So

$$(4-8) \quad c_1 \frac{\sin r}{\sin R} \leq \frac{\sin |px'|}{\sin |px|} \leq c_1^{-1} \frac{\sin r}{\sin R}.$$

Let $\theta = \angle xpy$. Since $|xy|/2 \leq \pi/2$, by the cosine law and inequalities (4-7), (4-8),

$$\begin{aligned} c_1^2 \lambda^2 &\leq \frac{\sin^2(|x'y'|/2)}{\sin^2(|xy|/2)} \\ &= \frac{\sin^2((|px'| - |py'|)/2) + \sin^2(\theta/2) \sin |px'| \sin |py'|}{\sin^2((|px| - |py|)/2) + \sin^2(\theta/2) \sin |px| \sin |py|} \leq c_1^{-2} \lambda^2. \end{aligned}$$

Case 2 ($\kappa = -1$). By Lemma 4.7(2),

$$\lambda \delta = \frac{\sinh r}{\sinh R} \cdot \frac{R}{\cosh R} < \frac{r}{R} R = r,$$

which, together with Lemma 4.7(4), gives

$$(4-9) \quad \lambda \geq \frac{\sinh(|px'| - |py'|/2)}{\sinh(|px| - |py|/2)} = \frac{\sinh(\lambda ||px| - |py||/2)}{\sinh(|px| - |py|/2)} \geq (1 - \delta)\lambda \geq c_{-1}\lambda,$$

since

$$\frac{\cosh R}{R} \geq \frac{1 + R^2/2}{R} > 1.$$

If $\delta < R/\cosh R < R$, we have

$$\frac{\lambda \delta}{2r} < \frac{r}{R} \cdot \frac{\delta}{2r} = \frac{\delta}{2R} < 1.$$

Hence we can apply Lemma 4.7(2) with $\lambda = (\sinh r)/\sinh R \leq r/R$, to get

$$\frac{\sinh r - \sinh(r - \lambda \delta)}{\sinh r} \leq \frac{2 \sinh(\lambda \delta/2) \cosh r}{\sinh r} \leq \frac{\lambda \delta}{r} \cosh r \leq \frac{\delta \cosh R}{R}.$$

Thus

$$\sinh(r - \lambda \delta) \geq \left(1 - \delta \frac{\cosh R}{R}\right) \sinh r.$$

For $x' \in \tilde{A}_{r-\lambda\delta}^r$,

$$\left(1 - \delta \frac{\cosh R}{R}\right) \sinh r \leq \sinh(r - \lambda \delta) \leq \sinh |px'| \leq \sinh r.$$

For $x \in \tilde{A}_{R-\lambda\delta}^R$,

$$\frac{\sinh R - \sinh(R - \delta)}{\sinh R} \leq \frac{2 \sinh(\delta/2) \cosh R}{\sinh R} \leq \frac{\delta \cosh R}{R},$$

and

$$\left(1 - \delta \frac{\cosh R}{R}\right) \sinh R \leq \sinh(R - \lambda \delta) \leq \sinh |px| \leq \sinh R.$$

Then

$$(4-10) \quad c_{-1} \frac{\sinh r}{\sinh R} \leq \frac{\sinh |px'|}{\sinh |px|} \leq c_{-1}^{-1} \frac{\sinh r}{\sinh R}.$$

By inequalities (4-9), (4-10), and the cosine law, we get

$$\begin{aligned} c_{-1}^2 \lambda^2 &\leq \frac{\sinh^2(|x'y'|/2)}{\sinh^2(|xy|/2)} \\ &= \frac{\sinh^2((|px'| - |py'|)/2) + \sin^2(\theta/2) \sinh |px'| \sinh |py'|}{\sinh^2((|px| - |py|)/2) + \sin^2(\theta/2) \sinh |px| \sinh |py|} \leq c_{-1}^{-2} \lambda^2. \end{aligned}$$

Case 3 ($\kappa = 0$). This is straightforward. \square

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Flag subdivisions and γ -vectors	257
CHRISTOS A. ATHANASIADIS	
Rays and souls in von Mangoldt planes	279
IGOR BELEGRADEK, ERIC CHOI and NOBUHIRO INNAMI	
Isoperimetric surfaces with boundary, II	307
ABRAHAM FRANDBEN, DONALD SAMPSON and NEIL STEINBURG	
Cyclic branched coverings of knots and quandle homology	315
YUICHI KABAYA	
On a class of semihereditary crossed-product orders	349
JOHN S. KAUTA	
An explicit formula for spherical curves with constant torsion	361
DEMETRE KAZARAS and IVAN STERLING	
Comparing seminorms on homology	373
JEAN-FRANÇOIS LAFONT and CHRISTOPHE PITTET	
Relatively maximum volume rigidity in Alexandrov geometry	387
NAN LI and XIAOCHUN RONG	
Properness, Cauchy indivisibility and the Weil completion of a group of isometries	421
ANTONIOS MANOUSSOS and POLYCHRONIS STRANTZALOS	
Theta lifts of strongly positive discrete series: the case of $(\tilde{Sp}(n), O(V))$	445
IVAN MATIĆ	
Tunnel one, fibered links	473
MATT RATHBUN	
Fusion symmetric spaces and subfactors	483
HANS WENZL	