Pacific Journal of Mathematics

THE SYNTOMIC REGULATOR FOR K_4 OF CURVES

AMNON BESSER AND ROB DE JEU

Volume 260 No. 2 December 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 260, No. 2, 2012
dx.doi.org/10.2140/pjm.2012.260.305

THE SYNTOMIC REGULATOR FOR K_4 OF CURVES

AMNON BESSER AND ROB DE JEU

Dedicated to the memory of Jon Rogawski

Let C be a curve defined over a complete discrete valuation subfield of $\mathbb{C}_p$. Assuming that C has good reduction over the residue field, we compute the syntomic regulator on a certain part of $K_4^{(3)}(C)$. The result can be expressed in terms of p-adic polylogarithms and Coleman integration. We also compute the syntomic regulator on a certain part of $K_4^{(3)}(F)$ for the function field F of C. The result can be expressed in terms of p-adic polylogarithms and Coleman integration, or by using a trilinear map ("triple index") on certain functions.

1. Introduction

Let K be a complete discrete valuation field of characteristic zero, R its valuation ring, and κ its residue field. Assume κ is of positive characteristic p. If $\mathscr{X}/R$ is a scheme, smooth and of finite type, then, after tensoring with $\mathbb{Q}$, one can decompose the algebraic K-theory of $\mathscr{X}$ according to the Adams weight eigenspaces, that is,

$$K_n(\mathscr{X}) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_j K_n^{(j)}(\mathscr{X}),$$

where $K_n^{(j)}(\mathscr{X})$ consists of those α in $K_n(\mathscr{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k; see [Soulé 1985, Proposition 5]. The cup product on $K_*(\mathscr{X})$ results in cup products $K_m^{(i)}(\mathscr{X}) \times K_n^{(j)}(\mathscr{X}) \to K_{m+n}^{(i+j)}(\mathscr{X})$. There is a regulator map

$$\mathrm{reg}_p : K_n^{(j)}(\mathscr{X}) \to H_{\mathrm{syn}}^{2j-n}(\mathscr{X}, j)\,;$$

see [Besser 2000b]. In many interesting cases the target group of the regulator is isomorphic to the rigid cohomology group of the special fiber $\mathscr{X}_\kappa$, in the sense of Berthelot, $H_{\mathrm{rig}}^{2j-n-1}(\mathscr{X}_\kappa/K)$. We shall be interested in the situation where $\mathscr{X}$ is a proper, irreducible, smooth curve $\mathscr{C}$ over R with a geometrically irreducible generic fiber C, and the K-group is $K_4^{(3)}(\mathscr{C})$. $K_4^{(3)}(C)$ is known to be isomorphic

MSC2010: primary 11G55, 14F42, 19E08, 19F27; secondary 11G20, 14F30, 19E20.
Keywords: algebraic K-theory, syntomic regulator, p-adic polylogarithm.

to $K_4^{(3)}(\mathscr{C})$ under localization; see Section 2.2. The target group for the regulator in this case is $H^1_{\mathrm{rig}}(\mathscr{C}_\kappa/K) \cong H^1_{\mathrm{dR}}(C/K)$. The cup product gives a pairing

$$H^1_{\mathrm{dR}}(C/K) \times H^1_{\mathrm{dR}}(C/K) \xrightarrow{\cup} H^2_{\mathrm{dR}}(C/K) \cong K,$$

where the last isomorphism is given by the trace map. We will denote this pairing by $\cup$ as well. If α is an element of $K_4^{(3)}(C)$ and ω is an element of $H^1_{\mathrm{dR}}(C/K)$, we want to compute $\omega \cup \mathrm{reg}_p(\alpha) \in K$.

To achieve this goal, we first of all need to be able to write elements in the above mentioned K-group. We do this using an integral version of the motivic complexes introduced by the second named author. The complex $\mathcal{M}_{(3)}(F)$ was defined in [de Jeu 1995, Section 3] for any field F of characteristic zero. It consists of three terms in cohomological degrees 1, 2 and 3:

$$M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 F^*_{\mathbb{Q}}, \tag{1.1}$$

with $F^*_{\mathbb{Q}} = F^* \otimes_{\mathbb{Z}} \mathbb{Q}$, and $M_n(F)$ a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F \setminus \{0, 1\}$, modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\begin{aligned} \mathrm{d}[x]_3 &= [x]_2 \otimes x, \\ \mathrm{d}\big([x]_2 \otimes y\big) &= (1-x) \otimes (x \wedge y). \end{aligned} \tag{1.2}$$

There are maps

$$H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F)$$

for $i = 2, 3$, and for $i = 3$ this is an isomorphism. Quotienting out by a suitable subcomplex (see Section 2.4.2) one obtains the complex

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_3(F) \to \widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}} \to \textstyle\bigwedge^3 F^*_{\mathbb{Q}}, \tag{1.3}$$

which is quasiisomorphic to $\mathcal{M}_{(3)}(F)$ in degrees 2 and 3. Its shape is more in line with conjectures (see for instance [Goncharov 1994, Conjecture 2.1]) and it is easier to work with for explicit examples. Each $\widetilde{M}_i(F)$ is a quotient of $M_i(F)$, and the image of $[x]_i$ in $\widetilde{M}_i(F)$ is still denoted $[x]_i$.

We can apply this with F the function field $K(C)$ of C, but as the syntomic regulator needs some information over the residue field, we have to use an analogous complex.

Notation 1.4. For the curve $\mathscr{C}$ as above with generic fiber C/K, we let $\mathbb{O} \subset F$ be the local ring consisting of functions that are generically defined on the special fiber $\mathscr{C}_\kappa$.

In Section 2.5.2 we shall construct a complex

$$\mathcal{M}_{(3)}(\mathbb{O}) : M_3(\mathbb{O}) \to M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to \mathbb{O}^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 \mathbb{O}^*_{\mathbb{Q}}, \tag{1.5}$$

with, in this case, $M_n(\mathbb{O})$ a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathbb{O}^\flat$, the *special units* of $\mathbb{O}$, namely those u in $\mathbb{O}^*$ for which $1-u$ is also in $\mathbb{O}^*$, again modulo nonexplicit relations depending on n, and $\mathbb{O}^*_{\mathbb{Q}} = \mathbb{O}^* \otimes_{\mathbb{Z}} \mathbb{Q}$. The maps in the complex are given by (1.2) as before, and there is a natural map $\mathcal{M}_{(3)}(\mathbb{O}) \to \mathcal{M}_{(3)}(F)$ of complexes. In fact, one may view $M_2(\mathbb{O}) \subseteq M_2(F)$; see Remark 2.45. The complex comes with maps

$$H^i(\mathcal{M}_{(3)}(\mathbb{O})) \to K^{(3)}_{6-i}(\mathbb{O}) \tag{1.6}$$

for $i = 2$ and 3.

Similar constructions, satisfying in particular (1.6), can be made in the following situation.

Notation 1.7. Suppose $k \subset K$ is a number field and let R' be the local ring $R \cap k$. For $\mathscr{C}'$ a smooth, proper, geometrically irreducible curve over R', let $\mathbb{O}'$ denote the local ring of rational functions on $\mathscr{C}'$ that are generically defined on the special fiber above the maximal ideal of R'.

In this case one has an additional map

$$M_2(\mathbb{O}') \otimes_{\mathbb{Q}} \mathbb{O}'^*_{\mathbb{Q}} \xrightarrow{\partial_1} \coprod_x \widetilde{M}_2(k(x)) \,,$$

where $M_2(\mathbb{O}')$ is now a $\mathbb{Q}$-vector space on symbols $[u]_2$ with u in $\mathbb{O}'^*$ such that $1-u$ is also in $\mathbb{O}'^*$, the coproduct is over all closed points of the generic fiber $C' = \mathscr{C}' \otimes_{R'} k$, given by

$$\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f) \cdot [g(x)]_2 \,,$$

with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

To explain the terms in which the formula for the regulator will be expressed, we need to introduce Coleman integration theory (see Section 4). Coleman [1982; Coleman and de Shalit 1988] defined an integration theory on curves over $\mathbb{C}_p$ with good reduction and on certain rigid analytic subdomains of these, which he termed "wide open spaces". One first needs to choose a branch of the p-adic logarithm, that is, a group homomorphism $\log : \mathbb{C}_p^* \to \mathbb{C}_p$, such that around $z = 1$, it is given by the usual power series expansions for $\log(1+y)$. This amounts to specifying $\log(p)$ in $\mathbb{C}_p$. Once this is done, the theory includes single valued iterated integrals on the appropriate domain, called "Coleman functions". In particular, we have the functions

$$\begin{aligned} \mathrm{Li}_2(z) &= -\textstyle\int_0^z \log(1-x)\,\mathrm{dlog}\,x, \\ \mathrm{L}_2(z) &= \mathrm{Li}_2(z) + \log(z)\log(1-z), \\ \mathrm{L}_{\mathrm{mod},2}(z) &= \mathrm{Li}_2(z) + \tfrac{1}{2}\log(z)\log(1-z). \end{aligned} \tag{1.8}$$

The function $\mathrm{Li}_2(z)$ is defined on $\mathbb{C}_p \setminus \{1\}$; see the beginning of Section VI in [Coleman 1982]. Consequently, all 3 functions are defined everywhere except possibly $0, 1, \infty$. They, and other Coleman functions, can be assigned a value at these points as follows.

For every point $y \in \mathbb{P}^1(\mathbb{C}_p)$, the residue disc U_y is the collection of points reducing to the same point as y. For each such y, and in terms of a local parameter $z = z_y$ on U_y, a Coleman function G can be expanded as $G(z) = \sum_{i=0}^{n} f_i(z) \log^i(z)$, where all $f_i(z)$ are in $\mathbb{C}_p[\![z, z^{-1}]\!]$. We define the *constant term* $c_z(G)$ at y with respect to the parameter z as the constant term of f_0; see Definition 7.7. In general the constant term will depend on the choice of the local parameter z, but there are many Coleman functions for which the constant term is independent of this choice. In such a case we will write $G(y)$ for this constant term. In particular, this is the case at all points y for $\mathrm{L}_{\mathrm{mod},2}(z)$ and $\int \mathrm{L}_2(g)\omega$ for any rational function g (it is in fact sufficient that ω is holomorphic at y), as well as for $\mathrm{Li}_2(z)$ and $\mathrm{L}_2(z)$ at all points except ∞ (see Lemmas 10.7 and 10.9 as well as Corollary 10.8). We further define all three functions from (1.8) to be 0 at 0 and ∞ (this is the constant term with respect to the standard parameter). For any Coleman function G, which is the integral of a form η, and divisor $D = \sum n_i y_i$, we will define

$$G(D) = \int_D \eta := \sum n_i G(y_i),$$

where we will be assuming that either G is defined at each y_i, or its constant term there is independent of the parameter.

We note that $\mathrm{L}_{\mathrm{mod},2}(z) + \mathrm{L}_{\mathrm{mod},2}(z^{-1}) = 0$ for z in $\mathbb{C}_p \setminus \{0, 1\}$ [Coleman 1982, Proposition 6.4(ii)], and that this extends to all values using constant terms. Similarly we have $\mathrm{L}_2(z) + \mathrm{L}_2(z^{-1}) = \frac{1}{2}\log^2(z)$.

We shall state the theorems in the introduction in a way that allows comparison with similar results in the classical case over $\mathbb{C}$. The formulas in that case can be easily transformed by using Stokes' theorem, whereas it seems the formulas in the syntomic case are not as flexible. Consequently, in the syntomic case we have to state a larger number of theorems. In order to enable a comparison in Remark 10.14 of the syntomic formulas below (especially those in Theorems 1.12 and 1.13) with those in the classical case, we recall and reformulate some of the classical results in Section 3.

We are now ready to state the first main theorem. In it, and the remaining theorems in the introduction, we assume that K is a closed subfield of $\mathbb{C}_p$ and evaluate Coleman functions at closed points of C by working over a finite extension of K over which all such points are rational. The result will be in K by Galois equivariance of Coleman integration.

Theorem 1.9. *Suppose, in the situation of Notation 1.4, that ω is a holomorphic form on C.*

(1) *The assignment*

$$[g]_2 \otimes f \mapsto 2 \int_{(f)} \mathrm{L}_2(g)\,\omega$$

gives a well-defined map $\Psi_{p,\omega} : M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$, *and this induces a map* $\Psi_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K$.

(2) *Suppose* $k \subset K$ *is a number field, and* $\mathscr{C}'$ *is a smooth, proper, geometrically connected curve over the local ring* $R' = R \cap k$. *Let* $\mathbb{O}'$ *be as in Notation 1.7 and put* $\mathscr{C} = \mathscr{C}' \otimes_{R'} R$. *Let* α' *in* $H^2(\mathcal{M}_{(3)}(\mathbb{O}'))$ *be such that* $\partial_1(\alpha') = 0$. *Then there exists a unique* β' *in* $K_4^{(3)}(\mathscr{C}')$ *whose image in* $K_4^{(3)}(\mathbb{O}')$ *under localization equals the image of* α' *under (1.6) modulo* $K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}}$. *If* β *is the image of* β' *under* $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathscr{C})$, *then we have*

$$\omega \cup \mathrm{reg}_p(\beta) = \Psi_{p,\omega}(\alpha),$$

where α *is the image of* α' *in* $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$.

Remark 1.10. The reader should compare the above formula for the regulator with the formula obtained by Coleman and de Shalit [1988], which is known to be the syntomic regulator by [Besser 2000c]. There, the regulator is obtained by sending the symbol $\{f, g\}$ in $K_2(F)$ to $\int_{(f)} \log(g)\omega$. The similarity with the present formula should be clear.

The rest of our results concern the K-theory of open curves over R and not over a number field. Thus, they are more general on the one hand, but progressively harder to state. Indeed, the first theorem is special because we are able to simplify matters by taking account of boundary terms over number fields.

As we are now computing on an open scheme, we no longer have a nontrivial cup product pairing, so we first need to explain what it is that we are computing. Under the regulator, each element of $K_4^{(3)}(\mathbb{O})$ maps to $H^1_{\mathrm{dR}}(U/K)$ for some wide open space U in C in the terminology of Coleman. There exists a canonical projection $H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/K)$, compatible with restriction to a smaller U; see [Besser 2000c, Proposition 4.8] and (9.13) below. We denote by reg'_p the composition

$$K_4^{(3)}(\mathbb{O}) \to H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/K).$$

Theorem 1.11. *Suppose* ω *is a holomorphic form on* C. *The assignment*

$$[g]_2 \otimes f \mapsto 2 \int_{(f)} \mathrm{L}_2(g)\omega - 2 \sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)),$$

where in the sum y *runs through the closed points of* C, *gives a well-defined map* $\Psi'_{p,\omega} : M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$. *It induces a map* $\Psi'_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K$, *which* *coincides with the composition*

$$H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

Over the complex numbers, it is known that computing the cup product of the regulator with holomorphic forms suffices to describe it completely in the case we are considering because those linear maps surject onto the dual of the target space of the regulator (see the beginning of Section 4 of [de Jeu 1996], especially Proposition 4.1). This is not true over the p-adics. It is therefore important to have formulas for the cup product of the regulator with a general cohomology class (such a class can be represented by a form of the second kind on C, that is, a meromorphic form all of whose residues are 0). This can be done at the cost of introducing further machinery — the notion of the triple index. It is a generalization of the "local index" that was introduced in [Besser 2000c, Section 4].

Informally speaking, working on an annulus e over $\mathbb{C}_p$, $e \cong \{r < |z| < 1\}$, the triple index associates to the integrals F, G and H of three rigid analytic 1-forms on e (in this case these forms are simply Laurent series converging on e multiplied by dz) together with choices of integrals for $F\,\mathrm{d}G$, $F\,\mathrm{d}H$ and $G\,\mathrm{d}F$, a number $\langle F, G; H\rangle_e$ in $\mathbb{C}_p$ that is supposed to be a generalization of $\mathrm{Res}_e\, FG\,\mathrm{d}H$. Note that the integrals appearing in the data for the triple index make perfect sense once one admits a log function to correspond to the integral of $\mathrm{d}z/z$, and are determined up to a constant by the form they integrate. Suppose now that $C/\mathbb{C}_p$ is a curve with good reduction and that C contains discs $D_i \cong \{|z| < 1\}$. The rigid analytic domain $U = C \setminus \bigcup_i (D_i - e_i)$, where $e_i \subset D_i$ is the annulus corresponding to $\{r < |z| < 1\}$, is called a wide open space by Coleman. The $e_i \subset U$ are called the ends of U. Suppose that F, G and H are Coleman functions defined on U such that restricted to the e_i they are of the type allowing us to compute the triple indices $\langle F|_{e_i}, G|_{e_i}; H|_{e_i}\rangle_{e_i}$. We may use auxiliary data composed of Coleman integrals restricted to e_i for computing these. It sometimes turns out that the sum of triple indices over all the e_i depends only on F, G, and H and not on the auxiliary data. This applies in particular to the sum of triple indices in the two theorems below. It is further known that this sum of triple indices behaves well with respect to shrinking the wide open space U. Finally, if everything is defined over a complete subfield K of $\mathbb{C}_p$ then this sum of triple indices is in K.

Theorem 1.12. *Let ω be a form of the second kind on C. The assignment*

$$[g]_2 \otimes f \mapsto \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g) \right\rangle_e,$$

where F_ω is any Coleman integral of ω, and the sum of triple indices is over all ends e of a wide open space U on which all f, g and $1-g$ are invertible and ω is *holomorphic, gives a well-defined map* $\Psi''_{p,\omega} : M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to K$. *It induces a map* $\Psi''_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K$, *which coincides with the composition*

$$H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

The complex $\widetilde{\mathcal{M}}_{(3)}(F)$ defined in (1.3) is easier to work with in explicit computations than the complex $\mathcal{M}_{(3)}(F)$. Therefore, just as in [de Jeu 1996, Remark 4.5], it is desirable to have a formula for the regulator using this complex. With that in mind, we define in Section 2.5.5 a complex

$$\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) : \widetilde{M}_3(\mathcal{O}) \to \widetilde{M}_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to \textstyle\bigwedge^3 \mathcal{O}^*_{\mathbb{Q}}$$

such that its cohomology in degrees 2 and 3 is isomorphic to that of the complex $\mathcal{M}_{(3)}(\mathcal{O})$ in (1.5). There is a natural map $\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(3)}(F)$ of complexes, and one may view $\widetilde{M}_2(\mathcal{O}) \subseteq \widetilde{M}_2(F)$. Corresponding to the statements in Theorems 1.11 and 1.12 for $\mathcal{M}_{(3)}(\mathcal{O})$, we have the following two expressions for the regulator in this case.

Theorem 1.13. 1. *Let ω be a form of the second kind on C. The assignment*

$$[g]_2 \otimes f \mapsto \tfrac{2}{3} \sum_e \Big\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g) \Big\rangle_e \\ - \tfrac{2}{3} \sum_e \Big\langle \log(f), \log(1-g); \int F_\omega \,\mathrm{dlog}(g) \Big\rangle_e$$

gives a well-defined map $\Psi'''_{p,\omega} : \widetilde{M}_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to K$. *It induces a map*

$$\Psi'''_{p,\omega} : H^2(\widetilde{\mathcal{M}}_{(3)}(\mathcal{O})) \to K,$$

which coincides with the composition of maps

$$H^2(\widetilde{\mathcal{M}}_{(3)}(\mathcal{O})) \xrightarrow{\simeq} H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K,$$

with the leftmost map being the isomorphism alluded to before.

2. *If ω is a holomorphic form on C, then the same holds for the assignment*

$$[g]_2 \otimes f \mapsto \tfrac{2}{3} \int_{(f)} (3\mathrm{L}_2(g) - \log(1-g)\log(g))\omega + \tfrac{2}{3} \int_{(g)} \log(f)\log(1-g)\omega \\ -2 \sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)) \,.$$

A key complex for doing computations is $\mathscr{C}^\bullet(\mathcal{O}) : \mathscr{C}^1(\mathcal{O}) \to \mathscr{C}^2(\mathcal{O})$ in cohomological degrees 1 and 2, which we shall construct in Section 2.5.4. The theorems in this introduction admit analogous results expressed in terms of this complex. We avoided these results for clarity in the introduction. However, they are very useful in applications since it is easier to find explicit examples to which these results apply, for instance, for certain elliptic curves; see [de Jeu 1996, Section 6].

We end the introduction with a conjecture. The regulator formulas that we obtain do not depend on any integrality assumptions. This is only required because the syntomic regulator is a map from the K-theory of an integral model. Thus we conjecture the following.

Conjecture 1.14. Theorems 1.9, 1.11, 1.12 and 1.13 hold, with the same formulas, with $\mathbb{O}$ replaced by F and $\mathscr{C}$ replaced by C.

Notation 1.15. Unless stated otherwise, throughout the paper, we will be working with the following notation.

K will be a discrete valuation field of characteristic zero with valuation ring R and residue field κ of positive characteristic p, which is a subfield of $\overline{\mathbb{F}}_p$. In various places, k will be a number field inside K. In that case we denote by $\mathbb{F} \subseteq \kappa$ the residue field of the local ring $R' = k \cap R$.

$\mathscr{C}$ will be a smooth, proper, geometrically irreducible curve over R. The generic fiber is denoted C, the special fiber is denoted $\mathscr{C}_\kappa$. We let $F = K(C)$, and $\mathbb{O} \subset F$ will be the valuation ring for the valuation on F corresponding to the generic point of $\mathscr{C}_\kappa$, which consists of those elements in F that are generically defined on $\mathscr{C}_\kappa$.

If $k \subset K$ is a number field, and $\mathscr{C}'$ is a smooth, proper, geometrically irreducible curve over $R' = R \cap k$, then the generic fiber is denoted C', the special fiber is denoted $\mathscr{C}'_{\mathbb{F}}$. We let $F' = k(C')$, and $\mathbb{O}' \subset F'$ will be the valuation ring for the valuation on F' corresponding to the generic point of $\mathscr{C}'_{\mathbb{F}}$. In particular, if $\mathscr{C} = \mathscr{C}' \otimes_{R'} R$, then $\mathbb{O}' = \mathbb{O} \cap F'$.

If S is a subset of a group, then we denote by $<S>$ the subgroup generated by S, and if S is a subset of a $\mathbb{Q}$-vector space, we denote by $<S>_{\mathbb{Q}}$ the $\mathbb{Q}$-vector subspace generated by S.

All tensor products will be over $\mathbb{Q}$, unless specified otherwise.

For the convenience of the reader, we give a commutative diagram that plays the role of a two-dimensional Leitfaden ("Leitteppich") for the proofs in this paper. In the left lower square we may also use $\mathbb{O}'$ instead of $\mathbb{O}$, in which case $C = C' \otimes_{R'} K$.

$$
\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \arrow[r] \arrow[d] & K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}} \arrow[d] & \\
H^2(\mathcal{M}_{(3)}(\mathbb{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathbb{O}) \arrow[r, "\mathrm{reg}'_p"] \arrow[d] & H^1_{\mathrm{dR}}(C/K) \arrow[d, "\omega\cup\cdot"] \\
H^1(\mathscr{C}^\bullet(\mathbb{O})) \arrow[r] & K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}} \arrow[r, dotted] & K
\end{tikzcd}
\tag{1.16}
$$

The constructions in algebraic K-theory will be carried out in Section 2. The top left square comes from the natural map $\mathcal{M}_{(3)}(\mathscr{C}') \to \mathcal{M}_{(3)}(\mathbb{O}')$ (see Section 2.5.3), and is justified by (2.58), whereas the bottom left square is (2.67). The map

$$K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K)$$

already factorizes through the quotient map $K_4^{(3)}(\mathbb{O}) \to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}}$ (see Corollary 9.5). The resulting composition in the bottom line of (1.16) is then computed in Section 9, using the techniques developed in the preceding sections. In Section 10 we then finish the proofs of the theorems above, based on this calculation.

2. *K*-theory

2.1. ***Introduction.*** Consider a proper, smooth, geometrically irreducible curve $\mathscr{C}$ over R as in Notation 1.4, or $\mathscr{C}'$ over R' as in Notation 1.7. We shall construct various cohomological complexes whose cohomologies are related to that of F, $\mathbb{O}$, F' or $\mathbb{O}'$. The main idea is the same as in [de Jeu 1996], but the fact that we shall be working with a discrete valuation ring rather than a field gives rise to some complications. In order to highlight the idea we start with a more gentle exposition. For the proofs of the statements that are used in the construction, we refer the reader to [de Jeu 1995], especially Sections 2.1 through 2.3 and 3. There most of the work was done over $\mathbb{Q}$, but in fact the proofs hold over our base $\mathbb{O}$, a discrete valuation ring of characteristic zero, without any change.

It will be clear from the constructions that the complexes are natural in terms of F, F', $\mathbb{O}$ and $\mathbb{O}'$, which we shall use later in this paper. In particular, if we start with $\mathscr{C}'$ over R' and let $\mathscr{C} = \mathscr{C}' \otimes_{R'} R$, then there are natural maps from the complexes for F' to those for F, and from those for $\mathbb{O}'$ to those for $\mathbb{O}$.

If B is a Noetherian scheme of finite Krull dimension d, then according to [Soulé 1985, Proposition 5], one can write

$$K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=\min\{2,n\}}^{n+d} K_n^{(j)}(B) \tag{2.1}$$

where $K_n^{(j)}(B)$ consists of all α in $K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k. (The regularity assumption at the beginning of Section 4 of [loc. cit.] is not necessary; see [Gillet and Soulé 1999, Proposition 8].) If in addition B is separated and regular, then the pullback $K_*(B) \to K_*(\mathbb{A}^1_B)$ is an isomorphism; see [Quillen 1973, §7]. The weight behaves naturally with respect to pullback, also giving us $K_m^{(j)}(B) \simeq K_m^{(j)}(\mathbb{A}^1_B)$ under pullback. And under suitable hypotheses for a closed embedding, there is a pushforward Gysin map with a shift in weights corresponding to the codimension; see, for instance, [de Jeu 1995, Proposition 2.3].

Let $X_B = \mathbb{P}^1_B \setminus \{t = 1\}$ with t the standard affine coordinate on $\mathbb{P}^1_B$. Write $\square^1_B$ for the closed subset $\{t = 0, \infty\}$ in $\mathbb{P}^1_B$. Then the relative exact sequence for the couple $(X_B; \square^1_B)$ gives us

$$\cdots \to K_{n+1}(X_B) \to K_{n+1}(\square^1_B) \to K_n(X_B; \square^1_B) \to K_n(X_B) \to K_n(\square^1_B) \to \cdots$$

for $n \geq 0$. Because the map pullback $K_{n+1}(B) \to K_{n+1}(X_B)$ is an isomorphism, combining it with the pullback $K_{n+1}(X_B) \to K_{n+1}(\square^1_B) = K_{n+1}(B)^2$ shows that the map $K_{n+1}(X_B) \to K_{n+1}(\square^1_B)$ corresponds to the diagonal embedding $K_{n+1}(B) \to K_{n+1}(B)^2$. As this holds for all $n \geq 0$, we get that we have an isomorphism $K_n(X_B; \square^1_B) \simeq K_{n+1}(B)$ for $n \geq 0$. Note that we have a choice of sign here in the isomorphism of the cokernel of $K_n(B) \to K_n(B)^2$ with $K_n(B)$. This results in similar choices of signs in the maps $H^i(\mathcal{M}_{(n)}(\mathbb{O})) \to K^{(n)}_{2n-i}(\mathbb{O})$ and $H^i(\widetilde{\mathcal{M}}_{(n)}(\mathbb{O})) \to K^{(n)}_{2n-i}(\mathbb{O})$ later on in this section.

We will have to go up one level in the relativity. If we let $\square^2_B$ be shorthand for $\{t_1 = 0, \infty\}; \{t_2 = 0, \infty\}$, then we can get a long exact sequence

$$\begin{aligned}&\cdots \to K_{n+1}(X^2_B; \{t_1 = 0, \infty\}) \to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\})\\ &\to K_n(X^2_B; \square^2_B) \to K_n(X^2_B; \{t_1 = 0, \infty\}) \to K_n(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \to \cdots .\end{aligned}$$

The composition

$$\begin{aligned}&K_{n+1}(X_B; \{t_1 = 0, \infty\}) \xrightarrow{\simeq} K_{n+1}(X^2_B; \{t_1 = 0, \infty\})\\ &\qquad \to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \simeq K_{n+1}(X_B; \{t_1 = 0, \infty\})^2\end{aligned}$$

(with the first the pullback along the projection $(t_1, t_2) \mapsto t_2$) is the diagonal embedding, hence we obtain an isomorphism $K_n(X^2_B; \square^2_B) \simeq K_{n+1}(X_B; \square^1_B)$ for $n \geq 0$. Therefore we get

$$K_n(X^2_B; \square^2_B) \simeq K_{n+1}(X_B; \square^1_B) \simeq K_{n+2}(B) \quad \text{for } n \geq 0.$$

A similar argument with weights gives us an isomorphism

$$K^{(j)}_n(X^2_B; \square^2_B) \simeq K^{(j)}_{n+2}(B) \quad \text{for } n \geq 0.$$

In order to get elements in $K_{n+2}(X^2_B; \square^2_B)$, we use localization sequences. We first explain the idea for $K_{n+1}(X_B; \square^1_B)$, because for $K_{n+2}(X^2_B; \square^2_B)$ the process involves a spectral sequence. If u is an element in our discrete valuation ring $\mathbb{O}$ such that both u and $1 - u$ are units, then we get an exact localization sequence

$$\cdots \to K_m(\mathbb{O}) \to K_m(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \to K_m(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_{m-1}(\mathbb{O}) \to \cdots$$

where $X_{\mathbb{O},\mathrm{loc}} = X_{\mathbb{O}} \setminus \{t = u\}$ and we identified $\{t = u\} \subset X_{\mathbb{O}}$ with $\mathbb{O}$ (or rather $\mathrm{Spec}(\mathbb{O})$). We used here that u and $1 - u$ are units in $\mathbb{O}$ so that $\{t = u\}$ does not meet $\square^1_{\mathbb{O}}$ or $\{t = 1\}$, and that $\mathbb{O}$ is regular in order to identify $K_m(\mathbb{O})$ with $K'_m(\mathbb{O})$. (If we want to leave out $\{t=u\}$ and $\{t=v\}$ simultaneously for two distinct elements u and v in $\mathbb{O}$ such that all of u, v, $1-u$ and $1-v$ are units, which we shall do below, this already becomes far more complicated and one is forced to use a spectral sequence.) The image of $K_2(\mathbb{O}) \to K_2(X_{\mathbb{O}}; \square^1_{\mathbb{O}})$ can be controlled by looking at the weights, which for the bit that we are interested in gives us

$$\cdots \to K_2^{(1)}(\mathbb{O}) \to K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \to K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O}) \to \cdots .$$

Because of weights in K-theory, one knows that $K_2^{(1)}(\mathbb{O}) = 0$, so that

$$K_3^{(2)}(\mathbb{O}) \simeq \mathrm{Ker}(K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O})),$$

and we can analyze $K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}})$ as a subgroup of $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}})$. In [de Jeu 1995, Section 3.2] universal elements $[S]_n$ were constructed, of which we want to use $[S]_2$ here. It gives rise to an element $[u]_2$ in $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}})$ with boundary $(1-u)^{-1}$ in $K_1^{(1)}(\mathbb{O})$. If we use this for various u (suitably modifying the localization sequence above into a spectral sequence) and also consider elements coming from the cup product

$$K_1^{(1)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \times K_1^{(1)}(\mathbb{O}) \to K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}},)$$

we can get part of $K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \simeq K_3^{(2)}(\mathbb{O})$ by intersecting the kernel of the map corresponding to $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O})$ with the space generated by the symbols $[u]_2$ and the image $K_1^{(1)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \cup K_1^{(1)}(\mathbb{O})$ of the cup product.

2.2. ***Preliminary material.*** We describe some basic facts about the various K-groups of F, $\mathbb{O}$, C and $\mathscr{C}$, or F', $\mathbb{O}'$, C' and $\mathscr{C}'$, including those mentioned in the introduction. The two cases are very similar so we shall treat them together.

We shall first consider the case where $F = k(C')$ for a smooth, projective curve $\mathscr{C}'$ over R' with geometrically irreducible generic fiber C'. Let $\mathscr{C}'_{\mathbb{F}}$ be the special fiber of $\mathscr{C}'$, which is a smooth, projective curve over the finite field $\mathbb{F}$. Because $\mathscr{C}'_{\mathbb{F}}$ is regular, there is an exact localization sequence

$$\cdots \to K_4^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to K_4^{(3)}(\mathbb{O}') \to K_4^{(3)}(F') \to K_3^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to \cdots . \tag{2.2}$$

By [Harder 1977, Korollar 2.3.2], $K_n(L)$ is torsion for $n \geq 2$ for all function fields L of curves over finite fields, so in particular, $K_4^{(3)}(\mathbb{O}') \xrightarrow{\simeq} K_4^{(3)}(F')$. If $F = K(C)$, then we get

$$\cdots \to K_4^{(2)}(\kappa(\mathscr{C}_\kappa)) \to K_4^{(3)}(\mathbb{O}) \to K_3^{(2)}(F) \to K_3^{(2)}(\kappa(\mathscr{C}_\kappa)) \to \cdots .$$

By our assumptions (see Notation 1.15), $\kappa \subseteq \overline{\mathbb{F}}_p$. According to [Quillen 1973, Proposition 2.2] or [Srinivas 1996, Lemma 5.9], $K_n(\kappa(\mathscr{C}_\kappa))$ is the direct limit of K_n of function fields of curves over finite fields, hence is torsion as well, and we find $K_4^{(3)}(\mathbb{O}) \simeq K_4^{(3)}(F)$.

From the exact localization sequence

$$\cdots \to \coprod_{x \in \mathscr{C}_{\mathbb{F}}^{\prime(1)}} K_n^{(1)}(\mathbb{F}(x)) \to K_n^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to K_n^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to \cdots$$

and the fact that $K_n^{(1)}(L)$ is zero for any field L for $n \geq 2$, we see that $K_n^{(2)}(\mathscr{C}'_{\mathbb{F}})$ is trivial for $n \geq 2$. From the exact localization sequence

$$\cdots \to K_4^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(C') \to K_3^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to \cdots$$

we see that $K_n^{(2)}(\mathscr{C}'_{\mathbb{F}})$ is trivial for $n \geq 2$, hence $K_4^{(3)}(\mathscr{C}') \simeq K_4^{(3)}(C')$. Using a direct limit argument as before, we then see that $K_4^{(3)}(\mathscr{C}) \simeq K_4^{(3)}(C)$ as well.

Remark 2.3. We now have two identifications fitting into a commutative diagram

$$\begin{array}{ccc} K_4^{(3)}(\mathscr{C}') & \longrightarrow & K_4^{(3)}(\mathbb{O}') \\ \| & & \| \\ K_4^{(3)}(C') & \longrightarrow & K_4^{(3)}(F') \end{array}$$

and similarly for F, $\mathbb{O}$, $\mathscr{C}$ and C. From the exact localization sequence

$$\cdots \to \coprod_{x \in C'^{(1)}} K_4^{(2)}(k(x)) \to K_4^{(3)}(C') \to K_4^{(3)}(F') \xrightarrow{\partial} \coprod_{x \in C'^{(1)}} K_3^{(2)}(k(x)) \to \cdots$$

we see that the map $K_4^{(3)}(F') \to K_4^{(3)}(C')$ is injective because $K_4^{(2)}(L) = 0$ for any number field L. Hence the map $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathbb{O}')$ is also injective.

Remark 2.4. We have

$$K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}} \quad \text{inside} \;\; K_4^{(3)}(F').$$

(This makes sense because $F'^{*}_{\mathbb{Q}} = K_1^{(1)}(F')$.) Namely, $K_4^{(3)}(C) = \mathrm{Ker}(\partial)$ in the localization sequence in Remark 2.3. On the other hand, for f in $F^*_{\mathbb{Q}}$ and α in $K_3^{(2)}(k)$, $\partial(\alpha \cup f) = \alpha \cup \mathrm{div}(f)$ in $\coprod_{x \in C^{(1)}} k(x)^*_{\mathbb{Q}}$, hence this is trivial only if f is in $k^*_{\mathbb{Q}}$. But

$$K_3^{(2)}(k) \cup k^*_{\mathbb{Q}} \subseteq K_4^{(3)}(k),$$

which is zero since k is a number field. Therefore $K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}$ injects into $\coprod_{x \in C^{(1)}} k(x)^*_{\mathbb{Q}}$ under ∂.

Remark 2.5. Note that a local parameter of R' is also a local parameter for $\mathbb{O}'$, so F'^{*} is generated by $\mathbb{O}'^{*}$ and that local parameter. This implies that

$$K_3^{(2)}(k) \cup \mathbb{O}'^{*}_{\mathbb{Q}} = K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}},$$

again because $K_3^{(2)}(k) \cup k^*_{\mathbb{Q}}$ is trivial.

We shall need the following result at several places later on.

Proposition 2.6. *For a discrete valuation ring $\mathcal{O}$, with residue field κ and field of fractions F, for all $n \geq 1$, the sequence ${\mathcal{O}^*_{\mathbb{Q}}}^{\otimes n} \to K_n^{(n)}(F) \to K_{n-1}^{(n-1)}(\kappa) \to 0$ is exact.*

Proof. Since $K_n^{(n)}(L) \simeq K_n^M(L)_{\mathbb{Q}}$ for any field L by [Soulé 1985, Théorème 2], with $K_n^M(L)$ the Milnor K-theory of L, it suffices to show that

$$(\mathcal{O}^*)^{\otimes_{\mathbb{Z}} n} \to K_n^M(F) \to K_{n-1}^M(\kappa) \to 0$$

is exact. If π is a uniformizer of $\mathcal{O}$, then $K_n^M(F)$ is generated by symbols $\{u_1, \dots, u_n\}$ and $\{u_1, \dots, u_{n-1}, \pi\}$, with all u_j in $\mathcal{O}^*$. The map $K_n^M(F) \to K_{n-1}^M(\kappa)$ is the tame symbol, which is trivial on the first type of generator, and maps the second to $\{\bar{u}_1, \dots, \bar{u}_{n-1}\}$. It is clearly surjective. So we only have to show that if α in $(\mathcal{O}^*)^{\otimes_{\mathbb{Z}}(n-1)}$ maps to the trivial element under the composition

$$(\mathcal{O}^*)^{\otimes_{\mathbb{Z}}(n-1)} \to (\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)} \to K_{n-1}^M(\kappa),$$

then the image of $\alpha \otimes \pi$ in $K_n^M(F)$ is in the image of $(\mathcal{O}^*)^{\otimes_{\mathbb{Z}} n}$. Noticing that the Steinberg relations $\cdots \otimes x \otimes \cdots \otimes (1-x) \otimes \cdots$ in $(\mathcal{O}^*)^{\otimes_{\mathbb{Z}}(n-1)}$ surject onto those in $(\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)}$, we see that we may assume that α is in the kernel of the map $(\mathcal{O}^*)^{\otimes_{\mathbb{Z}}(n-1)} \to (\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)}$. From the exact sequence

$$1 \to 1 + \mathcal{O}\pi \to \mathcal{O}^* \to \kappa \to 1$$

and the fact that, if we have exact sequences $0 \to A_i \to B_i \to C_i \to 0$ $(i = 1, \dots, m)$ of Abelian groups, then the kernel of $B_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m \to C_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} C_m$ is the image of $A_1 \otimes_{\mathbb{Z}} B_2 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m + B_1 \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_3 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m + \cdots$, we see α lies in the image of

$$(1 + \mathcal{O}\pi) \otimes_{\mathbb{Z}} \mathcal{O}^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{O}^* + \mathcal{O}^* \otimes_{\mathbb{Z}} (1 + \mathcal{O}\pi) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{O}^* + \cdots .$$

But each element $\{u_1, \dots, u_{n-1}, \pi\}$ with all u_i in $\mathcal{O}^*$ and at least one of them in $1 + \mathcal{O}\pi$ lies in the image of $(\mathcal{O}^*)^{\otimes_{\mathbb{Z}} n}$. Namely, an element in $1 + \mathcal{O}\pi$ is of the form $1 - \pi^d u$ for some u in $\mathcal{O}^*$, $d > 0$. If $d = 1$ we can rewrite $\{\dots, 1 - \pi u, \dots, \pi\} = -\{\dots, 1 - \pi u, \dots, u\}$. If $d > 1$, then using that

$$\frac{1 - \pi^d u}{1 - \pi} = 1 - \pi \frac{\pi^{d-1} u - 1}{1 - \pi},$$

we find that $\{\dots, 1 - \pi^d u, \dots, \pi\} = \{\dots, 1 - \pi \frac{\pi^{d-1} u - 1}{1 - \pi}, \dots, \pi\}$, which reduces to the case $d = 1$ as $(\pi^{d-1} u - 1)/(1 - \pi)$ is in $\mathcal{O}^*$. □

Assumption 2.7. Throughout the construction of the complexes in the various subsections below, we let F be a field of characteristic zero. In the constructions for complexes for $\mathcal{O}$, $\mathcal{O}$ will be a discrete valuation ring, with residue field κ and field of fractions F, which we assume to be of characteristic zero. We shall always assume that $|\kappa| > 2$, so that $\mathbb{O}^\flat$ is nonempty and $\langle \mathbb{O}^\flat \rangle = \mathbb{O}^*$.

2.3. ***A few more preliminaries.*** It will be convenient to introduce the notation $F^\flat = F^* \setminus \{1\}$, as well as $\mathbb{O}^\flat = \{u \text{ in } \mathbb{O}^* \text{ such that } 1-u \text{ is in } \mathbb{O}^*\}$, and $\kappa^\flat = \kappa^* \setminus \{1\}$.

Throughout the remainder of Section 2, we shall let X_F^{loc} be the scheme obtained from $X_F = \mathbb{P}_F^1 \setminus \{t=1\}$ by removing all points $t = u$ with u in $F^\flat$. We write $X_F^{2,\mathrm{loc}}$ for $(X_F^{\mathrm{loc}})^2$. Similarly, we let $X_\mathbb{O} = \mathbb{P}_\mathbb{O}^1 \setminus \{t=1\}$, we write $X_\mathbb{O}^{\mathrm{loc}}$ for the scheme obtained from $X_\mathbb{O}$ by removing all subschemes $t = u$ with u in $\mathbb{O}^\flat$, and we write $X_\mathbb{O}^{2,\mathrm{loc}}$ for $(X_\mathbb{O}^{\mathrm{loc}})^2$. Finally, for κ, we let $K_\kappa = \mathbb{P}_\kappa^1 \setminus \{t=1\}$, we write X_κ^{loc} for the scheme obtained from X_κ by removing all subschemes $t = u$ with u in $\kappa^\flat$, and we write $X_\kappa^{2,\mathrm{loc}}$ for $(X_\kappa^{\mathrm{loc}})^2$. (Of course, we would have to remove such a closed subscheme for only a finite set of u's first, and then take a direct limit. But by [Quillen 1973, Proposition 2.4] and some exact sequences in relative K-theory this will give us the K-theory of X_κ^{loc} anyway. Moreover, as such a direct limit over finite subsets of $\mathbb{O}^\flat$ or $F^\flat$ is clearly filtered, hence exact, this procedure will commute with taking spectral sequences, etc., below, so that we work directly in the direct limit.)

Since writing $\{t = 0, \infty\}$ or $\{t_1 = 0, \infty\}$; $\{t_2 = 0, \infty\}$ can be rather too long in places, we often abbreviate the first by writing $\square$, and the second by writing $\square^2$.

Let $(1+I)^* = K_1^{(1)}(X_F^{\mathrm{loc}}; \square)$. From the exact sequence

$$\cdots \to K_2^{(1)}(\square) \to K_1^{(1)}(X_F^{\mathrm{loc}}; \square) \to K_1^{(1)}(X_F^{\mathrm{loc}}) \to K_1^{(1)}(\square) \to \cdots$$

we see that $(1+I)^* \subset K_1^{(1)}(X_F^{\mathrm{loc}})$ as $K_2^{(1)}(\square) \simeq K_2^{(1)}(F)^{\oplus 2} = 0$. So we can describe $(1+I)^*$ explicitly as those elements in $K_1^{(1)}(X_F^{\mathrm{loc}})$ that restrict to 1 at $t = 0$ and $t = \infty$. Because $K_1(X_F^{\mathrm{loc}})$ is given by the units in the ring corresponding to a localization of the affine line, we find that

$$(1+I)^* = \left\{ \prod_j \left(\frac{t-u_j}{t-1} \right)^{n_j} \text{ with } u_j \text{ in } F^\flat, n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j} = 1 \right\} \otimes_\mathbb{Z} \mathbb{Q}.$$

Note that in particular the divisor map

$$(1+I)^* \to \coprod_{t \in F^\flat} K_0^{(0)}(F) \tag{2.8}$$

is an injection.

Note that, if A is any $\mathbb{Q}$-subspace of $K_n^{(l)}(X_F^{\mathrm{loc}}; \square)$, and we use the cup product $(1+I)^* \cup A \to K_{n+1}^{(l+1)}(X_F^{2,\mathrm{loc}}; \square^2)$ by pulling $(1+I)^*$ back along the first projection, and A along the second, then $\mathrm{d}((1+I)^* \cup A) = (\mathrm{d}(1+I)^*) \cup A - (1+I)^* \cup (\mathrm{d}A)$, and $\coprod_{t_1 \in F^\flat} A/(\mathrm{d}(1+I)^*) \cup A \simeq A \otimes F_\mathbb{Q}^*$ because $F^\flat$ generates F^*, and the functions in $(1+I)^*$ (without $\cdots \otimes_\mathbb{Z} \mathbb{Q}$) give exactly the multiplicative relations among the elements in $F^\flat$. Of course, by reversing the role of the projections we can do this with t_2 instead of t_1 instead. This will be used in order to change $\coprod_{t\in F^\flat}\cdots$ into $\cdots\otimes_{\mathbb{Q}} F^*_{\mathbb{Q}}$ in localization sequences or spectral sequences below.

Under Assumption 2.7, we can do the same for $\mathbb{O}$. Namely, define

$$(1+I)^*_{\mathbb{O}} = K_1^{(1)}(X^{\mathrm{loc}}_{\mathbb{O}};\square).$$

Because $K_2^{(1)}(\mathbb{O})=0$ and $K_1^{(1)}(\mathbb{O})=\mathbb{O}^*_{\mathbb{Q}}$, one sees by exactly the same argument as for $(1+I)^*$ that

$$(2.9)\quad (1+I)^*_{\mathbb{O}} = \left\{\prod_j\left(\frac{t-u_j}{t-1}\right)^{n_j}\;\middle|\; u_j \text{ in } \mathbb{O}^\flat,\ n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j}=1\right\}\otimes_{\mathbb{Z}}\mathbb{Q}\,.$$

In particular, we have $(1+I)^*_{\mathbb{O}}\subseteq(1+I)^*$ under localization of the base from $\mathbb{O}$ to F. Note that we used here that $(1+I)^*_{\mathbb{O}}$ gives us exactly the relations needed to turn $\coprod_{t\in\mathbb{O}^\flat}\cdots$ into $\cdots\otimes\mathbb{O}^*_{\mathbb{Q}}$, as $(1+I)^*_{\mathbb{O}}$ (without $\cdots\otimes_{\mathbb{Z}}\mathbb{Q}$) gives the multiplicative relations among elements in $\mathbb{O}^\flat$, and $\mathbb{O}^\flat$ generates $\mathbb{O}^*$.

Finally, we like to mention that for x in F, under the map

$$K_0^{(0)}(F)_{|t=x}\to K_0^{(1)}(X_F;\square)\simeq F^*_{\mathbb{Q}},$$

1 is mapped to $x^{\pm1}$; see [de Jeu 1995, Lemma 3.14]. The same holds for $\mathbb{O}$ instead of F, and this is compatible with products.

2.4. ***Construction of the complexes for F and C'.*** Several parts of the constructions of the complexes in this section and in Section 2.5 below were carried out in earlier papers [de Jeu 1995; 1996; Besser and de Jeu 2003], but we review them so that we can refer to the relevant details in some new constructions for $\mathbb{O}$ and in the calculations relating to regulators in later sections. Also, in various cases the constructions were carried out more generally, in which case they tend to become dependent on assumptions on weights in K-theory, and our exposition below will avoid such assumptions.

2.4.1. *Construction of the complexes $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$.* The principle of the construction of the complex $\mathcal{M}_{(2)}(F)$ was first used in Bloch's Irvine notes (finally published as [Bloch 2000]). The construction of $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$ can be found in [de Jeu 1995, Section 3].

We start with the localization sequence

$$(2.10)\quad \cdots\to\coprod_{t\in F^\flat}K_2^{(1)}(F)\to K_2^{(2)}(X_F;\square)\to K_2^{(2)}(X_F^{\mathrm{loc}};\square)\to \\ \coprod_{t\in F^\flat}K_1^{(1)}(F)\to K_1^{(2)}(X_F;\square)\to\cdots.$$

Because $K_2^{(1)}(F)=0$ for any field F by (2.1), this means that the cohomological complex (in degrees 1 and 2)

$$RC_{(2)}(F) : K_2^{(2)}(X_F^{\mathrm{loc}}; \square) \to \coprod_{t \in F^\flat} K_1^{(1)}(F) \tag{2.11}$$

has cohomology groups $H^1(RC_{(2)}(F)) \simeq K_3^{(2)}(F)$ and $H^2(RC_{(2)}(F)) \simeq K_2^{(2)}(F)$.

In [de Jeu 1995, Section 3.2] (see also [Bloch 1990]), for every x in $F^\flat$ an element $[x]_2$ was constructed in $K_2^{(2)}(X_F^{\mathrm{loc}}; \square)$ with the property that its boundary in $\coprod K_1^{(1)}(F)$ is $(1-x)^{-1}_{|t=x}$. Let

$$\begin{aligned} \mathrm{Symb}_1(F) &= K_1^{(1)}(F) = F^*_{\mathbb{Q}}, \\ \mathrm{Symb}_2(F) &= \langle [x]_2 \text{ with } x \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F). \end{aligned}$$

Then we get a subcomplex of (2.11):

$$\mathit{Symb}_2(F) : \mathrm{Symb}_2(F) \to \coprod_{t \in F^\flat} \mathrm{Symb}_1(F). \tag{2.12}$$

Letting $F^*_{\mathbb{Q}}$ act on the right in (2.8) gives the subcomplex

$$(1+I)^* \cup F^*_{\mathbb{Q}} \to \mathrm{d}(\cdots), \tag{2.13}$$

which is acyclic by [de Jeu 1995, Lemma 3.7]. Taking the quotient of (2.12) by (2.13), we obtain the complex

$$\mathcal{M}_{(2)}(F) : M_2(F) \to F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}},$$

where we used that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t \in F^\flat} \cdots$ into $\cdots \otimes F^*_{\mathbb{Q}}$, as $F^\flat$ generates F^*, and

$$M_2(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup \mathrm{Symb}_1(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup F^*_{\mathbb{Q}}.$$

Then $M_2(F)$ is a $\mathbb{Q}$ -vector space generated by the $[x]_2$, x in $F^\flat$, and the boundary of $[x]_2$ is $(1-x) \otimes x$.

Note that from the maps $\mathcal{M}_{(2)}(F) \leftarrow \mathit{Symb}_2(F) \to RC_{(2)}(F)$, with the left one a quasiisomorphism, we obtain maps

$$H^i(\mathcal{M}_{(2)}(F)) \to K_{4-i}^{(2)}(F)$$

for $i = 1$ and 2. The map for $i = 1$ is an injection as the corresponding statement holds for $RC_{(2)}(F)$ and $\mathit{Symb}_2(F)$ is a subcomplex, and we are in the lowest degree. For $i = 2$ the map is an isomorphism because $K_2^{(2)}(F)$ is the quotient of $F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$ by $\langle x \otimes (1-x) \text{ with } x \text{ in } F^\flat \rangle$.

We shall quotient out the complex $\mathcal{M}_{(2)}(F)$ in order to end up with a second term $\bigwedge^2 F^*_{\mathbb{Q}}$ rather than $F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$. The shape of the quotient complexes $\widetilde{\mathcal{M}}_{(2)}(F)$ here and $\widetilde{\mathcal{M}}_{(3)}(F)$ in Section 2.4.2 is more in line with conjectures; see for instance

[Goncharov 1994, Conjecture 2.1]. Besides, the definition of complex $\mathcal{M}_{(3)}(C')$ depends on the complexes $\widetilde{\mathcal{M}}_{(2)}(L)$ for number fields L.

Namely, consider the subcomplex of $\mathcal{M}_{(2)}(F)$

$$N_2(F) \to \mathrm{d}(\cdots) \tag{2.14}$$

with

$$N_2(F) = \langle [u]_2 + [u^{-1}]_2 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} \subseteq M_2(F). \tag{2.15}$$

As $\mathrm{d}([x]_2 + [x^{-1}]_2) = x \otimes x$, the second term is in fact $\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. By the proof of [de Jeu 1995, Corollary 3.22], (2.14) is acyclic. Taking the quotient complex we get

$$\widetilde{\mathcal{M}}_{(2)}(F) : \widetilde{M}_2(F) \to \textstyle\bigwedge^2 F^*_{\mathbb{Q}}, \tag{2.16}$$

with $\widetilde{M}_2(F) = M_2(F)/N_2(F)$, and $\mathrm{d}[x]_2 = (1-x) \wedge x$.

Because $\widetilde{\mathcal{M}}_{(2)}(F)$ is quasiisomorphic to $\mathcal{M}_{(2)}(F)$ we have maps

$$H^i(\widetilde{\mathcal{M}}_{(2)}(F)) \to K^{(2)}_{4-i}(F). \tag{2.17}$$

Again this map is an injection for $i = 1$ and an isomorphism for $i = 2$.

There are essentially two ways of generalizing the complex $\mathcal{M}_{(2)}(F)$. The first one is to look at another part of the localization sequence (2.10), the other to replace X_F by X^n_F for $n \geq 2$, and use localization there, which will give a spectral sequence. The first will be used to construct the complex $\mathscr{C}^\bullet(F)$ in Section 2.4.4 below, the second (with $n = 2$) will be used for constructing the complex $\mathcal{M}_{(3)}(F)$ below.

2.4.2. *Construction of the complexes $\mathcal{M}_{(3)}(F)$ and $\widetilde{\mathcal{M}}_{(3)}(F)$.* Those complexes were also defined in [de Jeu 1995, Section 3]. The complex $\mathcal{M}_{(3)}(F)$ consists of three terms in cohomological degrees 1, 2 and 3,

$$M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 F^*_{\mathbb{Q}}, \tag{2.18}$$

and comes equipped with maps

$$H^2(\mathcal{M}_{(3)}(F)) \to K^{(3)}_4(F) \quad \text{and} \quad H^3(\mathcal{M}_{(3)}(F)) \to K^{(3)}_3(F).$$

The last of those two maps is in fact an isomorphism.

Although we shall need a similar complex $\mathcal{M}_{(3)}(\mathbb{O})$ in order to have information about the special fiber, we describe the complex $\mathcal{M}_{(3)}(F)$ first, as it is notationally easier. Moreover, in the part of the complex we are interested in, we can view $\mathcal{M}_{(3)}(\mathbb{O})$ as a subcomplex of $\mathcal{M}_{(3)}(F)$ (see Remark 2.45).

Consider the divisors on X^2_F defined by putting $t_i = u_j$ for some u_j in $F^\flat$ for $i = 1$ or 2. Then there is a spectral sequence (see [de Jeu 1996, page 257; de Jeu

1995, page 221])

(2.19)

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_0^{(1)}(F) \\
K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F) \\
K_4^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_2^{(1)}(F) \\
\vdots & \vdots & \vdots
\end{array}
$$

converging to $K_*^{(3)}(X_F^2;\square^2) \simeq K_{*+2}^{(3)}(F)$. The only terms in it that contribute to $K_4^{(3)}(F)$ are

$$K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) \quad \text{and} \quad \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)$$

because $\coprod_{t_1,t_2\in F^\flat} K_1^{(2)}(F)$ is trivial. Let $RC_{(3)}(F)$ be the cohomological complex in degrees 1, 2 and 3, consisting of the row in (2.19) that begins with $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$:

$$RC_{(3)}(F) : K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) \to \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F). \tag{2.20}$$

This complex was denoted $C_{(3)}$ in [de Jeu 1995, Section 3.1], but considering the notational overload of the letter C in this paper, we prefer to think of it as a row complex rather than just a complex.

Note that $K_1^{(2)}(F)$ equals zero, so for $i = 2$ and 3 there is a map

$$H^i(RC_{(3)}(F)) \to K_{6-i}^{(3)}(F)\,. \tag{2.21}$$

For x in $F^\flat$, in addition to the element $[x]_2$ in $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ of Section 2.4.1, there is also an element $[x]_3$ in $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$ (see [de Jeu 1995, Section 3.2]) with boundary

$$-[x]_{2|t_1=x} + [x]_{2|t_2=x} \quad \text{in} \quad \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)$$

in (2.19). Let us define $\mathrm{Symb}_n(F) \subseteq K_n^{(n)}(X_F^{n-1,\mathrm{loc}}; \square^{n-1})$ for $n = 1$, 2 and 3 by setting

$$\begin{aligned}
\mathrm{Symb}_1(F) &= F_{\mathbb{Q}}^*,\\
\mathrm{Symb}_2(F) &= \langle [u]_2 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F),\\
\mathrm{Symb}_3(F) &= \langle [u]_3 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F).
\end{aligned}$$

For $n \leq 2$, those are the definitions given in Section 2.4.1, and for $n = 3$, by $\tilde{\cup}$ we mean the following. In the projection X_F^2 to X_F, we can use one of the factors to pull back $(1+I)^*$, the other to pull back $\mathrm{Symb}_2(F)$ and then take the product to land in $\mathrm{Symb}_3(F)$, giving us two cup products. The $\tilde{\cup}$ indicates that we take the sum of the images of both possibilities for those cup products.

Because, in (2.20), $\mathrm{d}[u]_2 = (1-u)_{|t=u}^{-1}$ and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$, it follows that

$$(2.22)\quad \mathit{Symb}_{(3)}(F) : \mathrm{Symb}_3(F) \to \coprod_{t_1 \in F^\flat} \mathrm{Symb}_2(F) \coprod \coprod_{t_2 \in F^\flat} \mathrm{Symb}_2(F) \to \coprod_{t_1, t_2 \in F^\flat} \mathrm{Symb}_1(F)$$

is a subcomplex of (2.20). It is shown in [de Jeu 1995, Lemma 3.9 and Remark 3.10] that the subcomplex

$$(2.23)\quad (1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F) \to \coprod_{t_1 \in F^\flat} (1+I)^* \cup F_{\mathbb{Q}}^* \coprod \coprod_{t_2 \in F^\flat} (1+I)^* \cup F_{\mathbb{Q}}^* + \mathrm{d}(\cdots) \to \mathrm{d}(\cdots)$$

of (2.22) is acyclic.

S_2 acts on the spectral sequence (2.19) by swapping t_1 and t_2. It therefore also acts on the complex (2.20) above. Because the symbol $[x]_3$ is alternating by construction (see [de Jeu 1995, Section 3.2]), we can take the alternating parts of (2.22) and (2.23), and form the quotient complex

$$\mathcal{M}_{(3)}(F) : M_3(F) \to M_2(F) \otimes F_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^* \otimes \textstyle\bigwedge^2 F_{\mathbb{Q}}^*,$$

where

$$\begin{aligned}
M_3(F) &= \mathrm{Symb}_3(F)/((1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F))^{\mathrm{alt}},\\
M_2(F) &= \mathrm{Symb}_2(F)/(1+I)^* \cup F_{\mathbb{Q}}^*,
\end{aligned}$$

as before in Section 2.4.1. Note that, for $n = 2$ and 3, $M_n(F)$ is a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by $\mathrm{d}[x]_3 = [x]_2 \otimes x$ and

$$(2.24)\qquad \mathrm{d}[x]_2 \otimes y = (1-x) \otimes (x \wedge y).$$

As before, we used here that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t\in F^\flat}\cdots$ into $\cdots\otimes F^*_{\mathbb{Q}}$, as $F^\flat$ generates F^*. As $Symb_{(3)}(F)$ is a subcomplex of $RC_{(3)}(F)$, this gives us maps

$$\mathcal{M}_{(3)}(F) \leftarrow Symb_{(3)}(F)^{\mathrm{alt}} \to RC_{(3)}(F)^{\mathrm{alt}} \to RC_{(3)}(F)$$

with the left map a quasiisomorphism. Combining this with (2.21) gives us a map

$$H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F) \tag{2.25}$$

for $i=2$ and 3. (For $i=1$, starting with $H^1(RC_{(3)}(F)) \to K_5^{(3)}(F)/K_4^{(2)}(F)\cup F^*_{\mathbb{Q}}$, we still obtain a map $H^1(\mathcal{M}_{(3)}(F)) \to K_5^{(3)}(F)/K_4^{(2)}(F)\cup F^*_{\mathbb{Q}}$.)

Finally, we quotient out $\mathcal{M}_{(3)}(F)$ in order to obtain $\widetilde{\mathcal{M}}_{(3)}(F)$, as follows. Let

$$N_3(F) = \langle [u]_3 - [u^{-1}]_3 \text{ with } u \text{ in } F^\flat\rangle_{\mathbb{Q}} \subseteq M_3(F)$$

(cf. (2.15); in general $N_n(F)$ is generated by the $[u]_n + (-1)^n[u^{-1}]_n$) and consider the subcomplex

$$N_3(F) \to N_2(F)\otimes F^*_{\mathbb{Q}} \to \mathrm{d}(\cdots) \tag{2.26}$$

of $\mathcal{M}_{(3)}(F)$. By the proofs of [de Jeu 1995, Proposition 3.20, Corollary 3.22] it is acyclic in degrees 2 and 3, hence for the quotient complex

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_3(F) \to \widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}} \to \textstyle\bigwedge^3 F^*_{\mathbb{Q}},$$

where $\widetilde{M}_3(F) = M_3(F)/N_3(F)$, we get a map

$$H^i(\widetilde{\mathcal{M}}_{(3)}(F)) \xleftarrow{\simeq} H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F)\,. \tag{2.27}$$

In $\widetilde{M}_3(F)$ we still denote the class of $[x]_i$ with $[x]_i$, so that the maps are now given by $\mathrm{d}[u]_3 = [u]_2\otimes u$ and $\mathrm{d}[u]_2\otimes v = (1-u)\wedge u\wedge v$.

The next remark, lemma, and corollary will be used in Section 10 to define the various maps in the theorems in the introduction.

Remark 2.28. Consider the map

$$\begin{aligned}\Phi : (F^*_{\mathbb{Q}})^{\otimes 3} &\to \mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}}\\ a\otimes b\otimes c &\mapsto \tfrac{2}{3}((a\cdot b)\otimes c - (a\cdot c)\otimes b),\end{aligned}$$

where $a_1\cdot a_2 = \frac{1}{2}(a_1\otimes a_2 + a_2\otimes a_1)$ in $\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. Up to scaling, Φ is the composition of antisymmetrizing in the last two factors, followed by symmetrizing in the first two factors, so it is trivial on $F^*_{\mathbb{Q}}\otimes \mathrm{Sym}^2(F^*_{\mathbb{Q}})$. It is easy to check that $\Phi\circ\Phi = \Phi$ and Φ maps a generator $(a\cdot a)\otimes c$ of $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}}$ to itself modulo $\mathrm{Sym}^3(F^*_{\mathbb{Q}})$. In particular, $\mathrm{id}-\Phi$ maps $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}} + F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$ to $F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$.

For $\tilde{\alpha}$ in $\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$, let α be a lift of $\tilde{\alpha}$ to $M_2(F)\otimes F^*_{\mathbb{Q}}$, so that $(\mathrm{d}\otimes\mathrm{id})(\alpha)$ is in $(F^*_{\mathbb{Q}})^{\otimes 3}$. Because of the statements just after (2.14), there is a unique β_α in

$N_2(F)\otimes F^*_{\mathbb{Q}} \subset M_2(F)\otimes F^*_{\mathbb{Q}}$ with $\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha) = (\mathrm{d}\otimes\mathrm{id})(\beta_\alpha)$. By definition, α is unique up to adding β' in $N_2(F)\otimes F^*_{\mathbb{Q}}$. But

$$\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha+\beta') = \Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha) + \Phi\circ(\mathrm{d}\otimes\mathrm{id})(\beta') = (\mathrm{d}\otimes\mathrm{id})(\beta_\alpha+\beta'+\gamma)$$

for some γ in $\mathrm{d}(N_3(F)) = \langle([h]_2+[h^{-1}]_2)\otimes h\rangle \subset N_2(F)\otimes F^*_{\mathbb{Q}}$ as $(\mathrm{d}\otimes\mathrm{id})(\beta')$ is in $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}}$, hence $(\Phi-\mathrm{id})\circ(\mathrm{d}\otimes\mathrm{id})(\beta')$ is in $\mathrm{Sym}^3(F^*_{\mathbb{Q}})$. So $\beta_{\alpha+\beta'} = \beta_\alpha+\beta'+\gamma$, hence the class of $\alpha-\beta_\alpha$ is well-defined in $M_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F))$.

Let

$$\begin{aligned}\Xi : \widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}} &\to M_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F))\\ \tilde{\alpha} &\mapsto \alpha-\beta_\alpha \text{ modulo } \mathrm{d}(N_3(F))\end{aligned}$$

be the resulting map, so α in $M_2(F)\otimes F^*_{\mathbb{Q}}$ lifts $\tilde{\alpha}$ and β_α in $N_2(F)\otimes F^*_{\mathbb{Q}}$ satisfies $\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha) = (\mathrm{d}\otimes\mathrm{id})(\beta_\alpha)$. Clearly, the quotient map $M_2(F)\otimes F^*_{\mathbb{Q}} \to \widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$ gives a quotient map $M_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F)) \to \widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$, and Ξ is a section of the latter. Hence

$$M_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F)) = \mathrm{im}(\Xi)\oplus N_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F)).$$

Now assume $\tilde{\alpha}$ is in the kernel of $\mathrm{d}:\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}} \to \bigwedge^3 F^*_{\mathbb{Q}}$. If α in $M_2(F)\otimes F^*_{\mathbb{Q}}$ lifts $\tilde{\alpha}$, then $(\mathrm{d}\otimes\mathrm{id})(\alpha)$ is in $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}} + F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. The same holds for $\eta = (\mathrm{d}\otimes\mathrm{id})(\alpha-\beta_\alpha)$ with $\alpha-\beta_\alpha$ any representative of $\Xi(\tilde{\alpha})$, so that α lifts $\tilde{\alpha}$ and $(\mathrm{d}\otimes\mathrm{id})(\beta_\alpha) = \Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)$. Therefore $\eta-\Phi(\eta)$ is in $F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. But $\Phi(\eta) = \Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha) - \Phi\circ\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha) = 0$, hence $\alpha-\beta_\alpha$ is in $(M_2(F)\otimes F^*_{\mathbb{Q}})^{\mathrm{d}=0}$, and therefore Ξ maps $(\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}})^{\mathrm{d}=0}$ to $(M_2\otimes F^*_{\mathbb{Q}})^{\mathrm{d}=0}/\mathrm{d}(N_3(F))$. It is easy to check that $\Xi(\mathrm{d}(\widetilde{M}_3(F))) = \mathrm{d}(M_3(F))/\mathrm{d}(N_3(F))$, so that Ξ induces the inverse to the natural isomorphism $H^2(\mathcal{M}_{(3)}(F)) \to H^2(\widetilde{\mathcal{M}}_{(3)}(F))$.

Lemma 2.29. *Let V be a $\mathbb{Q}$-vector space.*

(1) *Suppose we have a linear map $G : (F^*_{\mathbb{Q}})^{\otimes 3} \to V$. Then the assignment*

$$[g]_2\otimes f \mapsto G((1-g)\otimes g\otimes f)$$

defines a linear map $\Psi : M_2(F)\otimes F^*_{\mathbb{Q}} \to V$.

(2) *If this Ψ is trivial on* $\mathrm{d}(N_3(F))$, *then $\Psi\circ\Xi$ maps $[g]_2\otimes f$ in $\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$ to*

$$G((1-g)\otimes g\otimes f) - \tfrac{2}{3}G(((1-g)\cdot g)\otimes f) + \tfrac{2}{3}G(((1-g)\cdot f)\otimes g),$$

where $a_1\cdot a_2 = \frac{1}{2}(a_1\otimes a_2 + a_2\otimes a_1)$ in $\mathrm{Sym}^2(F^*_{\mathbb{Q}}) \subset (F^*_{\mathbb{Q}})^{\otimes 2}$.

(3) *Suppose that we have linear maps $\Psi : M_2(F)\otimes F^*_{\mathbb{Q}}/\mathrm{d}(N_3(F)) \to V$ and $H : \mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}} \to V$, such that $\Psi(([a]_2+[a^{-1}]_2)\otimes b) = H((a\cdot a)\otimes b)$. Then $\Psi\circ\Xi$ maps $[g]_2\otimes f$ in $\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$ to*

$$\Psi(g,f) - \tfrac{2}{3}H((1-g)\cdot g)\otimes f) + \tfrac{2}{3}H((1-g)\cdot f)\otimes g).$$

Proof. (1) The map Ψ is the composition of G with $\mathrm{d}\otimes\mathrm{id}$, with $\mathrm{d}: M_2(F) \to F_{\mathbb{Q}}^{*\otimes 2}$ the differential in $\mathcal{M}_{(2)}(F)$. For (2) and (3) we lift $\alpha = [g]_2 \otimes f$ in $\widetilde{M}_2(F) \otimes F_{\mathbb{Q}}^*$ to $[g]_2 \otimes f$ in $M_2(F) \otimes F_{\mathbb{Q}}^*$ to find $\Psi \circ \Xi([g]_2 \otimes f) = \Psi([g]_2 \otimes f) - \Psi(\beta)$, with $\beta = \beta_\alpha$, so it suffices to compute $\Psi(\beta)$. For (2) we find

$$\begin{aligned}\Psi(\beta) &= G(\mathrm{d}\otimes\mathrm{id}(\beta)) = G(\Phi(\mathrm{d}\otimes\mathrm{id}([g]_2 \otimes f))) = G(\Phi((1-g)\otimes g \otimes f))\\ &= \tfrac{2}{3}G(((1-g)\cdot g)\otimes f) - \tfrac{2}{3}G(((1-g)\cdot f)\otimes g)\end{aligned}$$

For (3) we find the formula in a similar way by noting that β can be written as a sum of elements of the form $[a]_2 + [a^{-1}]_2$ and that $\mathrm{d}([a]_2 + [a^{-1}]_2) = a \cdot a$. $\square$

Corollary 2.30. *Under the assumptions in* (2) *and* (3) *of Lemma 2.29, the composition $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to H^2(\mathcal{M}_{(3)}(F)) \to V$ is given by the corresponding formulas.*

2.4.3. *Construction of the complex $\mathcal{M}_{(3)}(C')$.* In this section we consider the situation where we have smooth, projective, geometrically irreducible curve C' over a number field k with function field $F' = k(C')$.

Because we are interested in finding elements in $K_4^{(3)}(C')$, we introduce yet another complex, $\mathcal{M}_{(3)}(C')$, which is the total complex associated to the double complex:

$$\begin{tikzcd} M_3(F') \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(F') \otimes_{\mathbb{Q}} F'^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"] & F'^*_{\mathbb{Q}} \otimes \bigwedge^2 F'^*_{\mathbb{Q}} \arrow[d, "\partial_2"] \\ 0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}} \end{tikzcd}$$

(Although not needed in this paper, one could define the complex $\widetilde{\mathcal{M}}_{(3)}(C')$ by using $\widetilde{\mathcal{M}}_{(3)}(F')$ in the top row.) Here the coproducts are over all closed points x of C'. The boundary maps are as follows. The d's in the top row are as in $\mathcal{M}_{(3)}(F')$. In the bottom row, $\mathrm{d}[z]_2 = (1-z)\wedge z$. For the vertical maps,

$$\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f)\cdot [g(x)]_2,$$

with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$. Finally, $\partial_{2,x}$ described as follows. Let π be a uniformizer at x, u_j units at x. Then $\partial_{2,x}$ is determined by

$$\pi \wedge u_1 \wedge u_2 \mapsto u_1(x)\wedge u_2(x) \quad \text{and} \quad u_1 \wedge u_2 \wedge u_3 \mapsto 0\,.$$

Therefore, an element $\sum_i [g_i]_2 \otimes f_i$ in $H^2(\mathcal{M}_{(3)}(F'))$ satisfies

$$\sum_i (1-g_i)\otimes (g_i \wedge f_i) = 0$$

in $F'^*_{\mathbb{Q}} \otimes \bigwedge^2 F'^*_{\mathbb{Q}}$. The additional condition for it to lie in $H^2(\mathcal{M}_{(3)}(C'))$ is that $\sum_i \mathrm{ord}_x(f_i)[g_i(x)]_2 = 0$ in $\widetilde{M}_2(k(x))$ for all closed points x in C', with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

We have an obvious map $\mathcal{M}_{(3)}(C') \to \mathcal{M}_{(3)}(F')$, corresponding to the localization map in (2.2). In [de Jeu 1996, Theorem 5.2], it is shown that this induces a commutative diagram:

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(C')) \arrow[r] \arrow[d] & H^2(\mathcal{M}_{(3)}(F')) \arrow[d] \\ K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \arrow[r] & K_4^{(3)}(F') \end{tikzcd} \tag{2.31}$$

Note that it was shown in Remark 2.4 that $K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}}$ is indeed a direct sum, and that the lower horizontal map is an injection.

Remark 2.32. If k is totally real then $K_3^{(2)}(k)$ is zero. But in general we can use the projection

$$K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \to K_4^{(3)}(C')$$

to get a map $H^2(\mathcal{M}_{(3)}(C')) \to K_4^{(3)}(C')$ as the composition

$$H^2(\mathcal{M}_{(3)}(C')) \to K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \to K_4^{(3)}(C').$$

2.4.4. *Construction of the complex* $\mathscr{C}^\bullet(F)$. The complex $\mathscr{C}^\bullet(F)$ is described in [de Jeu 1996, Section 3], but it was first constructed in [Bloch 1990]. We recall its construction in order to clarify the construction of the corresponding complex for $\mathbb{O}$ in Section 2.5.4.

One starts with another part of the exact localization sequence (2.10) in relative K-theory.

$$\begin{aligned} \cdots \to \coprod_{t \in F^\flat} K_3^{(2)}(F) \to K_3^{(3)}(X_F; \square) \to K_3^{(3)}(X_F^{\mathrm{loc}}; \square) \\ \to \coprod_{t \in F^\flat} K_2^{(2)}(F) \to K_2^{(3)}(X_F; \square) \to \cdots . \end{aligned} \tag{2.33}$$

Because $K_2^{(3)}((X_F; \square)) \simeq K_3^{(3)}(F) \simeq K_3^M(F)_{\mathbb{Q}}$, so that the map

$$\coprod_{t \in F^\flat} K_2^{(2)}(F) \to K_2^{(3)}(X_F; \square)$$

is surjective, this shows that the cohomological complex in degrees 1 and 2,

$$AC_{(3)}(F) : K_3^{(3)}(X_F^{\mathrm{loc}}; \square) \to \coprod_{t \in F^\flat} K_2^{(2)}(F),$$

has maps $H^1(AC_{(3)}(F)) \simeq K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}$ and $H^2(AC_{(3)}(F)) \simeq K_3^{(3)}(F)$. (Here AC stands for "auxiliary complex".)

Again we have an acyclic subcomplex

$$(1+I)^* \cup K_2^{(2)}(F) \to \mathrm{d}(\cdots),$$

and therefore the quotient complex $\mathscr{C}^\bullet(F)$ is a cohomological complex in degree 1 and 2,

$$\mathscr{C}^\bullet(F) : \mathscr{C}^1(F) \to \mathscr{C}^2(F),$$

with

$$\mathscr{C}^1(F) = \frac{K_3^{(3)}(X_F^{\mathrm{loc}};\square)}{(1+I)^* \cup K_2^{(2)}(F)} \quad \text{and} \quad \mathscr{C}^2(F) = K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^*.$$

It comes with maps

$$H^1(\mathscr{C}^\bullet(F)) \simeq K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^* \tag{2.34}$$

and $H^2(\mathscr{C}^\bullet(F)) \simeq K_3^{(3)}(F.)$

Note that if g is in $F^\flat$, and f is in F^*, then $[g]_2 \cup f$ lies in $K_3^{(3)}(X_F^{\mathrm{loc}};\square)$. In fact, if we take the class of $[g]_2$ in $M_2(F)$ instead, then we do get a well-defined class in $\mathscr{C}^1(F)$, as $(1+I)^* \cup F_{\mathbb{Q}}^* \cup f$ goes to zero in $\mathscr{C}^1(F)$ by definition. Under the differential in the complex, $[g]_2 \cup (f)$ is mapped to

$$\{(1-g)^{-1}, f\} \otimes g = -\{1-g, f\} \otimes g,$$

so the condition for an element $\sum_i [g_i]_2 \cup (f_i)$ to be in $H^1(\mathscr{C}^\bullet(F))$ is that

$$\sum_i \{1-g_i, f_i\} \otimes g_i = 0 \quad \text{in } K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^*.$$

The map $M_{(2)}(F) \otimes F_{\mathbb{Q}}^* \to \mathscr{C}^1(F)$ given by $[g]_2 \otimes f \mapsto [g]_2 \cup f$ fits into a commutative diagram

$$\begin{tikzcd} M_3(F) \arrow[r] \arrow[d] & M_2(F) \otimes F_{\mathbb{Q}}^* \arrow[r] \arrow[d] & F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^* \arrow[d] \\ 0 \arrow[r] & \mathscr{C}^1(F) \arrow[r] & \mathscr{C}^2(F) \end{tikzcd} \tag{2.35}$$

where we map $f \otimes g \wedge h$ to $\{f, g\} \otimes h - \{f, h\} \otimes g$. Multiplying the map $H^2(\mathscr{M}_{(3)}(F)) \to K_4^{(3)}(F)$ by -1 if necessary, we obtain a commutative diagram (see [de Jeu 1996, Proposition 3.2]):

$$\begin{tikzcd} H^2(\mathscr{M}_{(3)}(F)) \arrow[r] \arrow[d] & K_4^{(3)}(F) \arrow[d] \\ H^1(\mathscr{C}^\bullet(F)) \arrow[r] & K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^* \end{tikzcd} \tag{2.36}$$

2.5. *Construction of the complexes for $\mathbb{O}$ and $\mathscr{C}'$.*

Remark 2.37. At various stages there will be some properties of the complexes for $\mathbb{O}$ that depend on $K_3^{(2)}(\kappa)$ being trivial. Clearly, this applies to $\mathbb{O}$ as in Section 1 by our remarks about the K-groups of $\kappa(\mathscr{C}_\kappa)$ and $\mathbb{F}(\mathscr{C}'_{\mathbb{F}})$ in Section 2.2.

2.5.1. *Construction of the complex $\mathcal{M}_{(2)}(\mathbb{O})$.* When we try to imitate the localization sequence (2.10) for $\mathbb{O}$ rather than F, we are dealing with the two dimensional scheme $X_{\mathbb{O}}$, and we end up with a spectral sequence instead,

$$\begin{array}{ccc}
\vdots & \vdots & \\
K_1^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_0^{(1)}(F) & \\
K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_1^{(1)}(F) & \coprod_{t\in\kappa^\flat} K_0^{(0)}(\kappa) \\
K_3^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_2^{(1)}(F) & \coprod_{t\in\kappa^\flat} K_1^{(0)}(\kappa) \\
\vdots & \vdots & \vdots
\end{array} \tag{2.38}$$

which converges to $K_*^{(2)}(X_{\mathbb{O}};\square)\simeq K_{*+1}^{(2)}(\mathbb{O})$.

Because $K_2^{(1)}(F)$, $K_1^{(0)}(\kappa)$ and $K_2^{(0)}(\kappa)$ are all trivial, if we let $RC_{(2)}(\mathbb{O})$ be the cohomological complex in degrees 1, 2 and 3, given by

$$K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)\to\coprod_{t\in\mathbb{O}^\flat} K_1^{(1)}(F)\to\coprod_{t\in\kappa^\flat} K_0^{(0)}(\kappa)\,, \tag{2.39}$$

then there are maps $H^1(RC_{(2)}(\mathbb{O}))\simeq K_3^{(2)}(\mathbb{O})$ and $H^2(RC_{(2)}(\mathbb{O}))\to K_2^{(2)}(\mathbb{O})$. The last map is surjective by Proposition 2.6 and the exact sequence

$$\cdots\to K_2^{(1)}(\kappa)\to K_2^{(2)}(\mathbb{O})\to K_2^{(2)}(F)\to K_1^{(1)}(\kappa)\to\cdots$$

as $K_2^{(1)}(\kappa)=0$. Note that the map $K_1^{(1)}(F)\to K_0^{(0)}(\kappa)$ is surjective, so that $H^3(RC_{(2)}(\mathbb{O}))$ is zero, as is $K_1^{(2)}(\mathbb{O})$.

Now let $A\subseteq K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ be the inverse image of $\coprod_{t\in\mathbb{O}^\flat}\mathbb{O}_{\mathbb{Q}}^*$ in $\coprod_{t\in\mathbb{O}^\flat}K_1^{(1)}(F)$. Because $K_1^{(1)}(\mathbb{O})=\mathbb{O}_{\mathbb{Q}}^*$ is equal to

$$\ker\bigl(K_1^{(1)}(F)\to K_0^{(0)}(\kappa)\bigr),$$

this means that the subcomplex

$$RC_{(2)}(\mathbb{O}):A\to\coprod_{t\in\mathbb{O}^\flat}\mathbb{O}_{\mathbb{Q}}^* \tag{2.40}$$

of (2.39) has maps $H^1(RC_{(2)}(\mathbb{O}))\to K_3^{(2)}(\mathbb{O})$ and $H^2(RC_{(2)}(\mathbb{O}))\to K_2^{(2)}(\mathbb{O})$.

We again use the element $[u]_2$ in $K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square)$ for every u in $\mathcal{O}^\flat$, and put

$$\begin{aligned}\mathrm{Symb}_1(\mathcal{O}) &= K_1^{(1)}(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^*,\\ \mathrm{Symb}_2(\mathcal{O}) &= \langle [u]_2 \text{ with } u \text{ in } \mathcal{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^*.\end{aligned}$$

(See (2.9) for the definition of $(1+I)_{\mathcal{O}}^*$.) Observe that, if u is in $\mathcal{O}^\flat$ and v is in $\mathcal{O}_{\mathbb{Q}}^*$, then $[u]_2$ and $(1+I)_{\mathcal{O}}^* \cup v$ are in A, so we get a subcomplex of (2.40)

$$\mathit{Symb}_2(\mathcal{O}) : \mathrm{Symb}_2(\mathcal{O}) \to \coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^*, \tag{2.41}$$

containing the acyclic subcomplex

$$(1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* \to \mathrm{d}(\cdots). \tag{2.42}$$

We take the quotient complex of (2.41) by (2.42) to obtain the complex

$$\mathcal{M}_{(2)}(\mathcal{O}) : M_2(\mathcal{O}) \to \mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^*, \tag{2.43}$$

with $M_2(\mathcal{O}) = \mathrm{Sym}_2(\mathcal{O})/(1+I)^* \cup \mathcal{O}_{\mathbb{Q}}^*$. Then $M_2(\mathcal{O})$ is a $\mathbb{Q}$-vector space generated by the $[u]_2$, u in $\mathcal{O}^\flat$, and $\mathrm{d}[u]_2 = (1-u)\otimes u$. (Again, we used that $\mathrm{d}(1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^*$ gives us exactly the right relations to change $\coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^*$ into $\mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^*$ because $\mathcal{O}^\flat$ generates $\mathcal{O}^*$.) Note that we now have maps

$$\mathcal{M}_{(2)}(\mathcal{O}) \leftarrow \mathit{Symb}_2(\mathcal{O}) \to \mathit{RC}_{(2)}(\mathcal{O}),$$

with the left one a quasiisomorphism, so we obtain maps

$$H^i(\mathcal{M}_{(2)}(\mathcal{O})) \to K_{4-i}^{(2)}(\mathcal{O}) \tag{2.44}$$

for $i=1$ and 2. Again the map for $i=1$ is an injection (cf. (2.17)). For $i=2$ the map is a surjection by Proposition 2.6 because $K_2^{(2)}(\mathcal{O}) = \ker\big(K_2^{(2)}(F) \to K_1^{(1)}(\kappa)\big)$.

Localizing the base from $\mathcal{O}$ to F in (2.38) gives us (2.19), so that we get a map of complexes $M_2(\mathcal{O}) \to M_2(F)$ since the various steps in the constructions of the two complexes are compatible.

Remark 2.45. The map $M_2(\mathcal{O}) \to M_2(F)$ is injective. Namely, because the construction of the complexes for $\mathcal{M}_{(2)}(\mathcal{O})$ and $\mathcal{M}_{(2)}(F)$ is compatible with the localization from $\mathcal{O}$ to F in (2.38), we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathcal{M}_{(2)}(\mathcal{O})) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^* \arrow[d] \\ 0 \arrow[r] & H^1(\mathcal{M}_{(2)}(F)) \arrow[r] & M_2(F) \arrow[r] & F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*, \end{tikzcd}$$

with $H^1(\mathcal{M}_{(2)}(\mathbb{O})) \subseteq K_3^{(2)}(\mathbb{O})$ and $H^1(\mathcal{M}_{(2)}(F)) \subseteq K_3^{(2)}(F)$. From the exact localization sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathbb{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to \cdots$$

we see that $K_3^{(2)}(\mathbb{O}) \simeq K_3^{(2)}(F)$, so that the map on H^1's must be injective. As $\mathbb{O}_{\mathbb{Q}}^* \otimes \mathbb{O}_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$ is clearly injective, $M_2(\mathbb{O}) \to M_2(F)$ must be injective as well. So we may think of $M_2(\mathbb{O})$ as the subspace of $M_2(F)$ generated by the $[u]_2$ with u in $\mathbb{O}^\flat \subset F^\flat$.

2.5.2. *Construction of the complex $\mathcal{M}_{(3)}(\mathbb{O})$.* In this subsection, we shall be making Assumption 2.7.

If we now try to imitate the construction of $\mathcal{M}_{(3)}(F)$ using $\mathbb{O}$ instead of F, we see some differences. For example, in the construction of the spectral sequence, in codimension one, we shall end up with copies of $\{t_i = u\}$ for u in $\mathbb{O}^\flat$, which look like $X_{\mathbb{O}}$, out of which we have to remove the intersections with all other such pieces of codimension one of the form $\{t_i = v\}$ for $i = 1$ and 2, and v in $\mathbb{O}^\flat$. Note that, in particular, we also cut out $t_i = v$ with u and v different elements in $\mathbb{O}^\flat$, but reducing to the same in the residue field. Then $t_i = v$ cuts out the bit in the special fiber in $t_i = u$. We therefore end up with copies of $X_F^{'\mathrm{loc}} = X_F \setminus \{t = u \text{ with } u \text{ in } \mathbb{O}^\flat\}$.

So if we do this for $\mathbb{O}$, we end up with the following spectral sequence, converging to $K_*^{(3)}(X_{\mathbb{O}}^2; \square^2) \simeq K_{*+2}^{(3)}(\mathbb{O})$; see [Besser and de Jeu 2003, (3.7)]. For typographical reasons, let us introduce the following abbreviations:

$$K_{n,\mathbb{O}}^{(j),m} := K_n^{(j)}(X_{\mathbb{O}}^m; \square^m), \quad K_{n,F}^{(j),1} := K_n^{(j)}(X_F^{'\mathrm{loc}}; \square), \quad K_{n,\kappa}^{(j),1} := K_n^{(j)}(X_\kappa; \square).$$

Then the spectral sequence is

(2.46)

$$\begin{array}{llll}
\vdots & \vdots & \vdots & \\
K_{2,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{1,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_0^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{0,\kappa}^{(1),1}\Big)^2 & \\
K_{3,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{2,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_1^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{1,\kappa}^{(1),1}\Big)^2 & \coprod_{t_1,t_2\in\kappa^\flat} K_0^{(0)}(\kappa) \\
K_{4,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{3,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_2^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{2,\kappa}^{(1),1}\Big)^2 & \coprod_{t_1,t_2\in\kappa^\flat} K_1^{(0)}(\kappa) \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Here the $(\cdots)^2$ corresponds to two copies, corresponding to a coproduct over t_1 in $\mathcal{O}^\flat$ or $\kappa^\flat$, and t_2 in $\mathcal{O}^\flat$ or $\kappa^\flat$. As explained before, in order to obtain $X_F^{\prime\mathrm{loc}}$ out of X_F, we only remove $t_i = u_j$ with u_j in $\mathcal{O}^\flat$.

Now notice that all $K_j^{(0)}(\kappa)$ are zero for $j \geq 1$, that $K_j^{(1)}(F)$ is zero for $j \geq 2$, and finally that $K_j^{(1)}(X_\kappa^{\mathrm{loc}}; \square)$ is zero as well for $j \geq 2$: we consider the exact localization sequence

$$\cdots \to K_j^{(1)}(X_\kappa^1; \square) \to K_j^{(1)}(X_\kappa^{\mathrm{loc}}; \square) \to \coprod K_{j-1}^{(0)}(\kappa) \to \cdots ,$$

and use that $K_j^{(1)}(X_\kappa^1; \square) \simeq K_{j+1}^{(1)}(\kappa)$, which is zero as $K_m^{(1)}(L) = 0$ for $m \geq 2$ for any field L, as well as that $K_{j-1}^{(0)}(\kappa) = 0$ because $j-1 \geq 1$. Therefore, with $RC_{(3)}(\mathcal{O})$ the following cohomological complex in degrees 1 through 4 (corresponding to the row in (2.46) starting with $K_3^{(3)}(X^2_{\mathcal{O},\mathrm{loc}}; \square^2)$):

$$RC_{(3)}(\mathcal{O}) : K_3^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}}; \square^2) \to \Big(\coprod_{t \in \mathcal{O}^\flat} K_2^{(2)}(X_F^{\mathrm{loc}}; \square)\Big)^2$$
$$\to \coprod_{t_1,t_2 \in \mathcal{O}^\flat} K_1^{(1)}(F) \coprod \Big(\coprod_{t \in \kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}}; \square)\Big)^2 \to \coprod_{t_1,t_2 \in \kappa^\flat} K_0^{(0)}(\kappa) \tag{2.47}$$

has maps

$$H^i(RC_{(3)}(\mathcal{O})) \to K_{6-i}^{(3)}(\mathcal{O}) \tag{2.48}$$

for $i = 2$, 3 and 4.

Remark 2.49. Note that for $i = 4$ this statement is vacuous since from the localization sequence

$$\cdots \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to K_2^{(3)}(\mathcal{O}) \to K_2^{(3)}(F) \to \cdots$$

and the facts that $K_2^{(3)}(F)$ is trivial and $K_3^{(3)}(F) \to K_2^{(2)}(\kappa)$ is surjective (see Proposition 2.6), it follows that $K_2^{(3)}(\mathcal{O})$ is zero.

Remark 2.50. The map $K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\mathrm{loc}}; \square)$ is injective. Namely, we have an exact localization sequence

$$\cdots \to K_2^{(1)}(X_\kappa^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to \cdots ,$$

and $K_2^{(1)}(X_\kappa^{\mathrm{loc}}; \square)$ equals zero, as seen above. Also, we have an exact localization sequence

$$\cdots \to \coprod_{t \in F^* \setminus F^\flat \bigcup \{1\}} K_2^{(1)}(F) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\mathrm{loc}}; \square) \to \cdots ,$$

and again $K_2^{(1)}(F)$ is zero.

Remark 2.51. Note that, because we can localize $\mathbb{O}$ to F, we have a natural map of the spectral sequence in (2.46) to the one in (2.19), which, at the level of the complexes (2.20) and (2.47), simply forgets the terms over κ, includes a coproduct over $\mathbb{O}^\flat$ into the corresponding coproduct over $F^\flat$, and uses the maps $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ and $K_3^{(3)}(X_{\mathbb{O}}^{2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{\prime 2,\mathrm{loc}};\square^2)$. By Remark 2.50, the first one is always injective, and the second is injective if $K_5^{(2)}(\kappa)$ and $K_4^{(2)}(F)$ are zero.

Let us try to create a jewel in the crown of the scary notation in (2.47). Define $\mathrm{Symb}_n(\mathbb{O}) \subseteq K_n^{(n)}(X_{\mathbb{O}}^{n-1,\mathrm{loc}};\square^{n-1})$ for $n=1$, 2 and 3 by setting

$$\begin{aligned}
&\mathrm{Symb}_1(\mathbb{O}) = \mathbb{O}_{\mathbb{Q}}^*,\\
&\mathrm{Symb}_2(\mathbb{O}) = \langle [u]_2 \text{ with } u \text{ in } \mathbb{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathbb{O}}^* \cup \mathrm{Symb}_1(\mathbb{O}),
\end{aligned}$$

as before, and

$$\mathrm{Symb}_3(\mathbb{O}) = \langle [u]_3 \text{ with } u \text{ in } \mathbb{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathbb{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathbb{O}).$$

Again, by $\tilde{\cup}$ we denote that we use both products, coming from the two ways of projecting $X_{\mathbb{O}}^2$ to $X_{\mathbb{O}}$.

Note that for $n=1$, $\mathrm{Symb}_1(\mathbb{O}) = \mathbb{O}_{\mathbb{Q}}^* \subseteq \mathrm{Symb}_1(F) = F_{\mathbb{Q}}^*$, and that for $n=2$, we can view $\mathrm{Symb}_2(\mathbb{O}) \subseteq \mathrm{Symb}_2(F)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ by Remark 2.50, as

$$K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) \subseteq K_2^{(2)}(X_F^{\mathrm{loc}};\square).$$

Because $\mathrm{d}[u]_2 = (1-u)^{-1}_{|t=u}$, and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$ (where both terms lie in a copy of $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}})$, again by Remark 2.50), it follows that

$$Symb_{(3)}(\mathbb{O}) : \mathrm{Symb}_3(\mathbb{O}) \to \Bigl(\coprod_{t\in\mathbb{O}^\flat} \mathrm{Symb}_2(\mathbb{O})\Bigr)^2 \to \coprod_{t_1,t_2\in\mathbb{O}^\flat} \mathbb{O}_{\mathbb{Q}}^* \tag{2.52}$$

is a subcomplex (in degrees 1, 2 and 3) of (2.47). Note that we used here that elements in $\mathbb{O}^\flat$ never give rise to a pole or zero over κ, so the map to $\coprod K_0^{(0)}(\kappa)$ is zero. Also, we used that an element $[u]_2$ with u in $\mathbb{O}^\flat$ under the localization (of its construction),

$$K_2^{(2)}(X_{\mathbb{O}} \setminus \{t=u\};\square) \to K_1^{(1)}(\mathbb{O}) \to \cdots$$

maps to $(1-u)^{-1}$, so under the boundary in (2.46) it never hits the $K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ components. Similarly, the elements in $(1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*$ never hit these components.

Again, one shows that the subcomplex of (2.52) given by

$$(1+I)_{\mathbb{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathbb{O}) \to \Bigl(\coprod_t (1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*\Bigr)^2 + (\cdots) \to \mathrm{d}(\cdots)$$

is acyclic; see [de Jeu 1995, Lemma 3.7 and Remark 3.10].

Taking the quotient complex and the alternating part for the action of S_2 under swapping the coordinates, we finally get a complex

$$M_3(\mathcal{O}) \to M_2(\mathcal{O}) \to \mathcal{O}^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 \mathcal{O}^*_{\mathbb{Q}} .$$

Here

$$M_3(\mathcal{O}) = \mathrm{Symb}_3(\mathcal{O})/\big((1+I)^*_{\mathcal{O}} \,\tilde{\cup}\, \mathrm{Symb}_2(\mathcal{O})\big)^{\mathrm{alt}}$$

and, as before,

$$M_2(\mathcal{O}) = \mathrm{Symb}_2(\mathcal{O})/(1+I)^*_{\mathcal{O}} \cup \mathcal{O}^*_{\mathbb{Q}} .$$

Note that $M_n(\mathcal{O})$ is a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathcal{O}^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by $\mathrm{d}[u]_3 = [u]_2 \otimes u$ and $\mathrm{d}[u]_2 \otimes v = (1-u) \otimes (u \wedge v)$.

In particular, the condition for an element $\sum_i [u_i] \otimes v_i$ in $M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}$ to lie in $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$ is that

$$\sum_i (1-u_i) \otimes (u_i \wedge v_i) = 0 \quad \text{in } \mathcal{O}^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 \mathcal{O}^*_{\mathbb{Q}} . \tag{2.53}$$

Again S_2 acts on the various complexes by swapping the coordinates, and we get maps

$$\mathcal{M}_{(3)}(\mathcal{O}) \leftarrow \mathit{Symb}_{(3)}(\mathcal{O})^{\mathrm{alt}} \to \mathit{RC}_{(3)}(\mathcal{O})^{\mathrm{alt}} \to \mathit{RC}_{(3)}(\mathcal{O})$$

with the left map a quasiisomorphism. Combining this with (2.48) gives us a map

$$H^i(\mathcal{M}_{(3)}(\mathcal{O})) \to K^{(3)}_{6-i}(\mathcal{O}) \tag{2.54}$$

for $i = 2$ and 3, where the map for $i = 3$ is a surjection if $K^{(2)}_3(\kappa) = 0$ by Proposition 2.6 and the localization sequence

$$\cdots \to K^{(2)}_3(\kappa) \to K^{(3)}_3(\mathcal{O}) \to K^{(3)}_3(F) \to K^{(2)}_2(\kappa) \to \cdots .$$

Remark 2.55. Notice that by construction (that is, by compatibility of everything we did with the localization of $\mathcal{O}$ to F), these maps for $i = 2$ or 3 fit into a commutative diagram:

$$\begin{tikzcd} H^i(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K^{(3)}_{6-i}(\mathcal{O}) \arrow[d] \\ H^i(\mathcal{M}_{(3)}(F)) \arrow[r] & K^{(3)}_{6-i}(F) \end{tikzcd} \tag{2.56}$$

We also note that it was proved in Remark 2.45 that the map $M_2(\mathcal{O}) \to M_2(F)$ is injective. Because we clearly have that $\mathcal{O}^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}}$ is an injection, this means that, in degrees 2 and 3, $\mathcal{M}_{(3)}(\mathcal{O})$ injects into $\mathcal{M}_{(3)}(F)$.

2.5.3. *Construction of the complex $\mathcal{M}_{(3)}(\mathscr{C}')$.* In this subsection we imitate the definition of the complex $\mathcal{M}_{(3)}(C')$ in Section 2.4.3, but using the complex $\mathcal{M}_{(3)}(\mathbb{O}')$ rather than $\mathcal{M}_{(3)}(F')$ in the top row. The advantage of using the complex $\mathcal{M}_{(3)}(\mathscr{C}')$ (just like the advantage of using any $\mathbb{O}'$-complex over the corresponding F'-complex) is that the syntomic regulator gets the input it needs on the special fiber of $\mathscr{C}'$.

We therefore put ourselves in the situation of Notation 1.7, so assume we have a number field $k \subset K$, a proper, smooth, irreducible curve $\mathscr{C}'$ over $R' = \mathbb{O} \cap k$, and that the generic fiber $C' = \mathscr{C}' \otimes_{R'} k$ is geometrically irreducible. We put $F' = k(C')$, and $\mathbb{O}'$ the discrete valuation ring in F' corresponding to the generic point of the special fiber of $\mathscr{C}'$. We have a commutative diagram as follows:

$$\begin{tikzcd} M_3(\mathbb{O}') \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(\mathbb{O}') \otimes_{\mathbb{Q}} \mathbb{O}'^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"] & \mathbb{O}'^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathbb{O}'^*_{\mathbb{Q}} \arrow[d, "\partial_2"] \\ 0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}}\,. \end{tikzcd}$$

The d's in the top row are as in $\mathcal{M}_{(3)}(\mathbb{O}')$. The vertical maps and the map in the bottom row are given by the same formulas as before (see 2.4.3), via the natural map $\mathcal{M}_{(3)}(\mathbb{O}') \to \mathcal{M}_{(3)}(F')$ corresponding to the localization from $\mathbb{O}'$ to F'.

We let $\mathcal{M}_{(3)}(\mathscr{C}')$ be the cohomological complex in degrees 1 through 4, given by the total complex associated to the double complex in the commutative diagram above. Note that therefore in particular, an element $\sum_i [u_i]_2 \otimes v_i$ in $M_2(\mathbb{O}') \otimes \mathbb{O}'^*_{\mathbb{Q}}$ is in $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$ if and only if it satisfies (2.53) as well as, for every closed point x in C',

$$\sum_i \operatorname{ord}_x(v_i)[u_i(x)]_2 = 0 \tag{2.57}$$

in $\widetilde{M}_2(k(x))$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

The map to K-theory is similar to the map for $\mathcal{M}_{(3)}(F')$, but now we get

$$H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to H^2(\mathcal{M}_{(3)}(\mathbb{O}')) \to K_4^{(3)}(\mathbb{O}'),$$

where the first arrow corresponds to forgetting the bottom row in $\mathcal{M}_{(3)}(\mathscr{C}')$. In fact, because this is compatible with the localization to F' (that is, with the map $\mathcal{M}_{(3)}(\mathbb{O}') \to \mathcal{M}_{(3)}(F')$), from (2.31) we find that we have a commutative diagram

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \arrow[r] \arrow[d] & K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}} \arrow[d, equal] \\ H^2(\mathcal{M}_{(3)}(C')) \arrow[r] & K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \end{tikzcd} \tag{2.58}$$

where the group on the right is contained in $K_4^{(3)}(\mathcal{O}') = K_4^{(3)}(F')$, and we used that $K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} = K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}}$ by Remarks 2.3 and 2.5. This proves that the top square in (1.16) exists and commutes.

Note that in Theorem 1.9(2), the condition $\partial_1(\alpha') = 0$ on α' in $H^2(\mathcal{M}_{(3)}(\mathcal{O}'))$ is exactly that α' satisfies (2.57), hence lies in the subspace $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$. Therefore we have proved the existence of β' in the theorem. Its uniqueness is clear because the direct sum above gives an injection $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathcal{O}')/K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}}$.

Remark 2.59. Just as in Remark 2.32, we can consider the projection

$$K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}} \to K_4^{(3)}(\mathscr{C}')$$

to get a map $H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to K_4^{(3)}(\mathscr{C}')$ as the composition

$$H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}} \to K_4^{(3)}(\mathscr{C}').$$

2.5.4. *Construction of the complex* $\mathscr{C}^\bullet(\mathcal{O})$. The remainder of the theorems in the introduction will be proved in Section 10. The necessary calculations will in fact depend heavily on the analogue of $\mathscr{C}^\bullet(F)$ for $\mathcal{O}$, $\mathscr{C}^\bullet(\mathcal{O})$.

Because we are dealing with the two dimensional scheme $X_{\mathcal{O}}$, the localization sequence (2.33) becomes a spectral sequence (cf. (2.38)):

$$\begin{array}{cccc}
 & \vdots & \vdots & \vdots \\
 & K_2^{(3)}(X_{\mathcal{O}}^{\text{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_1^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_0^{(1)}(\kappa) \\
(2.60) & K_3^{(3)}(X_{\mathcal{O}}^{\text{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa) \\
 & K_4^{(3)}(X_{\mathcal{O}}^{\text{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_3^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_2^{(1)}(\kappa) \\
 & \vdots & \vdots & \vdots
\end{array}$$

converging to $K_*^{(3)}(X_{\mathcal{O}};\square) \simeq K_{*+1}^{(3)}(\mathcal{O})$. Let us notice that $K_2^{(1)}(\kappa)$ and $K_3^{(1)}(\kappa)$ are zero, and that the exact localization sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathcal{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to K_2^{(2)}(\mathcal{O}) \to K_2^{(2)}(F) \to \cdots$$

tells us that $K_2^{(2)}(\mathcal{O}) \subseteq K_2^{(2)}(F)$ and $K_3^{(2)}(\mathcal{O}) \simeq K_3^{(2)}(F)$. Therefore we get an exact sequence

$$0 \to \frac{K_4^{(3)}(\mathcal{O})}{K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}} \to K_3^{(3)}(X_{\mathcal{O}}^{\text{loc}};\square) \to \ker\Big(\coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(F) \to \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa)\Big).$$

In the middle row of the spectral sequence (2.60) above, let $B \subseteq K_3^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}}; \square)$ be the inverse image of $\coprod K_2^{(2)}(\mathbb{O})$ (with the coproduct over all of $\mathbb{O}^\flat$). Then we have a cohomological complex in degrees 1 and 2,

$$AC_{(3)}(\mathbb{O}) : B \to \coprod_{t \in \mathbb{O}^\flat} K_2^{(2)}(\mathbb{O}), \tag{2.61}$$

an isomorphism

$$H^1(AC_{(3)}(\mathbb{O})) \simeq \frac{K_4^{(3)}(\mathbb{O})}{K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}_{\mathbb{Q}}^*},$$

and a map $H^2(AC_{(3)}(\mathbb{O})) \to K_3^{(3)}(\mathbb{O})$.

Remark 2.62. If $K_3^{(2)}(\kappa) = 0$, or more generally, the map $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ is surjective, then from the exact localization sequence

$$\cdots \to K_4^{(3)}(F) \to K_3^{(2)}(\kappa) \to K_3^{(3)}(\mathbb{O}) \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to \cdots,$$

Proposition 2.6 and (2.44), we see that the map $\coprod_{t \in \mathbb{O}^\flat} K_2^{(2)}(\mathbb{O}) \to K_3^{(3)}(\mathbb{O})$, and hence the map $H^2(AC_{(3)}(\mathbb{O})) \to K_3^{(3)}(\mathbb{O})$, are surjective.

Remark 2.63. Because $K_1^{(2)}(F)$ and $K_2^{(1)}(\kappa)$ are zero, and $K_2^{(2)}(F) \to K_1^{(1)}(\kappa)$ is surjective, from (2.60) we get that there is an exact sequence

$$\operatorname{Ker}\Bigg(\coprod_{t \in \mathbb{O}^\flat} K_2^{(2)}(F) \to \coprod_{t \in \kappa^\flat} K_1^{(1)}(\kappa)\Bigg) \to K_2^{(3)}(X_{\mathbb{O}}; \square) \to K_2^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}}; \square) \to 0.$$

If $K_3^{(2)}(\kappa)$ is zero, or, more generally, the map $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ surjective, then Proposition 2.6 tells us that $\coprod_{t \in \mathbb{O}^\flat} K_2^{(2)}(\mathbb{O})$ surjects onto $K_2^{(3)}(X_{\mathbb{O}}; \square) \simeq K_3^{(3)}(\mathbb{O})$, and we can conclude that $K_2^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}}; \square)$ is zero.

Now we consider the acyclic subcomplex $(1+I)_{\mathbb{O}}^* \cup K_2^{(2)}(\mathbb{O}) \to \mathrm{d}(\cdots)$ of (2.61), and quotient out to find a complex $\mathscr{C}^\bullet(\mathbb{O}) : \mathscr{C}^1(\mathbb{O}) \to \mathscr{C}^2(\mathbb{O})$, where

$$\mathscr{C}^1(\mathbb{O}) = \frac{B}{(1+I)_{\mathbb{O}}^* \cup K_2^{(2)}(\mathbb{O})} \tag{2.64}$$

and $\mathscr{C}^2(\mathbb{O}) = K_2^{(2)}(\mathbb{O}) \otimes \mathbb{O}_{\mathbb{Q}}^*$. We still have an isomorphism

$$H^1(\mathscr{C}^\bullet(\mathbb{O})) \simeq K_4^{(3)}(\mathbb{O}) / K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}_{\mathbb{Q}}^* \tag{2.65}$$

and a map $H^2(\mathscr{C}^\bullet(\mathbb{O})) \to K_3^{(3)}(\mathbb{O})$, which by Proposition 2.6 and (2.44) is a surjection if $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ is surjective, for example, if $K_3^{(2)}(\kappa) = 0$.

Observe that if g is in $\mathcal{O}^\flat$, and f is in $\mathcal{O}^*_{\mathbb{Q}}$, then $[g]_2 \cup (f)$ is in $\mathcal{C}^1(\mathcal{O})$, and has boundary $\{(1-g)^{-1}, f\} \otimes g = -\{(1-g), f\} \otimes g$ in $\mathcal{C}^2(\mathcal{O})$. The condition for $\sum_i [g_i]_2 \cup (f_i)$ to be in $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ is therefore that

$$\sum_i \{1-g_i, f_i\} \otimes g_i = 0 \quad \text{in } \mathcal{C}^2(\mathcal{O}) = K_2^{(2)}(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}.$$

Note that because the construction of the spectral sequence in (2.60) is compatible with localizing the base from $\mathcal{O}$ to F and enlarging the coproduct from being over $\mathcal{O}^\flat$ to $F^\flat$ (in which case it becomes the localization sequence in (2.33)), and that $(1+I)^*_{\mathcal{O}}$ is contained in $(1+I)^*$, and $K_2^{(2)}(\mathcal{O}) \subseteq K_2^{(2)}(F)$, we have an obvious map of complexes, $\mathcal{C}^\bullet(\mathcal{O}) \to \mathcal{C}^\bullet(F)$, which fits into the commutative diagram

$$\begin{tikzcd} H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[d] \\ H^1(\mathcal{C}^\bullet(F)) \arrow[r] & K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_{\mathbb{Q}} \end{tikzcd} \tag{2.66}$$

and similarly for H^2.

Finally, we have a commutative diagram

$$\begin{tikzcd} M_3(\mathcal{O}) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}^1(\mathcal{O}) \arrow[r] & \mathcal{C}^2(\mathcal{O}) \end{tikzcd}$$

as follows. We map $[u]_2 \otimes v$ to $[u]_2 \cup v$, and $u \otimes v \wedge w$ to $\{u, v\} \otimes w - \{u, w\} \otimes v$. This gives rise to a commutative diagram

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O}) \arrow[d] \\ H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}, \end{tikzcd} \tag{2.67}$$

which is the bottom left square of (1.16). Obviously, the two diagrams above are compatible with (2.35) and (2.36) under the localization from $\mathcal{O}$ to F.

2.5.5. *Construction of the complexes $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ and $\widetilde{\mathcal{M}}_{(3)}(\mathcal{O})$.* For $n = 2$ and 3, let $N_n(\mathcal{O}) = \langle [u]_n + (-1)^n [u^{-1}]_n \text{ with } u \text{ in } \mathcal{O}^\flat \rangle_{\mathbb{Q}} \subseteq M_n(\mathcal{O})$. Consider the subcomplex of $\mathcal{M}_{(2)}(\mathcal{O})$ given by $N_2(\mathcal{O}) \to \mathrm{d}(\cdots)$. Because the corresponding subcomplex (2.14) of $\mathcal{M}_{(2)}(F)$ is acyclic and the natural map $M_2(\mathcal{O}) \to M_2(F)$ is an injection (see Remark 2.45), this subcomplex is acyclic. The second term is $\mathrm{Sym}^2(\mathcal{O}^*_{\mathbb{Q}})$, and the resulting quotient complex of $\mathcal{M}_{(2)}(\mathcal{O})$ is

$$\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) : \widetilde{M}_2(\mathcal{O}) \to \textstyle\bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \,, \tag{2.68}$$

with $\widetilde{M}_2(\mathcal{O}) = M_2(\mathcal{O})/N_2(\mathcal{O})$, and $\mathrm{d}[u]_2 = (1-u) \wedge u$.

Because $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ is quasiisomorphic to $\mathcal{M}_{(2)}(\mathcal{O})$ we have maps

$$H^i(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \to K_{4-i}^{(2)}(\mathcal{O}) \,.$$

For $i = 1$ this is again an injection. There is a map $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(2)}(F)$ obtained by localizing the construction from $\mathcal{O}$ to F, and for $i = 1, 2$ a commutative diagram

$$\begin{tikzcd}
H^i(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \arrow[d] & H^i(\mathcal{M}_{(2)}(\mathcal{O})) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & K_{4-i}^{(2)}(\mathcal{O}) \arrow[d] \\
H^i(\widetilde{\mathcal{M}}_{(2)}(F)) & H^i(\mathcal{M}_{(2)}(F)) \arrow[l, "\simeq"'] \arrow[r] & K_{4-i}^{(2)}(F) \,.
\end{tikzcd}$$

In this diagram for $i = 1$ the central vertical map is injective by the discussion in Remark 2.45. Hence the same holds for the map $H^1(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \to H^1(\widetilde{\mathcal{M}}_{(2)}(F))$, the map $\widetilde{M}_2(\mathcal{O}) \to \widetilde{M}_2(F)$ is an injection, and $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ is a subcomplex of $\widetilde{\mathcal{M}}_{(2)}(F)$.

By Remark 2.45, in the commutative diagram

$$\begin{tikzcd}
M_3(\mathcal{O}) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^* \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{Q}}^* \otimes \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \arrow[d] \\
M_3(F) \arrow[r] & M_2(F) \otimes F_{\mathbb{Q}}^* \arrow[r] & F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^*
\end{tikzcd}$$

the two right-most maps are injective. (If we knew (as part of the rigidity conjecture) that $H^1(\mathcal{M}_{(3)}(\mathcal{O})) \to H^1(\mathcal{M}_{(3)}(F))$ were injective, then this would also hold for the left-most map.) We can quotient out the complex $\mathcal{M}_{(3)}(\mathcal{O})$ in the first row by the subcomplex

$$N_3(\mathcal{O}) \to N_2(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^* \to \mathrm{d}(\cdots),$$

which maps to the subcomplex (2.26) of the second row. We saw earlier that $d : N_2(\mathcal{O}) \to \mathrm{Sym}^2(\mathcal{O}_{\mathbb{Q}}^*)$ is an isomorphism, so as in the proof of [de Jeu 1995, Corollary 3.22] one sees that this subcomplex is acyclic in degrees 2 and 3. The quotient complex is

$$\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) : \widetilde{M}_3(\mathcal{O}) \to \widetilde{M}_2(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^* \to \textstyle\bigwedge^3 \mathcal{O}_{\mathbb{Q}}^* \,,$$

where $\widetilde{M}_3(\mathcal{O}) = M_3(\mathcal{O})/N_3(\mathcal{O})$, and the natural map $\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(3)}(F)$ is an injection in degrees 2 and 3 because, as we saw earlier, $\widetilde{M}_2(\mathcal{O})$ injects into $\widetilde{M}_2(F)$. Still denoting the class of $[x]_i$ with $[x]_i$, the maps are now given by $\mathrm{d}[u]_3 = [u]_2 \otimes u$ and

$$\mathrm{d}[u]_2 \otimes v = (1-u) \wedge u \wedge v. \tag{2.69}$$

Using (2.54) we see that for $i = 2, 3$ we have a commutative diagram

$$\begin{tikzcd} H^i(\widetilde{\mathcal{M}}_{(3)}(\mathcal{O})) \arrow[d] & H^i(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & K^{(3)}_{6-i}(\mathcal{O}) \arrow[d] \\ H^i(\widetilde{\mathcal{M}}_{(3)}(F)) & H^i(\mathcal{M}_{(3)}(F)) \arrow[l, "\simeq"'] \arrow[r] & K^{(3)}_{6-i}(F)\,. \end{tikzcd}$$

Remark 2.70. Using the statements just before (2.68), the arguments in Remark 2.28 can also be given for $\mathcal{O}$ instead of F. This way we obtain a map

$$\widetilde{M}_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathcal{O})),$$

which we still denote by Ξ. It yields a decomposition

$$M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathcal{O})) = \mathrm{im}(\Xi) \oplus N_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathcal{O})),$$

and induces the inverse to the natural isomorphism $H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to H^2(\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}))$. The formulas in Lemma 2.29 and Corollary 2.30 apply in this case as well.

2.6. ***A diagram.*** For the convenience of the reader, we give in Figure 1 a commutative diagram summarizing the cohomology groups of most of the complexes introduced, and the maps. We have kept the layout of the diagram in the same spirit as the relativity in the plane. Note that the outer square is only relevant in the situation of Notation 1.7, and that we may replace F and $\mathcal{O}$ elsewhere in the diagram with F' and $\mathcal{O}'$ in this case.

The top half of this diagram is the top of the one in (1.16). The vertical maps correspond to the maps from constructions over $\mathcal{O}$ to the corresponding constructions over F. The horizontal maps are the maps on cohomology of complexes constructed in the previous subsections, and the diagonal maps correspond to the maps in (2.35), (2.56), (2.58) and (2.66).

Note that by Remarks 2.3 and 2.5 the rightmost vertical map is an isomorphism.

3. The classical case

In Proposition 3.1 below, we rephrase the results in Theorem 4.2 and Remarks 4.3 and 4.5 of [de Jeu 1996], which concern a curve C over $\mathbb{C}$ with function field $F = \mathbb{C}(C)$ and associated analytic manifold C_{an}, in a way that resembles the formulas in Theorems 1.12 and 1.13(1). (See Remark 10.14 for some thoughts on this comparison.) In fact, Sections 7 and 8 grew out of attempts to obtain syntomic analogues of those results of [loc. cit.], but the resulting formulas seem to be less flexible than the classical ones so we rephrase the latter.

In this section we let $H^1_{\mathrm{dR}}(F, \mathbb{R}(2)) = \lim_{\substack{\to \\ U}} H^1_{\mathrm{dR}}(U, \mathbb{R}(2))$ where the limit is over U with $C_{\mathrm{an}} \setminus U$ finite, and similarly for other cohomology groups, or forms. Here

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathcal{C}')) \arrow[rrrrr] \arrow[dr] \arrow[ddddd] & & & & & K_4^{(3)}(\mathcal{C}') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}} \arrow[dl] \arrow[ddddd] \\
& H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[rrr] \arrow[dr] \arrow[ddd] & & & K_4^{(3)}(\mathcal{O}) \arrow[dl] \arrow[ddd] & \\
& & H^1(\mathscr{C}^\bullet(\mathcal{O})) \arrow[r] \arrow[d] & \frac{K_4^{(3)}(\mathcal{O})}{K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}} \arrow[d] & & \\
& & H^1(\mathscr{C}^\bullet(F)) \arrow[r] & \frac{K_4^{(3)}(F)}{K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}} & & \\
& H^2(\mathcal{M}_{(3)}(F)) \arrow[rrr] \arrow[ur] & & & K_4^{(3)}(F) \arrow[ul] & \\
H^2(\mathcal{M}_{(3)}(C')) \arrow[rrrrr] \arrow[ur] & & & & & K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \arrow[ul]
\end{tikzcd}$$

Figure 1. Diagram summarizing cohomology groups (see previous page).

$\mathbb{R}(m) = (2\pi i)^m \mathbb{R} \subset \mathbb{C}$. If ω is holomorphic on C_{an}, then by [loc. cit., Proposition 4.6] one has a well-defined map $H^1_{\mathrm{dR}}(F, \mathbb{R}(2)) \to \mathbb{C}$ by taking a representative β of a class in $H^1_{\mathrm{dR}}(F, \mathbb{R}(2))$ satisfying [loc. cit., (9)], and computing $\int_{C_{\mathrm{an}}} \omega \wedge \beta$.

The signs of the maps in the following proposition are normalized to be compatible with the ones in the theorems in the introduction (see Remark 3.3).

Proposition 3.1. *Let C be a smooth, proper, irreducible curve over $\mathbb{C}$ with function field $F = \mathbb{C}(C)$, and let C_{an} be the analytic manifold associated to $C(\mathbb{C})$. For a holomorphic 1-form ω on C_{an}, the maps*

$$\begin{aligned}
\Psi''_{\infty,\omega} &: M_2(F) \otimes F^*_{\mathbb{Q}} \to \mathbb{C} \\
&\qquad [g]_2 \otimes f \mapsto 4 \int_{C_{\mathrm{an}}} \log|f| \log|g| \operatorname{dlog}|1-g| \wedge \omega, \\
\Psi'''_{\infty,\omega} &: \widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}} \to \mathbb{C} \\
&\qquad [g]_2 \otimes f \mapsto \tfrac{8}{3} \int_{C_{\mathrm{an}}} \log|f| (\log|g| \operatorname{dlog}|1-g| - \log|1-g| \operatorname{dlog}|g|) \wedge \omega
\end{aligned}$$

are well-defined, and induce maps $H^2(\mathcal{M}_{(3)}(F)) \to \mathbb{C}$ and $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to \mathbb{C}$, respectively. Moreover, with $\mathrm{reg}_{\mathbb{C}} : K_4^{(3)}(F) \to H^2_{\mathscr{D}}(F, \mathbb{R}(3)) \simeq H^1_{\mathrm{dR}}(F, \mathbb{R}(2))$ the

Beilinson regulator map, the compositions

$$H^2(\mathcal{M}_{(3)}(F)) \xrightarrow{(2.25)} K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C},$$
$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \xrightarrow{(2.27)} K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C}$$

coincide with these induced maps.

Proof. Since $d\otimes\mathrm{id}: M_2(F)\otimes F_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^*\otimes F_{\mathbb{Q}}^*\otimes F_{\mathbb{Q}}^*$ maps $[g]_2\otimes f$ to $(1-g)\otimes g\otimes f$, $\Psi''_{\infty,\omega}$ is well-defined. That it induces the stated map on $H^2(\mathcal{M}_{(3)}(F))$, and that this induced map has the stated property, follows from Proposition 3.2 and (the proof of) Theorem 4.2 of [de Jeu 1996], where we normalize the maps as explained in Remark 3.3 below. (The condition in [loc. cit.] that C is defined over a number field is not used in the proof of Theorem 4.2. The same holds for the condition with respect to complex conjugation on ω, which guaranteed only that the value of the integral was in $\mathbb{R}(1)\subset\mathbb{C}$.)

Applying Corollary 2.30 shows that $\Psi'''_{\infty,\omega}$ maps $[g]_2\otimes f$ to

$$\frac{4}{3}\int_{C_{\mathrm{an}}}(3\log|f|\log|g|\,\mathrm{d}\log|1-g|+\log|1-g|(\log|g|\,\mathrm{d}\log|f|-\log|f|\,\mathrm{d}\log|g|))\wedge\omega.$$

Using a limit version of Stokes' theorem we may subtract $0=\int_{C_{\mathrm{an}}}\mathrm{d}(\alpha\wedge\omega)$ for $\alpha=\frac{4}{3}\log|g|\log|1-g|\log|f|$, which gives the formula in the proposition. $\square$

Remark 3.2. The Bloch–Wigner dilogarithm $D(z):\mathbb{P}^1_{\mathbb{C}}\setminus\{0,1,\infty\}\to(2\pi i)\mathbb{R}\subset\mathbb{C}$ satisfies $\mathrm{d}D(z)=\log|z|\mathrm{d}i\arg(1-z)-\log|1-z|\mathrm{d}i\arg(z)$ and extends to a continuous function on $\mathbb{P}^1_{\mathbb{C}}$. It is the function in the classical case that corresponds to $\mathrm{L}_{\mathrm{mod},2}(z)$ in the sense that they have similar functional equations, for example, $D(z)+D(z^{-1})=0$. Because $\mathrm{d}\log(g)\wedge\omega=\mathrm{d}\log(1-g)\wedge\omega=0$, we find $\mathrm{d}(P_{2,\mathrm{Zag}}(g)\log|f|\omega)$ equals

$$P_{2,\mathrm{Zag}}(g)\,\mathrm{d}\log|f|\wedge\omega+\log|f|(\log|1-g|\,\mathrm{d}\log|g|-\log|g|\,\mathrm{d}\log|1-g|)\wedge\omega.$$

Hence $\Psi'''_{\infty,\omega}$ is also given by mapping $[g]_2\otimes f$ to $\frac{8}{3}\int_{C_{\mathrm{an}}}\log|f|D(g)\omega$.

Remark 3.3. The signs in Proposition 3.1 and Remark 3.2 are chosen in a way that is compatible with the ones in the p-adic case in Remark 5.25 below. In [de Jeu 1996] it is shown that, for a holomorphic 1-form ω, the map

$$K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C}$$

factorizes through the quotient map $K_4^{(3)}(F)\to K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_{\mathbb{Q}}^*$ in (2.36), giving maps $H^2(\mathcal{M}_{(3)}(F))\to H^1(\mathscr{C}^\bullet(F))\simeq K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_{\mathbb{Q}}^*\to\mathbb{C}$. This composition is the one used in Proposition 3.1, and there is a choice of sign in the isomorphism here, which we normalize as follows.

The regulator map gives us

$$\mathrm{reg}_{\mathbb{C}} : K_3^{(3)}(X_F^{\mathrm{loc}}; \square) \to H^3_{\mathscr{D}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(3)) \simeq H^2_{\mathrm{dR}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(2)).$$

Computing the last cohomology group here as

$$\frac{\bigl\{(\varepsilon, \varepsilon_\infty, \varepsilon_0) \bigm| \varepsilon \in A^2(X_F^{\mathrm{loc}}),\, \varepsilon_s \in A^1(F),\, \mathrm{d}\varepsilon = 0,\, \mathrm{d}\varepsilon_s = \varepsilon_{|t=s} (s = 0, \infty)\bigr\}}{\bigl\{(\mathrm{d}\psi, \psi_{|t=\infty} + \mathrm{d} f_\infty, \psi_{|t=0} + \mathrm{d} f_0) \bigm| \psi \in A^1(X_F^{\mathrm{loc}}),\, f_s \in A^0(F)\bigr\}}, \tag{3.4}$$

we can map the class of $(\varepsilon, \varepsilon_\infty, \varepsilon_0)$ to

$$\frac{1}{2\pi i} \int_{X \times C_{\mathrm{an}}} \omega \wedge \mathrm{dlog}(t) \wedge \varepsilon - \int_{C_{\mathrm{an}}} \omega \wedge (\varepsilon_\infty - \varepsilon_0), \tag{3.5}$$

where the integral is taken with the product orientation on $X \times C_{\mathrm{an}}$, because this is trivial on $(\mathrm{d}\psi, \psi_{|t=\infty}, \psi_{|t=0})$ with ψ in $A^1(F)$. The calculations in [de Jeu 1996] are carried out using ε in $A^*(X_F^{\mathrm{loc}})$ that restrict to 0 for $t = 0$ or ∞, which yield the same cohomology group. The calculations in the proof of Proposition 3.1 therefore use the first term in (3.5).

The connecting map

$$H^1_{\mathrm{dR}}(F; \mathbb{R}(2))_{|t=\infty} \oplus H^1_{\mathrm{dR}}(F; \mathbb{R}(2))_{t=0} \to H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2))$$

in the long exact sequence for relative cohomology maps $(\varepsilon_\infty, \varepsilon_0)$ to $(0, \varepsilon_\infty, \varepsilon_0)$. The map in (3.5) therefore factorizes the composition

$$H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2)) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(F; \mathbb{R}(2)) \xrightarrow{\int_{C_{\mathrm{an}}} \omega \wedge \cdot} \mathbb{C}$$

(with one of the two natural choices of isomorphism in the first map) over the localization map $H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2)) \to H^2_{\mathrm{dR}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(2))$.

For this choice of isomorphism we have a commutative diagram

$$\begin{tikzcd} K_3^{(3)}(X_C; \square) \arrow[r, "\simeq"] \arrow[d, "\mathrm{reg}_{\mathbb{C}}"] & K_4^{(3)}(C) \arrow[d, "\mathrm{reg}_{\mathbb{C}}"] \\ H^2_{\mathrm{dR}}(X_{C_{\mathrm{an}}}; \square; \mathbb{R}(2)) \arrow[r, "\simeq"] & H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(2)) \end{tikzcd} \tag{3.6}$$

by normalizing the isomorphism at the top in the same way, and using the same convention in all localizations. This fixes the choice of sign in (2.34). Finally, there is a choice in the sign of the map $H^2(\mathscr{M}_{(3)}(F)) \to H^1(\mathscr{C}^\bullet(F))$ (see (2.36)), but we choose this so that the formulas in Proposition 3.1 hold.

For $\mathbb{O}$, one can give a similar discussion on the K-theory side using the diagram (2.67), and this is compatible with the one here by the commutativity of (2.66) and the compatibility of (2.67) with (2.36). In particular, the choices of signs on the K-theory side for $\mathbb{O}$ are compatible with those for F.

In Remark 5.25 we give a description of the maps on the p-adic side, using the description of syntomic cohomology in (5.5) that matches our description above. Comparing the sign of the term $\varepsilon_\infty - \varepsilon_0$ in both cases, and taking into account that we are ultimately cupping on the left with ω in the p-adic case as well (see Proposition 5.22), it is then clear that we have normalized the formulas in Proposition 3.1 and those in the theorems in the introduction in the same way.

4. Coleman integration

In this short section we briefly discuss Coleman's integration theory in the one-dimensional case only. The interested reader may refer to [Besser 2000b] for more details.

Coleman theory is done on wide open spaces in the sense of Coleman [1988]. In general these are the overconvergent spaces described in Section 5. In the one-dimensional case these can be described concretely in the following way. Let X be a curve over $\mathbb{C}_p$ with good reduction (there is a minor assumption that it is obtained by extension of coefficients from a curve over a complete discretely valued subfield, which will always be satisfied in our applications). The rigid analytic space $X(\mathbb{C}_p)$ is set-theoretically decomposed as the union $X = \bigcup_x U_x$ where x varies over the points in the reduction of X and U_x is the residue disc (tube in the language of Berthelot) of points reducing to x. By the assumption of good reduction each residue disc is isomorphic to a disc $|z| < 1$. A wide open space U is obtained from X by fixing a finite and nonempty set of points S in the reduction and throwing away the discs inside the residue discs U_x, $x \in S$, isomorphic to $|z| < r$ for arbitrarily large $r < 1$. The space U should be thought of as the inverse limit of the corresponding spaces U_r.

Coleman theory associates to U the $\mathbb{C}_p$-algebra $A_{\mathrm{col}}(U)$ and the $A_{\mathrm{col}}(U)$-modules $\Omega^i_{\mathrm{col}}(U)$ with differentials forming a complex. The key property is that this complex is exact at the one and zero forms, that is, there is an exact sequence

$$0 \to \mathbb{C}_p \to A_{\mathrm{col}}(U) \to \Omega^1_{\mathrm{col}}(U) \to \Omega^2_{\mathrm{col}}(U).$$

The space $\Omega^1_{\mathrm{col}}(U)$ contains the space $\Omega^1(U)$ of overconvergent forms on U, that is, those forms that are rigid analytic on some U_r. Similarly, the space $A_{\mathrm{col}}(U)$ contains the space $A(U)$ of overconvergent functions. The differential extends the usual differential on the subspaces.

The whole picture extends to higher dimensions. We shall only need the case where U is one-dimensional. In this case the space $\Omega^2_{\mathrm{col}}(U)$ is already 0.

Coleman functions may be interpreted as locally analytic functions on U. More precisely, again in the one-dimensional case, for $x \notin S$, the intersection of the residue disc U_x with U is U_x, while for $x \in S$ it is an annulus e_x isomorphic to an annulus of the form $r < |z| < 1$. A Coleman function is analytic on each disc U_x and is in the polynomial algebra $A(e_x)[\log(z)]$ where z is a local parameter on an annulus U_x (here, there is an implicit global choice of a branch of the p-adic logarithm).

We define the space $A_{\mathrm{col},1}(U)$ to be the inverse image of $\Omega^1(U) \subseteq \Omega^1_{\mathrm{col}}(U)$ under the differential d. The space of differentials $\Omega^1_{\mathrm{col},1}(U)$ is $A_{\mathrm{col},1}(U) \cdot \Omega^1(U)$.

If $\omega \in \Omega^1(U_r)$ and $y, z \in U_r$ the integral $\int_z^y \omega$ is clearly well-defined as $f(y)-f(z)$ where $f \in A_{\mathrm{col}}(U_r)$ and $\mathrm{d}f = \omega$. It is a basic property of Coleman integration that if X, U, ω, z, y are all defined over the complete subfield K, then so is the integral $\int_z^y \omega$.

For $f \in A(U)$ the function $\log(f)$ is in $A_{\mathrm{col},1}(U)$. Pullback by a rigid analytic endomorphism ϕ of U (such as the Frobenius endomorphisms that will appear in the next section) preserves Coleman functions and in particular $A_{\mathrm{col},1}(U)$.

5. Regulators

In this section we compute the regulator on $\mathscr{C}^1(\mathbb{O})$ in (modified) syntomic cohomology. In case the element lies in the subspace $H^1(\mathscr{C}^\bullet(\mathbb{O}))$, we also explain how we wish to interpret the cup product of this regulator with the cohomology class of a form ω of the second kind on C, and what are the obstacles for doing so, thus paving the way for constructions in the next sections.

We first write down the relevant spaces and the (modified) syntomic complexes computing their cohomology. For the full story the reader should consult [Besser 2000b].

We begin with a smooth proper relative curve $\mathscr{C}/R$. Related to that is the space $X_{\mathscr{C}} := \mathbb{P}^1_{\mathscr{C}} \setminus \{t = 1\}$. The superscript loc will denote various localizations, obtained by removing the image of a finite number of R-sections. We note that the computations in this section can be done after a finite base change, so we may easily get from more general localizations into this situation by further localization. We shall use localizations $\mathscr{C}^{\mathrm{loc}}$ of $\mathscr{C}$ or $X^{\mathrm{loc}}_{\mathscr{C}}$ of $X_{\mathscr{C}}$. If the localization is nontrivial, and we may and do assume this, then all localized schemes are affine.

Our goal is to compute the syntomic regulator $K_4^{(3)}(\mathscr{C}) \to H^2_{\mathrm{syn}}(\mathscr{C}, 3)$. According to [Besser 2000b, Proposition 8.6.3] there is an isomorphism, commuting with the regulator, $H^2_{\mathrm{syn}}(\mathscr{C}, 3) \xrightarrow{\simeq} \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}, 3)$, where $\tilde{H}_{\mathrm{ms}}$ is the Gros style modified rigid syntomic cohomology, in the sense of loc. cit. From now on we shall therefore concentrate on modified syntomic cohomology. We shall refer to it simply as syntomic cohomology.

Let us recall one of the possible models for modified syntomic cohomology for affine schemes. Let A be an affine R-scheme. We assume we have an open embedding $A \hookrightarrow \overline{A}$, where $\overline{A}$ is proper. From the embedding $A \hookrightarrow \overline{A}$ one obtains the overconvergent space $A^\dagger$. This space can be made sense of in Grosse-Klönne's

[2000] theory of overconvergent spaces as the space whose affine ring, $\mathcal{O}(A^\dagger)$, is the weak completion, in the sense of Monsky–Washnitzer, of $\mathcal{O}(A)$. However, here we shall simply think of $A^\dagger$ formally as the inverse system of strict neighborhoods of the special fiber of A in that of $\overline{A}$.

We further assume that we have an R-linear endomorphism $\phi : A^\dagger \to A^\dagger$ whose reduction is a power of Frobenius, say of degree $q = p^r$. We call ϕ a Frobenius endomorphism. Standard results [Coleman 1985, Theorem A-1; van der Put 1986, Theorem 2.4.4.ii] imply one always has such ϕ.

With the above data, we have

$$\tilde{H}^n_{\mathrm{ms}}(A, j) = H^n(\mathrm{MF}(F^j\Omega^\bullet(A^\dagger) \xrightarrow{1-\phi^*/q^j} \Omega^\bullet(A^\dagger))).$$

Here, the filtration is the stupid filtration on the space of differentials and MF denotes the mapping fiber (cone shifted by -1). To be more precise, one really needs to take the limit of these cohomology groups with respect to powers of ϕ, in a way explained in [Besser 2000b], but it is also explained there that one can ignore this point.

The cohomology groups $\tilde{H}_{\mathrm{ms}}$ are in fact functorial with respect to arbitrary maps of schemes. This functoriality is not at all obvious from the definition except in the case where the maps extend to the dagger spaces and commute with ϕ. Fortunately, this will always be the case for us. In this situation, one may also construct relative cohomology in the obvious way (the reader is advised to look at [Besser and de Jeu 2003, Section 5] for constructions of complexes computing relative syntomic cohomology).

To end this general review we recall that the corresponding syntomic regulator is defined by the formula

$$f \in \mathcal{O}(A)^* \subset K_1(A) \mapsto (\mathrm{dlog}(f), \log(f_0)/q) \in \tilde{H}^1_{\mathrm{ms}}(A, 1), \tag{5.1}$$

where $f_0 = f^q/\phi^*(f)$ and has the property that $\log(f_0)$ is in $\mathcal{O}(A^\dagger)$. We also recall from[Besser 2000b, Definition 6.5] that the cup product

$$\tilde{H}^\bullet_{\mathrm{ms}}(A, i) \times \tilde{H}^\bullet_{\mathrm{ms}}(A, j) \to \tilde{H}^\bullet_{\mathrm{ms}}(A, i + j)$$

is given by

$$\begin{aligned}(\omega_1, \varepsilon_1) \cup (\omega_2, \varepsilon_2) \\ = \Big(\omega_1\wedge\omega_2,\ \varepsilon_1\wedge\Big(\gamma+(1-\gamma)\frac{\phi^*}{q^j}\Big)\omega_2+(-1)^{\deg\omega_1}\Big(\Big((1-\gamma)+\gamma\frac{\phi^*}{q^i}\Big)\omega_1\Big)\wedge\varepsilon_2\Big)\end{aligned} \tag{5.2}$$

for some constant γ, which can be taken arbitrarily (producing homotopic products).

We now write these constructions for the affine schemes we are considering. To simplify notation we write U for $(\mathscr{C}^{\mathrm{loc}})^\dagger$, U' for $(X^{\mathrm{loc}}_{\mathscr{C}})^\dagger$, and X_U for $(X_{\mathscr{C}^{\mathrm{loc}}})^\dagger$. We may localize so that $U' \subset X_U$. We fix a Frobenius endomorphism $\phi : U \to U$. We can then take the Frobenius endomorphism for X_U to be the product of ϕ with the map $t \mapsto t^q$ and for U' the restriction of this endomorphism to U'. Since $t \mapsto t^q$ fixes 0 and ∞ we can use the embedding of U in U' at $t = 0$ and $t = \infty$ to create the complex computing relative cohomology. With this we have the following models for syntomic cohomology.

$$(5.3)\quad \tilde{H}^i_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, i) = \frac{\left\{(\omega, \varepsilon), \omega \in \Omega^i(U'), \varepsilon \in \Omega^{i-1}(U'), \mathrm{d}\omega = 0, \mathrm{d}\varepsilon = \left(1 - \frac{\phi^*}{q^i}\right)(\omega)\right\}}{\{(0, \mathrm{d}\varepsilon), \varepsilon \in \Omega^{i-2}(U')\}}$$

for $i = 1, 2$. Now, for relative syntomic cohomology one we can write, by throwing away terms which are forced to be 0,

$$(5.4)\quad \tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) = \frac{\left\{(\omega, \varepsilon, \varepsilon_\infty, \varepsilon_0),\ \begin{array}{l} \omega \in \Omega^2(U'),\ \varepsilon \in \Omega^1(U'),\ \varepsilon_s \in \mathbb{O}(U), s = 0, \infty, \\ \mathrm{d}\omega = 0,\ \mathrm{d}\varepsilon = \left(1 - \frac{\phi^*}{q^2}\right)(\omega),\ \mathrm{d}\varepsilon_s = \varepsilon_{|t=s},\ s = 0, \infty \end{array}\right\}}{\left\{\left(0, \mathrm{d}\varepsilon, \varepsilon_{|t=\infty}, \varepsilon_{|t=0}\right),\ \varepsilon \in \mathbb{O}(U')\right\}}.$$

The map between $\tilde{H}^2_{\mathrm{ms}}(X_{\mathscr{C}}, \square, 2)$ and $\tilde{H}^2_{\mathrm{ms}}(X_{\mathscr{C}}, 2)$ remembers only ω and ε. Since U' is two dimensional and therefore does not support forms of degree 3, we also have

$$(5.5)\quad \tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3) = \frac{\{(\varepsilon, \varepsilon_\infty, \varepsilon_0), \varepsilon \in \Omega^2(U'), \varepsilon_s \in \Omega^1(U), \mathrm{d}\varepsilon = 0, \mathrm{d}\varepsilon_s = \varepsilon_{|t=s}(s = 0, \infty)\}}{\{(\mathrm{d}\varepsilon, \varepsilon_{|t=\infty} + \mathrm{d}\varepsilon_\infty, \varepsilon_{|t=0} + \mathrm{d}\varepsilon_0), \varepsilon \in \Omega^1(U'), \varepsilon_\infty, \varepsilon_0 \in \mathbb{O}(U')\}}.$$

If we replace U' by X_U we obtain a model for $\tilde{H}^3_{\mathrm{ms}}(X_{\mathscr{C}^{\mathrm{loc}}}, \square, 3)$.

The last model is

$$(5.6)\qquad \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3) = \frac{\{\varepsilon \in \Omega^1(U), \mathrm{d}\varepsilon = 0\}}{\{\mathrm{d}\varepsilon, \varepsilon \in \mathbb{O}(U)\}}.$$

This is of course just the first de Rham cohomology of U. However, the "correct" isomorphism with this cohomology is not the obvious one but rather the one twisted by $1 - \phi^*/q^3$, that is,

$$(5.7)\qquad H^1_{\mathrm{dR}}(U/K) \to \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3), \quad [\eta] \mapsto [(1 - \phi^*/q^3)\eta]$$

(for an explanation of this see [Besser 2000b, Proposition 10.1.3]). Here, and in what follows, we denote the cohomology class of an element in square brackets.

At this point, we are able to make more precise the definition of the p-adic regulator for open curves that was hinted at in the introduction before stating Theorem 1.11. As explained there, for each U as above, one has a canonical projection $H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)$. This is the unique Frobenius equivariant splitting of the natural restriction map in the other direction. These projections are compatible in the obvious way when restricting to a smaller U.

Definition 5.8. The regulator map

$$\mathrm{reg}'_p : K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to H^1_{\mathrm{dR}}(C/K)$$

is the composition

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)\,.$$

Using the compatibility of the maps $\mathbf{p}$ mentioned above for all possible $\mathscr{C}^{\mathrm{loc}}$, from $K_4^{(3)}(\mathscr{O}) = \varinjlim_{\mathscr{C}^{\mathrm{loc}}} K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ (see [Quillen 1973, Proposition 2.2; Srinivas 1996, Lemma 5.9]) we also obtain a well defined regulator map

$$\mathrm{reg}'_p : K_4^{(3)}(\mathscr{O}) \to H^1_{\mathrm{dR}}(C/K).$$

We need a formula for the cup product

$$\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) \times \tilde{H}^1_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 1) \to \tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3)$$

in terms of the models (5.4), (5.3) and (5.5) respectively. Using the formula for a cup product between a cone and a complex and (5.2) with $\gamma = 0$ we find the following formula:

$$(\omega, \varepsilon, \varepsilon_\infty, \varepsilon_0) \cup (\eta, h) = \Big(h\omega + \varepsilon \wedge \frac{\phi^*}{q}\eta,\, \varepsilon_\infty\eta,\, \varepsilon_0\eta\Big). \tag{5.9}$$

Suppose now that f and g are in $\mathscr{O}^*(\mathscr{C}^{\mathrm{loc}})$ (see Section 2.5.4). To compute the regulator of $[g]_2 \cup (f)$ we start with $[g]_2$ in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$. It maps in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathscr{C}})$ to $-((t-g)/(t-1)) \cup (1-g)$, by pulling back along g the corresponding result for the universal elements [Besser and de Jeu 2003, Proposition 6.7].

Lemma 5.10. *We have in $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 2)$ that*

$$-\mathrm{reg}_p\Big(\frac{t-g}{t-1} \cup (1-g)\Big) = (\omega_g, \varepsilon_g)$$

in the model (5.3) with

$$\omega_g = -\,\mathrm{dlog}\Big(\frac{t-g}{t-1}\Big) \wedge \mathrm{dlog}(1-g)$$
$$\varepsilon_g = \frac{1}{q}\log(1-g)_0\,\mathrm{dlog}\Big(\frac{t-g}{t-1}\Big) - \frac{1}{q^2}\log\Big(\frac{t-g}{t-1}\Big)_0\,\mathrm{dlog}\,\phi^*(1-g)$$

Proof. This follows from the formula (5.1) for the regulators of functions, the compatibility of reg_p with cup products and the cup product formula (5.2). $\square$

In what follows, the notation $[a_1, \ldots, a_i]$ will denote the class of $(a_1, \ldots, a_i)$ in (5.4) or (5.5), depending on whether $i = 3$ or 4.

Proposition 5.11. *We have in* $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2)$, *using the model* (5.4),

$$\mathrm{reg}_p([g]_2) = [\omega_g, \varepsilon_g, 0, \Theta(g)]$$

where

$$\mathrm{d}\Theta(g) = \varepsilon_g|_{t=0} = \frac{1}{q}\log(1-g)_0 \,\mathrm{dlog}\, g - \frac{1}{q^2}\log g_0 \,\mathrm{dlog}\, \phi^*(1-g). \tag{5.12}$$

Proof. We are looking for a closed four-tuple, whose first two coordinates represent the cohomology class of $(\omega_g, \varepsilon_g)$. It is easy to see that we may assume that the first two coordinates are indeed $(\omega_g, \varepsilon_g)$. Then the closedness condition implies that the differentials of the next two coordinates give the restrictions to $t = \infty$ and $t = 0$, respectively, of ε_g. These are, respectively, 0 and $\varepsilon_g|_{t=0}$, so the result is clear. $\square$

Remark 5.13.

1. One can show that there exists a function Θ on $\mathbb{P}^1$ such that $\Theta(g)$ is indeed the composition of Θ and g, but we shall not need to use this.
2. The determination of the regulator at this stage is incomplete, since we have only determined $\Theta(g)$ up to a constant. It will turn out that for the regulator computation this is irrelevant. For the computation of the boundary this becomes much trickier. We in fact failed to determine the boundary of the regulator directly. When we need this towards the end of Section 10 for the proof of Theorem 1.9, we shall use a trick to overcome this difficulty, which in particular forces us to assume working over a number field at that stage.

Proposition 5.14. *The regulator of* $[g]_2 \cup (f)$ *in* $\tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3)$ *is represented by the following element in the model* (5.5),

$$\varepsilon(g, f) := \Big(\frac{1}{q}\log f_0 \omega_g + \frac{1}{q}\varepsilon_g \wedge \phi^* \,\mathrm{dlog}\, f, 0, \frac{1}{q}\Theta(g)\phi^* \,\mathrm{dlog}\, f\Big).$$

Proof. This follows again from the compatibility of the regulator with cup products and from the formulas for the cup product in relative syntomic cohomology (5.9). $\square$

Suppose now that $\alpha = \sum_i [g_i]_2 \cup (f_i)$ belongs to

$$H^1(\mathscr{C}^\bullet(\mathbb{O})) \simeq K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}};$$

see (2.65). Note that α is only determined up to an element in $(1+I)^*_{\mathbb{O}} \cup \mathbb{O}^*_{\mathbb{Q}}$; see (2.61) and (2.64). A term in the latter space consists explicitly of elements of the form

$$\delta = \sum_j \delta_{1,j} \cup \delta_{2,j}, \tag{5.15}$$

with $\delta_{1,j} \in K_1^{(1)}(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$ and $\delta_{2,j} \in K_2^{(2)}(\mathscr{C}^{\mathrm{loc}})$, for all possible localizations. Therefore, for an appropriately chosen $\mathscr{C}^{\mathrm{loc}}$, there exists $\beta \in K_3^{(3)}(X_{\mathscr{C}^{\mathrm{loc}}}, \square)$ whose restriction to $(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$ is $\alpha + \delta$, where δ is as in (5.15). If we write $\mathrm{reg}_p(\beta) = [\varepsilon, \varepsilon_\infty, \varepsilon_0]$, with the ε's on X_U, then we have $[\varepsilon, \varepsilon_\infty, \varepsilon_0]|_{(X^{\mathrm{loc}}_{\mathscr{C}}, \square)} = \sum[\varepsilon(g_i, f_i)] + \mathrm{reg}_p(\delta)$. Writing this explicitly, this means that

$$(\varepsilon, \varepsilon_\infty, \varepsilon_0)|_{(U', \square)} = \sum \varepsilon(g_i, f_i) + \mathrm{reg}_p(\delta) + (\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0})$$

for some $\lambda \in \Omega^1(U')$ and where now $\mathrm{reg}_p(\delta)$ means any form representing this class.

The isomorphism $T_0^\infty : \tilde{H}^3_{\mathrm{ms}}(X_{\mathscr{C}^{\mathrm{loc}}}, \square, 3) \cong \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3)$ is obtained by integration from 0 to ∞. More precisely it is given by

$$[\varepsilon, \varepsilon_\infty, \varepsilon_0] \mapsto \left[\left(\int_0^\infty \varepsilon \right) - (\varepsilon_\infty - \varepsilon_0) \right] \tag{5.16}$$

where the integration is only with respect to the variable t;

$$\int_0^\infty (f(x,t)\,\mathrm{d}t \wedge \mathrm{d}x) = \left(\int_0^\infty f(x,t)\,\mathrm{d}t \right) \mathrm{d}x. \tag{5.17}$$

Note that we are integrating forms on X_U. For forms on U' we may do Coleman integration instead (Section 4). This technique was introduced in [Besser and de Jeu 2003, Section 5]. Note that we only discussed Coleman integration over $\mathbb{C}_p$. The extension of scalars of U and the fibers of $U' \to U$, to $\mathbb{C}_p$ are wide open space in the sense of Coleman so one can do Coleman integration on them. By abuse of notation we shall continue to denote this extension of scalars by the same letters. Coleman integration will be the same as ordinary integration if the forms extend to X_U. The theory of Coleman integration is not sufficiently developed yet to tell us that what we do makes sense in general, so we must be careful to check that it makes sense for the particular forms we are working with.

Now we check what happens to the term $\varepsilon(g, f)$ under this integration, which we continue to denote by T_0^∞. The integral of the first term is

$$\begin{aligned}\int_0^\infty \frac{1}{q} \log f_0 \omega_g + \frac{1}{q} \varepsilon_g \wedge \phi^* \,\mathrm{dlog}\, f &= \frac{1}{q} \log f_0 \int_0^\infty \omega_g + \frac{1}{q} \left(\int_0^\infty \varepsilon_g \right) \wedge \phi^* \,\mathrm{dlog}\, f \\ &= \frac{1}{q} \log f_0 \log g \,\mathrm{dlog}(1-g) - \frac{1}{q^2} \log(1-g)_0 \log g \phi^* \,\mathrm{dlog}\, f.\end{aligned}$$

The last equality follows because $\int_0^\infty \mathrm{dlog}((t-g)/(t-1)) = -\log g$ and the term involving $\log((t-g)/(t-1))_0$ vanishes because it does not involve a $\mathrm{d}t$. Subtracting the term $\varepsilon_\infty - \varepsilon_0$ we obtain

$$\begin{aligned}T_0^\infty \varepsilon(g, f) = \frac{1}{q} &\log f_0 \log g \,\mathrm{dlog}(1-g) \\ &- \frac{1}{q^2} \log(1-g)_0 \log g \phi^* \,\mathrm{dlog}\, f + \frac{1}{q} \Theta(g) \phi^* \,\mathrm{dlog}\, f.\end{aligned} \tag{5.18}$$

Note that this integral belongs to $\Omega^1_{\mathrm{col},1}(U)$, in the notation of Section 4.

Lemma 5.19. *For δ in $(1+I)^*_{\mathbb{O}} \cup K_2^{(2)}(\mathbb{O})$ we have $T_0^\infty(\mathrm{reg}_p(\delta)) = 0$.*

Proof. As in (5.15) δ is a sum of terms of the form $\delta_1 \cup \delta_2$ with δ_1 in $K_1^{(1)}(X_{\mathscr{C}}^{\mathrm{loc}}, \square)$ and δ_2 in $K_2^{(2)}(\mathscr{C}^{\mathrm{loc}})$. That T_0^∞ vanishes on these elements follows from the proof of [Besser and de Jeu 2003, Proposition 7.2]. $\square$

Now we deal with the term $(\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0})$.

Proposition 5.20. *Suppose that $X_{\mathscr{C}}^{\mathrm{loc}}$ is obtained from $X_{\mathscr{C}^{\mathrm{loc}}}$ by removing the graphs of $t = h_j(x)$ for $j = 1, \dots, n$. Assume further that the reductions of those graphs are either disjoint or identical (which we can achieve by shrinking $\mathscr{C}^{\mathrm{loc}}$). Then there are $a_j(x), a(x) \in \mathbb{O}(U)$ such that we have*

$$T_0^\infty(\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0}) = \mathrm{d}\Big(a + \sum_j a_j \log(h_j)\Big),$$

where, if there are two h_j with identical reduction, one may take just one of them. In particular, it belongs to $\Omega^1_{\mathrm{col},1}(U)$.

Proof. We have global coordinates x and t on U' so we can write $\lambda = f(x,t)\,\mathrm{d}x + g(x,t)\,\mathrm{d}t$. Then

$$\mathrm{d}\lambda = \Big(\frac{\partial f}{\partial t} - \frac{\partial g}{\partial x}\Big)\,\mathrm{d}t \wedge \mathrm{d}x.$$

Therefore

$$\int_{t=0}^{t=\infty} \mathrm{d}\lambda = (f(x,\infty) - f(x,0))\,\mathrm{d}x - \Big(\int_{t=0}^{t=\infty} \frac{\partial g}{\partial x}\,\mathrm{d}t\Big)\,\mathrm{d}x.$$

But the first term is exactly $\lambda|_{t=\infty} - \lambda|_{t=0}$ so we find

$$T_0^\infty\,[\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0}] = -\mathrm{d}\Big(\int_{t=0}^{t=\infty} g(x,t)\,\mathrm{d}t\Big).$$

Consider now the two-form $\gamma = g(x,t)\mathrm{d}x \wedge \mathrm{d}t \in \Omega^2(U')$. This is closed so represents a cohomology class in $H^2_{\mathrm{rig}}((X_{\mathscr{C}}^{\mathrm{loc}})_\kappa/K)$. We have a short exact sequence

$$H^2_{\mathrm{rig}}((X_{\mathscr{C}^{\mathrm{loc}}})_\kappa/K) \to H^2_{\mathrm{rig}}((X_{\mathscr{C}}^{\mathrm{loc}})_\kappa/K \xrightarrow{\mathrm{Res}} \oplus_i H^1_{\mathrm{rig}}((\mathscr{C}^{\mathrm{loc}})_\kappa/K),$$

where the map $\mathrm{Res} = \oplus_j \mathrm{Res}_j$ is the sum of the boundary maps on the reductions of $t = h_j(x)$, composed with the pullback under the isomorphisms of these graphs with $(\mathscr{C}^{\mathrm{loc}})_\kappa$. Suppose that $\mathrm{Res}_j(\gamma)$ is the cohomology class of $a_j(x)\,\mathrm{d}x \in \Omega^1(U)$. Let $\gamma_j := a_j(x)\,\mathrm{d}x \wedge \mathrm{dlog}(t - h_j(x))$. Clearly $\mathrm{Res}_l(\gamma_j) = 0$ if $l \neq j$. We claim that $\mathrm{Res}_j(\gamma_j) = \mathrm{Res}_j(\gamma)$. This can be seen easily by applying the map $(x,t) \to (x, t - h_j(x))$, transforming γ_j to $a_j(x)\,\mathrm{d}x \wedge \mathrm{dlog}(t)$. Thus, $\gamma - \sum_j \gamma_j$ extends to $H^2_{\mathrm{rig}}((X_{\mathscr{C}^{\mathrm{loc}}})_\kappa/K)$ and its integral is a holomorphic one form on U. Let this form be $a(x)\,\mathrm{d}x$. Since $\int_{t=0}^{t=\infty} \gamma_j = \pm a_j(x)\log(h_j(x))\mathrm{d}x$

we find $\pm\int_{t=0}^{t=\infty}\gamma=(a(x)+\sum a_j(x)\log(h_j(x)))\,\mathrm{d}x$ and dividing by $\mathrm{d}x$ we find $\int_{t=0}^{t=\infty} g(x,t)\,\mathrm{d}t=\pm(a(x)+\sum a_j(x)\log(h_j(x)))$. This completes the proof. $\square$

These results give us a strategy for breaking the regulator into a sum of terms, each depending on the pairs (g_i, f_i), as follows. Suppose that ω is a form of the second kind on C and let $[\omega]$ be its cohomology class in $H^1_{\mathrm{dR}}(C/K)$.

Definition 5.21. A functional $L_\omega : \Omega^1_{\mathrm{col},1}(U)\to\mathbb{C}_p$ will be called good if it has the following properties:

- it kills terms of the forms $\mathrm{d}a$ and $\mathrm{d}(a\log f)$ for $a, f\in\mathbb{O}(U)$,
- if η is in $\Omega^1(U)$ then we have $L_\omega(\eta)=[\omega]\cup\mathbf{p}([\eta])$.

Proposition 5.22. *Suppose that an element β in $K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ maps to $\sum_i[g_i]_2\cup(f_i)$ in $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ under the natural map*

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})\to K_4^{(3)}(\mathbb{O})\to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup\mathbb{O}^*_{\mathbb{Q}}$$

(*see* (2.65)), *and that* $\mathrm{reg}_p(\beta)=[\eta_0]$ *in the model* (5.6). *Then we have, for a good functional L_ω,*

$$[\omega]\cup\mathbf{p}([\eta_0])=\sum_i L_\omega\big(T_0^\infty\varepsilon(g_i, f_i)\big).$$

Proof. We must first show that the map

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})\xrightarrow{\mathrm{reg}_p}\tilde H^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}},3)\xrightarrow{\eta_0\mapsto L_\omega(\eta_0)}\mathbb{C}_p$$

factors via $K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup\mathbb{O}^*_{\mathbb{Q}}$. By further localizing, it suffices to show that the map above vanishes on elements of the form $\gamma\cup f$ with $\gamma\in K_3^{(2)}(\mathscr{C}^{\mathrm{loc}})$ and $f\in\mathbb{O}^*(\mathscr{C}^{\mathrm{loc}})$. We have

$$\tilde H^1_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}},2)=\{(0,\varepsilon),\,\varepsilon\in\mathbb{O}(U),\ \mathrm{d}\varepsilon=0\}=\{(0,\varepsilon),\,\varepsilon\in K\}.\tag{5.23}$$

Thus $\mathrm{reg}_p(\gamma)=(0,\alpha)$ for some $\alpha\in K$. On the other hand, by (5.1) we have $\mathrm{reg}_p(f)=(\mathrm{dlog}\, f,\log(f_0)/q)$ (here it does not matter what f_0 is). Using (5.2) we obtain, in the model (5.6)

$$\mathrm{reg}_p(\gamma\cup f)=(0,\alpha)\cup(\mathrm{dlog}\, f,\log(f_0)/q)=\alpha\,\mathrm{dlog}\, f.$$

The factorization thus follows from first property of the good functional. Next, by Proposition 5.20 the first property also implies that L_ω kills all terms of the form $T_0^\infty[\mathrm{d}\lambda,\lambda_{|t=\infty},\lambda_{|t=0}]$. The result now follows immediately from the discussion above. $\square$

Remark 5.24. There is a final wrinkle here because of the normalization (5.7) for the syntomic regulator. For β as in the corollary, the regulator of β is in fact $[\eta]$ with $(1-(\phi^*/q^3))[\eta]=[\eta_0]$. Thus, once we have the functional L_ω we shall be able to compute $[\omega] \cup \mathbf{p}(\eta_0)$ but will in fact want $[\omega] \cup \mathbf{p}(\eta)$. Fortunately, it is easy to see (and will be explained) that if we know $[\omega] \cup \mathbf{p}(\eta_0)$ for *all* ω, then we also know $[\omega] \cup \mathbf{p}(\eta)$ for all ω. In fact, as in previous computations, the result with η is much simpler than with η_0, confirming the "correctness" of our normalization.

Remark 5.25. As with some of our previous works on syntomic regulators, one can ask about the sign compatibility between the p-adic and classical regulators; see [Besser et al. 2009, Remark 4.16]. As explained in Remark 3.3, the signs in the various isomorphisms induced by using relative K-theory and relative Deligne or de Rham cohomology are normalized by choosing one of the natural isomorphisms $H^2_{\mathrm{dR}}(X_{C_{\mathrm{an}}}; \square; \mathbb{R}(2)) \simeq H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \square; \mathbb{R}(2))$, in this case by choosing (3.5), and then demanding that (3.6) commutes. The same approach works for the syntomic regulator, using (5.16) and the analogue of (3.6) for syntomic cohomology.

Because the descriptions of relative cohomology in 3.3 and (5.5) and the signs in front of the term $\varepsilon_\infty - \varepsilon_0$ in (3.5) and (5.16) are the same (note that just as in Section 3 we are ultimately cupping on the left with ω; see Proposition 5.22), we have chosen the "relativity isomorphism" for Deligne (or de Rham) cohomology and syntomic cohomology in a compatible way. Therefore (3.6) and its analogue for syntomic cohomology lead to the same sign for the K-theory (under the compatibility of the constructions for $\mathbb{C}$ and F as explained in Section 2).

6. Wishes

This section is highly speculative. It contains no formal proofs. Nevertheless, we feel it is vital for the understanding of a significant portion of the computations to come. It also suggests interesting research directions into a more canonical representation of syntomic cohomology, one that would make the computations in the syntomic case equivalent to the complex case.

We want to follow a strategy that proved very successful in computing syntomic regulators on K_2 of curves; see the discussion after Proposition 5.2 in [Besser 2000c]. We argue heuristically, in some make-believe world where syntomic cohomology looks much more like Deligne cohomology from the computational standpoint, and get a formula for the regulator. Then we try to relate this formula with the formula we obtained in the previous section and see what needs to be proved to show that the two formulas are equivalent. That the make-believe formula turns out to be correct is a strong indication that one should be able to turn the make-believe computation into a rigorous one.

The make-believe computation is based on the following assumptions:

- The "cohomology" is given by the pairs (ω, h) where ω is an i-form and h is an $i-1$ form with $\mathrm{d}h = \omega$. Of course h is not an actual form but something like a Coleman form, for example a Coleman function.

- The "regulator" of a function f is the pair $(\operatorname{dlog}(f), \log(f))$.
- The cup product is given by $(\omega_1, h_1) \cup (\omega_2, h_2) = (\omega_1 \wedge \omega_2, \omega_1 \wedge h_2 \text{ or } h_1 \wedge \omega_2)$.

With these rules, we can redo the computation from the previous section in this make-believe language: We have in $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 2)$ that

$$-\operatorname{reg}_p\Big(\frac{t-g}{t-1} \cup (1-g)\Big) = (\omega_g, \varepsilon_g)$$

with ω_g as in Lemma 5.10 and

$$\varepsilon_g = -\log(1-g)\operatorname{dlog}\Big(\frac{t-g}{t-1}\Big).$$

Since the restriction of ε_g to $t = 0$ is $-\log(1-g)\operatorname{dlog}(g) = \mathrm{d}\,\mathrm{Li}_2(g)$ we have, following the proof of Proposition 5.11, that

$$\operatorname{reg}_p([g]_2) \in \tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) \quad \text{equals } [\omega_g, \varepsilon_g, 0, \mathrm{Li}_2(g)].$$

Cupping with $(\operatorname{dlog}(f), \log(f))$ we get

$$\tilde{\varepsilon}(g, f) := \operatorname{reg}_p([g]_2 \cup (f)) = \Big[-\log(f)\operatorname{dlog}\Big(\frac{t-g}{t-1}\Big) \wedge \operatorname{dlog}(1-g)), 0, 0\Big].$$

Applying T_0^∞ we find $T_0^\infty(\tilde{\varepsilon}(g, f)) = \log(f)\log(g)\operatorname{dlog}(1-g)$.

We now compare this with $T_0^\infty\varepsilon(g, f)$ of (5.18). Continuing to mimic the discussion of the K_2 in [Besser 2000c], the former version should be an untwisted version of the latter, that is, without the "twist" by $(1-(\phi^*/q^3))$. To see this, we use the formalism described in [Besser 2000c, Remark 3.1] to get

$$\begin{aligned}(6.1)\quad \Big(1-\frac{\phi^*}{q^3}\Big)&[\log(f)\log(g)\operatorname{dlog}(1-g)] = \\ &\frac{1}{q}\log(f_0)\log(g)\operatorname{dlog}(1-g) + \frac{1}{q^2}\log\phi^*(f)\log(g)\operatorname{dlog}(1-g)_0 \\ &\qquad + \frac{1}{q^3}\log(g_0)\log\phi^*(f)\phi^*\operatorname{dlog}(1-g).\end{aligned}$$

This already begins to look similar to $T_0^\infty\varepsilon(g, f)$, but there are differences. We want to argue that the difference is "exact". This cannot be taken to simply mean being the differential of something, since in Coleman's theory every form is integrable. Experience has shown that things are exact if they are the differential of a product of functions. We shall use two such assertions. Each one will correspond to a precise statement in the following sections, which will be justified by the techniques we shall introduce. To remind ourselves where these occurred, we shall call them "Wishes", and mark them explicitly.

Wish 6.2. We have in cohomology that $\Theta(g)\operatorname{dlog}\phi^*(f) = -\log\phi^*(f)\,\mathrm{d}\Theta(g)$.

Using this wish we can write the term $\frac{1}{q}\Theta(g)\,\mathrm{dlog}\,\phi^*(f)$ in (5.18) as

$$
\begin{aligned}
-\frac{1}{q}&\mathrm{d}\Theta(g)\log\phi^*(f)\\
&=-\frac{1}{q}\left(\frac{1}{q}\log(1-g)_0\,\mathrm{dlog}\,g-\frac{1}{q^2}\log g_0\,\mathrm{dlog}\,\phi^*(1-g)\right)\log\phi^*(f)\\
&=-\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}(g)\log\phi^*(f)+\frac{1}{q^3}\log(g_0)\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f),
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
T_0^{\infty}\varepsilon(g,f)=\frac{1}{q}\log(f_0)\log(g)\,\mathrm{dlog}(1-g)-\frac{1}{q^2}\log(1-g)_0\log(g)\phi^*\,\mathrm{dlog}(f)\\
-\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}(g)\log\phi^*(f)+\frac{1}{q^3}\log(g_0)\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f).
\end{aligned}
$$

Comparing this with $(1-(\phi^*/q^3))(\log(f)\log(g)\,\mathrm{dlog}(1-g))$ given in (6.1) we see that the first and last terms are the same, and that therefore we get our desired equality, "twisted" by $1-(\phi^*/q^3)$ if we get our second wish to come true.

Wish 6.3. We have in cohomology that

$$
\begin{aligned}
\log(1-g)_0\log(g)\phi^*(\mathrm{dlog}(f))+\log(1-g)_0\log\phi^*(f)\,\mathrm{dlog}(g)\\
+\log(g)\log\phi^*(f)\,\mathrm{dlog}(1-g)_0
\end{aligned}
$$

is trivial.

In Sections 7 and 8 we shall introduce triple indices. The wishes described above correspond to precise results stated in terms of triple indices, which we can indeed prove.

7. The triple index, local theory

We first briefly recall the theory of the "local index" from [Besser 2000c, Section 4]. In our new context this should be called the double index. To make things slightly simpler, we work in an algebraic context. The transition to working with annuli is straightforward.

Let K be a field of characteristic 0. We consider the algebra $A_{\log}:=K((z))[\log(z)]$ of polynomials over the formal variable $\log(z)$, over the field of finite to the left Laurent power series in z. We further consider the module of differentials $A_{\log}\cdot\mathrm{d}z$. It is an easy exercise in integration by parts to see that every form in $A_{\log}\cdot\mathrm{d}z$ has an integral in $A_{\log}$ in a unique way up to a constant. We distinguish in $A_{\log}$ the subfield $\mathsf{Mer}:=K((z))$ of meromorphic functions and the subspace $A_{\log,1}=\mathsf{Mer}+K\cdot\log(z)$ consisting exactly of all functions whose differential is in $\mathsf{Mer}\cdot\mathrm{d}z$. To $F\in A_{\log,1}$

we can associated the residue of its differential $\operatorname{Res} \mathrm{d}F \in K$. If $F \in A_{\log,1}$, then $F \in \mathsf{Mer}$ if and only if $\operatorname{Res} \mathrm{d}F = 0$.

Definition 7.1 [Besser 2000c, Proposition 4.5]. The double index is the unique antisymmetric bilinear form $\langle \cdot, \cdot \rangle : A_{\log,1} \times A_{\log,1} \to K$ such that $\langle F, G \rangle = \operatorname{Res} F\, \mathrm{d}G$ whenever this last expression makes sense.

We recall that the construction of this index is essentially trivial: one notices that the antisymmetry forces $\langle \log(z), \log(z) \rangle = 0$ and that $\langle F, G \rangle = -\operatorname{Res} G\, \mathrm{d}F$ whenever this expression makes sense. Then one writes $F = \alpha \log(z) + f$, $G = \beta \log(z) + g$ with $f, g \in \mathsf{Mer}$ and then one uses the bilinearity to write $\langle F, G \rangle$ as a sum of terms that can be computed.

The triple index turns out to be a bit more complicated. First of all we need to explain on which data it is evaluated:

- three functions F, G, H in $A_{\log,1}$,
- for each two functions R and S out of F, G, H a choice of $\int R\, \mathrm{d}S$ (that is, a function in $A_{\log}$ whose differential is $R\, \mathrm{d}S$) and of $\int S\, \mathrm{d}R$ in such a way that

$$\int R\, \mathrm{d}S + \int S\, \mathrm{d}R = RS. \tag{7.2}$$

As it will turn out this information is a bit redundant: clearly $\int R\, \mathrm{d}S$ determines $\int S\, \mathrm{d}R$. Also it will turn out that the index will be independent of $\int F\, \mathrm{d}G$. Still, these symmetric data are very convenient. To not carry around too much notation, we shall simply denote these data by $(F, G; H)$, where the additional choices should be understood from the context. In particular, any permutation of F, G, H induces an obvious permutation of the additional data. Also, if $(F_i, G; H)$, $i = 1, 2$ are given with all their additional data then there is a natural choice of data for $(F_1 + F_2, G; H)$, and similarly in the second and third positions. If we do need to indicate a change in the auxiliary data we shall write this as $(F, G; H | I_{F\mathrm{d}G}, \dots)$, where the subscript $F\, \mathrm{d}G$ indicates that I is an integral of $F\, \mathrm{d}G$.

Proposition 7.3. *There exists a unique function from data as above to K, denoted $(F, G; H) \mapsto \langle F, G; H \rangle$, called the triple index, such that the following conditions are satisfied.*

(1) *Trilinearity: the triple index is linear in each of the three variables, which means that $\langle \alpha_1 F_1 + \alpha_2 F_2, G; H \rangle = \alpha_1 \langle F_1, G; H \rangle + \alpha_2 \langle F_2, G; H \rangle$ provided that all auxiliary data are chosen in the way indicated above, and similarly for linearity in G and H.*

(2) *Symmetry: we have $\langle F, G; H \rangle = \langle G, F; H \rangle$, again with the choice of auxiliary data indicated above.*

(3) *Triple identity*: *we have*, *again with the obvious additional choices*,

$$\langle F, G; H\rangle + \langle F, H; G\rangle + \langle G, H; F\rangle = 0.$$

(4) *Reduction to the double index*: *if* $G \in \mathsf{Mer}$ *then* $\langle F, G; H\rangle = \langle F, \int G\,\mathrm{d}H\rangle$, *where* $\int G\,\mathrm{d}H$ *is taken from the auxiliary data and is in* $A_{\log,1}$ *because by assumption* $G\,\mathrm{d}H \in \mathsf{Mer}\cdot \mathrm{d}z$.

Proof. We first show that the dependency on the choices of integrals is forced by the properties of the triple index.

Lemma 7.4. *Suppose that the triple index exists. We then have the following change of constant formulas*:

(1) *If* C *is a constant*, *then*

$$\langle F, G; H|(I+C)_{G\mathrm{d}H}, (J-C)_{H\mathrm{d}G}\rangle = \langle F, G; H|I_{G\mathrm{d}H}, J_{H\mathrm{d}G}\rangle - C\cdot \operatorname{Res}\mathrm{d}F,$$
$$\langle F, G; H|(I+C)_{F\mathrm{d}H}, (J-C)_{H\mathrm{d}F}\rangle = \langle F, G; H|I_{F\mathrm{d}H}, J_{H\mathrm{d}F}\rangle - C\cdot \operatorname{Res}\mathrm{d}G.$$

(2) *The triple index is independent of the integral* $\int F\,\mathrm{d}G$.

Proof. We use the trilinearity. Consider the data $(F, 0; H)$, where the additional data are the same for F and H but we take the integral of $0\,\mathrm{d}H$ to be C, hence we are forced to take that of $H\,\mathrm{d}0$ to be $-C$. We take $\int 0\,\mathrm{d}F = 0$. The trilinearity implied that $\langle F, G; H\rangle$ and $\langle F, 0; H\rangle$ gives the left-hand side of the formula. But reduction to the double index means that $\langle F, 0; H\rangle = \langle F, C\rangle = -\operatorname{Res} C\,\mathrm{d}F$. An identical argument proves the second case. Finally, if in the above argument we take instead $\int 0\,\mathrm{d}F = D$ and $\int 0\,\mathrm{d}H = 0$, we see from exactly the same argument that the integral is independent of the auxiliary choice $\int F\,\mathrm{d}G$. □

We now check that the triple index is uniquely defined on all data where at least one of F, G, H is in Mer. Clearly in this case we can use reduction to the double index together with symmetry and the triple formula to compute the index, so it is clearly unique. The following lemma gives existence.

Lemma 7.5. *Consider the following recipe*:

(1) *if* $G \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = \langle F, \int G\mathrm{d}H\rangle$,

(2) *if* $F \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = \langle G, F; H\rangle$ *where the last expression is defined as in* (1),

(3) *if* $H \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = -(\langle F, H; G\rangle + \langle G, H; F\rangle)$ *where each of these terms is defined as in* 1.

Then this recipe gives a well-defined $\langle F, G; H\rangle$ *in all cases where at least one of* F, G *and* H *is in* Mer *and restricted to this subset it satisfies all properties of the triple index.*

Proof. To show that this expression is well-defined we need to consider what happens when two of F, G, H are in Mer: If $F, G \in \mathsf{Mer}$ we check that $\langle F, \int G\, \mathrm{d}H\rangle = \langle G, \int F\, \mathrm{d}H\rangle$. This follows because by the definition of the double index both expressions equal $\operatorname{Res} FG\, \mathrm{d}H$. Next we check that if $G, H \in \mathsf{Mer}$ then

$$\begin{aligned}
&\langle F, \textstyle\int G\, \mathrm{d}H\rangle + \langle F, \textstyle\int H\, \mathrm{d}G\rangle + \langle G, \textstyle\int H\, \mathrm{d}F\rangle \\
&\qquad = \langle F, GH\rangle + \langle G, \textstyle\int H\, \mathrm{d}F\rangle \quad \text{by bilinearity of the double index and (7.2)} \\
&\qquad = -\operatorname{Res} GH\, \mathrm{d}F + \operatorname{Res} GH\, \mathrm{d}F = 0.
\end{aligned}$$

Thus we find that we have a well-defined expression. We need to check that all properties of the expected triple index hold in this case. Trilinearity is essentially clear from the bilinearity of the double index. Symmetry is also easy: if F or G are in Mer then symmetry follows from the first two rules. If H is in Mer then the expression in (3) is clearly symmetric in F and G. The triple identity is forced by (3) and the reduction to the double index is an immediate consequence of our check that the triple index is well-defined. $\square$

Note that the proof of Lemma 7.4 applies verbatim for this partial triple index, so we know the dependency on the choices of integrals.

To extend the triple index to all F, G and H we first check the case where $F = G = H = \log(z)$. Then we can arrange that all auxiliary data equal $\frac{1}{2}\log^2(z)$. The triple formula implies immediately that (with these data)

$$\langle \log(z), \log(z); \log(z)\rangle = 0. \tag{7.6}$$

We can now demonstrate uniqueness for the triple index. Suppose $F_i = \alpha_i \log(z) + f_i$, $i = 1, 2, 3$ where $\alpha_i \in K$ and $f_i \in \mathsf{Mer}$. Choose some auxiliary data $\int R\, \mathrm{d}S$ for any two R and S out of f_i and $\alpha_i \log(z)$, where we continue to take $\int \log(z)\, \mathrm{dlog}(z) = \frac{1}{2}\log^2(z)$. Using trilinearity and (7.6) we can write $\langle F_1, F_2; F_3\rangle$, with some choice of auxiliary data, as the sum with some coefficients of triple indices where at least one of the entries is in Mer, which are therefore computable by previous considerations. Now we can use change of constant to write $\langle F_1, F_2; F_3\rangle$ with arbitrary auxiliary data. This shows uniqueness and gives a formula for the general index. We need to check that this formula is well-defined, which, given the fact that all the summands are well-defined thanks to Lemma 7.5, amounts to checking independence of the choices of the auxiliary data. This is just a tedious formal check: suppose for example that we add C to $\int \alpha_1 \log(z)\, \mathrm{d}f_3$, and correspondingly subtract C from $\int f_3 \alpha_1\, \mathrm{dlog}\, z$. This will have the effect that $\int F_1\, \mathrm{d}F_3$ will have C added to it and $\int F_3 \mathrm{d}F_1$ will have C subtracted from it. This procedure will subtract $\alpha_2 C = C \operatorname{Res} \mathrm{d}F_2$ from $\langle \alpha_1, \alpha_2 \log(z); f_3\rangle$ and will not change any of the other indices. This shows that the change does not alter the index.

It remains to check that our formula satisfies all the properties for the triple index. First the change of constant formula of Lemma 7.4 is clear because we used it in the definition and we showed that the formula we get is well-defined. Now given change of constant it easy to see that it is enough to check trilinearity, symmetry and triple identity for one choice of auxiliary data. The derivation of these three formulas is then completely formal. Finally, reduction to the double index can only occur if at least one α_i is 0. But in this case we clearly get the triple index for the case where $F_i \in \mathsf{Mer}$ so we know this formula already. □

To compute the triple index in some concrete situations, which will be needed later, we introduce the notion of the constant term.

Definition 7.7. The constant term with respect to the variable z is the linear functional $c_z : A_{\log} \to K$, first defined on Mer by

$$c_z\left(\sum a_n z^n\right) = a_0,$$

and then in general by

$$c_z\left(\sum_{i=0}^{\infty} f_i(z)\log^i(z)\right) = c_z(f_0).$$

Note that the unlike the triple index, the constant term definitely depends on the choice of the local parameter z. For example, for $\alpha \in K$ and the function $f(z) = \log(z) = \log(\alpha z) - \log(\alpha)$ we have $c_z(f) = 0$ but $c_{\alpha z}(f) = -\log(\alpha)$.

Proposition 7.8. *Let F, G and H be 3 functions in $A_{\log,1}$ whose differentials (which are in $\mathsf{Mer}\,\mathrm{d}z$) have at most simple poles at 0. The choice of integrals $\int F\,\mathrm{d}H$ and $\int G\,\mathrm{d}H$ gives auxiliary data for the computation of $\langle F, G; H\rangle$ and with respect to this choice we have*

$$\langle F, G; H\rangle = c_z(F)\cdot c_z(G)\cdot \operatorname{Res}\mathrm{d}H - \operatorname{Res}\mathrm{d}F\cdot c_z\left(\int G\,\mathrm{d}H\right) - \operatorname{Res}\mathrm{d}G\cdot c_z\left(\int F\,\mathrm{d}H\right).$$

Proof. We have a bilinear map

$$(F, H) \to \int' F\,\mathrm{d}H := \text{ unique } \int F\,\mathrm{d}H \text{ with } c_z\left(\int F\,\mathrm{d}H\right) = 0.$$

Therefore, we see that the map

$$(F, G, H) \to \langle F, G; H\rangle' := \left\langle F, G; H \;\middle|\; \int' F\,\mathrm{d}H_{F\mathrm{d}H}, \int' G\,\mathrm{d}H_{G\mathrm{d}H}\right\rangle$$

is trilinear and symmetric in F and G. By Lemma 7.4 it suffices to prove that

$$\langle F, G, H\rangle' = c_z(F)\cdot c_z(G)\cdot \operatorname{Res}\mathrm{d}H\ , \tag{7.9}$$

and as both sides are trilinear and symmetric in F and G, and as $F = a\log(z) + f(z)$ with $f(z)$ holomorphic and similarly for G and H, it suffices to treat the following cases:

(1) When f, g and h are holomorphic we have

$$\langle f, g, h\rangle' = \operatorname{Res} fg\,\mathrm{d}h = 0 = c_z(f)c_z(g)\operatorname{Res}\mathrm{d}h$$

since $\operatorname{Res}\mathrm{d}h = 0$.

(2) Suppose $F = G = H = \log(z)$. Since $c_z(\log^2(z)/2) = 0$ we see that the local index computed with all auxiliary data set equal to $\log^2(z)/2$ is given by $\langle\log(z), \log(z); \log(z)\rangle'$, and this we know is 0 by (7.6). On the other hand, the right-hand side of (7.9) is also zero since $c_z(\log(z)) = 0$.

(3) If g and h are holomorphic we have

$$\langle\log(z), g; h\rangle' = \langle\log(z), \int' g\,\mathrm{d}h\rangle = -\operatorname{Res}\Bigl(\int' g\,\mathrm{d}h\Bigr)\operatorname{dlog} z = \Bigl(\int' g\,\mathrm{d}h\Bigr)(0) = 0,$$

which equals $c_z(\log(z))c_z(g)\operatorname{Res}\mathrm{d}h$ as required.

(4) If f and g are holomorphic we find

$$\langle f, g; \log(z)\rangle' = \operatorname{Res} fg\operatorname{dlog} z = fg(0) = c_z(f)c_z(g)\operatorname{Res}\operatorname{dlog} z\,.$$

(5) If g is holomorphic and $a = c_z(g)$ we see that

$$\int'(g-a)\operatorname{dlog} z = \int' g\operatorname{dlog} z - a\log(z).$$

Using this we find

$$\begin{aligned}\langle\log(z), g; \log(z)\rangle &= \Bigl\langle\log(z), \int' g\operatorname{dlog} z\Bigr\rangle = \Bigl\langle\log(z), \int'(g-a)\operatorname{dlog} z\Bigr\rangle\\ &= -\operatorname{Res}\Bigl(\int'(g-a)\operatorname{dlog} z\Bigr)\operatorname{dlog} z = 0,\end{aligned}$$

since $\int'(g-a)\operatorname{dlog} z$ is holomorphic and has constant term 0. This again equals the right-hand side.

(6) The final case is for $\langle\log(z), \log(z); h\rangle$ with h holomorphic. As $c_z(h\log(z)) = 0$, we have the equation $\int' h\operatorname{dlog} z + \int'\log(z)\,\mathrm{d}h = h\log(z)$. We therefore immediately deduce this case from the previous one and the triple identity. □

8. The triple index, global theory

At this point we shall switch for convenience to assuming that our ground field is $\mathbb{C}_p$. Suppose now that we consider an open annulus $V \cong \{r < |z| < s\}$ with a parameter z. Then exactly the same analysis as in Section 7 gives us a triple index on V. Note that while a parameter is used for proving the existence of the index, the uniqueness statement is parameter-free, hence so is the index.

The uniqueness of the triple index immediately implies the following result (cf. [Besser 2000c, Lemma 4.6]).

Lemma 8.1. *If $\phi : V \to V$ is an endomorphism of degree n, let $\phi^*(F, G; H)$ be defined in the obvious way, pulling back by ϕ all the auxiliary data. Denote these data simply by $(\phi^* F, \phi^* G; \phi^* H)$. Then we have the formula*

$$\langle \phi^* F, \phi^* G; \phi^* H\rangle = n\langle F, G; H\rangle.$$

Consider now a wide open space U over $\mathbb{C}_p$, with set of ends $\mathsf{End}(U)$. We shall denote the triple index with respect to the end e by the subscript e. When we are given Coleman functions F, G and H in $A_{\mathrm{col},1}(U)$, in other words, such that their differentials are in $\Omega^1(U)$, we may choose Coleman integrals for all forms $R\,\mathrm{d}S$ when R and S are among F, G and H, and we may do so in such a way that $\int R\,\mathrm{d}S + \int S\,\mathrm{d}R = RS$ globally. This allows us to compute $\langle F, G; H\rangle_e$ at each end e and we may consider the global triple index

$$\langle F, G; H\rangle_{\mathrm{gl}} = \sum_{e\in \mathsf{End}(U)} \langle F, G; H\rangle_e.$$

Lemma 8.2. *For $F, G, H \in A_{\mathrm{col},1}(U)$, the expression $\langle F, G; H\rangle_{\mathrm{gl}}$ is independent of the auxiliary choices, so depends only on F, G and H.*

Proof. Since the possible integrals differ from one another by a global constant, if we change for example $\int G\,\mathrm{d}H$ by a constant C, the change of constant formula implies that the global triple index changes by

$$\sum_e C\,\mathrm{Res}_e\,\mathrm{d}F = C\sum_e \mathrm{Res}_e\,\mathrm{d}F = C\cdot 0 = 0. \qquad \square$$

Unlike the global double index, the global triple index does not depend solely on the cohomology classes of $\mathrm{d}F, \dots,$ and not even just on the differentials of the functions. For example, if C is a constant we have the formula

$$\langle F, C; H\rangle_{\mathrm{gl}} = \sum_e \left\langle F, \int C\,\mathrm{d}H\right\rangle_e = C\sum_e \langle F, H\rangle_e.$$

However, we do have the following.

Lemma 8.3. *If $F, G \in A_{\mathrm{col},1}(U)$ and C is a constant then $\langle F, G; C\rangle_{\mathrm{gl}} = 0$.*

Proof. Indeed,

$$\begin{aligned}\langle F, G; 1\rangle_{\mathrm{gl}} &= -\langle F, 1, G\rangle_{\mathrm{gl}} - \langle G, 1, F\rangle_{\mathrm{gl}} && \text{by the triple identity}\\ &= -\langle F, \textstyle\int \mathrm{d}G\rangle_{\mathrm{gl}} - \langle G, \textstyle\int \mathrm{d}F\rangle_{\mathrm{gl}} && \text{by reduction to the double index}\\ &= -\langle F, G\rangle_{\mathrm{gl}} - \langle G, F\rangle_{\mathrm{gl}} = 0,\end{aligned}$$

where the last two equalities follow because the global double index is independent of the choice of the integral and by the antisymmetry of the double index. □

The lemma suggests that the global triple index is quite an interesting creature. It deserves further study. For our purposes we only need the following results:

Proposition 8.4. *Let F, G, H in $A_{\mathrm{col}}(U)$ have $\mathrm{d}F, \mathrm{d}G, \mathrm{d}H$ in $\Omega^1(U)$, and suppose that the classes $[\mathrm{d}F]$ and $[\mathrm{d}G]$ in $H^1_{\mathrm{dR}}(U/K)$ are eigenvectors for Frobenius with eigenvalue q. Then $\langle F, G; H\rangle_{\mathrm{gl}} = 0$.*

Proof. We begin by establishing the following formulas. If $r \in A(U)$ then

$$\langle F, r, H\rangle_{\mathrm{gl}} = \sum_e \left\langle F, \int r\,\mathrm{d}H \right\rangle_e = 0, \tag{8.5}$$

where the last equality follows from [Besser 2000c, Corollary 4.11]. Similarly we find that if also $s \in A(U)$ then $\langle s, G, H\rangle_{\mathrm{gl}} = 0$. Now if $h \in A(U)$, then

$$\langle F, G; h\rangle_{\mathrm{gl}} = -\langle F, h; G\rangle_{\mathrm{gl}} - \langle G, h; F\rangle_{\mathrm{gl}} = 0,$$

by application of (8.5). This last formula shows that for fixed F and G the function $H \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ depends only on the cohomology class of $\mathrm{d}H$, $[\mathrm{d}H] \in H^1_{\mathrm{dR}}(U/K)$. Let ϕ be a Frobenius lift on U. The assumption on F and G implies the existence of $r, s \in A(U)$ such that $\phi^*F = qF + r$ and $\phi^*G = qG + s$. Using this we can compute

$$\begin{aligned} q\langle F, G; H\rangle_{\mathrm{gl}} &= \langle \phi^*F, \phi^*G; \phi^*H\rangle_{\mathrm{gl}}\\ &= \langle qF + r, qG + s; \phi^*H\rangle_{\mathrm{gl}} = q^2\langle F, G; \phi^*H\rangle_{\mathrm{gl}},\end{aligned}$$

using bilinearity and (8.5). This shows that the functional $[\mathrm{d}H] \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ is an eigenvector for the action of ϕ^* with eigenvalue $1/q$. Such a functional must be 0 because the eigenvalues of ϕ^* on $H^1_{\mathrm{dR}}(U/K)$ are either q or Weil numbers of weight 1. □

Note that this proposition applies in particular when F and G are of the form $r + \log(f)$ where $r, f \in A(U)$. This follows since by [Coleman and de Shalit 1988, Lemma 2.5.1], $\log(f^q/\phi^*(f))$ is in $A(U)$.

Proposition 8.6. *Suppose ω in $\Omega^1(U)$ has trivial residues on all ends, so that its Coleman integral F_ω is in fact analytic on the ends. Let F, G, H be Coleman functions on U whose differentials are holomorphic and represent eigenvectors for*

Frobenius with eigenvalue q on $H^1_{\mathrm{dR}}(U/K)$. Then, choosing the integrals globally as Coleman integrals,

(8.7)
$$\sum_e \left\langle F, G; \int F_\omega \,\mathrm{d}H \right\rangle_e + \sum_e \left\langle H, F; \int F_\omega \,\mathrm{d}G \right\rangle_e + \sum_e \left\langle G, H; \int F_\omega \,\mathrm{d}F \right\rangle_e = 0.$$

Proof. Note that the expression above makes sense since on each end e the form $F_\omega \,\mathrm{d}H$ is analytic, so the corresponding triple index is defined, and similarly with H replaced by F and G. Note also that this is of course not a global index in the sense of this section, since $F_\omega \,\mathrm{d}H$ is not holomorphic. The strategy for the proof is the same as for Proposition 8.4. First we notice that if F_ω is in fact holomorphic, then the identity holds by Proposition 8.4. It follows that the expression factors via the cohomology class $[\omega]$. Suppose now that we replace F by a holomorphic function u. We then have

$$\sum_e \left\langle u, G; \int F_\omega \,\mathrm{d}H \right\rangle_e = \sum_e \left\langle G, \int F_\omega u \,\mathrm{d}H \right\rangle_e,$$
$$\sum_e \left\langle u, H; \int F_\omega \,\mathrm{d}G \right\rangle_e = \sum_e \left\langle H, \int F_\omega u \,\mathrm{d}G \right\rangle_e,$$

by reduction to the double index, and

$$\begin{aligned}
\sum_e \left\langle G, H; \int F_\omega \mathrm{d}u \right\rangle_e & \\
&= \sum_e \left\langle G, H; F_\omega u - \int u\omega \right\rangle_e = \sum_e \langle G, H; F_\omega u\rangle_e \quad \text{by Proposition 8.4} \\
&= -\sum_e \langle G, F_\omega u; H\rangle - \sum_e \langle H, F_\omega u; G\rangle \qquad \text{by the triple identity} \\
&= -\sum_e \left\langle G, \int F_\omega u \,\mathrm{d}H \right\rangle_e - \sum_e \left\langle H, \int F_\omega u \,\mathrm{d}G \right\rangle_e
\end{aligned}$$

by reducing to the double index again as F_ω is analytic. This shows that if we replace F by u in the formula to be proved we indeed get 0. Similarly we get the same result if we replace G by a holomorphic v, H by a holomorphic w, or if we do 2 or 3 of these replacements at the same time. Now, exactly as in the proof of Proposition 8.4, writing the left-hand side of (8.7) as $T(F, G, H, \omega)$, we easily get from the previous computation that

$$qT(F, G, H, \omega) = T(\phi^* F, \phi^* G, \phi^* H, \phi^* \omega) = q^3 T(F, G, H, \phi^* \omega).$$

Defining the functional γ by $\gamma([\omega]) = T(F, G, H, \omega)$, this shows that γ satisfies $\gamma(\phi^*[\omega]) = q^{-2}\gamma([\omega])$, so that $\gamma((q^2\phi^* - \mathrm{id})[\omega]) = 0$. By the theory of Weil numbers, it follows that $\gamma = 0$. This proves what we want. □

9. A formula for the regulator

In this section we obtain our first explicit regulator formula, Theorem 9.10, using the theory of the triple index. For technical reasons, the syntomic regulator itself must be developed over a discretely valued field. However, since we have formulas for the regulator that make sense over $\mathbb{C}_p$ as well, we work from now until the end of this paper over $\mathbb{C}_p$.

Now that we have at our disposal the triple index, we can interpret our make-believe computation of Section 6 in such a way that it will become true. We continue with the notation of the previous section, so U is a wide open space over $\mathbb{C}_p$.

The first thing that the triple index allows us to do is to extend the cup product to some Coleman differential forms. We first need a lemma.

Lemma 9.1. *The map $\Omega^1_{\mathrm{col},1}(U) \to H^1(U) \otimes \Omega^1(U)$ given by*

$$\sum F_{\omega_i}\eta_i \mapsto \sum [\omega_i] \otimes \eta_i$$

is well-defined.

Proof. This is [Besser 2002, Corollary 6.2]. □

Proposition 9.2. *There is a unique bilinear map*

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : A_{\mathrm{col},1}(U) \otimes \Omega^1_{\mathrm{col},1}(U) \to \mathbb{C}_p$$

such that we have, for any F, G, H in $A_{\mathrm{col},1}(U)$,

$$\langle\!\langle F, G\,\mathrm{d}H \rangle\!\rangle = \langle F, G; H\rangle_{\mathrm{gl}}. \tag{9.3}$$

Proof. By definition, $\Omega^1_{\mathrm{col},1}(U)$ is generated by forms like $G\,\mathrm{d}H$ so uniqueness is clear. To show the existence we first note that by Lemma 8.3 the right-hand side depends only on $\mathrm{d}H$. This shows that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is well-defined as a map $A_{\mathrm{col},1}(U) \otimes A_{\mathrm{col},1}(U) \otimes \Omega^1(U) \to \mathbb{C}_p$, where the tensors are taken over $\mathbb{C}_p$. Lemma 9.1 shows that the kernel of the map $G \otimes \mathrm{d}H \to G\,\mathrm{d}H$ from $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ to $\Omega^1_{\mathrm{col},1}(U)$ is contained in $A(U) \otimes \Omega^1(U)$ so it is enough to observe that if g in $A(U)$ then $\langle F, g; H\rangle_{\mathrm{gl}} = \langle F, \int g\,\mathrm{d}H\rangle_{\mathrm{gl}}$ indeed depends only on the form $g\,\mathrm{d}H$. □

The interest in the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is justified by the fact that its restriction to $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ is given by $\langle\!\langle F, \mathrm{d}G \rangle\!\rangle = \langle F, G\rangle_{\mathrm{gl}}$. The pairing on the right was studied in [Besser 2000c].

Let us now fix ω in $\Omega^1(U)$ such that $[\omega]$ extends to C, or equivalently, that it has trivial residues on all ends, and let $F = F_\omega$ in $A_{\mathrm{col},1}(U)$ be a Coleman integral of ω. Let $\mathbf{p}([\omega])$ be the canonical projection of $[\omega]$ on $H^1_{\mathrm{dR}}(C/K)$.

Proposition 9.4. *The functional $L_{\mathbf{p}([\omega])}(\eta) = \langle\!\langle F, \eta \rangle\!\rangle$ on $\Omega^1_{\mathrm{col},1}(U)$ is good in the sense of Definition 5.21.*

Proof. Note that we are not claiming that this functional is independent of the choice of the constant of integration. We first need to prove that $L_{\mathbf{p}([\omega])}$ vanishes on forms of type $\mathrm{d}(a \log f)$, with a and f in $A(U)$. This is easily established:

$$\begin{aligned}\langle\!\langle F, \mathrm{d}(a \log f) \rangle\!\rangle &= \langle\!\langle F, a \,\mathrm{dlog}\, f \rangle\!\rangle + \langle\!\langle F, \log f \,\mathrm{d}a \rangle\!\rangle \\ &= \langle F, a; \log f \rangle_{\mathrm{gl}} + \langle F, \log f\, ; a \rangle_{\mathrm{gl}} \\ &= \langle a, \log f ; F \rangle_{\mathrm{gl}} = 0\end{aligned}$$

by Proposition 8.4. The second property of a good functional is immediate from the formula $\langle F, G \rangle_{\mathrm{gl}} = \mathbf{p}([\mathrm{d}F]) \cup \mathbf{p}([\mathrm{d}G])$ [Besser 2000c, Proposition 4.10]. $\square$

We will henceforth denote the above functional simply by L_ω. This is literally the case if ω is of the second kind on C, as in this case $\mathbf{p}(\omega) = [\omega]$.

Corollary 9.5. *The p-adic regulator $K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K)$ factors through the quotient map $K_4^{(3)}(\mathcal{O}) \to K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}$.*

Proof. By Proposition 5.22 and the normalization (5.7), the fact that a good functional for any cohomology class $\alpha \in H^1_{\mathrm{dR}}(C/K)$ exists implies that the composition

$$K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{1-\phi^*/q^3} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\alpha\cup} K$$

factors. As this is true for any α it follows that

$$K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{1-\phi^*/q^3} H^1_{\mathrm{dR}}(C/K)$$

factors, but $1 - \phi^*/q^3$ is invertible on $H^1_{\mathrm{dR}}(C/K)$ so the result follows. $\square$

Propositions 9.4 and 5.22 suggest that in order to get an explicit formula for reg'_p we need to compute $\langle\!\langle F, T_0^\infty \varepsilon(g, f) \rangle\!\rangle$, where the $\varepsilon(g, f)$ are computed in (5.18). We shall manipulate this by "making our wishes come true" in the form of the following proposition.

Proposition 9.6. *Let F be as in Proposition 9.4 and let $g, f \in \mathcal{O}^*(\mathcal{C}^{\mathrm{loc}})$ with $g \neq 1$. Let $T_0^\infty \varepsilon(g, f)$ be as in (5.18). Then we have*

$$\langle\!\langle F, T_0^\infty \varepsilon(g, f) \rangle\!\rangle = \sum_e \mathcal{T}(g, f, F)_e\,, \tag{9.7}$$

where (*choosing the integrals globally as Coleman integrals*)

$$\mathcal{T}(g, f, F)_e = \frac{1}{q}\,\langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e \\ +\frac{1}{q^2}\,\langle \log \phi^*(f), \log(g); \textstyle\int F\,\mathrm{dlog}(1-g)_0\rangle_e \\ +\frac{1}{q^3}\,\langle \log \phi^*(f), \log(g_0); \textstyle\int F\phi^*\,\mathrm{dlog}(1-g)\rangle_e\,. \tag{9.8}$$

Proof. We have by (5.18) and (9.3)

$$\begin{aligned}&\langle\!\langle F, T_0^\infty \varepsilon(g, f)\rangle\!\rangle \\ &\quad= \sum_e \Big(\frac{1}{q}\,\langle F, \log g; \textstyle\int \log f_0\,\mathrm{dlog}(1-g)\rangle_e \\ &\qquad\qquad -\frac{1}{q^2}\,\langle F, \log g; \textstyle\int \log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e + \frac{1}{q}\langle F, \Theta(g); \log \phi^*(f)\rangle_e\Big).\end{aligned}$$

Note that $\mathrm{d}F = \omega$ is in $\Omega^1(U)$ and has trivial residues along all ends. It follows that F is holomorphic on each end.

At every annulus e we obtain the identities

$$\begin{aligned}\langle F, \log g; \textstyle\int \log f_0\,\mathrm{dlog}(1-g)\rangle_e &= \langle \log(g), \textstyle\int F \log f_0\,\mathrm{dlog}(1-g)\rangle_e \\ &= \langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e,\end{aligned}$$

$$\begin{aligned}\langle F, \log g; \textstyle\int \log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e &= \langle \log g, \textstyle\int F \log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e \\ &= \langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e\,,\end{aligned}$$

$$\begin{aligned}\langle F, \Theta(g); \log\phi^*(f)\rangle_e &= \mathrm{Res}_e\, F\Theta(g)\,\mathrm{dlog}\,\phi^*(f) \\ &= -\langle \log\phi^*(f), \Theta(g)F\rangle_e,\end{aligned}$$

so we obtain

$$\begin{aligned}&\langle\!\langle F, T_0^\infty \varepsilon(g, f)\rangle\!\rangle \\ &\quad= \sum_e \Big(\,\frac{1}{q}\langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e \\ &\qquad\qquad -\frac{1}{q^2}\langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e - \frac{1}{q}\langle \log\phi^*(f), \Theta(g)F\rangle_e\Big).\end{aligned}$$

To equate this with the right-hand side of (9.7) we now realize our wishes one by one. First we notice that the first summands in each expression are identical. The realization of the first wish corresponds to the formula

$$\begin{aligned}\sum_e \langle \log\phi^*(f),\, \Theta(g)F\rangle_e &= \sum_e \langle \log\phi^*(f),\, \textstyle\int F\,\mathrm{d}\Theta(g)\rangle_e + \sum_e \langle \log\phi^*(f),\, \textstyle\int \Theta(g)\,\mathrm{d}F\rangle_e \\ &= \sum_e \langle \log\phi^*(f),\, \textstyle\int F\,\mathrm{d}\Theta(g)\rangle_e\,,\end{aligned}$$

as the second sum on the second line vanishes by [Besser 2000c, Corollary 4.11]. Now we may use the formula (5.12) for $\mathrm{d}\Theta(g)$ to write this as

$$\sum_e \Big(\frac{1}{q}\langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e - \frac{1}{q^2}\langle \log\phi^*(f), \log(g_0);\, \textstyle\int F\phi^*\,\mathrm{dlog}(1-g)\rangle_e\Big),$$

so the left-hand side of (9.7) becomes

$$\begin{aligned}\sum_e \Big(\frac{1}{q}\,\langle \log f_0, \log g;\, \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e - \frac{1}{q^2}\langle \log g,\, F\log(1-g)_0;\, \log\phi^*(f)\rangle_e \\ - \frac{1}{q^2}\langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e \\ + \frac{1}{q^3}\,\langle \log\phi^*(f), \log(g_0);\, \textstyle\int F\phi^*\,\mathrm{dlog}(1-g)\rangle_e\Big).\end{aligned}$$

Now the first and last terms both agree with those on the right-hand side of (9.7) and we are left with verifying the realization of the second wish in the form of

$$\begin{aligned}\sum_e \Big(\langle \log g,\, F\log(1-g)_0;\, \log\phi^*(f)\rangle_e + \langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e \\ + \langle \log\phi^*(f), \log(g);\, \textstyle\int F\,\mathrm{dlog}(1-g)_0\rangle_e\Big) = 0\,.\end{aligned}$$

If we could replace the last triple index by

$$\langle \log\phi^*(f), \log(g);\, F\log(1-g)_0\rangle_e$$

the result would be an immediate consequence of the triple identity; indeed,

$$\begin{aligned}\sum_e \langle \log\phi^*(f), \log(g);\, \textstyle\int F\,\mathrm{dlog}(1-g)_0\rangle_e &= \sum_e \langle \log\phi^*(f), \log(g);\, F\log(1-g)_0\rangle_e \\ &\quad - \sum_e \langle \log\phi^*(f), \log(g);\, \textstyle\int \log(1-g)_0\,\mathrm{d}F\rangle_e,\end{aligned}$$

and the last sum is 0 by Proposition 8.4. □

Proposition 9.9. *Let G be such that $\mathrm{d}G$ lies in $\Omega^1(U)$ and G is holomorphic on ends. Then, with the notation of Proposition 9.6, we have*

$$\mathcal{T}(g,f,\phi^*G)_e=\Big\langle \log(f),\log(g);\int\Big(\phi^*-\frac{1}{q^2}\Big)G\,\mathrm{dlog}(1-g)\Big\rangle_e.$$

Proof. Let $F=\phi^*G$. We replace each term of the form h_0 by $q\log(h)-\log\phi^*(h)$ in (9.8). Then we get

$$\begin{aligned}\mathcal{T}(g,f,F)_e=\frac{1}{q}&\,\langle q\log(f)-\log\phi^*(f),\log g;\textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e\\ &+\frac{1}{q^2}\,\langle\log\phi^*(f),\log(g);q\textstyle\int F\,\mathrm{dlog}(1-g)-\int F\,\mathrm{dlog}\,\phi^*(1-g)\rangle_e\\ &+\frac{1}{q^3}\,\langle\log\phi^*(f),q\log(g)-\log\phi^*(g);\textstyle\int F\phi^*\,\mathrm{dlog}(1-g)\rangle_e,\end{aligned}$$

which after some cancellations equals

$$\langle\log(f),\log(g);\textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e-\dfrac{1}{q^3}\,\langle\log\phi^*(f),\log\phi^*(g);\int F\,\mathrm{dlog}\,\phi^*(1-g)\rangle_e.$$

After substituting ϕ^*G for F and noting that

$$\begin{aligned}&\langle\log\phi^*(f),\log\phi^*(g);\textstyle\int\phi^*G\,\mathrm{dlog}\,\phi^*(1-g)\rangle_e\\ &\qquad\qquad=q\,\langle\log(f),\log(g);\textstyle\int G\,\mathrm{dlog}(1-g)\rangle_e\end{aligned}$$

by Lemma 8.1, this becomes

$$\begin{aligned}&\langle\log(f),\log(g);\textstyle\int\phi^*G\,\mathrm{dlog}(1-g)\rangle_e-\dfrac{1}{q^2}\langle\log(f),\log(g);\int G\,\mathrm{dlog}(1-g)\rangle_e\\ &\qquad=\Big\langle\log(f),\log(g);\int\Big(\phi^*-\frac{1}{q^2}\Big)G\,\mathrm{dlog}(1-g)\Big\rangle_e,\end{aligned}$$

as required. □

We now proceed to apply this theory to elements in K-theory.

Theorem 9.10. 1. *Suppose that an element $\beta\in K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ maps to $\sum_i[g_i]_2\cup f_i$ in $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ under the composition (with the last isomorphism from (2.65))*

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})\to K_4^{(3)}(\mathbb{O})\to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup\mathbb{O}_{\mathbb{Q}}^*\xrightarrow{\simeq}H^1(\mathscr{C}^\bullet(\mathbb{O}))\,,\tag{9.11}$$

and that $\mathrm{reg}_p(\beta)\in\tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}},3)$ is the image of $[\eta]\in H^1_{\mathrm{dR}}(U/K)$ under the isomorphism (5.7). Let ω in $\Omega^1(U)$ have trivial residues along all ends of U. Then

$$\langle F_\omega,F_\eta\rangle_{\mathrm{gl}}=\sum_i\sum_e\langle\log(f_i),\log(g_i);\textstyle\int F_\omega\,\mathrm{dlog}(1-g_i)\rangle_e,\tag{9.12}$$

where F_ω and F_η are any Coleman integrals of ω and η respectively.

2. *In particular, the composition*

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \xrightarrow{\mathrm{reg}_p} \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3) \xrightarrow{[\eta]\mapsto\langle F_\omega, F_\eta\rangle_{\mathrm{gl}}} \mathbb{C}_p$$

factors via (9.11).

Proof. First one easily checks that the validity of the formula depends only on the cohomology class of ω. Since the operator $\phi^* - 1/q^2$ is invertible on $H^1(U)$ we can assume that $\omega = (\phi^* - 1/q^2)\mu$ with μ in $\Omega^1(U)$ and that $F_\omega = (\phi^* - 1/q^2)G$ with G a Coleman integral of μ. Notice that G satisfies the condition of Proposition 9.9. Let η_0 be $\mathrm{reg}_p(\beta) \in \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3)$ in the model (5.6) so that by (5.7) we have $\eta_0 = (1 - \phi^*/q^3)\eta$ (up to an exact form, but this is irrelevant for global index computations). We can take the Coleman integral of η_0 to be $F_{\eta_0} = (1 - \phi^*/q^3)F_\eta$. Let $F = \phi^* G$. By Proposition 9.4 the functional $L_\omega(\eta) = \langle\!\langle F, \eta\rangle\!\rangle$ is good in the sense of Definition 5.21. It follows that we may apply Proposition 5.22 to obtain

$$\begin{aligned}
\langle\!\langle F, \eta_0\rangle\!\rangle &= \sum_i \langle\!\langle F, T_0^\infty \varepsilon(g_i, f_i)\rangle\!\rangle \\
&= \sum_i \sum_e \mathscr{T}(g_i, f_i, F)_e \quad \text{by Proposition 9.6} \\
&= \sum_i \sum_e \left\langle \log(f), \log(g); \int \left(\phi^* - \frac{1}{q^2}\right) G \,\mathrm{dlog}(1-g)\right\rangle_e \\
&= \sum_i \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g)\right\rangle_e
\end{aligned}$$

by Proposition 9.9. On the other hand, we have

$$\begin{aligned}
\langle\!\langle F, \eta_0\rangle\!\rangle = \langle F, F_{\eta_0}\rangle_{\mathrm{gl}} &= \left\langle F, \left(1 - \frac{\phi^*}{q^3}\right) F_\eta\right\rangle_{\mathrm{gl}} = \left\langle \phi^* G, F_\eta - \frac{\phi^*}{q^3} F_\eta\right\rangle_{\mathrm{gl}} \\
&= \langle \phi^* G, F_\eta\rangle_{\mathrm{gl}} - \left\langle \frac{1}{q^2} G, F_\eta\right\rangle_{\mathrm{gl}} = \left\langle \left(\phi^* - \frac{1}{q^2}\right) G, F_\eta\right\rangle_{\mathrm{gl}} = \langle F_\omega, F_\eta\rangle_{\mathrm{gl}}\,.
\end{aligned}$$

The last two equations immediately give the result. □

We can restate the first part of Theorem 9.10 in a form that is more convenient for the rest of this paper. As explained in the introduction, one has a canonical projection $H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)$. This is the unique Frobenius equivariant splitting of the natural restriction map in the other direction.

Recall now the Definition 5.8 of the regulator map reg'_p, using the projection map $\mathbf{p}$. It follows from [Besser 2000c, Proposition 4.10] that $\mathbf{p}$ can be described in the following way. It is the unique map such that for any $\eta \in \Omega^1(U)$ and for any form of the second kind ω on C, which is holomorphic on U, one has

$$[\omega] \cup (\mathbf{p}\eta) = \langle F_\omega, F_\eta, \rangle_{\mathrm{gl}}\,. \tag{9.13}$$

Corollary 9.14. *Suppose that an element $\beta \in K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ maps to $\sum_i [g_i]_2 \cup f_i$ in $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ under (9.11). Let ω be a form of the second kind on C that is holomorphic on U. Then $[\omega] \cup \mathrm{reg}'_p(\beta)$ is given by the right-hand side of (9.12).*

10. End of the proofs

In this section we prove our main theorems. These will all follow from manipulations of Theorem 9.10 and Corollary 9.14.

Fix a form ω of the second kind on C and a Coleman integral F_ω of ω. We begin with the proof of Theorem 1.12.

Lemma 10.1. *The assignment*

$$[g]_2 \otimes f \mapsto \sum_e \left\langle \log(f), \log(g); \int F_\omega \, \mathrm{dlog}(1-g) \right\rangle_e$$

extends to a well-defined map $\Psi''_{p,\omega} : M_2(F) \otimes F^*_{\mathbb{Q}} \to K$.

Proof. For functions $f, g, h \in F$ the map

$$G(h, g, f) = \sum_e \left\langle \log(f), \log(g); \int F_\omega \, \mathrm{dlog}(h) \right\rangle_e \tag{10.2}$$

is trilinear by the properties of the triple index. The result follows from Lemma 2.29. □

Lemma 10.3. *The restriction of $\Psi''_{p,\omega}$ to $(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0}$ coincides with the composition*

$$(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0} \to H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

Proof. This is an immediate consequence of diagram (2.67), noting the vertical map on the left there is $[g]_2 \otimes f \mapsto [g]_2 \cup f$, and of Corollary 9.14. □

Proof of Theorem 1.12. The only part of the theorem not proven already in Lemmas 10.1 and 10.3 is that the map $\Psi''_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$, but this follows immediately from Lemma 10.3. □

Proof of part 1 of Theorem 1.13. By Corollary 2.30, which applies with F replaced with $\mathbb{O}$ by Remark 2.70, the fact that $\Psi''_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$ implies that $\Psi''_{p,\omega} \circ \Xi : H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O})) \to K$ is induced by the following map, with G as in (10.2):

$$\begin{aligned}
[g]_2 \otimes f \mapsto G((1-g) \otimes g \otimes f) - \tfrac{1}{3} G((1-g) \otimes g \otimes f) - \tfrac{1}{3} G(g \otimes (1-g) \otimes f) \\
+ \tfrac{1}{3} G((1-g) \otimes f \otimes g) + \tfrac{1}{3} G(f \otimes (1-g) \otimes g) \\
= G((1-g) \otimes g \otimes f) - \tfrac{1}{3} G(g \otimes (1-g) \otimes f) + \tfrac{1}{3} G(f \otimes (1-g) \otimes g) \\
= \tfrac{2}{3} G((1-g) \otimes g \otimes f) - \tfrac{2}{3} G(g \otimes (1-g) \otimes f),
\end{aligned}$$

where we used that G is symmetric in the last two positions by Proposition 7.3(2), and that $G(f\otimes(1-g)\otimes g) = -G((1-g)\otimes g\otimes f) - G(g\otimes f\otimes(1-g))$ by Proposition 8.6. This is the formula in the first part of Theorem 1.13 by (10.2). □

For the proofs of Theorems 1.11 and 1.9, as well as part 2 of Theorem 1.13, we now assume that ω is a holomorphic form on C.

Lemma 10.4. *The associations*

$$[g]_2\otimes f \mapsto \int_{(1-g)} \log(g) F_\omega \operatorname{dlog}(f) - \int_{(g)} \log(1-g) F_\omega \operatorname{dlog}(f)$$
$$[g]_2\otimes f \mapsto \int_{(f)} \mathrm{L}_2(g)\omega$$
$$[g]_2\otimes f \mapsto \sum_y \operatorname{ord}_y(f) F_\omega(y)\, \mathrm{L}_{\mathrm{mod},2}(g(y))$$

induce well-defined maps on $\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$ (first) and $M_2(F)\otimes F^*_{\mathbb{Q}}$ (last two).

Proof. All three assertions follow from Lemma 2.29. This is essentially clear for the first association. For the second association, observe that $\mathrm{dL}_2 = \log(z)\operatorname{dlog}(1-z)$ by (1.8). Consider the association

$$(h, g, f) \mapsto \int_{(f)} \left(\omega \cdot \int \log(g) \operatorname{dlog}(h)\right).$$

Here, the integral $\int \log(g)\operatorname{dlog}(h)$ is a Coleman integral defined only up to a constant. However, if the constant changes, the entire expression changes by the same constant multiplied by $\int_{(f)} \omega$, which equals 0 as it is the p-adic Abel–Jacobi map applied to the principal divisor (f); see [Besser 2000a]. This association is therefore well-defined, clearly trilinear, and we obtain the required result again by Lemma 2.29. For the third association, one first needs to note that $\mathrm{L}_{\mathrm{mod},2}(g(y))$ is the value of $\mathrm{L}_{\mathrm{mod},2}(g)$ at y (this is not obvious in general because we are using the generalized way of assigning values to Coleman functions by taking constant terms, discussed in the introduction) as we shall see in Corollary 10.8, so the entire expression can be written as $F_\omega \cdot \mathrm{L}_{\mathrm{mod},2}(g)$ evaluated at the divisor of f. It is now possible to proceed as in the previous case, given that

$$\mathrm{dL}_{\mathrm{mod},2}(g) = (\log(g)\operatorname{dlog}(1-g) - \log(1-g)\operatorname{dlog}(g))/2,$$

by associating to f, g, h the value of $F_\omega \cdot \int(\log(g)\operatorname{dlog}(h) - \log(h)\operatorname{dlog}(g))$ at (f), where the constant of integration does not matter for exactly the same reason it did not in the previous case. □

By Lemma 10.4, the maps $\Psi_{p,\omega}$ in Theorem 1.9 and $\Psi'_{p,\omega}$ in Theorem 1.11 from $M_2(\mathbb{O})\otimes\mathbb{O}^*_{\mathbb{Q}}$ to K exist. (The existence of the maps in Theorem 1.13 will be deduced from those in Theorems 1.11 and 1.12 later.)

Next, we shall derive the formulas for the regulator. In all cases, we already have a formula for the regulator, expressed in terms of a sum of local indices on annuli. We can use the argument in the proof of [Besser 2000c, Proposition 5.5] using Proposition 8.4 to replace the sum over ends by a sum over points.

Let $\alpha = \sum_i [g_i]_2 \otimes f_i$ be an element of $(M_2(\mathbb{O}) \otimes \mathbb{O}_{\mathbb{Q}}^*)^{\mathrm{d}=0}$. By the above we have

$$\Psi''_{p,\omega}(\alpha) = \sum_i \sum_{y \in C} \left\langle \log(f_i), \log(g_i); \int F_\omega \, \mathrm{dlog}(1-g_i) \right\rangle_y .$$

We again extend scalars to $\mathbb{C}_p$, so in particular points are $\mathbb{C}_p$ valued. Fix a local parameter at each point y, which we shall call z_y, or, whenever there is no risk of confusion, simply z. Consider a single point y in C. We recall that with respect to the local parameter z at y we define, for a rational function f, $\bar{f}(y) = (f/z^{\mathrm{ord}_y(f)})(y)$. For such a function f we have $c_z(\log(f)) = \log(\bar{f}(y))$. We also have $\mathrm{Res}_y(F_\omega \, \mathrm{dlog}(f)) = \mathrm{ord}_y(f) \cdot F_\omega(y)$. Thus, using Proposition 7.8, we obtain

$$\begin{aligned}(10.5)\quad \Psi''_{p,\omega}(\alpha) = \sum_i \sum_{y \in C} \Big[& \mathrm{ord}_y(1-g_i) F_\omega(y) \log \bar{f}_i(y) \log \bar{g}_i(y) \\ & - \mathrm{ord}_y(f_i) c_z\Big(\int \log(g_i) F_\omega \, \mathrm{dlog}(1-g_i) \Big) \\ & - \mathrm{ord}_y(g_i) c_z\Big(\int \log(f_i) F_\omega \, \mathrm{dlog}(1-g_i) \Big) \Big].\end{aligned}$$

Let A (respectively B) be the subgroup of $k(C)^*$ generated by the f_i and g_i (respectively by the $1-g_i$). By choosing bases for A and B and then choosing appropriate integrals we can arrange it so that for each f in A and h in B an integral $\int \log(f) F_\omega \, \mathrm{dlog}\, h$ is chosen such that the map $(f, h) \mapsto \int \log(f) F_\omega \, \mathrm{dlog}\, h$ is bilinear. Since the overall sum in (10.5) is independent of the choice of integrals, we may and do assume from now on that the integrals there are chosen as above.

Lemma 10.6. *If $\sum_i [g_i]_2 \otimes f_i$ is in $(M_2(F) \otimes F_{\mathbb{Q}}^*)^{\mathrm{d}=0}$, then for every y in C we have*

$$\sum_i \mathrm{ord}_y(f_i) \, c_z\Big(\int \log(g_i) F_\omega \, \mathrm{dlog}(1-g_i) \Big) = \sum_i \mathrm{ord}_y(g_i) \, c_z\Big(\int \log(f_i) F_\omega \, \mathrm{dlog}(1-g_i) \Big).$$

Proof. With the choices above the map

$$(f, g, h) \mapsto \mathrm{ord}_y(f) c_z\Big(\int \log(g) F_\omega \, \mathrm{dlog}(h) \Big) - \mathrm{ord}_y(g) c_z\Big(\int \log(f) F_\omega \, \mathrm{dlog}(h) \Big)$$

is trilinear and antisymmetric with respect to f and g. The lemma follows since $\sum(1-g_i)\otimes(g_i\wedge f_i)=0$ by (2.24). □

We recall that the function $\mathrm{L}_2(z)$ is defined by $\mathrm{L}_2(z)=\mathrm{Li}_2(z)+\log(z)\log(1-z)$ and that we have $\mathrm{dL}_2(z)=\log(z)\,\mathrm{dlog}(1-z)$. Note that this last form is holomorphic in the residue disc of 1 and as a consequence so is $\mathrm{L}_2(z)$.

Lemma 10.7. *Let g be a rational function. The constant term at y of $\mathrm{L}_2(g)$ equals $\mathrm{L}_2(g(y))$ if $g(y)\neq 0,\infty$, equals 0 if $g(y)=0$ or 1 and equals $\log^2(\bar{g}(y))/2$ if $g(y)=\infty$, where $\bar{g}$ is computed with respect to the same local parameter as the constant term. In addition, the expansion of $\mathrm{L}_2(g)$ with respect to any local parameter z contains no summands of the form $Const\cdot z^n$ with $n<0$.*

Proof. This is clear if $g(y)\neq 0,\infty$. Suppose $g(y)=0$. Since Li_2 is holomorphic near 0 and has value 0 there, we see that the constant term and terms of the form z^n for $n<0$ are the same as in $\log(g)\log(1-g)$. Near y, $\log(g(z))=\mathrm{ord}_y(g)\log(z)+$, a holomorphic function in z. Also, $\log(1-g)$ is holomorphic near y with value 0 there. Thus the result is clear. Finally, by [Coleman 1982, Proposition 6.4], we have $\mathrm{L}_2(g)+\mathrm{L}_2(1/g)=\log^2(g)/2$ (from which it also follows that $\mathrm{L}_2(1)=0$) so the result at $g(y)=\infty$ is deduced from that of $1/g$ when $g(y)=\infty$. □

Corollary 10.8. *The constant term of $\mathrm{L}_{\mathrm{mod},2}(z)$ at 0, 1 and ∞ is 0, regardless of parameter. Furthermore, setting the value of $\mathrm{L}_{\mathrm{mod},2}$ at these points to be the above constant term, we have that for any rational function g the constant term of $\mathrm{L}_{\mathrm{mod},2}(g)$ at any point y equals $\mathrm{L}_{\mathrm{mod},2}(g(y))$.*

Proof. Since $\mathrm{L}_{\mathrm{mod},2}(z)=\mathrm{L}_2(z)-\log(z)\log(1-z)/2$ it is easy to check that the constant term of $\mathrm{L}_{\mathrm{mod},2}(g)$ is 0 at either $g(y)=0,\,1,\,\infty$, and the result easily follows. □

Lemma 10.9. *For any point y in C and for any choice of a Coleman integral $\int\mathrm{L}_2(g)\omega$ the quantity $c_z(\int\mathrm{L}_2(g)\omega)$ is independent of the choice of the local parameter z at y.*

Proof. Let f_ω be the unique Coleman integral of ω that vanishes at y. We may choose a Coleman integral $\int f_\omega d\mathrm{L}_2(g)$ in such a way that the integration by parts formula

$$\int\mathrm{L}_2(g)\omega=\mathrm{L}_2(g)\,f_\omega-\int f_\omega\,\mathrm{dL}_2(g)$$

holds. It is therefore sufficient to show that the constant term of each of the summands on the right is independent of the parameter. From the last assertion in Lemma 10.7 and the fact that $f_\omega(y)=0$ it is easy to see that the constant term of the first summand is 0. For the second summand we have

$$\int f_\omega \,\mathrm{dL}_2(g) = \int f_\omega \log(g)\,\mathrm{dlog}(1-g)$$
$$= \log(g) \int f_\omega \,\mathrm{dlog}(1-g) - \int \Big(\int f_\omega \,\mathrm{dlog}(1-g) \Big) \,\mathrm{dlog}(g)$$

for appropriate choices of integrals. As $f_\omega \,\mathrm{dlog}(1-g)$ is holomorphic at y, we may arrange it so that $\int f_\omega \,\mathrm{dlog}(1-g)$ vanishes at y. Then in the last formula the first term has constant term 0 while the second term is holomorphic at y hence its constant term is independent of z. □

Using the last lemma we may set

$$\int \mathrm{L}_2(g)\omega|_y := c_z\Big(\int \mathrm{L}_2(g)\omega \Big)$$

with respect to any parameter z at y. Using this we can define $\int_D \mathrm{L}_2(g)\omega$ for any divisor D of degree zero. If we change $\int \mathrm{L}_2(g)\omega$ by a constant, its value at y in the above sense will change by the same constant. Thus when D has degree 0 the integral $\int_D \mathrm{L}_2(g)\omega$ does not depend on the constant of integration even if D and the divisor of g have a common support. This explains the general definition of the integral in Theorem 1.9.

Lemma 10.10. *Choose integrals such that the integration by parts formula*

$$\int \log(g) F_\omega \,\mathrm{dlog}(1-g) = F_\omega \mathrm{L}_2(g) - \int \mathrm{L}_2(g)\omega$$

is satisfied. Then we have at a point y and with respect to the local parameter z,

$$c_z\Big(\int \log(g) F_\omega \,\mathrm{dlog}(1-g) \Big) = F_\omega(y) c_z(\mathrm{L}_2(g)) - \int \mathrm{L}_2(g)\omega|_y \,.$$

Proof. One just applies c_z to the integration by parts formula and observes that by Lemma 10.7 we have $c_z(F_\omega \mathrm{L}_2(g)) = F_\omega(y) c_z(\mathrm{L}_2(g))$. □

Proof of Theorem 1.11. We already saw that the association gives a well-defined map on $M_2(\mathbb{C}) \otimes \mathbb{C}^*_{\mathbb{Q}}$. It therefore suffices to show that it gives the same map on $(M_2(\mathbb{C}) \otimes \mathbb{C}^*_{\mathbb{Q}})^{\mathrm{d}=0}$ as $\Psi''_{p,\omega}$ in Theorem 1.12. Consider (10.5). By Lemma 10.6 we can choose our integrals such that for each point y the sum over i of each of the last two terms is identical. The term

$$\mathrm{ord}_y(f_i) c_z\big(\int \log(g_i) F_\omega \,\mathrm{dlog}(1-g_i)\big)$$

is computed in Lemmas 10.10 and 10.7. Substituting the results we see that we have the equation

$$\Psi''_{p,\omega}(\alpha) = \sum_i \Bigg[\sum_{y\in C} (\mathrm{ord}_y(1-g_i)F_\omega(y)\log \bar{f}_i(y)\log \bar{g}_i(y)) + 2\int_{(f_i)} \mathrm{L}_2(g_i)\omega \\ - \sum_{y\in C} \mathrm{ord}_y(f_i)F_\omega(y) \times \begin{cases} 0 & g_i(y)=0, \\ 2\mathrm{L}_2(g_i(y)) & g_i(y)\neq 0,\infty, \\ \log^2(\bar{g}_i(y)) & g_i(y)=\infty. \end{cases} \Bigg].$$

In the first sum over y, only terms with $g_i(y)=\infty$ can be nonzero. Thus neither sum over y contributes for $g_i(y)=0$, and the right-hand side becomes

$$\sum_i \Bigg[2\int_{(f_i)} \mathrm{L}_2(g_i)\omega - 2\sum_{g_i(y)\neq 0,\infty} \mathrm{ord}_y(f_i)F_\omega(y)\mathrm{L}_2(g_i(y)) \\ + \sum_{g_i(y)=\infty} F_\omega(y)\lambda_y(f_i,g_i) \Bigg] \tag{10.11}$$

with

$$\begin{aligned} \lambda_y(f,g) &= \mathrm{ord}_y(1-g)\log \bar{f}(y)\log \bar{g}(y) - \mathrm{ord}_y(f)\log^2 \bar{g}(y) \\ &= \log \bar{g}(y)(\mathrm{ord}_y(1-g)\log \bar{f}(y) - \mathrm{ord}_y(f)\log \bar{g}(y)) \\ &= \log \overline{1-g}(y)(\mathrm{ord}_y(g)\log \bar{f}(y) - \mathrm{ord}_y(f)\log \bar{g}(y)) \end{aligned}$$

because $g(y)=\infty$ implies $\mathrm{ord}_y(1-g)=\mathrm{ord}_y(g)$ and $\bar{g}(y) = -\overline{1-g}(y)$.

For y in C, the function

$$\mu_y(f,g,h) = \log \bar{h}(y)(\mathrm{ord}_y(g)\log \bar{f}(y) - \mathrm{ord}_y(f)\log \bar{g}(y))$$

is trilinear in f, g and h and antisymmetric in f and g. As $\sum_i (1-g_i)\otimes(g_i\wedge f_i)=0$ by (2.53), we find

$$\sum_i \mu_y(f_i, g_i, 1-g_i) = 0\,. \tag{10.12}$$

If $g_i(y)=0$ then $\mu_y(f_i,g_i,1-g_i)=0$, while if $g(y)\neq 0,\infty$ then

$$\mu_y(f_i,g_i,1-g_i) = -\mathrm{ord}_y(f_i)\log g_i(y)\log(1-g_i(y)),$$

where we set the value of $\log(y)\log(1-y)$ at 1 to be 0, which is its constant term. Thus, summing (10.12) multiplied by $F_\omega(y)$ over all y in C we see that

$$\sum_i \sum_{g_i(y)=\infty} F_\omega(y)\lambda_y(f_i,g_i) = \sum_i \sum_{g_i(y)\neq 0,\infty} \mathrm{ord}_y(f_i)F_\omega(y)\log g_i(y)\log(1-g_i(y)).$$

Substituting this into (10.11), and using that $\mathrm{L}_2(z)-\log(z)\log(1-z)/2=\mathrm{L}_{\mathrm{mod},2}(z)$ by definition, we obtain

$$\Psi''_{p,\omega}(\alpha)=2\sum_i\int_{(f_i)}\mathrm{L}_2(g_i)\omega-2\sum_i\sum_{g_i(y)\neq 0,\infty}\mathrm{ord}_y(f_i)F_\omega(y)\mathrm{L}_{\mathrm{mod},2}(g_i(y)).$$

This formula finishes the proof of Theorem 1.11 as $\mathrm{L}_{\mathrm{mod},2}(0)=\mathrm{L}_{\mathrm{mod},2}(\infty)=0$. □

Proof of Theorem 1.9. That the assignment is well-defined is part of Lemma 10.4. In order to see that it vanishes on $[f]_2\otimes f$, we note that we already know this is true for the assignment in Theorem 1.11, and that the second term in that assignment is trivial on such terms because $\mathrm{L}_{\mathrm{mod},2}(z)$ vanishes at 0 and ∞.

For part (2), consider (1.16). That $\partial_1(\alpha')=0$ means that α' satisfies (2.57), which is equivalent with α' being in $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$ inside $H^2(\mathcal{M}_{(3)}(\mathbb{C}'))$ (recall from Section 2.5.3 that the two vertical maps at the top in this diagram are injections if we use $\mathbb{C}'$ instead of $\mathbb{C}$ everywhere). The existence and uniqueness of β' was therefore proven just after (2.58). In fact, β' is the $K_4^{(3)}(\mathscr{C}')$ component of the image of α' in $K_4^{(3)}(\mathscr{C}')\oplus K_3^{(2)}(k)\cup\mathbb{C}'^*_{\mathbb{Q}}$, and the images of α' and β' in $K_4^{(3)}(\mathbb{C}')$ differ by some γ' in the image of $K_3^{(2)}(k)\cup\mathbb{C}'^*_{\mathbb{Q}}$. But $\omega\cup\mathrm{reg}'_p(\gamma')=0$ by the commutativity of the bottom right square, so that, after extending from $\mathbb{C}'$ to $\mathbb{C}$, we have $\omega\cup\mathrm{reg}'_p(\beta)=\Psi'_{p,\omega}(\alpha)$ by Theorem 1.11. It therefore suffices to show that the contribution of each $\mathrm{ord}_y(f)F_\omega(g(y))\mathrm{L}_{\mathrm{mod},2}(g(y))$ in $\Psi'_{p,\omega}(\alpha)$ is trivial.

Note that in Theorem 1.11 this sum has to be computed after a suitable finite extension $\tilde{K}$ of K that makes the relevant y rational, but that further extending the field to $\mathbb{C}_p$ as we are using here gives the same result. In fact, because we start over the number field k, the relevant y become rational over some number field $L\subset\tilde{K}$ containing k. The $\widetilde{M}_2(\cdot)$ are compatible with field extensions, and clearly the same holds for ∂_1. Therefore (2.57) gives us that for each closed point y of C'_L, $\partial_{1,y}(\alpha')$ is trivial in $\widetilde{M}_2(L)$. Because $F_\omega(y)$ is just a constant, comparing with the definition of $\partial_{1,y}$ in Section 2.4.3, we see that it suffices to show that the map

$$\begin{aligned}H^1(\widetilde{\mathcal{M}}_{(2)}(L))&\to\tilde{K}\\ \sum_i[a_i]_2&\mapsto\sum_i\mathrm{L}_{\mathrm{mod},2}(a_i)\end{aligned}$$

is well-defined. It is conjectured in [Besser and de Jeu 2003, Conjecture 1.14] that this map is the syntomic regulator map on as composition (with $\mathbb{C}_L$ the ring of integers in L)

$$H^1(\widetilde{\mathcal{M}}_{(2)}(L))\to K_3^{(2)}(L)\simeq K_3^{(2)}(\mathbb{C}_L)\to H^1_{\mathrm{syn}}(\mathbb{C}_L,2)\simeq K,$$

which would imply what we need. However, extending the domain of the map, we can show by more basic means that the map

$$\begin{aligned}\widetilde{M}_2(L) &\to \tilde{K}\\ [a]_2 &\mapsto \mathrm{L}_{\mathrm{mod},2}(a)\end{aligned}$$

is well-defined, which will prove what we want.

Namely, for any field L of characteristic zero, let $B_2'(L)$ be the free $\mathbb{Q}$-vector space on elements $\{b\}_2$ with b in F, $b \neq 0, 1$, modulo the five term relation

$$\{b\}_2 + \{c\}_2 + \left\{\frac{1-b}{1-bc}\right\}_2 + \{1-bc\}_2 + \left\{\frac{1-c}{1-bc}\right\}_2 = 0. \tag{10.13}$$

It is shown in [de Jeu 2000, Lemma 5.2] that there is a map $B_2'(L) \to \widetilde{M}_2(L)$, given by sending $\{b\}_2$ to $[b]_2$. In the case where L is a number field, this was already done on page 240 of [de Jeu 1995] (where the relations were not made explicit and the group was called $B_2(L)$), and the map was shown to be an isomorphism in that case. Finally, in [Coleman 1982, Corollaries 6.4(ii), (iii) and 6.5b] Coleman shows that $\mathrm{L}_{\mathrm{mod},2}$ (which is called D there) satisfies

$$\begin{aligned}\mathrm{L}_{\mathrm{mod},2}(z^{-1}) &= -\mathrm{L}_{\mathrm{mod},2}(z)\\ \mathrm{L}_{\mathrm{mod},2}(1-z) &= -\mathrm{L}_{\mathrm{mod},2}(z)\end{aligned}$$

as well as (with signs corrected)

$$\mathrm{L}_{\mathrm{mod},2}(z_1z_2) = \mathrm{L}_{\mathrm{mod},2}(z_1) + \mathrm{L}_{\mathrm{mod},2}(z_2) + \mathrm{L}_{\mathrm{mod},2}\left(\frac{z_1(1-z_2)}{z_1-1}\right) + \mathrm{L}_{\mathrm{mod},2}\left(\frac{z_2(1-z_1)}{z_2-1}\right).$$

Substituting $z_1 = (bc)^{-1}$, $z_2 = c$ in the last relation and using the first two, one sees that $\mathrm{L}_{\mathrm{mod},2}$ satisfies the relation corresponding to (10.13). Therefore it induces a map

$$\widetilde{M}_2(L) \simeq B_2'(L) \to K$$

mapping $[b]_2$ to $\mathrm{L}_{\mathrm{mod},2}(b)$. This finishes the proof of Theorem 1.9. □

Proof of part 2 of Theorem 1.13. Since by Theorem 1.11, the map $\Psi'_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$, we may again use Corollary 2.30, which applies with F replaced by $\mathbb{O}$ by Remark 2.70. Recall that $\Psi'_{p,\omega}$ is induced by

$$[g]_2 \otimes f \mapsto 2\int_{(f)} \mathrm{L}_2(g)\omega - 2\sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)).$$

Since $\mathrm{L}_{\mathrm{mod},2}(z) + \mathrm{L}_{\mathrm{mod},2}(z^{-1}) = 0$, while $\mathrm{L}_2(z) + \mathrm{L}_2(z^{-1}) = \frac{1}{2}\log^2(z)$, we see that we are in the situation of part (3) of Lemma 2.29 with $H(a \cdot b \otimes c) = \int_{(c)} \log(a)\log(b)\omega$.

Applying the corollary, the composition $\Psi'_{p,\omega} \circ \Xi : H^2(\widetilde{\mathcal{M}}_{(2)}(\mathbb{O})) \to K$ is given by

$$[g]_2 \otimes f \mapsto 2 \int_{(f)} \mathrm{L}_2(g)\omega - 2 \sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)) \\ - \tfrac{2}{3} \int_{(f)} \log(1-g)\log(g)\omega + \tfrac{2}{3} \int_{(g)} \log(f)\log(1-g)\omega,$$

as required. □

Remark 10.14. We would like to explain a bit of the heuristics suggesting that Theorem 1.13 gives a formula which is the p-adic analogue of the complex analytic formula for the regulator in Section 3.

Experience has taught us that complex surface integrals translate in the p-adic world to a similar formula involving local indices. For example, the complex analytic formula for the regulator of the symbol $\{f, g\}$ in $K_2(F)$,

$$\int_C \log|g| \, \overline{\mathrm{dlog}\, f} \wedge \omega = 2 \int_C \log|g| \, \mathrm{dlog}\,|f| \wedge \omega,$$

where ω is holomorphic, translates in the p-adic world into the formula

$$\langle \log f, F_\omega; \log g \rangle_{\mathrm{gl}}.$$

Note that, using the rules for the triple index, this is the same as the formula $\sum_e \langle \log f, \int (F_\omega \, \mathrm{dlog}(g)) \rangle_e$ obtained in [Besser 2000c, Propositon 5.1]. This corresponds to the regulator on an open curve using the same projection on $H^1_{\mathrm{dR}}(C/K)$ we have been using in this paper. For a sum $\{f_i, g_i\}$ in the kernel of the tame symbol, we may, for every pair $(f, g) = (f_i, g_i)$, replace $\langle \log f, F_\omega; \log g \rangle_{\mathrm{gl}}$ with $\int_{(f)} \log(g) \cdot \omega$, obtaining the formula of Coleman and de Shalit [1988, (1)]. This is similar to Theorem 1.11 specializing to Theorem 1.9.

Relying on these considerations, the maps $\Psi''_{p,\omega}$ and $\Psi'''_{p,\omega}$ in Theorems 1.12 and 1.13 are precise analogues, up to a factor of 4, of the maps $\Psi''_{\infty,\omega}$ and $\Psi'''_{\infty,\omega}$ in Proposition 3.1. Factors that are powers of 2 appear in comparison with other regulator formulas; see for example the introduction of [Besser 2012].

Acknowledgments

The authors would like to thank the European Community for support through the RTN network *Arithmetic Algebraic Geometry*, which enabled them to meet on various occasions during the long gestation period of this paper. Rob de Jeu would like to thank the Newton Institute in Cambridge, where part of this paper was written, for a conducive atmosphere.

This paper is dedicated to the memory of Jon Rogawski. One of us (AB) still remembers Jon's help and advice as a young postdoc at UCLA. He was a great mentor with his calm and assured guidance. He will be greatly missed.

References

[Besser 2000a] A. Besser, "A generalization of Coleman's p-adic integration theory", *Invent. Math.* **142**:2 (2000), 397–434. MR 2001i:14032 Zbl 1053.14020

[Besser 2000b] A. Besser, "Syntomic regulators and p-adic integration, I: Rigid syntomic regulators", pp. 291–334 in *Proceedings of the Conference on p-adic Aspects of the Theory of Automorphic Representations* (Jerusalem, 1998), vol. 120, 2000. part B. MR 2002c:14035 Zbl 1001.19003

[Besser 2000c] A. Besser, "Syntomic regulators and p-adic integration, II: K_2 of curves", pp. 335–359 in *Proceedings of the Conference on p-adic Aspects of the Theory of Automorphic Representations* (Jerusalem, 1998), vol. 120, 2000. part B. MR 2002c:14036 Zbl 1001.19004

[Besser 2002] A. Besser, "Coleman integration using the Tannakian formalism", *Math. Ann.* **322**:1 (2002), 19–48. MR 2003d:11176 Zbl 1013.11028

[Besser 2012] A. Besser, "On the syntomic regulator for K_1 of a surface", *Israel J. Math.* **190** (2012), 29–66. MR 2956231

[Besser and de Jeu 2003] A. Besser and R. de Jeu, "The syntomic regulator for the K-theory of fields", *Ann. Sci. École Norm. Sup.* (4) **36**:6 (2003), 867–924. MR 2005f:11133 Zbl 1106.11024

[Besser et al. 2009] A. Besser, P. Buckingham, R. de Jeu, and X.-F. Roblot, "On the p-adic Beilinson conjecture for number fields", *Pure Appl. Math. Q.* **5**:1 (2009), 375–434. MR 2010k:19006 Zbl 1192.19003

[Bloch 1990] S. Bloch, Letter to C. Deninger, 1990.

[Bloch 2000] S. J. Bloch, *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, CRM Monograph Series **11**, American Mathematical Society, Providence, RI, 2000. MR 2001i:11082 Zbl 0958.19001

[Coleman 1982] R. F. Coleman, "Dilogarithms, regulators and p-adic L-functions", *Invent. Math.* **69**:2 (1982), 171–208. MR 84a:12021 Zbl 0516.12017

[Coleman 1985] R. F. Coleman, "Torsion points on curves and p-adic abelian integrals", *Ann. of Math.* (2) **121**:1 (1985), 111–168. MR 86j:14014 Zbl 0578.14038

[Coleman and de Shalit 1988] R. Coleman and E. de Shalit, "p-adic regulators on curves and special values of p-adic L-functions", *Invent. Math.* **93**:2 (1988), 239–266. MR 89k:11041 Zbl 0655.14010

[Gillet and Soulé 1999] H. Gillet and C. Soulé, "Filtrations on higher algebraic K-theory", pp. 89–148 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR 2001i:19005 Zbl 0951.19003

[Goncharov 1994] A. B. Goncharov, "Polylogarithms and motivic Galois groups", pp. 43–96 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR 94m:19003 Zbl 0842.11043

[Grosse-Klönne 2000] E. Grosse-Klönne, "Rigid analytic spaces with overconvergent structure sheaf", *J. Reine Angew. Math.* **519** (2000), 73–95. MR 2001b:14033 Zbl 0945.14013

[Harder 1977] G. Harder, "Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern", *Invent. Math.* **42** (1977), 135–175. MR 57 #12780 Zbl 0391.20036

[de Jeu 1995] R. de Jeu, "Zagier's conjecture and wedge complexes in algebraic K-theory", *Compositio Math.* **96**:2 (1995), 197–247. MR 96h:19005 Zbl 0868.19002 [de Jeu 1996] R. de Jeu, "On $K_4^{(3)}$ of curves over number fields", *Invent. Math.* **125**:3 (1996), 523–556. MR 97d:19006 Zbl 0864.11059

[de Jeu 2000] R. de Jeu, "Towards regulator formulae for the K-theory of curves over number fields", *Compositio Math.* **124**:2 (2000), 137–194. MR 2002e:19005 Zbl 0985.19002

[van der Put 1986] M. van der Put, "The cohomology of Monsky and Washnitzer", pp. 4, 33–59 in *Introductions aux cohomologies p-adiques* (Luminy, 1984), Mém. Soc. Math. France (N.S.) **23**, Société Mathématique de France, Paris, 1986. MR 88a:14022 Zbl 0606.14018

[Quillen 1973] D. Quillen, "Higher algebraic K-theory, I", pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, Berlin, 1973. MR 49 #2895 Zbl 0292.18004

[Soulé 1985] C. Soulé, "Opérations en K-théorie algébrique", *Canad. J. Math.* **37**:3 (1985), 488–550. MR 87b:18013 Zbl 0575.14015

[Srinivas 1996] V. Srinivas, *Algebraic K-theory*, 2nd ed., Progress in Mathematics **90**, Birkhäuser, Boston, MA, 1996. MR 97c:19001 Zbl 0860.19001

Received July 21, 2012.

AMNON BESSER
DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O.B. 653
84105 BE'ER-SHEVA
ISRAEL
bessera@math.bgu.ac.il
http://www.math.bgu.ac.il/~bessera/

ROB DE JEU
FACULTEIT DER EXACTE WETENSCHAPPEN, AFDELING WISKUNDE
VU UNIVERSITY AMSTERDAM
DE BOELELAAN 1081A
1081 HV AMSTERDAM
NETHERLANDS
r.m.h.de.jeu@vu.nl
http://www.few.vu.nl/~jeu/

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US \$420/year for the electronic version, and \$485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 260 No. 2 December 2012

Special issue
devoted to the memory of Jonathan Rogawski