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We give a correct proof for the existence of the essential vector of an irreducible admissible generic representation of the general linear group over a p -adic field.

Nadir Matringe has indicated to me that the paper “Conducteur des représentations du groupe linéaire” [Jacquet et al. 1981a; 1981b] contains an error. Since the result therein has applications (see [Jacquet and Shalika 1985] for instance), it may be useful to correct the error. In any case, the correct proof is actually simpler than the erroneous proof. Separately, Matringe [2011] has given a different proof, which is of independent interest.

First, I recall the result in question. Let F be a non-Archimedean local field. We denote by α or $|\cdot|$ the absolute value of F , by q the cardinality of the residual field and finally by v the valuation function on F . Thus, $\alpha(x) = |x| = q^{-v(x)}$. Let ψ be an additive character of F whose conductor is the ring of integers \mathbb{O}_F . Let G_r be the group $GL(r)$ regarded as an algebraic group. We denote by w_r the permutation matrix whose antidiagonal entries are 1. For instance,

$$w_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by dg the Haar measure of $G_r(F)$ for which the compact group $G_r(\mathbb{O}_F)$ has volume 1. Let N_r be the subgroup of upper triangular matrices with unit diagonal and A_r the group of diagonal matrices. We define a character

$$\theta_{r,\psi} : N_r(F) \rightarrow \mathbb{C}^\times$$

by the formula

$$\theta_{r,\psi}(u) = \psi \left(\sum_{1 \leq i \leq r-1} u_{i,i+1} \right).$$

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We denote by du the Haar measure on $N_r(F)$ for which $N_r(\mathbb{C}_F)$ has measure 1. We have then an invariant quotient measure on $N_r(F) \backslash G_r(F)$.

Let S_r be the algebra of symmetric polynomials in

$$(X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_r, X_r^{-1}).$$

Let H_r be the Hecke algebra of $G_r(F)$, that is, the convolution algebra of compactly supported, complex-valued functions that are bi-invariant under the maximal compact group $G_r(\mathbb{C}_F)$. Let $\mathcal{S}_r : H_r \rightarrow S_r$ be the Satake isomorphism. Thus, for any r -tuple of nonzero complex numbers (x_1, x_2, \dots, x_r) we have a homomorphism of algebras $\mathcal{S}_r(x_1, x_2, \dots, x_r) : H_r \rightarrow \mathbb{C}$, defined by

$$\mathcal{S}_r(x_1, x_2, \dots, x_r) : \phi \mapsto \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r).$$

Concretely, it is defined in the following way. Let $t = (t_1, t_2, \dots, t_r)$ be a tuple of complex numbers such that $x_i = q^{-t_i}$ for each i . We denote by $\pi(t_1, t_2, \dots, t_r)$ the corresponding principal series representation of $G_{r-1}(F)$. It is the representation induced by the character

$$a = (a_1, a_2, \dots, a_r) \mapsto |a_1|^{t_1} |a_2|^{t_2} \dots |a_r|^{t_r}$$

of $A_r(F)$. Its space $I(t_1, t_2, \dots, t_r)$ is the space of smooth functions $\phi : G_r(F) \rightarrow \mathbb{C}$ such that

$$\phi \left[\begin{pmatrix} a_1 & * & \dots & \dots & * \\ 0 & a_2 & \dots & \dots & * \\ 0 & 0 & \dots & \dots & a_r \end{pmatrix} g \right] = |a_1|^{t_1 + \frac{r-1}{2}} |a_2|^{t_2 + \frac{r-1}{2} - 1} \dots |a_r|^{t_r - \frac{r-1}{2}} \phi(g).$$

The space $I(t_1, t_2, \dots, t_r)$ contains a unique vector ϕ_0 equal to 1 on $G_r(\mathbb{C}_F)$ and thus invariant under $G_r(\mathbb{C}_F)$. Under convolution, it is an eigenfunction of H_r with eigenvalue $\mathcal{S}_r(x_1, x_2, \dots, x_r)$, that is,

$$\int_{G_r(F)} \phi_0(gh) \phi(h) dh = \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r) \phi_0(g)$$

for every ϕ in H_r .

There is a unique function $W : G_r(F) \rightarrow \mathbb{C}$ satisfying the following properties:

- $W(gk) = W(g)$ for $k \in G_r(\mathbb{C}_F)$,
- $W(ug) = \theta_\psi(u) W(g)$ for $u \in N_r(F)$,
- for all (x_1, x_2, \dots, x_r) and all $\phi \in H_r$,

$$\int_{G_r(F)} W(gh) \phi(h) dh = \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r) W(g),$$

- $W(e) = 1$.

Thus, W is an eigenfunction of H_r with eigenvalue $S_r(x_1, x_2, \dots, x_r)$. We will denote this function by $W(x_1, x_2, \dots, x_r; \psi)$ and its value at g by

$$W(g; x_1, x_2, \dots, x_r; \psi).$$

Let (π, V) be an irreducible admissible representation of $G_r(F)$. We assume that π is *generic*, that is, there is a nonzero linear form $\lambda : V \rightarrow \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \theta_{r,\psi}(u) \lambda(v)$$

for all $u \in N_r(F)$ and all $v \in V$. Recall that such a form is unique within a scalar factor. We denote by ${}^{\circ}W(\pi; \psi)$ the space of functions of the form

$$g \mapsto \lambda(\pi(g)v)$$

with $v \in V$. It is the *Whittaker model* of π . On the other hand, we have the L -factor $L(s, \pi)$ [Godement and Jacquet 1972]. We denote by $P_\pi(X)$ the polynomial defined by $L(s, \pi) = P_\pi(q^{-s})^{-1}$. The main result of [Jacquet et al. 1981a] is the following theorem:

Theorem 1. *There is an element $W \in {}^{\circ}W(\pi; \psi)$ such that, for any $(r - 1)$ -tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$,*

$$\int_{N_{r-1}(F) \backslash G_{r-1}(F)} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; x_1, x_2, \dots, x_{r-1}; \bar{\psi}) |\det g|^{s-1/2} dg = \prod_{1 \leq i \leq r-1} P_\pi(q^{-s} x_i)^{-1}.$$

In [Jacquet et al. 1981a] it is shown that if we impose the extra condition

$$W \begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix} = W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

for all $h \in G_{r-1}(\mathbb{O}_F)$ and $g \in G_{r-1}(F)$, then W is unique. The vector W is then called the *essential vector* of π , and further properties of this vector are obtained in [Jacquet et al. 1981a].

The proof of this theorem is incorrect in that paper. We give a correct proof here.

1. Review of the properties of the L -factor

Let $r \geq 2$ be an integer. Let $t = (t_1, t_2, \dots, t_{r-1})$ be an $(r - 1)$ -tuple of complex numbers. We assume that

$$\operatorname{Re}(t_1) \geq \operatorname{Re}(t_2) \geq \dots \geq \operatorname{Re}(t_{r-1}).$$

Again, we consider the representation $\pi(t_1, t_2, \dots, t_{r-1})$ that acts on the space $I(t_1, t_2, \dots, t_{r-1})$. As before, let ϕ_0 be the unique vector of that space that is

equal to 1 on $G_{r-1}(\mathbb{O}_F)$. Recall it is invariant under $G_{r-1}(\mathbb{O}_F)$. We recall a standard result.

Lemma 1. *For each tuple t satisfying the above inequalities the vector ϕ_0 is a cyclic vector for the representation $\pi(t_1, t_2, \dots, t_{r-1})$.*

Proof. Indeed, if $\text{Re}(t_1) = \text{Re}(t_2) = \dots = \text{Re}(t_{r-1})$, the representation is irreducible and our assertion is trivial. If not, we use Langlands' construction [Silberger 1978]. For each root α of A_{r-1} we denote by N_α the corresponding subgroup of N_{r-1} or \bar{N}_{r-1} and by $\check{\alpha}$ the corresponding co-root. Thus, if α is a positive root, we have

$$\alpha(a_1, a_2, \dots, a_{r-1}) = a_i/a_j$$

with $i < j$ and

$$\langle t, \check{\alpha} \rangle = t_i - t_j.$$

Let $P(t)$ be the set of positive roots α such that $\text{Re}\langle t, \check{\alpha} \rangle > 0$. Let U be the unipotent group generated by the subgroups $N_{-\alpha}$ with $\alpha \in P(t)$. The intertwining operator

$$N\phi(g) = \int_{U(F)} \phi(ug) du$$

is defined by a convergent integral, and its kernel is a maximal invariant subspace. The formula of [Gindikin and Karpelevič 1966; Gindikin 1961] gives

$$N\phi_0(e) = \prod_{\alpha \in P(t)} \frac{1 - q^{-\langle t, \check{\alpha} \rangle - 1}}{1 - q^{-\langle t, \check{\alpha} \rangle}}.$$

Thus, $N\phi_0 \neq 0$, and our assertion follows. □

The representation $I(t_1, t_2, \dots, t_{r-1})$ admits a nonzero linear form λ such that, for $u \in N_{r-1}(F)$ and ϕ in the space of the representation,

$$\lambda(\pi(u)\phi) = \theta_{r-1, \bar{\psi}}(u) \lambda(\phi).$$

We denote by ${}^{\circ}\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \bar{\psi})$ the space spanned by the functions of the form

$$g \mapsto W_\phi(g), \quad W_\phi(g) = \lambda(\pi(t_1, t_2, \dots, t_{r-1})(g)\phi)$$

with $\phi \in I(t_1, t_2, \dots, t_{r-1})$. We recall the following result:

Lemma 2 [Jacquet and Shalika 1983]. *The map $\phi \mapsto W_\phi$ is injective.*

It follows that the image W_0 of ϕ_0 is a cyclic vector in ${}^{\circ}\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \bar{\psi})$. Up to a multiplicative constant, the function W_0 is equal to the function

$$W_0 = W(x_1, x_2, \dots, x_{r-1}; \bar{\psi}).$$

Now let π be an irreducible generic representation of $G_r(F)$. For $W \in {}^{\circ}\mathcal{W}(\pi, \psi)$ and $W' \in {}^{\circ}\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \bar{\psi})$ we consider the integral

$$\Psi(s, W, W') = \int_{N_{r-1} \backslash G_{r-1}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g) |\det g|^{s-1/2} dg.$$

The integral converges absolutely if $\operatorname{Re}(s) \gg 0$ and extends to a meromorphic function of s . In any case, it has a meaning as a formal Laurent series in the variable q^{-s} (see below). We recall a result from [Jacquet et al. 1983].

Lemma 3. *There are functions $W_j \in {}^{\circ}\mathcal{W}(\pi; \psi)$ and $W'_j \in {}^{\circ}\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \bar{\psi})$, $1 \leq j \leq k$, such that*

$$\sum_{1 \leq j \leq k} \Psi(s, W_j, W'_j) = \prod_{1 \leq i \leq r-1} L(s + t_i, \pi).$$

Since W_0 is a cyclic vector, after a change of notations, we see that there are functions $W_j \in {}^{\circ}\mathcal{W}(\pi; \psi)$ and integers n_j , $1 \leq j \leq k$, such that

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \dots, x_{r-1}; \bar{\psi})) = \prod_{1 \leq i \leq r-1} L(s + t_i, \pi).$$

In our discussion $|x_1| \leq |x_2| \leq \dots \leq |x_{r-1}|$. However, the functions

$$W(x_1, x_2, \dots, x_{r-1}; \bar{\psi})$$

are symmetric in the variables x_i . Thus, we have the following result:

Lemma 4. *Given an $(r - 1)$ -tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$ there are functions $W_j \in {}^{\circ}\mathcal{W}(\pi; \psi)$ and integers n_j , $1 \leq j \leq k$, such that*

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \dots, x_{r-1}; \bar{\psi})) = \prod_{1 \leq i \leq r-1} P_{\pi}(q^{-s} x_i)^{-1}.$$

2. The ideal I_{π}

We review the construction of [Jacquet et al. 1981a], adding a little more detail to some formal computations. First, we introduce a function

$$W(X_1, X_2, \dots, X_{r-1}; \bar{\psi}) : G_{r-1}(F) \rightarrow S_{r-1}$$

whose value at a point $g \in G_{r-1}(F)$ is denoted $W(g; X_1, X_2, \dots, X_{r-1}; \bar{\psi})$. It is defined by the following property: for every $(r - 1)$ -tuple $(x_1, x_2, \dots, x_{r-1})$ and every g , the scalar $W(g; x_1, x_2, \dots, x_{r-1}; \bar{\psi})$ is the value of the polynomial

$$W(g; X_1, X_2, \dots, X_{r-1}; \bar{\psi})$$

at the point $(x_1, x_2, \dots, x_{r-1})$. For g in a set compact modulo $N_{r-1}(F)$, the polynomials $W(g; X_1, X_2, \dots, X_{r-1}; \bar{\psi})$ remain in a finite dimensional vector subspace of S_{r-1} . We have the relation

$$|\det g|^s W(g; x_1, x_2, \dots, x_{r-1}; \bar{\psi}) = W(g; q^{-s}x_1, q^{-s}x_2, \dots, q^{-s}x_{r-1}; \bar{\psi}).$$

It follows that if $|\det g| = q^{-n}$, then the polynomial

$$W(g; X_1, X_2, \dots, X_{r-1}; \bar{\psi})$$

is homogeneous of degree n , that is,

$$W(g; XX_1, XX_2, \dots, XX_{r-1}; \bar{\psi}) = X^n W(g; X_1, X_2, \dots, X_{r-1}; \bar{\psi}).$$

For each integer n , we now define the integral

$$\begin{aligned} \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi) \\ := \int_{|\det g|=q^{-n}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g, X_1, X_2, \dots, X_{r-1}; \bar{\psi}) |\det g|^{-1/2} dg. \end{aligned}$$

The support of the integrand is contained in a set compact modulo $N_{r-1}(F)$, which depends on W . In addition, there is an integer $N(W)$ (depending on W) such that the support of the integrand is empty if $n < N(W)$. The polynomial

$$\Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi)$$

is homogeneous of degree n . We consider the following formal Laurent series with coefficients in S_{r-1} :

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_n X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

Hence, in fact

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_{n \geq N(W)} X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

If we multiply this Laurent series by $\prod_{1 \leq i \leq r-1} P_\pi(XX_i)$, we obtain a new Laurent series with coefficients in S_{r-1} , namely,

$$\begin{aligned} \Psi(X; W, X_1, X_2, \dots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) \\ = \sum_{n \geq N_1(W)} X^n a_n(X_1, X_2, \dots, X_{r-1}; \psi), \end{aligned}$$

where $N_1(W)$ is another integer (depending on W) and $a_n \in S_{r-1}$. Each a_n is homogeneous of degree n . We can replace π by the contragredient representation $\tilde{\pi}$,

ψ by $\bar{\psi}$ and the function W by the function \tilde{W} defined by

$$\tilde{W}(g) = W(w_r {}^t g^{-1}).$$

The function \tilde{W} belongs to ${}^{\mathcal{W}}(\tilde{\pi}, \bar{\psi})$. We define similarly

$$\Psi(\tilde{W}; X_1, X_2, \dots, X_{r-1}; \bar{\psi}).$$

We have then the following functional equation [Jacquet et al. 1983]:

$$\begin{aligned} \Psi(q^{-1} X^{-1}; \tilde{W}; X_1^{-1}, X_2^{-1}, \dots, X_{r-1}^{-1}; \bar{\psi}) & \prod_{i=1}^{r-1} P_{\tilde{\pi}}(q^{-1} X^{-1} X_i^{-1}) \\ & = c_{\pi} \prod_{i=1}^{r-1} \epsilon_{\pi}(X X_i, \psi) \Psi(X; W, X_1, X_2, \dots, X_{r-1}; \psi) \prod_{i=1}^{r-1} P_{\pi}(X X_i). \end{aligned}$$

The ϵ factors are monomials and $c_{\pi} = \pm 1$. Thus, there is another integer $N_2(W)$ such that in fact

$$\begin{aligned} \Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) & \prod_{1 \leq i \leq r-1} P_{\pi}(X X_i) \\ & = \sum_{N_2(W) \geq n \geq N_1(W)} X^n a_n(X_1, X_2, \dots, X_{r-1}). \end{aligned}$$

From now on we drop the dependence on ψ from the notation.

From the above considerations it follows that the product

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}) \prod_{1 \leq i \leq r-1} P_{\pi}(X X_i)$$

is in fact a polynomial in X with coefficients in S_{r-1} . Moreover, because the a_n are homogeneous of degree n , there is a polynomial $\Xi(W; X_1, X_2, \dots, X_{r-1})$ in S_{r-1} such that

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}) \prod_{1 \leq i \leq r-1} P_{\pi}(X X_i) = \Xi(W; X X_1, X X_2, \dots, X X_{r-1}).$$

In a precise way, let us write

$$\prod_{1 \leq i \leq r-1} P_{\pi}(X_i) = \sum_{m=0}^R P_m(X_1, X_2, \dots, X_{r-1}),$$

where each P_m is homogeneous of degree m . Then

$$\begin{aligned} \Psi(X; W; X_1, X_2, \dots, X_{r-1}) & \prod_{1 \leq i \leq r-1} P_{\pi}(X X_i) \\ & = \sum_n X^n \sum_{m=0}^R \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1}) P_m(X_1, X_2, \dots, X_{r-1}). \end{aligned}$$

The polynomial $\Xi(W; X_1, X_2, \dots, X_{r-1})$ is then determined by the condition that its homogeneous component of degree n noted $\Xi_n(W; X_1, X_2, \dots, X_{r-1})$ be given by

$$\begin{aligned} \Xi_n(W; X_1, X_2, \dots, X_{r-1}) &= \sum_{m=0}^R \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1}) P_m(X_1, X_2, \dots, X_{r-1}). \end{aligned}$$

The theorem amounts to saying there is a W such that $\Xi(W; X_1, X_2, \dots, X_{r-1})$ equals 1.

Let I_π be the subvector space of S_{r-1} spanned by the polynomials

$$\Xi(W; X_1, X_2, \dots, X_{r-1}).$$

Lemma 5. *In fact I_π is an ideal of the algebra S_{r-1} .*

Proof. Let Q be an element of S_{r-1} . Let ϕ be the corresponding element of H_{r-1} . Then

$$\begin{aligned} \int W(gh; X_1, X_2, \dots, X_{r-1}) \phi(h) dh &= W(g; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}). \end{aligned}$$

Let W be an element of ${}^{\mathfrak{W}}(\pi, \psi)$. Define another element W_1 of ${}^{\mathfrak{W}}(\pi, \psi)$ by

$$W_1(g) = \int_{G_{r-1}} W \left[g \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] \phi(h) |\det h|^{1/2} dh.$$

We claim that

$$\Xi(W_1; X_1, X_2, \dots, X_{r-1}) = \Xi(W; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}).$$

This will imply the Lemma.

By linearity, it suffices to prove our claim when Q is homogeneous of degree t . Then ϕ is supported on the set of h such that $|\det h| = q^{-t}$. We have then, for every n ,

$$\begin{aligned} \Psi_n(W_1; X_1, \dots, X_{r-1}) &= \int_{|\det g|=q^{-n}} W_1 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1}) |\det g|^{-1/2} dg \\ &= \int_{|\det g|=q^{-n}} \int W \begin{pmatrix} gh^{-1} & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1}) \phi(h) |\det h|^{1/2} dh \\ &\quad \times |\det g|^{-1/2} dg \\ &= \int_{|\det g|=q^{-n+t}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \int W(gh; X_1, \dots, X_{r-1}) \phi(h) dh |\det g|^{-1/2} dg \end{aligned}$$

$$\begin{aligned}
 &= \int_{|\det g|=q^{-n+t}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1}) |\det g|^{-1/2} dg Q(X_1, \dots, X_{r-1}) \\
 &= \Psi_{n-t}(W; X_1, \dots, X_{r-1}) Q(X_1, \dots, X_{r-1}).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \Xi_n(W_1; X_1, \dots, X_{r-1}) &= \sum_{m=0}^R \Psi_{n-m}(W_1; X_1, \dots, X_{r-1}) P_m(X_1, \dots, X_{r-1}) \\
 &= \sum_{m=0}^R \Psi_{n-m-t}(W; X_1, \dots, X_{r-1}) P_m(X_1, \dots, X_{r-1}) Q(X_1, \dots, X_{r-1}) \\
 &= \Xi_{n-t}(W; X_1, \dots, X_{r-1}) Q(X_1, \dots, X_{r-1}).
 \end{aligned}$$

Since Q is homogeneous of degree t our assertion follows. □

3. Proof of the theorem

Proof. Given an $(r - 1)$ -tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$, Lemma 4 shows that we can find W_j and integers n_j such that, for all s ,

$$\sum_{1 \leq j \leq k} (q^{-s})^{n_j} \Xi(W_j, q^{-s}x_1, q^{-s}x_2, \dots, q^{-s}x_{r-1}) = 1.$$

In particular,

$$\sum_{1 \leq j \leq k} \Xi(W_j, x_1, x_2, \dots, x_{r-1}) = 1.$$

Thus, the element

$$\sum_{1 \leq j \leq k} \Xi(W_j; X_1, X_2, \dots, X_{r-1})$$

of I_π does not vanish at $(x_1, x_2, \dots, x_{r-1})$. By the theorem of zeros of Hilbert we have then $I_\pi = S_{r-1}$. In particular, there is W such that

$$\Xi(W; X_1, X_2, \dots, X_{r-1}) = 1.$$

This implies the theorem. □

Remark 1. The proof in [Jacquet et al. 1981a] is correct if $L(s, \pi)$ is identically 1. In general, the proof there only shows that the polynomials in I_π cannot all vanish on a coordinate hyperplane $X_i = x$.

Remark 2. Consider an induced representation π of the form

$$\pi = I(\sigma_1 \otimes \alpha^{s_1}, \sigma_2 \otimes \alpha^{s_2}, \dots, \sigma_k \otimes \alpha^{s_k}),$$

where the representations $\sigma_1, \sigma_2, \dots, \sigma_k$ are tempered and s_1, s_2, \dots, s_k are real numbers such that

$$s_1 > s_2 > \dots > s_k.$$

The representation π may fail to be irreducible. But, in any case, it has a Whittaker model [Jacquet and Shalika 1983], and Theorem 1 is valid for the Whittaker model of π .

Remark 3. The proof of Matringe uses the theory of derivatives of a representation. The present proof appears simple only because we use Lemma 3, the proof of which is quite elaborate (and can be obtained from the theory of derivatives as in [Cogdell and Piatetski-Shapiro 2011]).

References

- [Cogdell and Piatetski-Shapiro 2011] J. Cogdell and I. Piatetski-Shapiro, “Derivatives and L-functions for $GL(n)$ ”, preprint, 2011.
- [Gindikin 1961] S. G. Gindikin, “Integral formulas for Siegel domains of second kind”, *Dokl. Akad. Nauk SSSR* **141** (1961), 531–534. In Russian; translated in *Sov. Math., Dokl.* **2** (1961), 1480–1483. MR 24 #A2053
- [Gindikin and Karpelevič 1966] S. G. Gindikin and F. I. Karpelevič, “An integral associated with Riemannian symmetric spaces of non-positive curvature”, *Izv. Akad. Nauk SSSR Ser. Mat.* **30** (1966), 1147–1156. In Russian; translated in *Translations* **2**, Am. Math. Soc., Providence, 1969. MR 36 #5271 Zbl 0165.55705
- [Godement and Jacquet 1972] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Springer, Berlin, 1972. MR 49 #7241 Zbl 0244.12011
- [Jacquet and Shalika 1983] H. Jacquet and J. Shalika, “The Whittaker models of induced representations”, *Pacific J. Math.* **109**:1 (1983), 107–120. MR 85h:22023 Zbl 0535.22017
- [Jacquet and Shalika 1985] H. Jacquet and J. Shalika, “A lemma on highly ramified ϵ -factors”, *Math. Ann.* **271**:3 (1985), 319–332. MR 87i:22048 Zbl 0541.12010
- [Jacquet et al. 1981a] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, “Conducteur des représentations du groupe linéaire”, *Math. Ann.* **256**:2 (1981), 199–214. MR 83c:22025 Zbl 0443.22013
- [Jacquet et al. 1981b] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, “Conducteur des représentations génériques du groupe linéaire”, *C. R. Acad. Sci. Paris Sér. I Math.* **292**:13 (1981), 611–616. MR 82f:22015 Zbl 0475.22012
- [Jacquet et al. 1983] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, “Rankin–Selberg convolutions”, *Amer. J. Math.* **105**:2 (1983), 367–464. MR 85g:11044 Zbl 0525.22018
- [Matringe 2011] N. Matringe, “Essential Whittaker functions for $GL(n)$ over a p -adic field”, preprint, 2011.
- [Silberger 1978] A. J. Silberger, “The Langlands quotient theorem for p -adic groups”, *Math. Ann.* **236**:2 (1978), 95–104. MR 58 #22413 Zbl 0362.20029

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Special issue devoted to the memory of Jonathan Rogawski

In memoriam: Jonathan Rogawski	257
DON BLASIUS, DINAKAR RAMAKRISHNAN and V. S. VARADARAJAN	
p -adic Rankin L -series and rational points on CM elliptic curves	261
MASSIMO BERTOLINI, HENRI DARMON and KARTIK PRASANNA	
The syntomic regulator for K_4 of curves	305
AMNON BESSER and ROB DE JEU	
Unique functionals and representations of Hecke algebras	381
BENJAMIN BRUBAKER, DANIEL BUMP and SOLOMON FRIEDBERG	
A relative trace formula for $\mathrm{PGL}(2)$ in the local setting	395
BROOKE FEIGON	
On the degrees of matrix coefficients of intertwining operators	433
TOBIAS FINIS, EREZ LAPID and WERNER MÜLLER	
Comparison of compact induction with parabolic induction	457
GUY HENNIART and MARIE-FRANCE VIGNERAS	
The functional equation and beyond endoscopy	497
P. EDWARD HERMAN	
A correction to <i>Conducteur des Représentations du groupe linéaire</i>	515
HERVÉ JACQUET	
Modular L -values of cubic level	527
ANDREW KNIGHTLY and CHARLES LI	
On occult period maps	565
STEPHEN KUDLA and MICHAEL RAPOPORT	
A prologue to “Functoriality and reciprocity”, part I	583
ROBERT LANGLANDS	
Truncation of Eisenstein series	665
EREZ LAPID and KEITH OUELLETTE	
Some comments on Weyl’s complete reducibility theorem	687
JONATHAN ROGAWSKI and V. S. VARADARAJAN	
On equality of arithmetic and analytic factors through local Langlands correspondence	695
FREYDOON SHAHIDI	



0030-8730(201212)260:2;1-A