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A CORRECTION TO *CONDUCTEUR DES REPRÉSENTATIONS DU GROUPE LINÉAIRE*

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We give a correct proof for the existence of the essential vector of an irreducible admissible generic representation of the general linear group over a $\mathfrak{p}$-adic field.

Nadir Matringe has indicated to me that the paper "Conducteur des représentations du groupe linéaire" [Jacquet et al. 1981a; 1981b] contains an error. Since the result therein has applications (see [Jacquet and Shalika 1985] for instance), it may be useful to correct the error. In any case, the correct proof is actually simpler than the erroneous proof. Separately, Matringe [2011] has given a different proof, which is of independent interest.

First, I recall the result in question. Let F be a non-Archimedean local field. We denote by α or $|\cdot|$ the absolute value of F, by q the cardinality of the residual field and finally by v the valuation function on F. Thus, $\alpha(x) = |x| = q^{-v(x)}$. Let ψ be an additive character of F whose conductor is the ring of integers $\mathbb{O}_F$. Let G_r be the group $GL(r)$ regarded as an algebraic group. We denote by w_r the permutation matrix whose antidiagonal entries are 1. For instance,

$$w_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by dg the Haar measure of $G_r(F)$ for which the compact group $G_r(\mathbb{O}_F)$ has volume 1. Let N_r be the subgroup of upper triangular matrices with unit diagonal and A_r the group of diagonal matrices. We define a character

$$\theta_{r,\psi} : N_r(F) \to \mathbb{C}^\times$$

by the formula

$$\theta_{r,\psi}(u) = \psi\Big(\sum_{1 \le i \le r-1} u_{i,i+1}\Big).$$

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We denote by du the Haar measure on $N_r(F)$ for which $N_r(\mathbb{O}_F)$ has measure 1. We have then an invariant quotient measure on $N_r(F)\backslash G_r(F)$.

Let S_r be the algebra of symmetric polynomials in

$$(X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_r, X_r^{-1}).$$

Let H_r be the Hecke algebra of $G_r(F)$, that is, the convolution algebra of compactly supported, complex-valued functions that are bi-invariant under the maximal compact group $G_r(\mathbb{O}_F)$. Let $\mathcal{S}_r : H_r \to S_r$ be the Satake isomorphism. Thus, for any r-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_r)$ we have a homomorphism of algebras $\mathcal{S}_r(x_1, x_2, \ldots, x_r) : H_r \to \mathbb{C}$, defined by

$$\mathcal{S}_r(x_1, x_2, \ldots, x_r) : \phi \mapsto \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r)\,.$$

Concretely, it is defined in the following way. Let $t = (t_1, t_2, \ldots, t_r)$ be a tuple of complex numbers such that $x_i = q^{-t_i}$ for each i. We denote by $\pi(t_1, t_2, \ldots, t_r)$ the corresponding principal series representation of $G_{r-1}(F)$. It is the representation induced by the character

$$a = (a_1, a_2, \ldots, a_r) \mapsto |a_1|^{t_1} |a_2|^{t_2} \cdots |a_r|^{t_r}$$

of $A_r(F)$. Its space $I(t_1, t_2, \ldots, t_r)$ is the space of smooth functions $\phi : G_r(F) \to \mathbb{C}$ such that

$$\phi\left[\begin{pmatrix} a_1 & * & \cdots & \cdots & * \\ 0 & a_2 & \cdots & \cdots & * \\ 0 & 0 & \cdots & \cdots & a_r \end{pmatrix} g\right] = |a_1|^{t_1 + \frac{r-1}{2}} |a_2|^{t_2 + \frac{r-1}{2} - 1} \cdots |a_r|^{t_r - \frac{r-1}{2}} \phi(g)\,.$$

The space $I(t_1, t_2, \ldots, t_r)$ contains a unique vector ϕ_0 equal to 1 on $G_r(\mathbb{O}_F)$ and thus invariant under $G_r(\mathbb{O}_F)$. Under convolution, it is an eigenfunction of H_r with eigenvalue $\mathcal{S}_r(x_1, x_2, \ldots, x_r)$, that is,

$$\int_{G_r(F)} \phi_0(gh)\, \phi(h)\, dh = \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r)\, \phi_0(g)$$

for every ϕ in H_r.

There is a unique function $W : G_r(F) \to \mathbb{C}$ satisfying the following properties:

- $W(gk) = W(g)$ for $k \in G_r(\mathbb{O}_F)$,
- $W(ug) = \theta_\psi(u)\, W(g)$ for $u \in N_r(F)$,
- for all $(x_1, x_2, \ldots, x_r)$ and all $\phi \in H_r$,

$$\int_{G_r(F)} W(gh)\, \phi(h)\, dh = \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r)\, W(g),$$

- $W(e) = 1$.

Thus, W is an eigenfunction of H_r with eigenvalue $S_r(x_1, x_2, \dots, x_r)$. We will denote this function by $W(x_1, x_2, \dots, x_r; \psi)$ and its value at g by

$$W(g; x_1, x_2, \dots x_r; \psi).$$

Let (π, V) be an irreducible admissible representation of $G_r(F)$. We assume that π is *generic*, that is, there is a nonzero linear form $\lambda : V \to \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \theta_{r,\psi}(u)\, \lambda(v)$$

for all $u \in N_r(F)$ and all $v \in V$. Recall that such a form is unique within a scalar factor. We denote by $\mathcal{W}(\pi; \psi)$ the space of functions of the form

$$g \mapsto \lambda(\pi(g)v)$$

with $v \in V$. It is the *Whittaker model* of π. On the other hand, we have the L-factor $L(s, \pi)$ [Godement and Jacquet 1972]. We denote by $P_\pi(X)$ the polynomial defined by $L(s, \pi) = P_\pi(q^{-s})^{-1}$. The main result of [Jacquet et al. 1981a] is the following theorem:

Theorem 1. *There is an element $W \in \mathcal{W}(\pi; \psi)$ such that, for any $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$,*

$$\int_{N_{r-1}(F)\backslash G_{r-1}(F)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; x_1, x_2, \dots, x_{r-1}; \bar{\psi}) |\det g|^{s-1/2}\, dg = \prod_{1 \le i \le r-1} P_\pi(q^{-s} x_i)^{-1}.$$

In [Jacquet et al. 1981a] it is shown that if we impose the extra condition

$$W\begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix} = W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

for all $h \in G_{r-1}(\mathcal{O}_F)$ and $g \in G_{r-1}(F)$, then W is unique. The vector W is then called the *essential vector* of π, and further properties of this vector are obtained in [Jacquet et al. 1981a].

The proof of this theorem is incorrect in that paper. We give a correct proof here.

1. Review of the properties of the L-factor

Let $r \ge 2$ be an integer. Let $t = (t_1, t_2, \dots, t_{r-1})$ be an $(r-1)$-tuple of complex numbers. We assume that

$$\mathrm{Re}(t_1) \ge \mathrm{Re}(t_2) \ge \cdots \ge \mathrm{Re}(t_{r-1}).$$

Again, we consider the representation $\pi(t_1, t_2, \dots, t_{r-1})$ that acts on the space $I(t_1, t_2, \dots, t_{r-1})$. As before, let ϕ_0 be the unique vector of that space that is equal to 1 on $G_{r-1}(\mathbb{O}_F)$. Recall it is invariant under $G_{r-1}(\mathbb{O}_F)$. We recall a standard result.

Lemma 1. *For each tuple t satisfying the above inequalities the vector ϕ_0 is a cyclic vector for the representation $\pi(t_1, t_2, \dots, t_{r-1})$.*

Proof. Indeed, if $\mathrm{Re}(t_1) = \mathrm{Re}(t_2) = \cdots = \mathrm{Re}(t_{r-1})$, the representation is irreducible and our assertion is trivial. If not, we use Langlands' construction [Silberger 1978]. For each root α of A_{r-1} we denote by N_α the corresponding subgroup of N_{r-1} or $\overline{N}_{r-1}$ and by $\check{\alpha}$ the corresponding co-root. Thus, if α is a positive root, we have

$$\alpha(a_1, a_2, \dots, a_{r-1}) = a_i / a_j$$

with $i < j$ and

$$\langle t, \check{\alpha} \rangle = t_i - t_j \,.$$

Let $P(t)$ be the set of positive roots α such that $\mathrm{Re}\langle t, \check{\alpha} \rangle > 0$. Let U be the unipotent group generated by the subgroups $N_{-\alpha}$ with $\alpha \in P(t)$. The intertwining operator

$$N\phi(g) = \int_{U(F)} \phi(ug)\, du$$

is defined by a convergent integral, and its kernel is a maximal invariant subspace. The formula of [Gindikin and Karpelevič 1966; Gindikin 1961] gives

$$N\phi_0(e) = \prod_{\alpha \in P(t)} \frac{1 - q^{-\langle t, \check{\alpha} \rangle - 1}}{1 - q^{-\langle t, \check{\alpha} \rangle}} \,.$$

Thus, $N\phi_0 \neq 0$, and our assertion follows. □

The representation $I(t_1, t_2, \dots, t_{r-1})$ admits a nonzero linear form λ such that, for $u \in N_{r-1}(F)$ and ϕ in the space of the representation,

$$\lambda(\pi(u)\phi) = \theta_{r-1, \overline{\psi}}(u)\, \lambda(\phi).$$

We denote by $\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$ the space spanned by the functions of the form

$$g \mapsto W_\phi(g), \quad W_\phi(g) = \lambda(\pi(t_1, t_2, \dots, t_{r-1})(g)\phi)$$

with $\phi \in I(t_1, t_2, \dots, t_{r-1})$. We recall the following result:

Lemma 2 [Jacquet and Shalika 1983]. *The map $\phi \mapsto W_\phi$ is injective.*

It follows that the image W_0 of ϕ_0 is a cyclic vector in $\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$. Up to a multiplicative constant, the function W_0 is equal to the function

$$W_0 = W(x_1, x_2, \dots, x_{r-1}; \overline{\psi}).$$

Now let π be an irreducible generic representation of $G_r(F)$. For $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$ we consider the integral

$$\Psi(s, W, W') = \int_{N_{r-1}\backslash G_{r-1}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g)|\det g|^{s-1/2}\, dg.$$

The integral converges absolutely if $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s. In any case, it has a meaning as a formal Laurent series in the variable q^{-s} (see below). We recall a result from [Jacquet et al. 1983].

Lemma 3. *There are functions $W_j \in \mathcal{W}(\pi; \psi)$ and $W'_j \in \mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$, $1 \le j \le k$, such that*

$$\sum_{1\le j\le k} \Psi(s, W_j, W'_j) = \prod_{1\le i\le r-1} L(s + t_i, \pi).$$

Since W_0 is a cyclic vector, after a change of notations, we see that there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that

$$\sum_j q^{-n_i s}\Psi(s, W_j, W(x_1, x_2, \dots, x_{r-1}; \overline{\psi})) = \prod_{1\le i\le r-1} L(s + t_i, \pi).$$

In our discussion $|x_1| \le |x_2| \le \cdots \le |x_{r-1}|$. However, the functions

$$W(x_1, x_2, \dots, x_{r-1}; \overline{\psi})$$

are symmetric in the variables x_i. Thus, we have the following result:

Lemma 4. *Given an $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$ there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that*

$$\sum_j q^{-n_j s}\Psi(s, W_j, W(x_1, x_2, \dots, x_{r-1}; \overline{\psi})) = \prod_{1\le i\le r-1} P_\pi(q^{-s}x_i)^{-1}.$$

2. The ideal I_π

We review the construction of [Jacquet et al. 1981a], adding a little more detail to some formal computations. First, we introduce a function

$$W(X_1, X_2, \dots, X_{r-1}; \overline{\psi}) : G_{r-1}(F) \to S_{r-1}$$

whose value at a point $g \in G_{r-1}(F)$ is denoted $W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$. It is defined by the following property: for every $(r-1)$-tuple $(x_1, x_2, \dots, x_{r-1})$ and every g, the scalar $W(g; x_1, x_2, \dots, x_{r-1}; \overline{\psi})$ is the value of the polynomial

$$W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$$

at the point $(x_1, x_2, \dots, x_{r-1})$. For g in a set compact modulo $N_{r-1}(F)$, the polynomials $W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$ remain in a finite dimensional vector subspace of S_{r-1}. We have the relation

$$|\det g|^s W(g; x_1, x_2, \dots, x_{r-1}; \overline{\psi}) = W(g; q^{-s}x_1, q^{-s}x_2, \dots, q^{-s}x_{r-1}; \overline{\psi}) .$$

It follows that if $|\det g| = q^{-n}$, then the polynomial

$$W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$$

is homogeneous of degree n, that is,

$$W(g; XX_1, XX_2, \dots, XX_{r-1}; \overline{\psi}) = X^n W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi}).$$

For each integer n, we now define the integral

$$\Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi) := \int_{|\det g| = q^{-n}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g, X_1, X_2, \dots, X_{r-1}; \overline{\psi}) |\det g|^{-1/2}\, dg.$$

The support of the integrand is contained in a set compact modulo $N_{r-1}(F)$, which depends on W. In addition, there is an integer $N(W)$ (depending on W) such that the support of the integrand is empty if $n < N(W)$. The polynomial

$$\Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi)$$

is homogeneous of degree n. We consider the following formal Laurent series with coefficients in S_{r-1}:

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_n X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

Hence, in fact

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_{n \geq N(W)} X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

If we multiply this Laurent series by $\prod_{1 \leq i \leq r-1} P_\pi(XX_i)$, we obtain a new Laurent series with coefficients in S_{r-1}, namely,

$$\Psi(X; W, X_1, X_2, \dots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = \sum_{n \geq N_1(W)} X^n a_n(X_1, X_2, \dots, X_{r-1}; \psi),$$

where $N_1(W)$ is another integer (depending on W) and $a_n \in S_{r-1}$. Each a_n is homogeneous of degree n. We can replace π by the contragredient representation $\widetilde{\pi}$,

ψ by $\bar{\psi}$ and the function W by the function $\widetilde{W}$ defined by

$$\widetilde{W}(g) = W(w_r {}^t g^{-1}).$$

The function $\widetilde{W}$ belongs to $\mathscr{W}(\tilde{\pi}, \bar{\psi})$. We define similarly

$$\Psi(\widetilde{W}; X_1, X_2, \dots, X_{r-1}; \bar{\psi}).$$

We have then the following functional equation [Jacquet et al. 1983]:

$$\Psi(q^{-1}X^{-1}; \widetilde{W}; X_1^{-1}, X_2^{-1}, \dots, X_{r-1}^{-1}; \bar{\psi}) \prod_{i=1}^{r-1} P_{\tilde{\pi}}(q^{-1}X^{-1}X_i^{-1}) \\ = c_\pi \prod_{i=1}^{r-1} \epsilon_\pi(XX_i, \psi)\, \Psi(X; W, X_1, X_2, \dots, X_{r-1}; \psi) \prod_{i=1}^{r-1} P_\pi(XX_i).$$

The ϵ factors are monomials and $c_\pi = \pm 1$. Thus, there is another integer $N_2(W)$ such that in fact

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) \prod_{1 \le i \le r-1} P_\pi(XX_i) \\ = \sum_{N_2(W) \ge n \ge N_1(W)} X^n a_n(X_1, X_2, \dots, X_{r-1}).$$

From now on we drop the dependence on ψ from the notation.

From the above considerations it follows that the product

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i)$$

is in fact a polynomial in X with coefficients in S_{r-1}. Moreover, because the a_n are homogeneous of degree n, there is a polynomial $\Xi(W; X_1, X_2, \dots, X_{r-1})$ in S_{r-1} such that

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i) = \Xi(W; XX_1, XX_2, \dots, XX_{r-1}).$$

In a precise way, let us write

$$\prod_{1 \le i \le r-1} P_\pi(X_i) = \sum_{m=0}^{R} P_m(X_1, X_2, \dots, X_{r-1}),$$

where each P_m is homogeneous of degree m. Then

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i) \\ = \sum_n X^n \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1})\, P_m(X_1, X_2, \dots, X_{r-1}).$$

The polynomial $\Xi(W; X_1, X_2, \dots, X_{r-1})$ is then determined by the condition that its homogeneous component of degree n noted $\Xi_n(W; X_1, X_2, \dots, X_{r-1})$ be given by

$$\Xi_n(W; X_1, X_2, \dots, X_{r-1}) = \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1}) P_m(X_1, X_2, \dots, X_{r-1}).$$

The theorem amounts to saying there is a W such that $\Xi(W; X_1, X_2, \dots, X_{r-1})$ equals 1.

Let I_π be the subvector space of S_{r-1} spanned by the polynomials

$$\Xi(W; X_1, X_2, \dots, X_{r-1}).$$

Lemma 5. *In fact I_π is an ideal of the algebra S_{r-1}.*

Proof. Let Q be an element of S_{r-1}. Let ϕ be the corresponding element of H_{r-1}. Then

$$\int W(gh; X_1, X_2, \dots, X_{r-1})\phi(h)\, dh = W(g; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}).$$

Let W be an element of $\mathcal{W}(\pi, \psi)$. Define another element W_1 of $\mathcal{W}(\pi, \psi)$ by

$$W_1(g) = \int_{G_{r-1}} W\left[g \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right] \phi(h) |\det h|^{1/2}\, dh.$$

We claim that

$$\Xi(W_1; X_1, X_2, \dots, X_{r-1}) = \Xi(W; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}).$$

This will imply the Lemma.

By linearity, it suffices to prove our claim when Q is homogeneous of degree t. Then ϕ is supported on the set of h such that $|\det h| = q^{-t}$. We have then, for every n,

$$\begin{aligned}
\Psi_n(W_1; X_1, \dots, X_{r-1}) &= \int_{|\det g| = q^{-n}} W_1\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1}) |\det g|^{-1/2}\, dg \\
&= \int_{|\det g| = q^{-n}} \int W\begin{pmatrix} gh^{-1} & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1})\, \phi(h) |\det h|^{1/2}\, dh \\
&\qquad \times |\det g|^{-1/2}\, dg \\
&= \int_{|\det g| = q^{-n+t}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \int W(gh; X_1, \dots, X_{r-1})\, \phi(h)\, dh\, |\det g|^{-1/2}\, dg
\end{aligned}$$

$$= \int_{|\det g|=q^{-n+t}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \ldots, X_{r-1}) |\det g|^{-1/2} \, dg \, Q(X_1, \ldots, X_{r-1})$$

$$= \Psi_{n-t}(W; X_1, \ldots, X_{r-1}) \, Q(X_1, \ldots, X_{r-1}).$$

Hence,

$$\begin{aligned}
&\Xi_n(W_1; X_1, \ldots, X_{r-1}) \\
&\qquad = \sum_{m=0}^{R} \Psi_{n-m}(W_1; X_1, \ldots, X_{r-1}) \, P_m(X_1, \ldots, X_{r-1}) \\
&\qquad = \sum_{m=0}^{R} \Psi_{n-m-t}(W; X_1, \ldots, X_{r-1}) \, P_m(X_1, \ldots, X_{r-1}) \, Q(X_1, \ldots, X_{r-1}) \\
&\qquad = \Xi_{n-t}(W; X_1, \ldots, X_{r-1}) \, Q(X_1, \ldots, X_{r-1}).
\end{aligned}$$

Since Q is homogeneous of degree t our assertion follows. □

3. Proof of the theorem

Proof. Given an $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_{r-1})$, Lemma 4 shows that we can find W_j and integers n_j such that, for all s,

$$\sum_{1 \le j \le k} (q^{-s})^{n_j} \, \Xi(W_j, q^{-s} x_1, q^{-s} x_2, \ldots, q^{-s} x_{r-1}) = 1.$$

In particular,

$$\sum_{1 \le j \le k} \Xi(W_j, x_1, x_2, \ldots, x_{r-1}) = 1.$$

Thus, the element

$$\sum_{1 \le j \le k} \Xi(W_j; X_1, X_2, \ldots, X_{r-1})$$

of I_π does not vanish at $(x_1, x_2, \ldots, x_{r-1})$. By the theorem of zeros of Hilbert we have then $I_\pi = S_{r-1}$. In particular, there is W such that

$$\Xi(W; X_1, X_2, \ldots, X_{r-1}) = 1\,.$$

This implies the theorem. □

Remark 1. The proof in [Jacquet et al. 1981a] is correct if $L(s, \pi)$ is identically 1. In general, the proof there only shows that the polynomials in I_π cannot all vanish on a coordinate hyperplane $X_i = x$.

Remark 2. Consider an induced representation π of the form

$$\pi = I(\sigma_1 \otimes \alpha^{s_1}, \sigma_2 \otimes \alpha^{s_2}, \dots, \sigma_k \otimes \alpha^{s_k}),$$

where the representations $\sigma_1, \sigma_2, \dots, \sigma_k$ are tempered and $s_1, s_2, \dots, s_k$ are real numbers such that

$$s_1 > s_2 > \cdots > s_k .$$

The representation π may fail to be irreducible. But, in any case, it has a Whittaker model [Jacquet and Shalika 1983], and Theorem 1 is valid for the Whittaker model of π.

Remark 3. The proof of Matringe uses the theory of derivatives of a representation. The present proof appears simple only because we use Lemma 3, the proof of which is quite elaborate (and can be obtained from the theory of derivatives as in [Cogdell and Piatetski-Shapiro 2011]).

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