A CORRECTION TO *CONDUCTEUR DES REPRÉSENTATIONS DU GROUPE LINÉAIRE*

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We give a correct proof for the existence of the essential vector of an irreducible admissible generic representation of the general linear group over a $p$-adic field.

Nadir Matringe has indicated to me that the paper “Conducteur des représentations du groupe linéaire” [Jacquet et al. 1981a; 1981b] contains an error. Since the result therein has applications (see [Jacquet and Shalika 1985] for instance), it may be useful to correct the error. In any case, the correct proof is actually simpler than the erroneous proof. Separately, Matringe [2011] has given a different proof, which is of independent interest.

First, I recall the result in question. Let $F$ be a non-Archimedean local field. We denote by $\alpha$ or $|\cdot|$ the absolute value of $F$, by $q$ the cardinality of the residual field and finally by $v$ the valuation function on $F$. Thus, $\alpha(x) = |x| = q^{-v(x)}$. Let $\psi$ be an additive character of $F$ whose conductor is the ring of integers $\mathcal{O}_F$. Let $G_r$ be the group $GL(r)$ regarded as an algebraic group. We denote by $w_r$ the permutation matrix whose antidiagonal entries are 1. For instance,

$$w_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by $dg$ the Haar measure of $G_r(F)$ for which the compact group $G_r(\mathcal{O}_F)$ has volume 1. Let $N_r$ be the subgroup of upper triangular matrices with unit diagonal and $A_r$ the group of diagonal matrices. We define a character $\theta_{r,\psi} : N_r(F) \to \mathbb{C}^\times$ by the formula

$$\theta_{r,\psi}(u) = \psi\left( \sum_{1 \leq i \leq r-1} u_{i,i+1} \right).$$

MSC2010: 11F70, 22E50.

Keywords: conductor, essential vector.
We denote by $du$ the Haar measure on $N_r(F)$ for which $N_r(\mathbb{C}_F)$ has measure 1. We have then an invariant quotient measure on $N_r(F)\backslash G_r(F)$.

Let $S_r$ be the algebra of symmetric polynomials in

$$(X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_r, X_r^{-1}).$$

Let $H_r$ be the Hecke algebra of $G_r(F)$, that is, the convolution algebra of compactly supported, complex-valued functions that are bi-invariant under the maximal compact group $G_r(\mathbb{C}_F)$. Let $\mathcal{S}_r : H_r \to S_r$ be the Satake isomorphism. Thus, for any $r$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_r)$ we have a homomorphism of algebras $\mathcal{S}_r(x_1, x_2, \ldots, x_r) : H_r \to \mathbb{C}$, defined by

$$\mathcal{S}_r(x_1, x_2, \ldots, x_r) : \phi \mapsto \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r).$$

Concretely, it is defined in the following way. Let $t = (t_1, t_2, \ldots, t_r)$ be a tuple of complex numbers such that $x_i = q^{-t_i}$ for each $i$. We denote by $\pi(t_1, t_2, \ldots, t_r)$ the corresponding principal series representation of $G_{r-1}(F)$. It is the representation induced by the character

$$a = (a_1, a_2, \ldots, a_r) \mapsto |a_1|^{t_1} |a_2|^{t_2} \cdots |a_r|^{t_r}$$

of $A_r(F)$. Its space $I(t_1, t_2, \ldots, t_r)$ is the space of smooth functions $\phi : G_r(F) \to \mathbb{C}$ such that

$$\left[ \begin{array}{ccc}
\begin{array}{cccc}
a_1 & * & \cdots & * \\
0 & a_2 & \cdots & * \\
0 & 0 & \cdots & a_r
\end{array}
\end{array} \right] g = |a_1|^{t_1+\frac{r-1}{2}} |a_2|^{t_2+\frac{r-1}{2}} \cdots |a_r|^{t_r-\frac{r-1}{2}} \phi(g).$$

The space $I(t_1, t_2, \ldots, t_r)$ contains a unique vector $\phi_0$ equal to 1 on $G_r(\mathbb{C}_F)$ and thus invariant under $G_r(\mathbb{C}_F)$. Under convolution, it is an eigenfunction of $H_r$ with eigenvalue $\mathcal{S}_r(x_1, x_2, \ldots, x_r)$, that is,

$$\int_{G_r(F)} \phi_0(gh) \phi(h) \, dh = \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r) \phi_0(g)$$

for every $\phi$ in $H_r$.

There is a unique function $W : G_r(F) \to \mathbb{C}$ satisfying the following properties:

- $W(gk) = W(g)$ for $k \in G_r(\mathbb{C}_F)$,
- $W(ug) = \theta_\psi(u) W(g)$ for $u \in N_r(F)$,
- for all $(x_1, x_2, \ldots, x_r)$ and all $\phi \in H_r$,

$$\int_{G_r(F)} W(gh) \phi(h) \, dh = \mathcal{S}_r(\phi)(x_1, x_2, \ldots, x_r) W(g),$$

- $W(e) = 1$. 

Thus, $W$ is an eigenfunction of $H_r$ with eigenvalue $S_r(x_1, x_2, \ldots, x_r)$. We will denote this function by $W(x_1, x_2, \ldots, x_r; \psi)$ and its value at $g$ by

$$W(g; x_1, x_2, \ldots, x_r; \psi).$$

Let $(\pi, V)$ be an irreducible admissible representation of $G_r(F)$. We assume that $\pi$ is generic, that is, there is a nonzero linear form $\lambda : V \to \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \theta_{r, \psi}(u) \lambda(v)$$

for all $u \in N_r(F)$ and all $v \in V$. Recall that such a form is unique within a scalar factor. We denote by $\mathcal{W}(\pi; \psi)$ the space of functions of the form

$$g \mapsto \lambda(\pi(g)v)$$

with $v \in V$. It is the Whittaker model of $\pi$. On the other hand, we have the $L$-factor $L(s, \pi)$ [Godement and Jacquet 1972]. We denote by $P_\pi(X)$ the polynomial defined by $L(s, \pi) = P_\pi(q^{-s})^{-1}$. The main result of [Jacquet et al. 1981a] is the following theorem:

**Theorem 1.** There is an element $W \in \mathcal{W}(\pi; \psi)$ such that, for any $(r - 1)$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_{r-1})$,

$$\int_{N_{r-1}(F) \setminus G_{r-1}(F)} W \left( \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right) W(g; x_1, x_2, \ldots, x_{r-1}; \psi) |\det g|^{s-1/2} \, dg$$

$$= \prod_{1 \leq i \leq r-1} P_\pi(q^{-s}x_i)^{-1}. $$

In [Jacquet et al. 1981a] it is shown that if we impose the extra condition

$$W \left( \begin{array}{cc} gh & 0 \\ 0 & 1 \end{array} \right) = W \left( \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right)$$

for all $h \in G_{r-1}(\mathbb{C}_F)$ and $g \in G_{r-1}(F)$, then $W$ is unique. The vector $W$ is then called the essential vector of $\pi$, and further properties of this vector are obtained in [Jacquet et al. 1981a].

The proof of this theorem is incorrect in that paper. We give a correct proof here.

**1. Review of the properties of the $L$-factor**

Let $r \geq 2$ be an integer. Let $t = (t_1, t_2, \ldots, t_{r-1})$ be an $(r - 1)$-tuple of complex numbers. We assume that

$$\text{Re}(t_1) \geq \text{Re}(t_2) \geq \cdots \geq \text{Re}(t_{r-1}).$$

Again, we consider the representation $\pi(t_1, t_2, \ldots, t_{r-1})$ that acts on the space $I(t_1, t_2, \ldots, t_{r-1})$. As before, let $\phi_0$ be the unique vector of that space that is
equal to 1 on $G_{r-1}(\mathbb{C}_F)$. Recall it is invariant under $G_{r-1}(\mathbb{C}_F)$. We recall a standard result.

**Lemma 1.** For each tuple $t$ satisfying the above inequalities the vector $\phi_0$ is a cyclic vector for the representation $\pi(t_1, t_2, \ldots, t_{r-1})$.

**Proof.** Indeed, if $\text{Re}(t_1) = \text{Re}(t_2) = \cdots = \text{Re}(t_{r-1})$, the representation is irreducible and our assertion is trivial. If not, we use Langlands’ construction [Silberger 1978]. For each root $\alpha$ of $A_{r-1}$ we denote by $N_{\alpha}$ the corresponding subgroup of $N_{r-1}$ or $\overline{N}_{r-1}$ and by $\tilde{\alpha}$ the corresponding co-root. Thus, if $\alpha$ is a positive root, we have

$$\alpha(a_1, a_2, \ldots, a_{r-1}) = a_i/a_j$$

with $i < j$ and

$$\langle t, \tilde{\alpha} \rangle = t_i - t_j .$$

Let $P(t)$ be the set of positive roots $\alpha$ such that $\text{Re}(\langle t, \tilde{\alpha} \rangle) > 0$. Let $U$ be the unipotent group generated by the subgroups $N_{-\alpha}$ with $\alpha \in P(t)$. The intertwining operator

$$N\phi(g) = \int_{U(F)} \phi(ug) \, du$$

is defined by a convergent integral, and its kernel is a maximal invariant subspace. The formula of [Gindikin and Karpelevič 1966; Gindikin 1961] gives

$$N\phi_0(e) = \prod_{\alpha \in P(t)} \frac{1 - q^{-\langle t, \tilde{\alpha} \rangle - 1}}{1 - q^{-\langle t, \tilde{\alpha} \rangle}} .$$

Thus, $N\phi_0 \neq 0$, and our assertion follows. \hfill \square

The representation $I(t_1, t_2, \ldots, t_{r-1})$ admits a nonzero linear form $\lambda$ such that, for $u \in N_{r-1}(F)$ and $\phi$ in the space of the representation,

$$\lambda(\pi(u)\phi) = \theta_{r-1, \overline{\psi}}(u) \lambda(\phi) .$$

We denote by $\mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$ the space spanned by the functions of the form

$$g \mapsto W_\phi(g), \quad W_\phi(g) = \lambda(\pi(t_1, t_2, \ldots, t_{r-1})(g)\phi)$$

with $\phi \in I(t_1, t_2, \ldots, t_{r-1})$. We recall the following result:

**Lemma 2 [Jacquet and Shalika 1983].** The map $\phi \mapsto W_\phi$ is injective.

It follows that the image $W_0$ of $\phi_0$ is a cyclic vector in $\mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$. Up to a multiplicative constant, the function $W_0$ is equal to the function

$$W_0 = W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi}) .$$
Now let $\pi$ be an irreducible generic representation of $G_r(F)$. For $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$ we consider the integral

$$\Psi(s, W, W') = \int_{N_{r-1} \backslash G_{r-1}} W \left( \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right) W'(g) |\det g|^{s-1/2} dg.$$ 

The integral converges absolutely if $\text{Re}(s) \gg 0$ and extends to a meromorphic function of $s$. In any case, it has a meaning as a formal Laurent series in the variable $q^{-s}$ (see below). We recall a result from [Jacquet et al. 1983].

**Lemma 3.** There are functions $W_j \in \mathcal{W}(\pi; \psi)$ and $W'_j \in \mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$, $1 \leq j \leq k$, such that

$$\sum_{1 \leq j \leq k} \Psi(s, W_j, W'_j) = \prod_{1 \leq i \leq r-1} L(s + t_i, \pi).$$

Since $W_0$ is a cyclic vector, after a change of notations, we see that there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers $n_j$, $1 \leq j \leq k$, such that

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})) = \prod_{1 \leq i \leq r-1} L(s + t_i, \pi).$$

In our discussion $|x_1| \leq |x_2| \leq \cdots \leq |x_{r-1}|$. However, the functions

$$W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})$$

are symmetric in the variables $x_i$. Thus, we have the following result:

**Lemma 4.** Given an $(r - 1)$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_{r-1})$ there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers $n_j$, $1 \leq j \leq k$, such that

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})) = \prod_{1 \leq i \leq r-1} P_\pi(q^{-s} x_i)^{-1}.$$ 

2. The ideal $I_\pi$

We review the construction of [Jacquet et al. 1981a], adding a little more detail to some formal computations. First, we introduce a function

$$W(X_1, X_2, \ldots, X_{r-1}; \overline{\psi}) : G_{r-1}(F) \rightarrow S_{r-1}$$

whose value at a point $g \in G_{r-1}(F)$ is denoted $W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})$. It is defined by the following property: for every $(r - 1)$-tuple $(x_1, x_2, \ldots, x_{r-1})$ and every $g$, the scalar $W(g; x_1, x_2, \ldots, x_{r-1}; \overline{\psi})$ is the value of the polynomial

$$W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})$$
at the point \((x_1, x_2, \ldots, x_{r-1})\). For \(g\) in a set compact modulo \(N_{r-1}(F)\), the polynomials \(W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})\) remain in a finite dimensional vector subspace of \(S_{r-1}\). We have the relation

\[
|\det g|^s W(g; x_1, x_2, \ldots, x_{r-1}; \overline{\psi}) = W(g; q^{-s}x_1, q^{-s}x_2, \ldots, q^{-s}x_{r-1}; \overline{\psi}).
\]

It follows that if \(|\det g| = q^{-n}\), then the polynomial

\[
W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})
\]

is homogeneous of degree \(n\), that is,

\[
W(g; XX_1, XX_2, \ldots, XX_{r-1}; \overline{\psi}) = X^n W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi}).
\]

For each integer \(n\), we now define the integral

\[
\Psi_n(W; X_1, X_2, \ldots, X_{r-1}; \psi) := \int_{|\det g| = q^{-n}} W(g, X_1, X_2, \ldots, X_{r-1}; \overline{\psi}) |\det g|^{-1/2} dg.
\]

The support of the integrand is contained in a set compact modulo \(N_{r-1}(F)\), which depends on \(W\). In addition, there is an integer \(N(W)\) (depending on \(W\)) such that the support of the integrand is empty if \(n < N(W)\). The polynomial

\[
\Psi_n(W; X_1, X_2, \ldots, X_{r-1}; \psi)
\]

is homogeneous of degree \(n\). We consider the following formal Laurent series with coefficients in \(S_{r-1}\):

\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}; \psi) = \sum_n X^n \Psi_n(W; X_1, X_2, \ldots, X_{r-1}; \psi).
\]

Hence, in fact

\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}; \psi) = \sum_{n \geq N(W)} X^n \Psi_n(W; X_1, X_2, \ldots, X_{r-1}; \psi).
\]

If we multiply this Laurent series by \(\prod_{1 \leq i \leq r-1} P_\pi(XX_i)\), we obtain a new Laurent series with coefficients in \(S_{r-1}\), namely,

\[
\Psi(X; W, X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = \sum_{n \geq N_1(W)} X^n a_n(X_1, X_2, \ldots, X_{r-1}; \psi),
\]

where \(N_1(W)\) is another integer (depending on \(W\)) and \(a_n \in S_{r-1}\). Each \(a_n\) is homogeneous of degree \(n\). We can replace \(\pi\) by the contragredient representation \(\overline{\pi}\),
ψ by \( \tilde{\psi} \) and the function \( W \) by the function \( \tilde{W} \) defined by
\[
\tilde{W}(g) = W(w_r, g^{-1}).
\]
The function \( \tilde{W} \) belongs to \( \mathcal{W}(\tilde{\pi}, \tilde{\psi}) \). We define similarly
\[
\Psi(\tilde{W}; X_1, X_2, \ldots, X_{r-1}; \tilde{\psi}).
\]
We have then the following functional equation [Jacquet et al. 1983]:
\[
\Psi(q^{-1}X^{-1}; \tilde{W}; X_1^{-1}, X_2^{-1}, \ldots, X_{r-1}^{-1}; \tilde{\psi}) \prod_{i=1}^{r-1} P_\pi(q^{-1}X^{-1}X_i^{-1}) = c_\pi \prod_{i=1}^{r-1} \epsilon_\pi(XX_i, \psi) \Psi(X; W, X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{i=1}^{r-1} P_\pi(XX_i).
\]
The \( \epsilon \) factors are monomials and \( c_\pi = \pm 1 \). Thus, there is another integer \( N_2(W) \) such that in fact
\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = \sum_{N_2(W) \geq n \geq N_1(W)} X^n a_n(X_1, X_2, \ldots, X_{r-1}).
\]
From now on we drop the dependence on \( \psi \) from the notation.

From the above considerations it follows that the product
\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i)
\]
is in fact a polynomial in \( X \) with coefficients in \( S_{r-1} \). Moreover, because the \( a_n \) are homogeneous of degree \( n \), there is a polynomial \( \Xi(W; X_1, X_2, \ldots, X_{r-1}) \) in \( S_{r-1} \) such that
\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = \Xi(W; XX_1, XX_2, \ldots, XX_{r-1}).
\]
In a precise way, let us write
\[
\prod_{1 \leq i \leq r-1} P_\pi(X_i) = \sum_{m=0}^{R} P_m(X_1, X_2, \ldots, X_{r-1}),
\]
where each \( P_m \) is homogeneous of degree \( m \). Then
\[
\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i)
\]
\[
= \sum_n X^n \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \ldots, X_{r-1}) P_m(X_1, X_2, \ldots, X_{r-1}).
\]
The polynomial $\Xi(W; X_1, X_2, \ldots, X_{r-1})$ is then determined by the condition that its homogeneous component of degree $n$ noted $\Xi_n(W; X_1, X_2, \ldots, X_{r-1})$ be given by

$$\Xi_n(W; X_1, X_2, \ldots, X_{r-1}) = \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \ldots, X_{r-1}) P_m(X_1, X_2, \ldots, X_{r-1}).$$

The theorem amounts to saying there is a $W$ such that $\Xi(W; X_1, X_2, \ldots, X_{r-1})$ equals 1.

Let $I_\pi$ be the subvector space of $S_{r-1}$ spanned by the polynomials $\Xi(W; X_1, X_2, \ldots, X_{r-1})$.

**Lemma 5.** In fact $I_\pi$ is an ideal of the algebra $S_{r-1}$.

**Proof.** Let $Q$ be an element of $S_{r-1}$. Let $\phi$ be the corresponding element of $H_{r-1}$. Then

$$\int W(g h; X_1, X_2, \ldots, X_{r-1}) \phi(h) \, dh = W(g; X_1, X_2, \ldots, X_{r-1}) Q(X_1, X_2, \ldots, X_{r-1}).$$

Let $W$ be an element of $\mathcal{W}(\pi, \psi)$. Define another element $W_1$ of $\mathcal{W}(\pi, \psi)$ by

$$W_1(g) = \int_{G_{r-1}} W \left[ g \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] \phi(h) |\det h|^{1/2} \, dh.$$}

We claim that

$$\Xi(W_1; X_1, X_2, \ldots, X_{r-1}) = \Xi(W; X_1, X_2, \ldots, X_{r-1}) Q(X_1, X_2, \ldots, X_{r-1}).$$

This will imply the Lemma.

By linearity, it suffices to prove our claim when $Q$ is homogeneous of degree $t$. Then $\phi$ is supported on the set of $h$ such that $|\det h| = q^{-t}$. We have then, for every $n$,

$$\Psi_n(W_1; X_1, \ldots, X_{r-1})$$

$$= \int_{|\det g| = q^{-n}} W_1 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \ldots, X_{r-1}) |\det g|^{-1/2} \, dg$$

$$= \int_{|\det g| = q^{-n}} \int W(\begin{pmatrix} gh^{-1} & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \ldots, X_{r-1}) \phi(h) |\det h|^{1/2} \, dh$$

$$\times |\det g|^{-1/2} \, dg$$

$$= \int_{|\det g| = q^{-n+t}} W(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g h; X_1, \ldots, X_{r-1}) \phi(h) \, dh |\det g|^{-1/2} \, dg$$
\[ = \int_{|\det g| = q^{-n+t}} W \left( \begin{array}{c} g \\ 0 \\ 1 \end{array} \right) W(g; X_1, \ldots, X_{r-1}) |\det g|^{-1/2} dg \ Q(X_1, \ldots, X_{r-1}) \]
\[ = \Psi_{n-t}(W; X_1, \ldots, X_{r-1}) \ Q(X_1, \ldots, X_{r-1}). \]

Hence,

\[ \Xi_n(W_1; X_1, \ldots, X_{r-1}) \]
\[ = \sum_{m=0}^{R} \Psi_{n-m}(W_1; X_1, \ldots, X_{r-1}) \ P_m(X_1, \ldots, X_{r-1}) \]
\[ = \sum_{m=0}^{R} \Psi_{n-m-t}(W; X_1, \ldots, X_{r-1}) \ P_m(X_1, \ldots, X_{r-1}) \ Q(X_1, \ldots, X_{r-1}) \]
\[ = \Xi_{n-t}(W; X_1, \ldots, X_{r-1}) \ Q(X_1, \ldots, X_{r-1}). \]

Since \( Q \) is homogeneous of degree \( t \) our assertion follows.

**3. Proof of the theorem**

**Proof.** Given an \((r-1)\)-tuple of nonzero complex numbers \((x_1, x_2, \ldots, x_{r-1})\), Lemma 4 shows that we can find \( W_j \) and integers \( n_j \) such that, for all \( s \),

\[ \sum_{1 \leq j \leq k} (q^{-s})^{n_j} \Xi(W_j, q^{-s}x_1, q^{-s}x_2, \ldots, q^{-s}x_{r-1}) = 1. \]

In particular,

\[ \sum_{1 \leq j \leq k} \Xi(W_j, x_1, x_2, \ldots, x_{r-1}) = 1. \]

Thus, the element

\[ \sum_{1 \leq j \leq k} \Xi(W_j; X_1, X_2, \ldots, X_{r-1}) \]

of \( I_\pi \) does not vanish at \((x_1, x_2, \ldots, x_{r-1})\). By the theorem of zeros of Hilbert we have then \( I_\pi = S_{r-1} \). In particular, there is \( W \) such that

\[ \Xi(W; X_1, X_2, \ldots, X_{r-1}) = 1. \]

This implies the theorem.

**Remark 1.** The proof in [Jacquet et al. 1981a] is correct if \( L(s, \pi) \) is identically 1. In general, the proof there only shows that the polynomials in \( I_\pi \) cannot all vanish on a coordinate hyperplane \( X_i = x \).
Remark 2. Consider an induced representation $\pi$ of the form
$$\pi = I(\sigma_1 \otimes \alpha^{s_1}, \sigma_2 \otimes \alpha^{s_2}, \ldots, \sigma_k \otimes \alpha^{s_k}),$$
where the representations $\sigma_1, \sigma_2, \ldots, \sigma_k$ are tempered and $s_1, s_2, \ldots, s_k$ are real numbers such that
$$s_1 > s_2 > \cdots > s_k.$$
The representation $\pi$ may fail to be irreducible. But, in any case, it has a Whittaker model [Jacquet and Shalika 1983], and Theorem 1 is valid for the Whittaker model of $\pi$.

Remark 3. The proof of Matringe uses the theory of derivatives of a representation. The present proof appears simple only because we use Lemma 3, the proof of which is quite elaborate (and can be obtained from the theory of derivatives as in [Cogdell and Piatetski-Shapiro 2011]).

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Received June 7, 2012. Revised July 18, 2012.

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