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**SOME COMMENTS ON WEYL'S
COMPLETE REDUCIBILITY THEOREM**

JONATHAN ROGAWSKI AND V. S. VARADARAJAN

SOME COMMENTS ON WEYL'S COMPLETE REDUCIBILITY THEOREM

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In memoriam: Jonathan Rogawski

In this note we discuss a purely algebraic proof of Weyl's theorem that all finite-dimensional representations of a complex semisimple Lie algebra are completely reducible. We give a simple and direct proof which is elementary in the sense that it does not use cohomology, and which is a synthesis of the older proofs of Casimir – van der Waerden and of Brauer.

1. Introduction

In the theory of semisimple Lie algebras, one of the central results is the theorem of Weyl that says that all finite-dimensional representations of a complex semisimple Lie algebra \mathfrak{g} are completely reducible, that is, direct sums of the irreducible ones. In this note we present a purely algebraic proof of this result. Our aim is pedagogical, and so we have made an effort to explain in detail facts that are usually taken for granted in expositions, so as to make this accessible to graduate students and advanced undergraduates. The result is valid over any field of characteristic 0; but this generalization can be deduced from the complex case by standard arguments. Hence we restrict ourselves to working over \mathbb{C} .

The complete reducibility theorem was first proved by Hermann Weyl [1968] in his great series of papers on the theory of representations of semisimple Lie groups. Although Elie Cartan had already obtained a complete description of the *irreducible* representations, he did not go seriously into the issue of how an arbitrary representation could be built out of irreducible representations.

Weyl's proof remains one of the most beautiful in the entire theory of representations. The basis of his proof is the following: there is a real form \mathfrak{u} of \mathfrak{g} , the so-called *compact form*, with the property that the simply connected group U corresponding to \mathfrak{u} is compact; here we recall that to say that \mathfrak{u} is a real form of \mathfrak{g} is to require that \mathfrak{u} is a real Lie subalgebra of \mathfrak{g} whose real dimension is the complex dimension of \mathfrak{g} .

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Once the existence of \mathfrak{u} is assumed, the proof is straightforward. The representations of \mathfrak{g} correspond naturally to the representations of \mathfrak{u} ; in one direction one restricts from \mathfrak{g} to \mathfrak{u} and in the other direction one extends from \mathfrak{u} to \mathfrak{g} by complexification. On the other hand, there is a natural correspondence between the representations of \mathfrak{u} and the continuous representations of U . Now the compactness of U means that any continuous representation of U is unitarizable, that is, is equivalent to a unitary representation, namely a continuous homomorphism of U into the group of unitary operators on a finite-dimensional complex Hilbert space. So, assuming the representation is unitary, its complete reducibility is immediate, because for any U -invariant subspace M , its orthogonal complement M^\perp is also U -invariant and is moreover complementary to M ; the complete irreducibility is then clear by induction on the dimension of the representation. The unitarizability of any representation of U means that if we start with any representation, we can find a U -invariant scalar product in the representation space. To do this, we start with an arbitrary scalar product in the representation space and then introduce the scalar product which is the group average of its transforms by elements of U ; this latter scalar product is invariant under U so that U is unitary with respect to it (see [Varadarajan 1984, Chapter 4, §4.11]; see also [Hawkins 2000, pp. 465–484]).

This proof is essentially transcendental: it uses invariant integration on the compact group and also the topological fact that the universal covering group of the adjoint group of the compact form is still compact. And neither of these is elementary. The reduction to the compact form \mathfrak{u} and the compact group U was named the *unitarian trick* by Weyl.

It must be mentioned that the idea of averaging on special compact groups (such as the orthogonal group) goes back to Hurwitz and Schur. For Hurwitz, the goal was to prove the finite generation of the invariants for actions of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. Although these groups are not compact, they contain the compact groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n, \mathbb{R})$, and invariance with respect to these will imply invariance with respect to the complex groups they are contained in (this is the first instance of the unitarian trick); this is because a holomorphic function on the complex groups which vanishes on the relevant compact subgroup is identically 0. For Schur, the goal was the representation theory of the compact groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n, \mathbb{R})$, and he used integration over the group to determine all the irreducible representations, their characters, and their dimensions. The work of Hurwitz and Schur triggered Weyl's imagination, and after he got the results for all semisimple compact groups, he described them in a letter to Schur (see [Borel 1986] for references to the work of Hurwitz and Schur and to Weyl's letter to Schur).

So in the years after Weyl's proof appeared, the question of a purely algebraic proof became a natural issue. The first such algebraic proof was given by Casimir

and van der Waerden [1935].¹ They give a beautiful purely algebraic argument which is elementary in the sense that it relies only on the known structure of irreducible modules as highest-weight modules, and the Casimir operator. A key element of their proof is a delicate calculation in $\mathfrak{sl}(2)$.

After the original proof, additional algebraic proofs appeared in [Brauer 1936] and [Rashevski 1953]. There was also a proof that was cohomological: here it is a question of first establishing that $H^1(\mathfrak{g}, M) = 0$ for any semisimple Lie algebra \mathfrak{g} and any finite-dimensional \mathfrak{g} -module M , from which the complete reducibility will follow via standard arguments (see [Varadarajan 1984, Chapter 3, §§3.12–3.13]).

In this note we shall present a proof which is in some sense a synthesis of the earlier proofs of [Casimir and van der Waerden 1935; Brauer 1936]. It is very short and very direct. It uses the Casimir operator in an essential manner (as do all other proofs), the fact that any irreducible module is the one with highest weight equal to some λ , and the fact that it is enough to lift invariant vectors from the quotient of a module to the module itself. All modules considered from now on are finite-dimensional.

For a detailed historical account of Weyl's theorem and the various proofs of it, see [Borel 1998].

2. The Casimir operator and the lifting of invariant vectors

Let \mathfrak{g} be a complex semisimple Lie algebra. If (X_i) is a basis for \mathfrak{g} and (X^i) is the dual basis with respect to the Cartan–Killing form (\cdot, \cdot) (that is, $(X_i, X^j) = \delta_{ij}$), then the Casimir operator is

$$\omega = \sum_i X_i X^i;$$

it is independent of the choice of the basis and lies in the center of the enveloping algebra of \mathfrak{g} . It commutes with the action of \mathfrak{g} and hence it goes into a scalar in any irreducible module for \mathfrak{g} . We write g_λ for the value of this scalar when the module is V_λ , the irreducible module of highest weight λ . Here we have chosen a Cartan subalgebra \mathfrak{h} , and a positive system of roots, so that $\lambda \in \mathfrak{h}^*$. The isomorphism between \mathfrak{h} and \mathfrak{h}^* carries the form on \mathfrak{h} to one on \mathfrak{h}^* which is denoted by the same symbol. It is known that (\cdot, \cdot) is real and positive definite on the real span of the roots, which includes all the highest weights. If λ is a highest weight, $(\lambda, \alpha) \geq 0$ for all roots $\alpha > 0$.

The first lemma is the calculation of g_λ . As usual, let

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha.$$

¹V. S. V. was fortunate to hear an exposition by Casimir himself of this proof in a conference in Utrecht in 1985. See the [Appendix](#).

Lemma 1 [Casimir and van der Waerden 1935]. *The scalar g_λ is given by*

$$g_\lambda = (\lambda + \rho, \lambda + \rho) - (\rho, \rho).$$

Moreover,

$$g_\lambda = 0 \iff \lambda = 0.$$

In other words, an irreducible module where ω is 0 is necessarily the trivial module.

Proof. We select an ON basis (K_i) for \mathfrak{h} and root vectors X_α with $(X_\alpha, X_{-\alpha}) = 1$ for all roots α . Then

$$\omega = \sum_i K_i^2 + \sum_{\alpha>0} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) = \sum_i K_i^2 + \sum_{\alpha>0} H_\alpha + 2 \sum_{\alpha>0} X_{-\alpha} X_\alpha.$$

If m_λ is a highest-weight vector for M_λ , then

$$\omega m_\lambda = g_\lambda m_\lambda.$$

Since the $X_\alpha (\alpha > 0)$ annihilate m_λ and $H m_\lambda = \lambda(H) m_\lambda$ for $H \in \mathfrak{h}$, it follows that

$$\begin{aligned} g_\lambda &= \sum_i \lambda(K_i)^2 + \sum_{\alpha>0} \lambda(H_\alpha) = (\lambda, \lambda) + \sum_{\alpha>0} (\lambda, H_\alpha) \\ &= (\lambda, \lambda) + 2(\lambda, \rho) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho). \end{aligned}$$

If $\lambda = 0$, then $g_\lambda = 0$. Conversely, suppose that $g_\lambda = 0$. Then $(\lambda, \lambda) + 2(\lambda, \rho) = 0$. As both (λ, λ) and (λ, ρ) are ≥ 0 , we must have $(\lambda, \lambda) = 0$, so that $\lambda = 0$. \square

The second lemma is that invariant vectors can be lifted. A vector v in a \mathfrak{g} -module is invariant if $\mathfrak{g}v = 0$.

Lemma 2. *Let*

$$M \rightarrow P \rightarrow 0$$

be an exact sequence of \mathfrak{g} -modules and let $p \in P$ be an invariant vector. Then we can lift p to an invariant vector in M , namely, find an invariant $u \in M$ such that u maps to p .

Proof. Replacing M by the preimage of the line $\mathbb{C}p$, we may assume that $P = \mathbb{C}p$. Now M is the direct sum of the generalized subspaces M_r of the Casimir operator, M_r being the largest subspace where ω has the single eigenvalue r . The M_r are stable under \mathfrak{g} , and as $\omega p = 0$, it follows that all the M_r for $r \neq 0$ map to 0 in P . Since M maps onto P , this means that $M_0 \neq 0$ and maps onto P . In other words, we may assume that $M = M_0$. But then, by Lemma 1, M has a Jordan composition series consisting only of trivial modules. Let $H_i, X_i, Y_i (1 \leq i \leq r = \text{rank of } \mathfrak{g})$ be the usual Chevalley generators for \mathfrak{g} . Then, for each i , H_i has the sole eigenvalue 0 in M . If $M[H_i : a]$ is the generalized eigensubspace of H_i for the eigenvalue a , it is standard that X_i (resp. Y_i) maps $M[H_i : a]$ into $M[H_i : a + 2]$ (resp. $M[H_i : a - 2]$).

But as 0 is the only eigenvalue for H_i in M , it follows that X_i and Y_i act as 0 on M . But then $H_i = [X_i, Y_i]$ also acts as 0, so that \mathfrak{g} acts trivially on M . This means that any vector u of M above p satisfies our requirements. \square

3. Proof of Weyl's theorem

Weyl's theorem can now be obtained by a standard general argument.

Weyl's theorem. *All \mathfrak{g} -modules are completely reducible.*

Proof. Suppose M is a \mathfrak{g} -module and N a proper submodule. It is a question of finding a submodule Q such that $M = N \oplus Q$. If $P = M/N$, we have an exact sequence of \mathfrak{g} -modules

$$0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0.$$

So we get the exact sequence \mathfrak{g} -modules

$$0 \rightarrow \text{Hom}(P, N) \rightarrow \text{Hom}(P, M) \rightarrow \text{Hom}(P, P) \rightarrow 0.$$

Now the identity I in $\text{Hom}(P, P)$ is an invariant element, and so, by [Lemma 2](#), can be lifted to an invariant element t in $\text{Hom}(P, M)$. Since t maps to I , we see that t splits the map $M \rightarrow P$. Since t is invariant, the range Q of t is a submodule of M complementary to N . \square

Remark. The reduction to trivial modules goes back to [\[Brauer 1936\]](#). It was resurrected by Chevalley [\[1955\]](#) in his proof of Weyl's theorem. Later on, when Mumford [\[1965\]](#) needed a characteristic p version of [Lemma 2](#), he formulated it as follows: if G is a connected semisimple group over an algebraically closed field K of characteristic $p > 0$, and V, W are G -modules with $V \rightarrow W \rightarrow 0$ exact, and if $w \in W$ is an invariant vector, there are an integer $d > 0$ and an invariant vector $v \in S_d(V)$ which maps to w^d (here S_d refers to the component of degree d in the symmetric algebra). It is known that the smallest value of d is a power p^m of p .

Appendix: A historical note on Casimir and his operator

Hendrik Brugt Gerhard Casimir (1909–2000) was a physicist whose Leiden thesis [\[1933\]](#) on the theory of diatomic molecules introduced the operator now known as the Casimir operator, as an element of the enveloping algebra of the rotation Lie algebra \mathfrak{o} . It commutes with all the elements of \mathfrak{o} and so is a scalar in any irreducible representation of \mathfrak{o} . Shortly afterwards, he discovered the analogue of this for any semisimple Lie algebra [\[1931\]](#). The corresponding operator in any representation of the Lie algebra eventually became known as the *Casimir operator*. It plays a central role in harmonic analysis and representation theory, even though for groups of higher dimension its value on an irreducible representation no longer determines the representation, unlike what happens for $\mathfrak{sl}(2)$. In [\[Casimir 1931\]](#), he himself pioneered

the idea that the Casimir operator should be viewed as a second-order differential operator on the group manifold, and that the matrix elements of irreducible representations of the group are eigenfunctions for this operator. The connection between representation theory and differential equations on the group manifold introduced by Casimir found its full force and scope only with the work of Harish-Chandra.

Casimir was one of the great Dutch physicists of the twentieth century who made significant contributions to both experimental and theoretical physics as well as to pure mathematics. In addition, he had a big influence on industry as the head of the research division of Philips. His mathematical contributions include the discovery and use of the Casimir operator as described above. In experimental physics, he predicted what is now known as the *Casimir effect*, which is a quantum mechanical attraction between conducting plates. His theoretical contributions are quite well known, such as his work on Lars Onsager's microscopic reversibility. One of us (V. S. V.) was present at a conference on semisimple Lie groups in Utrecht in 1985 when Casimir gave a talk on the history of the proof of Weyl's complete reducibility theorem and presented a brief sketch of the algebraic proof given in [Casimir and van der Waerden 1935]. His autobiography [Casimir 1983] is a wonderful document of great interest and humanity.

From the modern perspective, the Casimir operator is an element in the center of the universal enveloping algebra of a semisimple Lie algebra. For simple Lie algebras of dimension greater than 3, the Casimir element does not generate the center of the enveloping algebra. Some of the additional elements of the center were written down by Giulio Racah (1909–1965), an Israeli physicist and mathematician. Racah [1965] determined in some implicit manner the full center of the enveloping algebra of an arbitrary semisimple Lie algebra. The generators of the center discovered by Racah are known to physicists as *generalized Casimir operators*. For many in the 1950's and early 1960's (including V. S. V.), the Racah notes [1965] were almost the only sources of information on the structure and representations of semisimple Lie algebras till the appearance of [Blanchard et al. 1955] and [Jacobson 1962]. From the mathematical side, the center of the enveloping algebra was first investigated by Harish-Chandra (1923–1983). Harish-Chandra constructed what is now known as the *Harish-Chandra isomorphism* of the center of the universal enveloping algebra of a general semisimple Lie algebra \mathfrak{g} with the algebra of Weyl group invariants of the algebra of polynomials on a Cartan subalgebra of \mathfrak{g} . The center is very closely related to the algebra of polynomial invariants on the Lie algebra, and this was determined by Claude Chevalley (1909–1984), who proved that it is isomorphic, via restriction to a Cartan subalgebra, to the Weyl group invariants of the algebra of polynomials on the Cartan subalgebra. The Harish-Chandra isomorphism, and its p -adic twin, the *Satake isomorphism*, play a fundamental role in harmonic analysis on semisimple groups.

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This paper had its origins in discussions between Jonathan Rogawski and myself. At some point in 2008, although we had a draft in existence, we decided to put it on hold, because of his medical commitments. After Jonathan's untimely passing away in 2011, Don Blasius suggested to me that perhaps this manuscript should see the light of day as an exposition, pedagogical in scope, of a beautiful aspect of semisimple Lie algebras. It is essentially the same as the earlier draft. I thank Don not only for his suggestion to write up these remarks, but also for reading the manuscript and suggesting a number of improvements.

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JONATHAN ROGAWSKI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095
UNITED STATES
jonr@math.ucla.edu

V. S. VARADARAJAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095
UNITED STATES

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EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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Special issue devoted to the memory of Jonathan Rogawski

In memoriam: Jonathan Rogawski	257
DON BLASIUS, DINAKAR RAMAKRISHNAN and V. S. VARADARAJAN	
<i>p</i> -adic Rankin <i>L</i> -series and rational points on CM elliptic curves	261
MASSIMO BERTOLINI, HENRI DARMON and KARTIK PRASANNA	
The syntomic regulator for K_4 of curves	305
AMNON BESSER and ROB DE JEU	
Unique functionals and representations of Hecke algebras	381
BENJAMIN BRUBAKER, DANIEL BUMP and SOLOMON FRIEDBERG	
A relative trace formula for $\mathrm{PGL}(2)$ in the local setting	395
BROOKE FEIGON	
On the degrees of matrix coefficients of intertwining operators	433
TOBIAS FINIS, EREZ LAPID and WERNER MÜLLER	
Comparison of compact induction with parabolic induction	457
GUY HENNIART and MARIE-FRANCE VIGNERAS	
The functional equation and beyond endoscopy	497
P. EDWARD HERMAN	
A correction to <i>Conducteur des Représentations du groupe linéaire</i>	515
HERVÉ JACQUET	
Modular <i>L</i> -values of cubic level	527
ANDREW KNIGHTLY and CHARLES LI	
On occult period maps	565
STEPHEN KUDLA and MICHAEL RAPOPORT	
A prologue to “Functoriality and reciprocity”, part I	583
ROBERT LANGLANDS	
Truncation of Eisenstein series	665
EREZ LAPID and KEITH OUELLETTE	
Some comments on Weyl’s complete reducibility theorem	687
JONATHAN ROGAWSKI and V. S. VARADARAJAN	
On equality of arithmetic and analytic factors through local Langlands correspondence	695
FREYDOON SHAHIDI	