CALOGERO–MOSER VERSUS KAZHDAN–LUSZTIG CELLS

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In 1979, Kazhdan and Lusztig developed a combinatorial theory associated with Coxeter groups, defining in particular partitions of the group in left and two-sided cells. In 1983, Lusztig generalized this theory to Hecke algebras of Coxeter groups with unequal parameters. We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero-Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg. We conjecture that these coincide with Kazhdan-Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino, and we provide here a version of left cell representations. The Calogero-Moser cells will be studied in details in a forthcoming paper, providing thus several results supporting our conjecture.

1. Introduction

Kazhdan and Lusztig [1979] developed a combinatorial theory associated with Coxeter groups. They defined in particular partitions of the group in left and two-sided cells. For Weyl groups, these have a representation-theoretic interpretation in terms of primitive ideals, and they play a key role in Lusztig’s description [1984] of unipotent characters for finite groups of Lie type. Lusztig [1983; 2003] generalized this theory to Hecke algebras of Coxeter groups with unequal parameters.

We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero–Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg [2002]. We conjecture that these coincide with Kazhdan–Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino [2009], and we provide here a version of left cell representations. The Calogero–Moser cells are studied in detail in [Bonnafé and Rouquier ≥ 2013].

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2. Calogero–Moser spaces and cells

Rational Cherednik algebras at $t = 0$. Let us recall some constructions and results from [Etingof and Ginzburg 2002]. Let $V$ be a finite-dimensional complex vector space and $W$ a finite subgroup of $GL(V)$. Let $\mathcal{S}$ be the set of reflections of $W$, that is, elements $g$ such that $\ker(g - 1)$ is a hyperplane. We assume that $W$ is a reflection group, that is, it is generated by $\mathcal{S}$.

We denote by $\mathcal{S}/\sim$ the quotient of $\mathcal{S}$ by the conjugation action of $W$ and we let $\{c_s\}_{s \in \mathcal{S}/\sim}$ be a set of indeterminates. We put $A = \mathbb{C}[\mathcal{S}/\sim] = \mathbb{C}[\{c_s\}_{s \in \mathcal{S}/\sim}]$. Given $s \in \mathcal{S}$, let $v_s \in V$ and $\alpha_s \in V^*$ be eigenvectors for $s$ associated to the nontrivial eigenvalue.

The 0-rational Cherednik algebra $H$ is the quotient of $A \otimes T(V \oplus V^*) \rtimes W$ by the relations

$$[x, x'] = [\xi, \xi'] = 0,$$

$$[\xi, x] = \sum_{s \in \mathcal{S}} c_s \frac{(v_s, x) \cdot (\xi, \alpha_s)}{(v_s, \alpha_s)} s \text{ for } x, x' \in V^* \text{ and } \xi, \xi' \in V.$$ 

We put $Q = Z(H)$ and $P = A \otimes S(V^*)^W \otimes S(V)^W \subset Q$. The ring $Q$ is normal. It is a free $P$-module of rank $|W|$.

Galois closure. Let $K = \text{Frac}(P)$ and $L = \text{Frac}(Q)$. Let $M$ be a Galois closure of the extension $L/K$ and $R$ the integral closure of $Q$ in $M$. Let $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. Let $\mathcal{P} = \text{Spec } P = \mathbb{A}_\mathbb{C}^{\mathcal{S}/\sim} \times V/W \times V^*/W$, $\mathcal{Q} = \text{Spec } Q$ the Calogero–Moser space, and $\mathcal{R} = \text{Spec } R$.

We denote by $\pi : \mathcal{R} \to \mathcal{Q}$ the quotient by $H$, and by $\gamma : \mathcal{Q} \to \mathcal{P}$ and $\phi : \mathcal{P} \to \mathbb{A}_\mathbb{C}^{\mathcal{S}/\sim}$ the canonical maps. We put $p = \gamma \pi : \mathcal{R} \to \mathcal{P}$ the quotient by $G$.

Ramification. Let $\mathfrak{r} \in \mathcal{R}$ be a prime ideal of $R$. We denote by $D(\mathfrak{r}) \subset G$ its decomposition group and by $I(\mathfrak{r}) \subset D(\mathfrak{r})$ its inertia group.

We have a decomposition into irreducible components

$$\mathcal{R} \times_{\mathcal{P}} \mathcal{Q} = \bigcup_{g \in G/H} \mathcal{O}_g, \text{ where } \mathcal{O}_g = \{(x, \pi(g^{-1}(x))) \mid x \in \mathcal{R}\},$$

inducing a decomposition into irreducible components

$$V(\mathfrak{r}) \times_{\mathcal{P}} \mathcal{Q} = \bigsqcup_{g \in I(\mathfrak{r}) \setminus G/H} \mathcal{O}_g(\mathfrak{r}), \text{ where } \mathcal{O}_g(\mathfrak{r}) = \{(x, \pi(g^{-1}g'(x))) \mid x \in V(\mathfrak{r}), g' \in I(\mathfrak{r})\}.$$

Undeformed case. Let $\mathfrak{p}_0 = \phi^{-1}(0) = \sum_{s \in \mathcal{S}/\sim} P c_s$. We have

$$P/\mathfrak{p}_0 = \mathbb{C}[V \oplus V^*]^W \times W, \quad Q/\mathfrak{p}_0 Q = \mathbb{C}[V \oplus V^*]^{\Delta W},$$

where $\Delta(W) = \{(w, w) \mid w \in W\} \subset W \times W$. A Galois closure of the extension of $\mathbb{C}(\mathfrak{p}_0 Q) = \mathbb{C}(V \oplus V^*)^{\Delta W}$ over $\mathbb{C}(\mathfrak{p}_0) = \mathbb{C}(V \oplus V^*)^{W \times W}$ is $\mathbb{C}(V \oplus V^*)^{\Delta Z(W)}$. 

Let \( r_0 \in \mathcal{R} \) above \( p_0 \). Since \( p_0Q \) is prime, we have \( G = D(r_0)H = HD(r_0), I(r_0) = 1 \), and \( \mathbb{C}(r_0) \) is a Galois closure of the extension \( \mathbb{C}(p_0Q)/\mathbb{C}(p_0) \). Fix an isomorphism \( \iota : \mathbb{C}(r_0) \cong \mathbb{C}(V \oplus V^*)^{\Delta Z(W)} \) extending the canonical isomorphism of \( \mathbb{C}(p_0Q) \) with \( \mathbb{C}(V \oplus V^*)^{\Delta W} \).

The application \( \iota \) induces an isomorphism \( D(r_0) \cong (W \times W)/\Delta Z(W) \), that restricts to an isomorphism \( D(r_0) \cap H \cong \Delta W/\Delta Z(W) \). This provides a bijection \( G/H \cong (W \times W)/\Delta W \). Composing with the inverse of the bijection

\[
W \cong (W \times W)/\Delta W, \quad w \mapsto (1, w),
\]

we obtain a bijection \( G/H \cong W \).

From now on, we identify the sets \( G/H \) and \( W \) through this bijection. Note that this bijection depends on the choices of \( r_0 \) and of \( \iota \). Since \( M \) is the Galois closure of \( L/K \), we have \( \bigcap_{g \in G} H^g = 1 \), hence the left action of \( G \) on \( W \) induces an injection \( G \subset \mathcal{S}(W) \).

**Calogero–Moser cells.**

**Definition 2.1.** Let \( r \in \mathcal{R} \). The \( r \)-cells of \( W \) are the orbits of \( I(r) \) in its action on \( W \).

Let \( c \in \mathbb{A}_C^{\mathcal{R}/\sim} \). Choose \( r_c \in \mathcal{R} \) with \( p(r_c) = \bar{c} \times 0 \times 0 \). The \( r_c \)-cells are called the two-sided Calogero–Moser c-cells of \( W \). Choose now \( r_c^{\text{left}} \in \mathcal{R} \) contained in \( r_c \) with \( p(r_c^{\text{left}}) = \bar{c} \times V/W \times 0 \in \mathcal{P} \). The \( r_c^{\text{left}} \)-cells are called the left Calogero–Moser c-cells of \( W \). We have \( I(r_c^{\text{left}}) \subset I(r_c) \). Consequently, every left cell is contained in a unique two-sided cell.

The map sending \( w \in W \) to \( \pi(w^{-1}(r_c)) \) induces a bijection from the set of two-sided cells to \( \Upsilon^{-1}(c \times 0 \times 0) \).

**Families and cell multiplicities.** Let \( E \) be an irreducible representation of \( \mathbb{C}[W] \). We extend it to a representation of \( S(V) \rtimes W \) by letting \( V \) act by 0. Let

\[
\Delta(E) = e \cdot \Ind_{S(V) \rtimes W}^{H} (A \otimes_C E), \quad \text{where} \quad e = \frac{1}{|W|} \sum_{w \in W} w,
\]

be the spherical Verma module associated with \( E \). It is a \( Q \)-module.

Let \( c \in \mathbb{A}_C^{\mathcal{R}/\sim} \) and let \( \Delta^{\text{left}}(E) = (R/r_c^{\text{left}}) \otimes_P \Delta(E) \).

**Definition 2.2.** Given a left cell \( \Gamma \), we define the cell multiplicity \( m_\Gamma(E) \) of \( E \) as the length of \( \Delta^{\text{left}}(E) \) at the component \( C_\Gamma(r_c^{\text{left}}) \).

Note that \( \sum_\Gamma m_\Gamma(E) \cdot [C_\Gamma(r_c^{\text{left}})] \) is the support cycle of \( \Delta^{\text{left}}(E) \).

There is a unique two-sided cell \( \Lambda \) containing all left cells \( \Gamma \) such that \( m_\Gamma(E) \neq 0 \). Its image in \( \mathbb{Y} \) is the unique \( q \in \Upsilon^{-1}(c \times 0 \times 0) \) such that \( (Q/q) \otimes_Q \Delta(E) \neq 0 \). The corresponding map \( \text{Irr}(W) \to \Upsilon^{-1}(c \times 0 \times 0) \) is surjective, and its fibers are the Calogero–Moser families of \( \text{Irr}(W) \), as defined by Gordon [2003].


**Dimension 1.** Let $V$ be a one-dimensional complex vector space, let $d \geq 2$ and let $W$ be the group of $d$-th roots of unity acting on $V$. Let $\zeta = \exp(2i\pi/d)$, let $s = \zeta \in W$ and $c_i = \zeta s_i$ for $1 \leq i \leq d - 1$. We have $A = \mathbb{C}[c_1, \ldots, c_{d-1}]$ and

$$
H = A \left\{ x, \xi, s \mid sx^{-1} = \xi^{-1}x, \ s\xi^{-1} = \xi \xi \text{ and } [\xi, x] = \sum_{i=1}^{d-1} c_i s^i \right\}.
$$

Let $eu = \xi x - \sum_{i=1}^{d-1} (1 - \zeta^{-i})^{-1} \zeta s^i$. We have $P = A[x^d, \xi^d]$ and $Q = A[x^d, \xi^d, eu]$. Define $\kappa_1, \ldots, \kappa_d = \kappa_0$ by $\kappa_1 + \cdots + \kappa_d = 0$ and $\sum_{i=1}^{d-1} c_i s^i = \sum_{i=0}^{d-1} (\kappa_i - \kappa_{i+1}) \varepsilon_i$, where $\varepsilon_i = \frac{1}{d} \sum_{j=0}^{d-1} \zeta^{ij} s^j$. We have $A = \mathbb{C}[\kappa_1, \ldots, \kappa_d]/(\kappa_1 + \cdots + \kappa_d)$.

The normalization of the Galois closure is described as follows. There is an isomorphism of $A$-algebras

$$
A[X, Y, Z]/\left( XY - \prod_{i=1}^{d} (Z - \kappa_i) \right) \xrightarrow{\sim} Q, \quad X \mapsto x^d, \quad Y \mapsto \xi^d \quad \text{and} \quad Z \mapsto eu.
$$

We have an isomorphism of $A$-algebras

$$
A[X, Y, \lambda_1, \ldots, \lambda_d]/\left( e_i(\lambda) = e_i(\kappa), \ i = 1, \ldots, d-1, \ e_d(\lambda) = e_d(\kappa) + (-1)^{d+1} XY \right) \xrightarrow{\sim} R,
$$

where $Z = \lambda_d$ and where $e_i$ denotes the $i$-th elementary symmetric function. We have $G = \mathfrak{S}_d$, acting by permuting the $\lambda_i$, and $H = \mathfrak{S}_{d-1}$.

Let $p_0 = (\kappa_1, \ldots, \kappa_d) \in \text{Spec} \, P$ and

$$
t_0 = (\kappa_1, \ldots, \kappa_d, \lambda_1 - \zeta \lambda_d, \ldots, \lambda_{d-1} - \zeta^{d-1} \lambda_d) \in \text{Spec} \, R.
$$

We have $D(t_0) = \langle (1, 2, \ldots, d) \rangle \subset \mathfrak{S}_d$ and

$$
\mathbb{C}(t_0) = \mathbb{C}(X, Y, \lambda_d = \frac{\sqrt{d}}{d} XY) = \mathbb{C}(X, Y, Z = \frac{\sqrt{d}}{d} XY).
$$

The composite bijection $D(t_0) \xrightarrow{\sim} G/H \xrightarrow{\sim} W$ is an isomorphism of groups given by $(1, \ldots, d) \mapsto s$.

Fix $c \in \mathbb{C}^{d-1}$ and let $\kappa_1, \ldots, \kappa_d \in \mathbb{C}$ corresponding to $c$. Consider $r = r_c$ or $\tau^\text{left}_c$ as in Section 2 (see right after Definition 2.1). Then $I(t)$ is the subgroup of $\mathfrak{S}_d$ stabilizing $(\kappa_1, \ldots, \kappa_d)$. The left $c$-cells coincide with the two-sided $c$-cells and two elements $s^i$ and $s^j$ are in the same cell if and only if $\kappa_i = \kappa_j$. Finally, the multiplicity $m_{\Gamma}(\det^j)$ is 1 if $s^j \in \Gamma$ and 0 otherwise.

### 3. Coxeter groups

**Kazhdan–Lusztig cells.** Following [Kazhdan and Lusztig 1979; Lusztig 1983; 2003], let us recall the construction of cells.

We assume here $V$ is the complexification of a real vector space $V_{\mathbb{R}}$ acted on by $W$. We choose a connected component $C$ of $V_{\mathbb{R}} - \bigcup_{s \in \mathbb{G}} \ker(s - 1)$ and we
We denote by $S$ the set of $s \in S$ such that $\ker(s - 1) \cap \tilde{C}$ has codimension 1 in $\tilde{C}$. This makes $(W, S)$ into a Coxeter group, and we denote by $l$ the length function.

Let $\Gamma$ be a totally ordered free abelian group and let $L : W \to \Gamma$ be a weight function, that is, a function such that

$$L(ww') = L(w) + L(w') \quad \text{if } l(ww') = l(w) + l(w').$$

We denote by $v^\gamma$ the element of the group algebra $\mathbb{Z}[\Gamma]$ corresponding to $\gamma \in \Gamma$.

We denote by $H$ the Hecke algebra of $W$: this is the $\mathbb{Z}[\Gamma]$-algebra generated by elements $T_s$ with $s \in S$ subject to the relations

$$(T_s - v^{L(s)})(T_s + v^{-L(s)}) = 0 \quad \text{and} \quad \underbrace{T_s T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}},$$

for $s, t \in S$ with $m_{st} \neq \infty$, where $m_{st}$ is the order of $st$. Given $w \in W$, we put $T_w = T_{s_1} \cdots T_{s_n}$, where $w = s_1 \cdots s_n$ is a reduced decomposition.

Let $i$ be the ring involution of $H$ given by $i(v^\gamma) = v^{-\gamma}$ for $\gamma \in \Gamma$ and $i(T_s) = T_s^{-1}$. We denote by $\{C_w\}_{w \in W}$ the Kazhdan–Lusztig basis of $H$. It is uniquely defined by the properties that $i(C_w) = C_w$ and $C_w - T_w \in \bigoplus_{w' \in W} \mathbb{Z}[\Gamma_{<0}]T_{w'}$.

We introduce the partial order $\prec_L$ on $W$. It is the transitive closure of the relation given by $w' \prec_L w$ if there is $s \in S$ such that the coefficient of $C_w'$ in the decomposition of $C_s C_w$ in the Kazhdan–Lusztig basis is nonzero. We define $w \sim_L w'$ to be the corresponding equivalence relation: $w \sim_L w'$ if and only if $w \prec_L w$ and $w' \prec_L w$. The equivalence classes are the left cells. We define $\prec_{LR}$ as the partial order generated by $w \prec_{LR} w'$ if $w \prec_L w'$ or $w^{-1} \prec_L w'^{-1}$. As above, we define an associated equivalence relation $\sim_{LR}$. Its equivalence classes are the two-sided cells.

When $\Gamma = \mathbb{Z}$, $L = l$, and $W$ is a Weyl group, a definition of left cells based on primitive ideals in enveloping algebras was proposed by Joseph [1980]: let $\mathfrak{g}$ be a complex semisimple Lie algebra with Weyl group $W$. Let $\rho$ be the half-sum of the positive roots. Given $w \in W$, let $I_w$ be the annihilator in $U(\mathfrak{g})$ of the simple module with highest weight $-w(\rho) - \rho$. Then, $w$ and $w'$ are in the same left cell if and only if $I_w = I_{w'}$.

Representations and families. Let $\Gamma$ be a left cell. Let $W_{\leq \Gamma}$ and $W_{< \Gamma}$ be the sets of $w \in W$ such that there is $w' \in \Gamma$ with $w \prec_L w'$ and, respectively, $w \prec_L w'$ and $w \not\in \Gamma$. The left cell representation of $W$ over $\mathbb{C}$ associated with $\Gamma$ [Kazhdan and Lusztig 1979; Lusztig 2003] is the unique representation, up to isomorphism, that deforms into the left $H$-module

$$\left( \bigoplus_{w \in W_{\leq \Gamma}} \mathbb{Z}[\Gamma]C_w \right) / \left( \bigoplus_{w \in W_{< \Gamma}} \mathbb{Z}[\Gamma]C_w \right).$$
Lusztig [1982; 2003] has defined the set of constructible characters of $W$ inductively as the smallest set of characters with the following properties: it contains the trivial character, it is stable under tensoring by the sign representation and it is stable under $J$-induction from a parabolic subgroup. Lusztig’s families are the equivalences classes of irreducible characters of $W$ for the relation generated by $\chi \sim \chi'$ if $\chi$ and $\chi'$ occur in the same constructible character. Lusztig has determined constructible characters and families for all $W$ and all parameters.

Lusztig has shown for equal parameters, and conjectured in general, that the set of left cell characters coincides with the set of constructible characters.

A conjecture. Let $c \in \mathbb{R}^\mathcal{G}/\sim$. Let $\Gamma$ be the subgroup of $\mathbb{R}$ generated by $\mathbb{Z}$ and $\{c_s\}_{s \in \mathcal{G}}$. We endow it with the natural order on $\mathbb{R}$. Let $L : W \to \Gamma$ be the weight function determined by $L(s) = c_s$ if $s \in S$.

The following conjecture is due to Gordon and Martino [2009]. A similar conjecture has been proposed independently by the second author.\(^\text{1}\) It is known to hold for types $A_n, B_n, D_n$ and $I_2(n)$ [Gordon 2008; Gordon and Martino 2009; Bellamy 2011; Martino 2010a; 2010b].

**Conjecture 3.1.** The Calogero–Moser families of irreducible characters of $W$ coincide with the Lusztig families.

We propose now a conjecture involving partitions of elements of $W$, via ramification. The part dealing with left cell characters could be stated in a weaker way, using $Q$ and not $R$, and thus not needing the choice of prime ideals, by involving constructible characters.

**Conjecture 3.2.** There is a choice of $r_c^\text{left} \subset r_c$ such that

- the Calogero–Moser two-sided cells and left cells coincide with the Kazhdan–Lusztig two-sided cells and left cells, respectively, and
- the representation $\sum_{E \in \text{Irr}(W)} m\Gamma(E)E$, where $\Gamma$ is a Calogero–Moser left cell, coincide with the left cell representation of the corresponding Kazhdan–Lusztig cell.

Various particular cases and general results supporting **Conjecture 3.2** are provided in [Bonnafé and Rouquier \(\geq 2013\)]. In particular, the conjecture holds for $W = B_2$, for all choices of parameters.

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