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JINGBO DOU AND CHANGZHENG QU

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CLASSIFICATION OF POSITIVE SOLUTIONS FOR AN ELLIPTIC SYSTEM WITH A HIGHER-ORDER FRACTIONAL LAPLACIAN

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We discuss properties of solutions to the following elliptic PDE system in $\mathbb{R}^n$:

$$\begin{cases}(-\Delta)^{\alpha/2}u = \lambda_1 u^{p_1} + \mu_1 v^{p_2} + \beta_1 u^{p_3} v^{p_4},\\ (-\Delta)^{\alpha/2}v = \lambda_2 u^{q_1} + \mu_2 v^{q_2} + \beta_2 u^{q_3} v^{q_4},\end{cases}$$

where $0 < \alpha < n$, λ_j, μ_j, β_j $(j = 1, 2)$ are nonnegative constants and p_i and q_i $(i = 1, 2, 3, 4)$ satisfy some suitable assumptions. It is shown that this PDE system is equivalent to the integral system

$$\begin{cases}u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y) v^{p_4}(y)}{|x-y|^{n-\alpha}}\,dy,\\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\lambda_2 u^{q_1}(y) + \mu_2 v^{q_2}(y) + \beta_2 u^{q_3}(y) v^{q_4}(y)}{|x-y|^{n-\alpha}}\,dy\end{cases}$$

in $\mathbb{R}^n$. The radial symmetry, monotonicity and regularity of positive solutions are proved via the method of moving plane in integral forms and a regularity lifting lemma. For the special case with

$$p_1 = p_2 = q_1 = q_2 = p_3 + p_4 = q_3 + q_4 = \frac{n+\alpha}{n-\alpha},$$

positive solutions of the integral system (or the PDE system) are classified. Furthermore, our symmetry results, together with some known results on nonexistence of positive solutions, imply that, under certain integrability conditions, the PDE system has no positive solution in the subcritical case.

1. Introduction

In this paper, we study positive solutions of the following higher-order elliptic system in $\mathbb{R}^n$:

$$\begin{cases}(-\Delta)^{\alpha/2}u = \lambda_1 u^{p_1} + \mu_1 v^{p_2} + \beta_1 u^{p_3} v^{p_4},\\ (-\Delta)^{\alpha/2}v = \lambda_2 u^{q_1} + \mu_2 v^{q_2} + \beta_2 u^{q_3} v^{q_4},\end{cases} \tag{1}$$

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where $(-\Delta)^{\alpha/2}$ is a higher-order fractional Laplacian, $0 < \alpha < n$, $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$ are constants, and p_i and q_i $(i = 1, 2, 3, 4)$ satisfy some suitable assumptions.

System (1) arises from N-coupled higher-order nonlinear Schrödinger systems

$$
\begin{cases} i\dfrac{\partial \Phi_j}{\partial t} - (-\Delta)^m \Phi_j + \sum\limits_{i=1}^{N} \beta_{ij} |\Phi_i|^2 \Phi_j = 0, & y \in \mathbb{R}^n,\ t > 0, \\ \Phi_j(y, t) \to 0, & \text{as } y \to \infty,\ t > 0,\ j = 1, 2, \dots, N, \end{cases} \tag{2}
$$

for $m \in \mathbb{N}$, $\beta_{ij} = \beta_{ji}$. System (2) appears in some physical problems, especially in nonlinear optics. When $m = 1$, $n \leq 3$, it describes physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photo refractive media in optics (see [Akhmediev and Ankiewicz 1999]) and Bose–Einstein condensates (see [Esry et al. 1997]). When the spatial dimension is one, i.e., $n = 1$, system (2) has applications in quantum mechanics (see [Liu et al. 2007; Fu et al. 2009]).

Letting $\Phi_j(y, t) = e^{-i\lambda_j t} u_j(y)$, system (2) is transformed into the elliptic system

$$
\begin{cases} (-\Delta)^m u_j = \lambda_j u_j + \sum\limits_{i=1}^{N} \beta_{ij} |u_i|^2 u_j & \text{in } \mathbb{R}^n, \\ u_j(x) \to 0, & \text{as } |x| \to \infty,\ j = 1, 2, \dots, N. \end{cases} \tag{3}
$$

Clearly, in some sense, system (1) extends system (3) with $N = 2$.

For further discussion, we need an integral form of system (1). In Section 5, we will show that system (1) is equivalent to the following system:

$$
\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y) v^{p_4}(y)}{|x - y|^{n-\alpha}}\, dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\lambda_2 u^{q_1}(y) + \mu_2 v^{q_2}(y) + \beta_2 u^{q_3}(y) v^{q_4}(y)}{|x - y|^{n-\alpha}}\, dy. \end{cases} \tag{4}
$$

In particular, when $\mu_1 = \lambda_2 = 1$ and $\lambda_1 = \mu_2 = \beta_1 = \beta_2 = 0$, system (4) reduces to

$$
\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{v^{p_2}(y)}{|x - y|^{n-\alpha}}\, dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u^{q_1}(y)}{|x - y|^{n-\alpha}}\, dy, \end{cases} \tag{5}
$$

which is closely related to the maximizer of the best constant in a Hardy–Littlewood–Sobolev (HLS) inequality; see [Chen et al. 2005; Chen and Li 2005].

In recent years many works have been devoted to the study of the special cases of system (1) or system (4). In the case of $\alpha = 2$, under certain assumptions, the existence of bound state solutions and radially symmetric solutions of (3) was studied in [Bartsch et al. 2007; 2010; Busca and Sirakov 2000; Dancer et al. 2010; Liu and Wang 2008; Guo and Liu 2008; Hioe 1999; Lin and Wei 2005; Maia et al. 2006; Sirakov 2007; Wei and Weth 2007; 2008]. In particular, for $\alpha = 2$ ($n \geq 3$) and $\lambda_i = \mu_i = 1$, $\beta_i \geq 0$ ($i = 1, 2$), de Figueiredo and Sirakov [2005] proved the nonexistence of positive solutions for system (1) under some subcritical exponent conditions. When $m = n = 1$, system (2) is integrable, and there are many analytical and numerical results on solitary wave solutions of higher-order nonlinear Schrödinger equations (e.g., see [Liu et al. 2007; Fu et al. 2009]).

In the case of $\alpha = 2m$ ($m = 1, 2, \dots$) and $\mu_1 = \lambda_2 = 1$, $\lambda_1 = \mu_2 = \beta_1 = \beta_2 = 0$, system (1) becomes

$$\begin{cases} (-\Delta)^m u = v^{p_2}, \\ (-\Delta)^m v = u^{q_1}, \end{cases} \tag{6}$$

in $\mathbb{R}^n$. This system is equivalent to the integral system (5) with $\alpha = 2m$ (see [Chen and Li 2009b]). Guo, Liu and Zhang [Liu et al. 2006; Zhang 2007] proved that any positive solutions of (6) are radially symmetric for critical exponents $p_2 = q_1 = \frac{n+2m}{n-2m}$. Moreover, they also showed that there are no positive solutions of (6) if $p_2, q_1 \geq 1$, but are not both equal to 1, and satisfy the following subcritical exponent condition:

$$\frac{1}{p_2+1} + \frac{1}{q_1+1} > \frac{n-2m}{n}.$$

Assuming that p_2 and q_1 satisfy $\frac{\alpha}{n-\alpha} < p_2, q_1 < \infty$, under natural integrability conditions on u and v, Chen, Li and Ou [Chen et al. 2005; Chen and Li 2005] and Hang [2007] discussed the symmetry, monotonicity and regularity of positive solutions of system (5) with the critical exponent condition

$$\frac{1}{p_2+1} + \frac{1}{q_1+1} = \frac{n-\alpha}{n}.$$

Furthermore, Chen and Li [2009b] proved the nonexistence of positive solutions of system (5) satisfying some subcritical exponents assumptions.

In [Dou et al. 2011], we studied the symmetry, monotonicity and regularity of positive solutions of integral system (5) with weighted functions for $\max\{1, \frac{\alpha}{n-\alpha}\} < p_2, q_1 < \infty$ and

$$\frac{1}{p_2+1} + \frac{1}{q_1+1} \geq \frac{n-\alpha}{n}.$$

In addition, the nonexistence result for positive solutions of system (5) with $0 < p_2, q_1 < \frac{n+\alpha}{n-\alpha}$ was established.

In the case of $\lambda_i = 1$, $\mu_i = \beta_i = 0$ and $u(x) = v(x)$, system (1) reduces to the single elliptic equation

$$(-\Delta)^{\alpha/2} u = u^p, \quad \text{in } \mathbb{R}^n. \tag{7}$$

For $p = \frac{n+\alpha}{n-\alpha}$, Chen et al. [2006] and Li [2004] proved that any positive solutions u of Equation (7) are radially symmetric and monotonic about some point. Indeed all the positive solutions are given by

$$u(x) = \left(\frac{C_\alpha}{d + |x - \bar{x}|^2} \right)^{(n-\alpha)/2}, \tag{8}$$

where $d > 0$ is a constant and $C_\alpha = \big(2^{-\alpha}\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{n-\alpha}{2})^{-1}d\big)^{1/2}$. When $\alpha = 2m$ is any even number, the above result was also proved by Wei and Xu [1999], and they showed that there exist no positive solutions of (7) with $0 < \tau < \frac{n+2m}{n-2m}$. Moreover, for $\alpha = 2$, the problem is the so-called Yamabe problem, and the radial symmetry of solutions was discussed by Gidas, Ni and Nirenberg [Gidas et al. 1981].

In this paper, we show that system (1) is equivalent to integral system (4). By the discussion of the symmetry, monotonicity and regularity of positive solutions of integral system (4), we are able to perform the classification of positive solutions to system (1).

Throughout the paper, we use the following notation:

$$\Pi_1 = \big\{ f(x) \mid x \in \mathbb{R}^n,\ f \in L^{s_{11}}(\mathbb{R}^n) \cap L^{s_{21}}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n) \big\},$$
$$\Pi_2 = \big\{ f(x) \mid x \in \mathbb{R}^n,\ f \in L^{s_{12}}(\mathbb{R}^n) \cap L^{s_{22}}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n) \big\},$$

where $s_{1i} = n(p_i - 1)/\alpha$, $s_{2i} = n(q_i - 1)/\alpha$, $i = 1, 2$, and $k_0 = n(p_3 + p_4 - 1)/\alpha = n(q_3+q_4-1)/\alpha$ with $n/(n-\alpha) < p_i, q_i, p_3+p_4, q_3+q_4 < \infty$, and $p_3+p_4 = q_3+q_4$.

We are now in a position to state our main results.

Theorem 1.1. *Assume that* $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$, *and they are not equal to zero simultaneously. Let* (u, v) *be a pair of solutions to system* (4) *with* $u \in \Pi_1$, $v \in \Pi_2$. *Then* $u, v \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *for any* $\frac{n}{n-\alpha} < s < \infty$. *Furthermore*, $u, v \in C^\infty$.

Theorem 1.2. *Assume that* $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$ *and they are not equal to zero at the same time. Let* $(u, v) \in \Pi_1 \times \Pi_2$ *be a pair of solutions to system* (4). *Then* u *and* v *are radially symmetric and decreasing about some point.*

For system (4) with critical exponents, i.e., $p_1 = p_2 = q_1 = q_2 = p_3 + p_4 = q_3 + q_4 = \frac{n+\alpha}{n-\alpha}$, we have:

Theorem 1.3. *Let* $(u, v) \in L^{2n/(n-\alpha)}(\mathbb{R}^n) \times L^{2n/(n-\alpha)}(\mathbb{R}^n)$ *be a pair of positive solutions to system* (4) *with* $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$ *but not equal to zero at the same time. If* $p_1 = p_2 = q_1 = q_2 = p_3 + p_4 = q_3 + q_4 = \frac{n+\alpha}{n-\alpha}$, *then* $u, v \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *for any* $\frac{n}{n-\alpha} < s < \infty$, *and* $u, v \in C^\infty$. *Moreover*, u *and* v *are radially symmetric and decreasing about some point, and* u, v *must be of the following forms*:

$$u(x) = \left(\frac{c_1}{d + |x - \bar{x}|^2} \right)^{(n-\alpha)/2}, \quad v(x) = \left(\frac{c_2}{d + |x - \bar{x}|^2} \right)^{(n-\alpha)/2}, \tag{9}$$

where $\bar{x} \in \mathbb{R}^n$, $c_1, c_2 > 0$, $d > 0$ *and satisfy the conditions*

$$C_\alpha^2 c_1^{(n-\alpha)/2} = \lambda_1 c_1^{(n+\alpha)/2} + \mu_1 c_2^{(n+\alpha)/2} + \beta_1 \big(c_1^{p_3} c_2^{p_4}\big)^{(n-\alpha)/2},$$
$$C_\alpha^2 c_2^{(n-\alpha)/2} = \lambda_2 c_1^{(n+\alpha)/2} + \mu_2 c_2^{(n+\alpha)/2} + \beta_2 \big(c_1^{q_3} c_2^{q_4}\big)^{(n-\alpha)/2}.$$

Theorem 1.4. *System* (1) *is equivalent to integral system* (4).

Combining our symmetry and equivalence results with the known results on nonexistence of positive solutions in the subcritical case (see [Dancer et al. 2010; de Figueiredo and Sirakov 2005]), we can obtain results on the nonexistence of positive solutions (u, v) of system (1) with some suitable conditions.

Theorem 1.5. (i) *Suppose* $n \geq 3$, $\alpha = 2$, $\lambda_i = \mu_i = 1$ *and* $\beta_i = 0$ *for* $i = 1, 2$, *and*

$$\frac{n}{n-2} < p_1, q_2 < \frac{n+2}{n-2}, \quad p_2 = \frac{p_1(q_2-1)}{p_1-1}, \quad q_1 = \frac{q_2(p_1-1)}{q_2-1}.$$

Then system (1) *has no positive solutions* (u, v) *satisfying* $u \in L^{n(p_1-1)/2}(\mathbb{R}^n) \cap L^{n(q_1-1)/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *and* $v \in L^{n(p_2-1)/2}(\mathbb{R}^n) \cap L^{n(q_2-1)/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

(ii) *Assume that* $n \geq 3$, $\alpha = 2$ *and* $\lambda_i, \mu_i > 0$, $\beta_i \geq 0$, $\beta_i \not\equiv 0$, *and* p_j, q_j *satisfy*

$$\frac{n}{n-2} < p_1, q_2 < \frac{n+2}{n-2} \quad \textit{with} \quad p_2 = \frac{p_1(q_2-1)}{p_1-1}, \quad q_1 = \frac{q_2(p_1-1)}{q_2-1},$$

and

$$\frac{p_3}{p_1} + \frac{p_4}{p_2} = \frac{q_3}{q_1} + \frac{q_4}{q_2} = 1$$

with $0 \leq p_3 \leq p_1$, $0 \leq p_4 \leq p_2$, $0 \leq q_3 \leq q_1$, $0 \leq q_4 \leq q_2$, $p_3 + p_4 = q_3 + q_4$. *Then system* (1) *has no positive solutions* (u, v) *satisfying* $u \in \Pi_1 \cap L^\infty(\mathbb{R}^n)$ *and* $v \in \Pi_2 \cap L^\infty(\mathbb{R}^n)$.

(iii) *Assume that* $n = 3$, $\alpha = 2$, $\lambda_2 = \mu_1 = 0$, $\beta_1 = \beta_2 > -\sqrt{\lambda_1 \mu_2}$. *Then system* (1) *has no positive solutions* (u, v) *satisfying* $u \in \Pi_1 \cap L^\infty(\mathbb{R}^n)$ *and* $v \in \Pi_2 \cap L^\infty(\mathbb{R}^n)$, *where* $p_1 = q_2 = 3$, $p_3 = q_4 = 1$, $p_4 = q_3 = 2$.

Remark 1.6. We can show that the results above hold for the more general system

$$\begin{cases} (-\Delta)^{\alpha/2} u = \lambda_1 u^{p_1} + \mu_1 v^{p_2} + \beta_1 u^{p_3} v^{p_4}, \\ (-\Delta)^{\kappa/2} v = \lambda_2 u^{q_1} + \mu_2 v^{q_2} + \beta_2 u^{q_3} v^{q_4} \end{cases} \tag{10}$$

in $\mathbb{R}^n$, where $0 < \alpha, \kappa < n$, $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$. That is, if

$$p_1, p_2, p_3+p_4 > \frac{n}{n-\alpha}, \quad q_1, q_2, q_3+q_4 > \frac{n}{n-\kappa}, \quad s_{1i} = \frac{n(p_i-1)}{\alpha}, \quad s_{2i} = \frac{n(q_i-1)}{\kappa}$$

for $i = 1, 2$, and $u, v \in L^{k_0}(\mathbb{R}^n) \cap L^{k_1}(\mathbb{R}^n)$, where

$$k_0 = \frac{n(p_3 + p_4 - 1)}{\alpha}, \quad k_1 = \frac{n(q_3 + q_4 - 1)}{\kappa},$$

then the results of Theorem 1.1, Theorem 1.2 and Theorem 1.4 are still valid.

We remark that a more general system of m equations has been discussed by Chen and Li [2009a]. That is,

$$(11)\qquad \begin{cases} u_j(x) = \displaystyle\int_{\mathbb{R}^n} \frac{f_j(u(y))}{|x-y|^{n-\alpha}}\,dy, & j = 1, 2, \dots, m, \\ u(x) = (u_1(x), u_2(x), \dots, u_m(x)), \end{cases} \quad \text{in } \mathbb{R}^n$$

where $f_j(u) \geq 0$ are continuous real-valued functions and homogeneous of degree $\frac{n+\alpha}{n-\alpha}$, and satisfy $\partial f_i / \partial u_j \geq 0$ for $i = 1, 2, \dots, m$. System (11) includes only the critical exponent case of system (4). It was shown in [Chen and Li 2009a] any positive solutions of (11) are radially symmetric under the assumptions $u_j \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore, based on the Kelvin transformation and the results in [Chen et al. 2006], any positive solutions of (11) must be the form of (8). In our proof of Theorem 1.3, a key calculus lemma due to Li and Zhu [1995] and the Kelvin transformation are used to show that all positive solutions of (4) are given by (9).

The main difficulty in our proof is the lack of a maximum principle for the higher-order fractional Laplace operator. Theorem 1.4 says that system (1) is equivalent to the integral system (4), which is helpful for our discussion since we can use the method of moving planes in integral forms (see [Chen et al. 2006]) to discuss the radial symmetry and monotonicity of positive solution of the integral system (4). Furthermore, the regularity of solutions to system (4) is proved by the regularity lifting lemma introduced in [Chen and Li 2010; Ma et al. 2011].

The paper is organized as follows. In Section 2, we prove the regularity of solutions of system (4) (Theorem 1.1). The radially symmetric property and monotonicity of solutions are studied in Section 3 (Theorem 1.2). In Section 4, positive solutions of system (4) with critical exponents are classified. Namely, Theorem 1.3 is proved. In Section 5, we obtain some nonexistence results by proving Theorems 1.4 and 1.5.

Throughout the paper, we always assume that $\lambda_i, \mu_i, \beta_i \geq 0$ $(i = 1, 2)$ and they are not equal to zero simultaneously. Moreover, for convenience of presentation we shall use c, c_1, C, etc. for a suitable positive constants unless indicated otherwise.

2. Regularity

In this section, we prove the regularity of solutions to system (4). To this end, we need the following regularity lifting lemma (see [Chen and Li 2010; Ma et al. 2011]). An earlier version was introduced in [Chen and Li 2005].

Let V be a topological vector space. Suppose there are two extended norms (i.e., the norm of an element in V might be infinity) defined on V,

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0, \infty].$$

Let

$$X := \{f \in V : \|f\|_X < \infty\} \quad \text{and} \quad Y := \{f \in V : \|f\|_Y < \infty\}.$$

Lemma 2.1. *Let T be a contraction map from X into itself and from Y into itself. Assume that for any $f \in X$ there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g$ in X. Then $f \in Z$.*

We also need an equivalent form of the HLS inequality (see [Chen and Li 2005; 2010]): let $C(n, \alpha, p)$ be a uniform positive constant and define

$$Tf(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Assume that $f \in L^p(\mathbb{R}^n)$ for $\frac{n}{n-\alpha} < p < \infty$. Then

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C(n, \alpha, p)\|f\|_{L^{\frac{np}{n+\alpha p}}(\mathbb{R}^n)}. \tag{12}$$

Denote

$$u_R(x) = \begin{cases} u(x), & |u(x)| > R, \\ 0, & |u(x)| \le R. \end{cases}$$

Assume that $\phi \in L^r(\mathbb{R}^n)$, $\varphi \in L^s(\mathbb{R}^n)$ for $\frac{n}{n-\alpha} < r, s < \infty$. Define

$$T_1(\phi, \varphi) = \int_{\mathbb{R}^n} \frac{\lambda_1 u_R^{p_1-1}(y)}{|x-y|^{n-\alpha}}\phi(y)\, dy + \int_{\mathbb{R}^n} \frac{\mu_1 v_R^{p_2-1}(y) + \beta_1 u_R^{p_3}(y) v_R^{p_4-1}(y)}{|x-y|^{n-\alpha}}\varphi(y)\, dy,$$

$$T_2(\phi, \varphi) = \int_{\mathbb{R}^n} \frac{\mu_2 v_R^{q_2-1}(y)}{|x-y|^{n-\alpha}}\varphi(y)\, dy + \int_{\mathbb{R}^n} \frac{\lambda_2 u_R^{q_1-1}(y) + \beta_2 v_R^{q_4}(y) u_R^{q_3-1}(y)}{|x-y|^{n-\alpha}}\phi(y)\, dy.$$

Let $u_b(x) = u(x) - u_R(x)$, and

$$f_R(x) = \int_{\mathbb{R}^n} \frac{\mu_1 v_b^{p_2-1}(y) + \beta_1 u_b^{p_3}(y) v_b^{p_4-1}(y)}{|x-y|^{n-\alpha}} v(y)\, dy + \int_{\mathbb{R}^n} \frac{\lambda_1 u_b^{p_1-1}(y)}{|x-y|^{n-\alpha}} u(y)\, dy,$$

$$g_R(x) = \int_{\mathbb{R}^n} \frac{\lambda_2 u_b^{q_1-1}(y) + \beta_2 v_b^{q_4}(y) u_b^{q_3-1}(y)}{|x-y|^{n-\alpha}} u(y)\, dy + \int_{\mathbb{R}^n} \frac{\mu_2 v_b^{q_2-1}(y)}{|x-y|^{n-\alpha}} v(y)\, dy.$$

Denote the norm in the cross product space $L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ by

$$\|(u, v)\|_{r\times s} = \|u\|_r + \|v\|_s,$$

and define the mapping $T : L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \to L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ by

$$T(\phi, \varphi) = \big(T_1(\phi, \varphi), T_2(\phi, \varphi)\big).$$

Throughout the paper, we use the notation $\|u\|_s = \|u\|_{L^s(\mathbb{R}^n)}$.

Consider the equation

$$(\phi, \varphi) = T(\phi, \varphi) + (f_R, g_R). \tag{13}$$

Notice that there is no intersection between the supports of u_R, v_R and u_b, v_b, so (u, v) is a pair of solutions of (13).

Proof of Theorem 1.1. The proof is divided into three steps.

Step 1. Firstly, we show $u, v \in L^s(\mathbb{R}^n)$ for all $s > \frac{n}{n-\alpha}$. To this end, we show that

(i) T is a contracting map from $L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ to itself for R large enough;

(ii) f_R and g_R belong to $L^s(\mathbb{R}^n)$.

We first show (i). For any $\phi, \varphi \in L^s(\mathbb{R}^n)$, using the HLS inequality (12) and the Minkowski inequality, we have

$$\|T_1(\phi, \varphi)\|_s \le C(n, \alpha, \gamma)\big(\lambda_1 \big\|u_R^{p_1-1}\phi\big\|_\theta + \mu_1 \big\|{v_R}^{p_2-1}\varphi\big\|_\theta + \beta_1 \big\|{u_R}^{p_3}{v_R}^{p_4-1}\varphi\big\|_\theta\big), \tag{14}$$

where $\theta = \frac{ns}{n+\alpha s}$. By the Hölder inequality, we have

$$\big\|u_R^{p_1-1}\phi\big\|_\theta \le \|u_R\|_{s_{11}}^{p_1-1}\|\phi\|_s, \quad \big\|{v_R}^{p_2-1}\varphi\big\|_\theta \le \|v_R\|_{s_{12}}^{p_2-1}\|\varphi\|_s, \tag{15}$$

where $s_{1j} = n(p_j - 1)/\alpha$, $j = 1, 2$, and

$$\begin{aligned}\big\|{u_R}^{p_3}{v_R}^{p_4-1}\varphi\big\|_\theta &\le \Big(\int_{\mathbb{R}^n} u_R^{p_3 t_1\theta}(y)\,dy\Big)^{\frac{1}{t_1\theta}}\Big(\int_{\mathbb{R}^n} v_R^{(p_4-1)t_2\theta}(y)\,dy\Big)^{\frac{1}{t_2\theta}}\Big(\int_{\mathbb{R}^n} \varphi^{t_3\theta}(y)\,dy\Big)^{\frac{1}{t_3\theta}}\\ &= \|u_R\|_{k_0}^{p_3}\|v_R\|_{k_0}^{p_4-1}\|\varphi\|_s.\end{aligned} \tag{16}$$

In the above inequality we have chosen $t_3 = (n+s\alpha)/n > 1$, so we take $1/t_1 + 1/t_2 = s\alpha/(n+s\alpha)$ with

$$k_0 = t_1 p_3\theta = t_2(p_4 - 1)\theta,$$

and then

$$\frac{p_3}{k_0} + \frac{p_4-1}{k_0} = \frac{\alpha}{n}.$$

Substituting (15) and (16) into (14), we deduce that

$$\|T_1(\phi, \varphi)\|_s \le c\|u_R\|_{s_{11}}^{p_1-1}\|\phi\|_s + c\big(\|v_R\|_{s_{12}}^{p_2-1} + \|u_R\|_{k_0}^{p_3}\|v_R\|_{k_0}^{p_4-1}\big)\|\varphi\|_s. \tag{17}$$

Since $u \in L^{s_{11}}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n)$, $v \in L^{s_{12}}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n)$, we may choose R large enough such that

$$\|u_R\|_{s_{11}}^{p_2-1} \le \tfrac14, \quad \|v_R\|_{s_{12}}^{p_2-1} + \|u_R\|_{k_0}^{p_3}\|v_R\|_{k_0}^{p_4-1} \le \tfrac14.$$

Hence, from (17) we obtain

$$\|T_1(\phi, \varphi)\|_s \le \tfrac14\big(\|\phi\|_s + \|\varphi\|_s\big). \tag{18}$$

Similarly, we have

$$\|T_2(\phi,\varphi)\|_s \le \tfrac{1}{4}\big(\|\phi\|_s+\|\varphi\|_s\big). \tag{19}$$

Combining (18) and (19), one obtains

$$\|T(\phi,\varphi)\|_{s\times s} \le \tfrac{1}{2}\big(\|\phi\|_s+\|\varphi\|_s\big).$$

It turns out that T is the contracting map from $L^s(\mathbb{R}^n)\times L^s(\mathbb{R}^n)$ to itself.

(ii) Next we estimate f_R and g_R. We write

$$\begin{aligned} f_R(x) &= \int_{\mathbb{R}^n} \frac{\mu_1 v_b^{p_2-1}(y)+\beta_1 u_b^{p_3}(y) v_b^{p_4-1}(y)}{|x-y|^{n-\alpha}} v(y)\,dy + \int_{\mathbb{R}^n} \frac{\lambda_1 u_b^{p_1-1}(y)}{|x-y|^{n-\alpha}} u(y)\,dy \\ &=: J_1+J_2. \end{aligned} \tag{20}$$

For any $s>\frac{n}{n-\alpha}$, we apply the HLS inequality, Minkowski inequality and Hölder inequality to get

$$\begin{aligned} \|J_1\|_s &\le c\big\|{v_b}^{p_2-1}v\big\|_\theta + c\big\|{u_b}^{p_3}{v_b}^{p_4-1}v\big\|_\theta \\ &\le c\|v_b\|_{k_1}^{p_2-1}\|v\|_{k_2} + c\|u_b\|_{k_3}^{p_3}\|v_b\|_{k_4}^{p_4-1}\|v\|_{k_5}, \end{aligned} \tag{21}$$

and

$$\|J_2\|_s \le c\big\|u_b^{p_1-1}u\big\|_\theta \le c\|u_b\|_{k_6}^{p_1-1}\|u\|_{k_7}, \tag{22}$$

where

$$\frac{p_2-1}{k_1}+\frac{1}{k_2} = \frac{p_3}{k_3}+\frac{p_4-1}{k_4}+\frac{1}{k_5} = \frac{p_1-1}{k_6}+\frac{1}{k_7} = \frac{n+\alpha s}{ns} = \frac{1}{s}+\frac{\alpha}{n}.$$

Since v_b, u_b are bounded, k_1,k_3,k_4,k_6 can be chosen arbitrarily. Notice that $\frac{n}{n-\alpha} < p_3+p_4 = q_3+q_4$, so it follows that $k_0 = n(p_3+p_4-1)/\alpha = n(q_3+q_4-1)/\alpha > n/(n-\alpha)$. In view of $u, v\in L^{k_0}(\mathbb{R}^n)$, we may choose $k_2=k_5=k_7=k_0$ such that

$$\begin{aligned} \frac{1}{s} &= \frac{p_2-1}{k_1}+\frac{1}{k_0}-\frac{\alpha}{n} = \frac{p_2-1}{k_1}+\frac{n-\alpha k_0}{nk_0} \\ &= \frac{p_3}{k_3}+\frac{p_4-1}{k_4}+\frac{n-\alpha k_0}{nk_0} \\ &= \frac{p_1-1}{k_6}+\frac{n-\alpha k_0}{nk_0}. \end{aligned}$$

Now, letting $k_1,k_3,k_4,k_6\to\infty$, the previous equation implies that

$$s \to \frac{nk_0}{n-\alpha k_0}.$$

We conclude that $f_R\in L^{nk_0/(n-\alpha k_0)-\epsilon}(\mathbb{R}^n)$ for any small $\epsilon>0$. Obviously, $nk_0/(n-\alpha k_0) > k_0$. Similarly, we can show $g_R\in L^{nk_0/(n-\alpha k_0)-\epsilon}(\mathbb{R}^n)$.

By Lemma 2.1, if $n \le \alpha k_0$, we are done. If $n > \alpha k_0$, we repeat the above process, and after a few steps, we obtain

$$u,\, v \in L^s(\mathbb{R}^n), \qquad \frac{n}{n-\alpha} < s < \infty.$$

Step 2. We show $u, v \in L^\infty(\mathbb{R}^n)$. We split $u(x)$ into two parts, i.e., $u(x)$ can be written as

$$\begin{aligned} u(x) = &\int_{B_1(x)} \frac{\lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y) v^{p_4}(y)}{|x-y|^{n-\alpha}}\, dy \\ &+ \int_{\mathbb{R}^n \setminus B_1(x)} \frac{\lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y) v^{p_4}(y)}{|x-y|^{n-\alpha}}\, dy \\ =:&\, J_3 + J_4. \end{aligned}$$

We estimate J_3 and J_4 separately. First consider J_4. Since $1/|x-y|^{n-\alpha} < 1$, and $u, v \in L^s(\mathbb{R}^n)$ for any $\frac{n}{n-\alpha} < s$, according to the assumptions that $\frac{n}{n-\alpha} < p_1, p_2, p_3 + p_4$, and using the Hölder inequality, we have

$$\begin{aligned} J_4 &\le c \int_{\mathbb{R}^n \setminus B_1(x)} u^{p_1}(y)\, dy + c \int_{\mathbb{R}^n \setminus B_1(x)} v^{p_2}(y)\, dy + c \int_{\mathbb{R}^n \setminus B_1(x)} u^{p_3}(y) v^{p_4}(y)\, dy \\ &< \infty. \end{aligned}$$

Next, we compute J_3:

$$\begin{aligned} J_3 \le\, & c\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}}\, dy\right)^{1/p} \left(\int_{B_1(x)} |u(y)|^{p_1 p/(p-1)}\, dy\right)^{(p-1)/p} \\ &+ c\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}}\, dy\right)^{1/p} \left(\int_{B_1(x)} |v(y)|^{p_2 p/(p-1)}\, dy\right)^{(p-1)/p} \\ &+ c\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}}\, dy\right)^{1/p} \left(\int_{B_1(x)} \bigl(|u^{p_3}(y) v^{p_4}(y)|\bigr)^{p/(p-1)}\, dy\right)^{(p-1)/p}. \end{aligned}$$

Choose the constant p such that $(n-\alpha)p < n$, and then

$$\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}}\, dy\right)^{1/p} < C.$$

Since $u, v \in L^s(\mathbb{R}^n)$ for any $\frac{n}{n-\alpha} < s < \infty$, we get

$$\left(\int_{B_1(x)} |u(y)|^{p_1 p/(p-1)}\, dy\right)^{(p-1)/p} < C, \qquad \left(\int_{B_1(x)} |v(y)|^{p_2 p/(p-1)}\, dy\right)^{(p-1)/p} < C,$$

and by the Hölder inequality, we obtain

$$\left(\int_{B_1(x)} |u^{p_3}(y)v^{p_4}(y)|^{\frac{p}{p-1}}\,dy\right)^{\frac{p-1}{p}} \leq \left(\int_{B_1(x)} |u(y)|^{p_3 l_1 \frac{p}{p-1}}\,dy\right)^{\frac{p-1}{l_1 p}} \left(\int_{B_1(x)} |v(y)|^{p_4 l_2 \frac{p}{p-1}}\,dy\right)^{\frac{p-1}{l_2 p}} < C,$$

where $l_1, l_2 > 1$ and $1/l_1 + 1/l_2 = 1$. (We may choose $l_1 = (p_3 + p_4)/p_3$ and $l_2 = (p_3 + p_4)/p_4$.) So $u \in L^\infty(\mathbb{R}^n)$. Arguing as above, it also follows that $v \in L^\infty(\mathbb{R}^n)$.

Step 3. Using the usual bootstrap method, as in [Li 2004], we conclude $u, v \in C^\infty(\mathbb{R}^n)$. □

3. Radial symmetry and monotonicity

In this section, we use the method of moving plane in integral form to prove Theorem 1.2. The moving plane method in integral form used here was introduced by Chen, Li and Ou [2006] and exploits global properties of integral equations instead of using the amount of local properties of differential operators as the traditional moving plane method (e.g., see [Guo and Liu 2008; de Figueiredo and Sirakov 2005; Liu et al. 2006; Zhang 2007; Wei and Xu 1999; Gidas et al. 1981]).

We first deduce two representation formulas related to $u(x)$ and $v(x)$, respectively. Let λ be a real number. Define

$$\Sigma_\lambda = \{x = (x_1 \cdots x_n) \mid x_1 \geq \lambda\},$$

and set

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n), \quad u_\lambda(x) = u(x^\lambda) \quad \text{and} \quad v_\lambda(x) = v(x^\lambda).$$

For convenience, we set $Q_y(u, v) = \lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y) v^{p_4}(y)$ and $K_y(u, v) = \lambda_2 u^{q_1}(y) + \mu_2 v^{q_2}(y) + \beta_2 u^{q_3}(y) v^{q_4}(y)$. In view of (4), we have

$$\begin{aligned} u_\lambda(x) &= \int_{\mathbb{R}^n} \frac{Q_y(u, v)}{|x^\lambda - y|^{n-\alpha}}\,dy \\ &= \int_{\Sigma_\lambda} \frac{Q_y(u, v)}{|x^\lambda - y|^{n-\alpha}}\,dy + \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{Q_y(u, v)}{|x^\lambda - y|^{n-\alpha}}\,dy \\ &= \int_{\Sigma_\lambda} \frac{Q_y(u, v)}{|x^\lambda - y|^{n-\alpha}}\,dy + \int_{\Sigma_\lambda} \frac{Q_y(u_\lambda, v_\lambda)}{|x^\lambda - y^\lambda|^{n-\alpha}}\,dy. \end{aligned}$$

We also have

$$v_\lambda(x) = \int_{\Sigma_\lambda} \frac{K_y(u, v)}{|x^\lambda - y|^{n-\alpha}}\,dy + \int_{\Sigma_\lambda} \frac{K_y(u_\lambda, v_\lambda)}{|x^\lambda - y^\lambda|^{n-\alpha}}\,dy.$$

Noting that $|x^\lambda - y^\lambda| = |x-y|$, it is easy to see that

$$
\begin{aligned}
(23)\quad & u_\lambda(x)-u(x)\\
&=\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)(Q_y(u_\lambda,v_\lambda)-Q_y(u,v))\,dy\\
&=\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)\big[\lambda_1\big(u_\lambda^{p_1}(y)-u^{p_1}(y)\big)+\mu_1\big(v_\lambda^{p_2}(y)-v^{p_2}(y)\big)\\
&\qquad\qquad +\beta_1\big(u_\lambda^{p_3}(y)v_\lambda^{p_4}(y)-u^{p_3}(y)v^{p_4}(y)\big)\big]\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
(24)\quad & v_\lambda(x)-v(x)\\
&=\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)(K_y(u_\lambda,v_\lambda)-K_y(u,v))\,dy\\
&=\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)\big[\lambda_2\big(u_\lambda^{q_1}(y)-u^{q_1}(y)\big)+\mu_2\big(v_\lambda^{q_2}(y)-v^{q_2}(y)\big)\\
&\qquad\qquad +\beta_2\big(u_\lambda^{q_3}(y)v_\lambda^{q_4}(y)-u^{q_3}(y)v^{q_4}(y)\big)\big]\,dy.
\end{aligned}
$$

The next lemma shows the plane can start moving from $x_1=-\infty$ to the right.

Lemma 3.1. *Let $(u,v)\in\Pi_1\times\Pi_2$ be a pair of positive solutions of* (4). *Then, for λ sufficiently negative*,

$$
(25)\qquad u(x)\ge u_\lambda(x)\quad\text{and}\quad v(x)\ge v_\lambda(x)\quad\text{for all } x\in\Sigma_\lambda.
$$

Proof. Define

$$
\Sigma_\lambda^u=\{x\in\Sigma_\lambda \mid u(x)<u_\lambda(x)\}\quad\text{and}\quad \Sigma_\lambda^v=\{x\in\Sigma_\lambda \mid v(x)<v_\lambda(x)\}.
$$

Let Σ_λ^c be the complement of Σ_λ. From (23) and the mean value theorem, we have

$$
\begin{aligned}
(26)\quad & u_\lambda(x)-u(x)\\
&=\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)\\
&\qquad\times\big[\lambda_1(u_\lambda^{p_1}(y)-u^{p_1}(y))+\mu_1(v_\lambda^{p_2}(y)-v^{p_2}(y))\\
&\qquad\qquad +\beta_1\big(u_\lambda^{p_3}(y)(v_\lambda^{p_4}(y)-v^{p_4}(y))+v^{p_4}(y)(u_\lambda^{p_3}(y)-u^{p_3}(y))\big)\big]\,dy\\
&\le\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^\lambda-y|^{n-\alpha}}\right)\\
&\qquad\times\big[p_1\lambda_1\phi_1^{p_1-1}(u)(u_\lambda(y)-u(y))+p_2\mu_1\phi_2^{p_2-1}(v)(v_\lambda(y)-v(y))\\
&\qquad\qquad +p_4\beta_1u_\lambda^{p_3}(y)\phi_4^{p_4-1}(v)(v_\lambda(y)-v(y))\\
&\qquad\qquad +p_3\beta_1v^{p_4}(y)\phi_3^{p_3-1}(u)(u_\lambda(y)-u(y))\big]\,dy,
\end{aligned}
$$

where $u(y) \le \phi_i(u) \le u_\lambda(y)$, $i = 1, 3$, on Σ_λ^u, and $v(y) \le \phi_j(v) \le v_\lambda(y)$, $j = 2, 4$ on Σ_λ^v. It follows that we can write

$$u_\lambda(x) - u(x) \le c(I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 := \int_{\Sigma_\lambda^u} \frac{u_\lambda^{p_1-1}(y)}{|x-y|^{n-\alpha}}(u_\lambda(y)-u(y))\,dy, \quad I_2 := \int_{\Sigma_\lambda^v} \frac{v_\lambda^{p_2-1}(y)}{|x-y|^{n-\alpha}}(v_\lambda(y)-v(y))\,dy,$$

$$I_3 := \int_{\Sigma_\lambda^v} \frac{u_\lambda^{p_3}(y)v_\lambda^{p_4-1}(y)}{|x-y|^{n-\alpha}}(v_\lambda(y)-v(y))\,dy,$$

$$I_4 := \int_{\Sigma_\lambda^u} \frac{v^{p_4}(y)u_\lambda^{p_3-1}(y)}{|x-y|^{n-\alpha}}(u_\lambda(y)-u(y))\,dy.$$

Using the HLS inequality and the Hölder inequality, we get

$$\left(\int_{\Sigma_\lambda^u} |I_1|^\gamma\right)^{1/\gamma} \le C(n,\alpha,\gamma)\left\|{u_\lambda}^{p_1-1}(u_\lambda - u)\right\|_{L^\theta(\Sigma_\lambda^u)} \tag{27}$$

for any $\frac{n}{n-\alpha} < \gamma < \infty$ and $\theta = \frac{n\gamma}{n+\alpha\gamma}$. Let $m_1 = \frac{n+\alpha\gamma}{\alpha\gamma} > 1$ and $m_2 = \frac{n+\alpha\gamma}{n} > 1$. Thus, we invoke the Hölder inequality to obtain

$$\begin{aligned} &\left\{\int_{\Sigma_\lambda^u} \left[{u_\lambda}^{p_1-1}(y)(u_\lambda(y) - u(y))\right]^\theta dy\right\}^{1/\theta} \\ &\quad \le \left\{\left[\int_{\Sigma_\lambda^u} (u_\lambda(y))^{\theta(p_1-1)m_1}\,dy\right]^{1/m_1}\left[\int_{\Sigma_\lambda^u} (u_\lambda(y) - u(y))^{\theta m_2}\,dy\right]^{1/m_2}\right\}^{1/\theta} \\ &\quad = \|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma_\lambda^u)}}\|u_\lambda - u\|_{L^\gamma_{(\Sigma_\lambda^u)}}, \end{aligned} \tag{28}$$

where $s_{11} = n(p_1-1)/\alpha$. Substituting (28) into (27), we get

$$\left(\int_{\Sigma_\lambda^u} |I_1|^\gamma\right)^{1/\gamma} \le C(n,\alpha,\gamma)\|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma_\lambda^u)}}\|u_\lambda - u\|_{L^\gamma_{(\Sigma_\lambda^u)}}. \tag{29}$$

Similarly, one has

$$\|I_2\|_{L^\gamma_{(\Sigma_\lambda^u)}} \le C(n,\alpha,\gamma)\|v_\lambda\|^{p_2-1}_{L^{s_{12}}_{(\Sigma_\lambda^v)}}\|v_\lambda - v\|_{L^\gamma_{(\Sigma_\lambda^v)}}, \tag{30}$$

where $s_{12} = n(p_2-1)/\alpha$.

Next, we estimate I_3 and I_4. By the HLS inequality we have

$$\|I_3\|_{L^\gamma_{(\Sigma_\lambda^u)}} \le C(n,\alpha,\gamma)\left\|{u_\lambda}^{p_3}{v_\lambda}^{p_4-1}(v_\lambda - v)\right\|_{L^\theta_{(\Sigma_\lambda^v)}}. \tag{31}$$

Letting $1/t_1 + 1/t_2 + 1/t_3 = 1$ for $t_1 > 1$, it follows that

$$\begin{aligned}(32)\quad &\left\| {u_\lambda}^{p_3} {v_\lambda}^{p_4-1}(v_\lambda - v)\right\|_{L^\theta_{(\Sigma^v_\lambda)}}\\ &\le \left[\int_{\Sigma^v_\lambda} u_\lambda^{p_3\theta t_1}(y)\,dy\right]^{\frac{1}{t_1\theta}} \left[\int_{\Sigma^v_\lambda} v_\lambda^{(p_4-1)\theta t_2}(y)\,dy\right]^{\frac{1}{t_2\theta}} \left[\int_{\Sigma^v_\lambda} (v_\lambda(y)-v(y))^{\theta t_3}\,dy\right]^{\frac{1}{t_3\theta}}\\ &\le \|u_\lambda\|^{p_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}.\end{aligned}$$

Arguing as Section 2, we choose $t_3 = (n+\alpha\gamma)/n > 1$, $t_1 = (n+\alpha\gamma)(p_3 + p_4 - 1)/(p_3\alpha\gamma)$ and $t_2 = (n+\alpha\gamma)(p_3+p_4-1)/((p_4-1)\alpha\gamma)$, satisfying $1/t_1+1/t_2+1/t_3=1$. Then $k_0 = t_1 p_3\theta = t_2(p_4 - 1)\theta$. Substituting (32) into (31), we conclude

$$(33)\qquad \|I_3\|_{L^\gamma_{(\Sigma^u_\lambda)}} \le C(n,\alpha,\gamma)\|u_\lambda\|^{p_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}.$$

In the same way, one has

$$(34)\qquad \|I_4\|_{L^\gamma_{(\Sigma^u_\lambda)}} \le C(n,\alpha,\gamma)\|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda\|^{p_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}}.$$

Now, we compute the norm $L^\gamma(\Sigma^u_\lambda)$ of (26) for any $\frac{n}{n-\alpha} < \gamma < \infty$. Using the Minkowski inequality and combining (29), (30), (33) and (34), we arrive at

$$\begin{aligned}(35)\quad &\|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}}\\ &\le c\|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^u_\lambda)}} \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} + c\|v_\lambda\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^v_\lambda)}} \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}\\ &\quad + c\|u_\lambda\|^{p_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} + c\|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda\|^{p_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}}\\ &\le c\Big(\|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^u_\lambda)}} + \|u_\lambda\|^{p_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\Big) \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}}\\ &\quad + c\Big(\|v_\lambda\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{p_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}}\Big) \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}.\end{aligned}$$

Along the same line, noting that $p_3 + p_4 = q_3 + q_4$, we have

$$\begin{aligned}(36)\quad \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} &\le c\Big(\|u_\lambda\|^{q_1-1}_{L^{s_{21}}_{(\Sigma^u_\lambda)}} + \|u_\lambda\|^{q_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\Big) \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}}\\ &\quad + c\Big(\|v_\lambda\|^{q_2-1}_{L^{s_{22}}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{q_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{q_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}}\Big) \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}},\end{aligned}$$

where $s_{21} = n(q_1 - 1)/\alpha$, $s_{22} = n(q_2 - 1)/\alpha$, $k_0 = n(q_3 + q_4 - 1)/\alpha$.

By adding (35) and (36), we obtain

$$
\begin{aligned}
(37)\quad & \|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} + \|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}} \\
&\le c\Big(\|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^u_\lambda)}} + \|u_\lambda\|^{q_1-1}_{L^{s_{21}}_{(\Sigma^u_\lambda)}} + \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u_\lambda\|^{p_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} + \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u_\lambda\|^{q_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}}\Big) \\
&\qquad \times \|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} \\
&\quad + c\Big(\|v_\lambda\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^v_\lambda)}} + \|v_\lambda\|^{q_2-1}_{L^{s_{22}}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{p_3}_{L^{k_0}_{(\Sigma^v_\lambda)}}\|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{q_3}_{L^{k_0}_{(\Sigma^v_\lambda)}}\|v_\lambda\|^{q_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}}\Big) \\
&\qquad \times \|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}} \\
&\le c\Big(\|u\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^c_\lambda)}} + \|u\|^{q_1-1}_{L^{s_{21}}_{(\Sigma^c_\lambda)}} + \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u\|^{p_3-1}_{L^{k_0}_{(\Sigma^c_\lambda)}} + \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u\|^{q_3-1}_{L^{k_0}_{(\Sigma^c_\lambda)}}\Big) \\
&\qquad \times \|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} \\
&\quad + c\Big(\|v\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^c_\lambda)}} + \|v\|^{q_2-1}_{L^{s_{22}}_{(\Sigma^c_\lambda)}} + \|u\|^{p_3}_{L^{k_0}_{(\Sigma^c_\lambda)}}\|v\|^{p_4-1}_{L^{k_0}_{(\Sigma^c_\lambda)}} + \|u\|^{q_3}_{L^{k_0}_{(\Sigma^c_\lambda)}}\|v\|^{q_4-1}_{L^{k_0}_{(\Sigma^c_\lambda)}}\Big) \\
&\qquad \times \|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}}.
\end{aligned}
$$

Since $u \in \Pi_1$ and $v \in \Pi_2$, we can choose λ sufficiently negative such that

$$
c\Big(\|u\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^c_\lambda)}} + \|u\|^{q_1-1}_{L^{s_{21}}_{(\Sigma^c_\lambda)}} + \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u\|^{p_3-1}_{L^{k_0}_{(\Sigma^c_\lambda)}} + \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}}\|u\|^{q_3-1}_{L^{k_0}_{(\Sigma^c_\lambda)}}\Big) \le \tfrac{1}{2},
$$

$$
c\Big(\|v\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^c_\lambda)}} + \|v\|^{q_2-1}_{L^{s_{22}}_{(\Sigma^c_\lambda)}} + \|u\|^{p_3}_{L^{k_0}_{(\Sigma^c_\lambda)}}\|v\|^{p_4-1}_{L^{k_0}_{(\Sigma^c_\lambda)}} + \|u\|^{q_3}_{L^{k_0}_{(\Sigma^c_\lambda)}}\|v\|^{q_4-1}_{L^{k_0}_{(\Sigma^c_\lambda)}}\Big) \le \tfrac{1}{2}.
$$

Hence

$$
\|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} + \|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}} \le \tfrac{1}{2}\|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} + \tfrac{1}{2}\|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}}.
$$

This implies that $\|u_\lambda - u\|_{L^{\gamma}_{(\Sigma^u_\lambda)}} = \|v_\lambda - v\|_{L^{\gamma}_{(\Sigma^v_\lambda)}} = 0$, and therefore Σ^u_λ and Σ^v_λ must be empty. Thus, (25) is proved. □

Next we define

$$
\lambda_0 = \sup\{\lambda \in \mathbb{R} \mid u_\mu(x) \le u(x),\, v_\mu(x) \le v(x) \text{ for all } \mu \le \lambda \text{ and all } x \in \Sigma_\mu\}.
$$

By the regularity of positive solutions to system (4), we observe the fact that u and v are bounded as $|x| \to \infty$. Combining this and noting $u, v > 0$, we conclude $\lambda_0 < \infty$. Thus, we will move the plane to the limiting position to derive symmetry. That is, we have the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 1.2, we have*

$$
(38)\qquad u_{\lambda_0}(x) \equiv u(x) \quad \textit{and} \quad v_{\lambda_0}(x) \equiv v(x) \quad \textit{for all } x \in \Sigma_{\lambda_0}.
$$

Proof. We use argument by contradiction. Assume that there exists a $\lambda_0 < 0$ such that $u(x) \ge u_{\lambda_0}(x)$, and $v_\lambda(x) \ge v_{\lambda_0}(x)$, but $u(x) \not\equiv u_{\lambda_0}(x)$ or $v_\lambda(x) \not\equiv v_{\lambda_0}(x)$ for any $x \in \Sigma_{\lambda_0}$.

We show that the plane can be moved further to the right. More precisely, there exists an ε depending on n, α, and the solution (u, v) itself such that

$$u(x) \ge u_\lambda(x) \quad \text{and} \quad v(x) \ge v_\lambda(x), \quad \text{on } \Sigma_\lambda$$

for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.

In the case of $v(x) \not\equiv v_{\lambda_0}(x)$ on Σ_{λ_0}, from (23) and (24) we obtain $u(x) \not\equiv u_{\lambda_0}(x)$, that is, $u(x) > u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Let

$$\overline{\Sigma^u_{\lambda_0}} = \{x \in \Sigma_{\lambda_0} \mid u(x) \le u_{\lambda_0}(x)\} \quad \text{and} \quad \overline{\Sigma^v_{\lambda_0}} = \{x \in \Sigma_{\lambda_0} \mid v(x) \le v_{\lambda_0}(x)\}.$$

Obviously, $\overline{\Sigma^u_{\lambda_0}}$ has measure zero, and $\lim_{\lambda \to \lambda_0} \Sigma^u_\lambda \subset \overline{\Sigma^u_{\lambda_0}}$. The same fact holds for that of v. Let $(\Sigma^u_\lambda)^*$ be the reflection of set Σ^u_λ about the plane $x_1 = \lambda$. Similarly to (37), we have

$$\begin{aligned}
(39)\quad & \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} + \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} \\
&\le c\Big(\|u_\lambda\|^{p_1-1}_{L^{s_{11}}_{(\Sigma^u_\lambda)}} + \|u_\lambda\|^{q_1-1}_{L^{s_{21}}_{(\Sigma^u_\lambda)}} + \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda\|^{p_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}} + \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u_\lambda\|^{q_3-1}_{L^{k_0}_{(\Sigma^u_\lambda)}}\Big) \\
&\qquad \times \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} \\
&\quad + c\Big(\|v_\lambda\|^{p_2-1}_{L^{s_{12}}_{(\Sigma^v_\lambda)}} + \|v_\lambda\|^{q_2-1}_{L^{s_{22}}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{q_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{p_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}} + \|u_\lambda\|^{q_3}_{L^{k_0}_{(\Sigma^v_\lambda)}} \|v_\lambda\|^{q_4-1}_{L^{k_0}_{(\Sigma^v_\lambda)}}\Big) \\
&\qquad \times \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} \\
&\le c\Big(\|u\|^{p_1-1}_{L^{s_{11}}_{((\Sigma^u_\lambda)^*)}} + \|u\|^{q_1-1}_{L^{s_{21}}_{((\Sigma^u_\lambda)^*)}} + \|v\|^{p_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u\|^{p_3-1}_{L^{k_0}_{((\Sigma^u_\lambda)*)}} + \|v\|^{q_4}_{L^{k_0}_{(\Sigma^u_\lambda)}} \|u\|^{q_3-1}_{L^{k_0}_{((\Sigma^u_\lambda)^*)}}\Big) \\
&\qquad \times \|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} \\
&\quad + c\Big(\|v\|^{p_2-1}_{L^{s_{12}}_{((\Sigma^v_\lambda)^*)}} + \|v\|^{q_2-1}_{L^{s_{22}}_{((\Sigma^v_\lambda)^*)}} + \|u\|^{q_3}_{L^{k_0}_{((\Sigma^v_\lambda)^*)}} \|v\|^{p_4-1}_{L^{k_0}_{((\Sigma^v_\lambda)^*)}} + \|u\|^{q_3}_{L^{k_0}_{((\Sigma^v_\lambda)^*)}} \|v\|^{q_4-1}_{L^{k_0}_{((\Sigma^v_\lambda)^*)}}\Big) \\
&\qquad \times \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}
\end{aligned}$$

for any $\frac{n}{n-\alpha} < \gamma < \infty$. Since $u \in \Pi_1$, $v \in \Pi_2$, we can choose ε small enough, such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, we have

$$\|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} + \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} \le \tfrac{1}{2}\Big(\|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} + \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}}\Big).$$

This implies that $\|u_\lambda - u\|_{L^\gamma_{(\Sigma^u_\lambda)}} = \|v_\lambda - v\|_{L^\gamma_{(\Sigma^v_\lambda)}} = 0$. So Σ^u_λ and Σ^v_λ must be empty. The proof of (38) is then completed. □

Proof of Theorem 1.2. From Lemma 3.1, it follows that $u(x) \geq u_\lambda(x)$ and $v(x) \geq v_\lambda(x)$ on Σ_λ for λ enough negative. This implies the possibility of moving the plane from near $x_1 = -\infty$, so we can invoke Step 2: move the plane to the limiting position to derive symmetry. Furthermore, it follows from Lemma 3.2 that if the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then $u(x)$ and $v(x)$ must be symmetric and monotonic about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to $x_1 = 0$. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and monotonically decreasing about some point. This completes the proof of Theorem 1.2. □

4. Classification of positive solutions to system (4) with critical exponents

In this section, we prove Theorem 1.3. Since we have established the regularity and radial symmetry of solutions to system (4) in previous sections, we may employ a proposition in [Li and Zhu 1995; Li and Zhang 2003] to show the form of radially symmetric solutions of (4) with critical exponents. Throughout this section, we always assume that $p_1 = p_2 = q_1 = q_2 = p_3 + p_4 = q_3 + q_4 = \frac{n+\alpha}{n-\alpha}$ in system (4) and $(u, v) \in L^{2n/(n-\alpha)}(\mathbb{R}^n) \times L^{2n/(n-\alpha)}(\mathbb{R}^n)$. It is well known that system (4) (or (1)) is invariant with respect to scaling, translation, and inversion transformations with the above exponent conditions.

For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider the Kelvin transformation of w:

$$w_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} w\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

To classify solutions, we need the following lemma.

Lemma 4.1. *Let* (u, v) *be a pair of solutions of system* (4) *with the assumptions of Theorem 1.3. Then there exist* $\lambda > 0$ *and* $x_0 \in \mathbb{R}^n$ *such that*

$$u_{x_0,\lambda}(y) = u(y), \tag{40}$$

$$v_{x_0,\lambda}(y) = v(y). \tag{41}$$

Proof. It suffices to prove (40). The proof of (41) is similar. Consider $x_0 = 0$, for otherwise we make a translation transform and a scaling transform on $u_{x_0,\lambda}(y)$. Let (u, v) be a pair of solutions of (4). By radial symmetry we assume without loss of generality that $u(x)$ and $v(x)$ are symmetric about the origin and $\lim_{|x|\to\infty} |x|^{n-\alpha} u(x) = u_\infty = u(0)$. Let $\lambda^{n-\alpha} = u_\infty / u(0)$ and e be any unit vector in $\mathbb{R}^n$. We define

$$w(y) = \frac{1}{|y|^{n-\alpha}} u\left(\frac{y}{|y|^2} - e\right).$$

Then

$$w(0) = u_\infty \quad \text{and} \quad w(e) = u(0).$$

Thus, w must be symmetric about $\frac{1}{2}e$.

Now, choosing $y = (\frac{1}{2} - h)e$ for any h, as in [Chen et al. 2006], it is easy to see

$$w\left((\tfrac{1}{2}-h)e\right) = \left(\frac{1}{\left|\frac{1}{2}-h\right|}\right)^{n-\alpha} u\left(\frac{\frac{1}{2}-h}{\left|\frac{1}{2}-h\right|^2}e - e\right) = \left(\frac{1}{\left|\frac{1}{2}-h\right|}\right)^{n-\alpha} u\left(\frac{\frac{1}{2}+h}{\frac{1}{2}-h}e\right),$$

where

$$\frac{\frac{1}{2}-h}{\left|\frac{1}{2}-h\right|^2}e - e = e\left(\frac{\left(\frac{1}{2}-h\right)-\left(\frac{1}{2}-h\right)^2}{\left|\frac{1}{2}-h\right|^2}\right) = e\frac{\left(\frac{1}{2}-h\right)\left(1-\frac{1}{2}+h\right)}{\left(\frac{1}{2}-h\right)^2} = e\frac{\frac{1}{2}+h}{\frac{1}{2}-h}.$$

Taking $y = (\frac{1}{2} - h)e$, we have

$$w\left((\tfrac{1}{2}+h)e\right) = \left(\frac{1}{\left|\frac{1}{2}+h\right|}\right)^{n-\alpha} u\left(\frac{\frac{1}{2}-h}{\frac{1}{2}+h}e\right).$$

Since w is symmetric about $\frac{1}{2}e$, by scaling we have

$$\frac{\lambda^{(n-\alpha)/2}}{\left|\frac{1}{2}-h\right|^{n-\alpha}} u\left(\lambda\frac{\frac{1}{2}+h}{\frac{1}{2}-h}e\right) = \frac{\lambda^{(n-\alpha)/2}}{\left|\frac{1}{2}+h\right|^{n-\alpha}} u\left(\lambda\frac{\frac{1}{2}-h}{\frac{1}{2}+h}e\right).$$

Letting $t = (\frac{1}{2} - h)/(\frac{1}{2} + h)$, it follows that

$$u\left(\frac{\lambda e}{t}\right) = t^{n-\alpha} u(\lambda t e).$$

Replacing t, e by $\lambda/|x-y|$, $y-x/|x-y|$, respectively, it follows that $u(y-x) = (\lambda/|y-x|)^{n-\alpha} u(\lambda^2(y-x)/|y-x|^2)$. Furthermore, we can take the translation transform to obtain (40). □

To prove our main result, we also need the following proposition from [Li and Zhang 2003]. Earlier versions with stronger assumptions were first proved by Li and Zhu [1995].

Proposition 4.2 [Li and Zhang 2003]. *Let $f \in C^1(\mathbb{R}^n)$, $\lambda > 0$ and $\mu > 0$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that*

$$\left(\frac{\lambda}{|y-x|}\right)^{\mu} f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) = f(y), \quad y \in \mathbb{R}^n \setminus \{x\}.$$

Then for some $a \geq 0$, $d > 0$ and $\bar{x} \in \mathbb{R}^n$, we have

$$f(x) \equiv \pm a\left(\frac{1}{d+|x-\bar{x}|^2}\right)^{\mu/2}.$$

Proof of Theorem 1.3. From Lemma 4.1 and Proposition 4.2, we obtain directly that the solution of system (4) must be of the form (9). □

5. Equivalence of system (1) and system (4)

In this section, we show the equivalence of system (1) and the integral system (4). The proof is similar to that in [Chen and Li 2011] which is based on properties and the Fourier transform of the Riesz potential. For completeness and convenience of the reader, the details will be included. However, by choosing a suitable cut-off function, we provide a different approach for the case of even numbers $\alpha = 2m$.

First, we define a positive solution of (1) in the distribution sense, i.e, $u, v \in H^{\alpha/2}(\mathbb{R}^n)$, and they satisfy

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \phi \, dx = \int_{\mathbb{R}^n} \big(\lambda_1 u^{p_1} + \mu_1 v^{p_2} + \beta_1 u^{p_3} v^{p_4}\big) \phi \, dx, \tag{42}$$

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} v (-\Delta)^{\alpha/4} \phi \, dx = \int_{\mathbb{R}^n} \big(\lambda_2 u^{q_1} + \mu_2 v^{q_2} + \beta_2 u^{q_3} v^{q_4}\big) \phi \, dx \tag{43}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x) > 0$. As usual, by the Fourier transform we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \phi \, dx = c_n \int_{\mathbb{R}^n} |\xi|^\alpha \hat{u}(\xi) \hat{\phi}(\xi) \, d\xi, \tag{44}$$

where $\hat{u}$ and $\hat{\phi}$ are the Fourier transforms of u and ϕ, respectively.

For $\alpha = 2m$, where m is a positive integer, we prove that every positive solution of PDE system (1) satisfies integral system (4). Here we don't use the maximum principles for higher-order elliptic operators; the method be used here comes from [Lu and Zhu 2011].

Lemma 5.1. *Any positive solutions of system* (1) *with $\alpha = 2m$ satisfy the integral system* (4).

Proof. We define the cut-off function on $B_R(0)$:

$$\eta(x) = \begin{cases} 1, & x \in B_1(0), \\ 0, & x \notin B_2(0), \end{cases}$$

and $0 < \eta^{(i)} < 2$ on $B_2(0)$ for $i = 1, 2, \dots, 2m$. Let $\eta_R(x - y) = \eta(\frac{|x-y|}{R})$ on $B_2(x)$ and choose $\phi(x - y) = \eta_R(x - y)/|x - y|^{n-2m}$. It is easy to check that $\phi \in C_0^\infty(\mathbb{R}^n)$. Hence, for any $u, v \in H^m(\mathbb{R}^n)$, by definition (42) and integration by parts, we have

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^{m/2} u (-\Delta)^{m/2} \phi \, dy &= \int_{\mathbb{R}^n} u (-\Delta)^m \phi \, dy \\ &= \int_{\mathbb{R}^n} u (-\Delta)^m \left(\frac{\eta_R(x-y)}{|x-y|^{n-2m}} \right) dy \\ &= \int_{\mathbb{R}^n} Q_y(u, v) \frac{\eta_R(x-y)}{|x-y|^{n-2m}} \, dy, \end{aligned} \tag{45}$$

where $Q_y(u, v)$ is defined in Section 3. Since

$$(-\Delta)^m\left(\frac{\eta_R(x-y)}{|x-y|^{n-2m}}\right) = (-\Delta)^m\left(\frac{1}{|x-y|^{n-2m}}\right)\eta_R(x-y) + \sum_{i=1}^{2m} c_i|x-y|^{-n+i}\eta_R^{(i)}R^{-i},$$

one has

$$\int_{\mathbb{R}^n} u(-\Delta)^m\left(\frac{\eta_R(x-y)}{|x-y|^{n-2m}}\right)dy = \int_{\mathbb{R}^n} u(-\Delta)^m\left(\frac{1}{|x-y|^{n-2m}}\right)\eta_R(x-y)\,dy + \sum_{i=1}^{2m} c_i\int_{\mathbb{R}^n} R^{-i}u|x-y|^{-n+i}\eta_R^{(i)}\,dy. \tag{46}$$

As in [Lu and Zhu 2011], for $u \in L^{2n/(n-2m)}(\mathbb{R}^n)$, using the Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^n} u(x-y)^{-n-i}\eta_R^{(i)}R^{-i}\,dy &\le c_i R^{-i}\left(\int_{\mathbb{R}^n} u^{2n/(n-2m)}dy\right)^{\frac{n-2m}{2n}}\left(\int_{B_{2R}\setminus B_R}|x-y|^{2n(i-n)/(n+2m)}dy\right)^{\frac{n+2m}{2m}}\\ &\le \frac{c_i}{R^i}\int_R^{2R} r^{2n(i-n)/(n+2m)}r^{n-1}dr \to 0,\end{aligned} \tag{47}$$

as $R \to \infty$. We also note that

$$\int_{\mathbb{R}^n} u(-\Delta)^m\left(\frac{\eta_R(x-y)}{|x-y|^{n-2m}}\right)dy = \int_{\mathbb{R}^n} \delta(x-y)u(y)\,dy = u(x). \tag{48}$$

Therefore, combining (45), (46), (47) with (48), we have

$$u(x) = \int \frac{\lambda_1 u^{p_1}(y) + \mu_1 v^{p_2}(y) + \beta_1 u^{p_3}(y)v^{p_4}(y)}{|x-y|^{n-2m}}\,dy.$$

In the same way, we obtain

$$v(x) = \int_{\mathbb{R}^n} \frac{\lambda_2 u^{q_1}(y) + \mu_2 v^{q_2}(y) + \beta_2 u^{q_3}(y)v^{q_4}(y)}{|x-y|^{n-2m}}\,dy.$$

The proof of the lemma is completed. □

Now, we consider the case that α is not even, that is, system (1) is equivalent to the integral system (4) for any α.

Proof of Theorem 1.4. (i) For any $\phi \in C_0^\infty(\mathbb{R}^n)$, set

$$\psi(x) = \int \frac{\phi(x)}{|x-y|^{n-\alpha}}\,dy,$$

so that $(-\Delta)^{\alpha/2}\psi=\phi$, and then $\psi\in H^{\alpha}(\mathbb{R}^n)\subset H^{\alpha/2}(\mathbb{R}^n)$, and satisfies

$$\int_{\mathbb{R}^n}(-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}\psi\,dx=\int_{\mathbb{R}^n}Q_x(u,v)\psi(x)\,dx.$$

This implies

$$\int_{\mathbb{R}^n}u(-\Delta)^{\alpha/2}\psi\,dx=\int_{\mathbb{R}^n}u\phi\,dx=\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\frac{Q_y(u,v)}{|x-y|^{n-\alpha}}dy\right\}\phi(x)\,dx$$

for any nonnegative $\phi\in C_0^{\infty}(\mathbb{R}^n)$. Thus, we get

$$u(x)=\int\frac{\lambda_1u^{p_1}(y)+\mu_1v^{p_2}(y)+\beta_1u^{p_3}(y)v^{p_4}(y)}{|x-y|^{n-\alpha}}\,dy.$$

Similarly, we have

$$v(x)=\int\frac{\lambda_2u^{q_1}(y)+\mu_2v^{q_2}(y)+\beta_2u^{q_3}(y)v^{q_4}(y)}{|x-y|^{n-\alpha}}\,dy.$$

(ii) Now we show that any positive solutions of the integral system (4) satisfy system (1). Assume that $u,v\in L^{2n/(n-2m)}(\mathbb{R}^n)$ are the solutions of the integral system (4). Invoking the Fourier transform on both sides of the first equation of (4), we have

$$\hat{u}(\xi)=c_n|\xi|^{-\alpha}\widehat{Q_\xi(u,v)}.$$

Then

$$|\xi|^{\alpha}\hat{u}(\xi)=c_n\widehat{Q_\xi(u,v)}(\xi).$$

Hence, for any $\phi\in C_0^{\infty}(\mathbb{R}^n)$, by (44) one has

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}\phi\,dx&=c_n\int_{\mathbb{R}^n}|\xi|^{\alpha}\hat{u}(\xi)\hat{\phi}(\xi)\,d\xi\\&=c_n\int_{\mathbb{R}^n}\widehat{Q_\xi(u,v)}\hat{\phi}(\xi)\,d\xi\\&=c_n\int_{\mathbb{R}^n}Q_x(u,v)\phi(x)\,dx.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^{\alpha/4}v(-\Delta)^{\alpha/4}\phi\,dx&=c_n\int_{\mathbb{R}^n}\widehat{K_\xi(u,v)}(\xi)\hat{\phi}(\xi)\,d\xi\\&=c_n\int_{\mathbb{R}^n}K_x(u,v)\phi(x)\,dx.\end{aligned}$$

This means that (u,v) is a pair of solutions of

$$\begin{cases}(-\Delta)^{\alpha/2}u=c_n(\lambda_1u^{p_1}+\mu_1v^{p_2}+\beta_1u^{p_3}v^{p_4}),\\(-\Delta)^{\alpha/2}v=c_n(\lambda_2u^{q_1}+\mu_2v^{q_2}+\beta_2u^{q_3}v^{q_4}),\end{cases}$$

for $x \in \mathbb{R}^n$, in the sense of distributions. This completes the proof of the theorem. □

Now, we can combine Theorems 1.2 and 1.4 to show the nonexistence results.

Proof of Theorem 1.5. It suffices to verify the condition for exponents.

(i) and (ii) Under conditions (i) and (ii), respectively, the nonexistence results have been proved in [de Figueiredo and Sirakov 2005]. Combining this with our symmetry results, we find that there exist no nontrivial positive solutions (u, v) with $u \in \Pi_1 \cap L^\infty(\mathbb{R}^n)$, $v \in \Pi_2 \cap L^\infty(\mathbb{R}^n)$ satisfying conditions (i) and (ii), respectively.

(iii) Combining the nonexistence results of Dancer, Wei and T. Weth [2010] and our symmetry results, we conclude that there exist no nontrivial positive solutions (u, v) with $u \in \Pi_1 \cap L^\infty(\mathbb{R}^n)$ and $v \in \Pi_2 \cap L^\infty(\mathbb{R}^n)$ with $p_1 = q_2 = 3$, $p_3 = q_4 = 1$, $p_4 = q_3 = 2$. □

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JINGBO DOU
SCHOOL OF STATISTICS
XI'AN UNIVERSITY OF FINANCE AND ECONOMICS
710100 XI'AN
CHINA
jbdou@xaufe.edu.cn

CHANGZHENG QU
CENTER FOR NONLINEAR STUDIES
NINGBO UNIVERSITY
315211 NINGBO
CHINA
quchangzheng@nbu.edu.cn

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