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We prove that every irreducible subrepresentation in the space of automorphic forms on $G(\mathbb{A})$, where G is a connected reductive group defined over a number field k, and $\mathbb{A}$ is the related ring of adeles, is a subrepresentation of the representation induced from a cuspidal automorphic representation of a Levi subgroup.

1. Introduction

In this note we prove the global (automorphic) version (over a number field k) of Casselman's subrepresentation theorem. We explain it in more detail: in the local theory (i.e., considering admissible representations of reductive groups over local fields) there is Harish-Chandra's subquotient theorem [1954], and then there is also Casselman's subrepresentation theorem [1980; 1995]; both of them state that every irreducible representation (in the appropriate category) of this given reductive group is a subquotient or (in the case of Casselman's theorem) a subrepresentation of a representation induced from a "simpler" one (of an appropriate subgroup). The global analog of the Harish-Chandra subquotient theorem would be Langlands' theorem which describes a general automorphic representation as a subquotient of a representation induced from a cuspidal representation of a Levi subgroup.

We prove the following global version of Casselman's subrepresentation theorem.

Theorem. *Let G be a connected reductive group defined over k. Let (Π, V) be an $((\mathfrak{g}_\infty, K_\infty)\times G(\mathbb{A}_f))$-irreducible subspace of automorphic forms in $A(G(k)\backslash G(\mathbb{A}))$. Then, there exists a parabolic subgroup $P = MU$ of G, an irreducible automorphic cuspidal representation π_0 of M (thus appearing in the space of cuspidal automorphic forms on M) such that, as abstract global representations, we have*

$$\Pi \hookrightarrow \mathrm{ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi_0,$$

where we consider the normalized parabolic induction (so we extend π_0 trivially on $U(\mathbb{A})$) and we take $\mathbf{K}$-finite vectors.

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We explain all the notation in the Preliminaries section.

We are sure that the experts in the field are aware of the above claim, but we were not able to find the reference for this statement, which is somewhat more precise than the aforementioned Langlands' result in his Corvallis lecture [Borel and Jacquet 1979]. The proof is a pretty straightforward application of the Langlands proof in his Corvallis lecture, with the decomposition results (on the spaces of automorphic forms) obtained (along with much stronger results) in [Mœglin and Waldspurger 1995]. We hope that this result will be very helpful for explicit calculations with automorphic forms, since it is explicitly applicable to the discrete (and **K**-finite) part of automorphic L^2 situation.

2. Preliminaries

Let k be a number field, and $\mathbb{A}$ its ring of adeles. Let G be a connected reductive group defined over k, and $G_\infty = \prod_v G(k_v)$, where the product is over archimedean places of k. We further denote $G(\mathbb{A}_f) = \prod'_{v<\infty} G(k_v)$. Let $\mathcal{U}$ be the enveloping algebra of the complexified Lie algebra $\mathfrak{g}$ of G_∞ (and $\mathfrak{g}_\infty$ is the Lie algebra of G_∞). We follow the notation of the first chapter of [Mœglin and Waldspurger 1995]. We denote by $\mathfrak{z}$ the center of $\mathcal{U}$ and by K_v a maximal compact subgroup of $G(k_v)$, where $K_v = G(O_{k_v})$ for almost all $v < \infty$. Here O_{k_v} is the ring of integers in k_v. We set $K_\infty = \prod_{v|\infty} K_v$ and $\mathbf{K} = \prod_v K_v$. We fix a minimal parabolic subgroup P_0 of G defined over k, and consequently, standard parabolic subgroups (defined over k) with respect to P_0. We denote by S a maximal k-split torus of G, chosen inside P_0 and by Δ the set of simple k-roots of G with respect to S (and P_0). We know that each standard k-parabolic subgroup corresponds to a subset θ of Δ. We denote this by putting $P = P_\theta$. We denote the modular function on P by δ_P. For a standard Levi k-subgroup M of G, we denote by $\mathfrak{z}^M$ the analogue of $\mathfrak{z}$ for group M. We denote by Z_M the center of M.

We use the following definition of an automorphic form: Let $P = MU$ be a standard k-parabolic subgroup of G and $\phi : U(\mathbb{A})M(k) \setminus G(\mathbb{A}) \to \mathbb{C}$ a function. We say that ϕ is automorphic if it satisfies the following conditions:

(1) ϕ has moderate growth (see [Mœglin and Waldspurger 1995, I.2.3]).

(2) ϕ is smooth (see [Mœglin and Waldspurger 1995, I.2.5]).

(3) ϕ is **K**-finite.

(4) ϕ is $\mathfrak{z}$-finite.

Note that the space $A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ of all automorphic forms as above can be related to the usual situation with the automorphic forms on $M(k) \setminus M(\mathbb{A})$ by attaching to each $k \in \mathbf{K}$ and ϕ as above a function $\phi_k : M(k) \setminus M(\mathbb{A}) \to \mathbb{C}$ defined by $\phi_k(m) = \delta_P^{-1/2}(m)\phi(mk)$ by noting that ϕ is automorphic if and only if it is smooth,

$\mathbf{K}$-finite, and for all $k \in \mathbf{K}$, ϕ_k is an automorphic form on $M(k) \setminus M(\mathbb{A})$. We denote by $A_0(U(\mathbb{A})M(k)\setminus G(\mathbb{A}))$ the cuspidal part of the space $A(U(\mathbb{A})M(k)\setminus G(\mathbb{A}))$; i.e., the space of all automorphic forms ϕ from $A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ with the property that for every standard k-parabolic subgroup $P' = M'U'$ such that $P_0 \subset P' \subsetneq P$ we have $\phi_{P'} = 0$ (the constant term along P', defined by $\phi_{P'}(g) = \int_{U'(k)\setminus U'(\mathbb{A})} \phi(ug)\mathrm{d}u$).

The space $A(U(\mathbb{A})M(k)\setminus G(\mathbb{A}))$ is a module for the action of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$, i.e., for the global idempotent Hecke algebra $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$, where $\mathcal{H}_\infty$ is related to $\mathcal{U}$ and finite measures on K_∞, and $\mathcal{H}_f = \otimes'_{v<\infty} \mathcal{H}_v$, where $\mathcal{H}_v,\ v < \infty$ is the Hecke algebra of compactly supported, locally constant functions on $G(k_v)$ (see [Borel and Jacquet 1979, Section 4]). Note that $A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ is a submodule of $A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ with this action. Note that the constant term (with respect to some standard k-parabolic subgroup $P = MU$) is an intertwining operator between $A(G(k) \setminus G(\mathbb{A}))$ and $A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ [Mœglin and Waldspurger 1995, I.2.6].

Let ξ be a character of $Z_M(k) \setminus Z_M(\mathbb{A})$, and let π be an irreducible submodule of $A(M(k) \setminus M(\mathbb{A}))$, for a standard k-Levi subgroup M of G. We denote by $A(M(k) \setminus M(\mathbb{A}))_\pi$ the isotypic submodule attached to π (in the theorem below we deal with cuspidal π, so the relevant subquotients are indeed subspaces). We set

$$A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\xi = \big\{\phi \in A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) : \\ \phi(zg) = \delta_P^{1/2}(z)\xi(z)\phi(g) \text{ for all } z \in Z_M(\mathbb{A}),\ g \in G(\mathbb{A})\big\},$$

$$A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi = \big\{\phi \in A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) : \\ \phi_k \in A(M(k) \setminus M(\mathbb{A}))_\pi \text{ for all } k \in \mathbf{K}\big\}.$$

Analogously, we define by $A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\xi$ and $A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi$ the cuspidal parts of the above spaces (i.e., the parts realized in the space of cuspidal automorphic forms).

Proposition 2.1. *Let ξ be a character of $Z_M(k) \setminus Z_M(\mathbb{A})$ and let $\Pi_0(M)_\xi$ denote the set of isomorphism classes of irreducible representations of $M(\mathbb{A})$ occurring as submodules in $A_0(M(k) \setminus M(\mathbb{A}))_\xi$. We have the decomposition*

$$A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\xi = \bigoplus_{\pi \in \Pi_0(M)_\xi} A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi .$$

Proof. This is explained in [Mœglin and Waldspurger 1995, p. 44]. □

Remark. By the proof of Lemma I.3.2 of [Mœglin and Waldspurger 1995], $\mathfrak{z}^M$ acts on $A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ by left translations; every automorphic form there is $\mathfrak{z}^M$-finite; analogously every element of that space is $Z_M(\mathbb{A})$-finite, again here $Z_M(\mathbb{A})$ acts by left translations (because we examine $\mathbf{K}$-finite automorphic forms). Also, it is easy to see that $A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))$ is $Z_M(\mathbb{A})$-invariant subspace with this $Z_M(\mathbb{A})$-action.

3. The theorem

In this section we prove the main theorem stated in Section 1. The proof follows directly from the next theorem, so our embedding from the main theorem is realized through the calculation of the constant term.

Theorem 3.1. *Let* (Π, V) *be an* $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$*-) irreducible subspace of automorphic forms inside* $A(G(k) \setminus G(\mathbb{A}))$ *such that some constant term of a function from* V *does not vanish along a* k*-parabolic subgroup* P_θ *of* G; *assume that* θ *is minimal (set of simple roots) with this property. Then, there exists an irreducible automorphic representation* π_0 *of* $M_\theta(\mathbb{A})$ *(appearing in* $A_0(M_\theta(k) \setminus M_\theta(\mathbb{A}))$ *such that the space of constant terms of* V *along* P_θ, *denoted by* V_0, *belongs (up to a left translation by an element from* $Z_{M_\theta}(\mathbb{A})$*) to the space* $A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_{\pi_0}$ *of cuspidal automorphic forms.*

Proof. Let $f \in V$. By definition, the constant term $f_{P_\theta}(g) = \int_{U_\theta(k)\setminus U_\theta(\mathbb{A})} f(ug)\mathrm{d}u$ belongs to $A(U_\theta(\mathbb{A})M_\theta(k)\setminus G(\mathbb{A}))$, more precisely, to the cuspidal part of this space (because of the minimality of θ; see [Mœglin and Waldspurger 1995, I.2.6, I.2.18]). By the remark above the Theorem, $Z_{M_\theta}(\mathbb{A})$ acts on $A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))$ by left translations, and every function from this space is $Z_{M_\theta}(\mathbb{A})$-finite. For every $z \in Z_{M_\theta}(\mathbb{A})$, let $V_0^z = l(z)V_0$ (the action by left translations). We know that taking the constant term is intertwining operator, so V_0 (and V_0^z) is (as an abstract $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$- representation) irreducible and isomorphic to V. Let $W = \sum_{z\in Z_{M_\theta}(\mathbb{A})} V_0^z$.

We prove that there exists $F \in W$, $F \neq 0$ such that $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}}\{l(z)F : z \in Z_{M_\theta}(\mathbb{A})\} = 1$. Firstly, let $F \neq 0$ be an element from W such that the dimension of the space $Y := \operatorname{span}_{\mathbb{C}}\{l(z)F : z \in Z_{M_\theta}(\mathbb{A})\}$ is minimal. We claim that this dimension is one. Indeed, let us assume that this dimension (of Y) is greater than one. If, for every $a \in Z_{M_\theta}(\mathbb{A})$ acting on Y, the whole space Y is an eigenspace for certain eigenvalue, it would mean that $l(a)$, for every a, acts as a scalar operator on Y, and then every one-dimensional subspace, (also the one spanned by F) would be $Z_{M_\theta}(\mathbb{A})$-invariant; a contradiction (this would mean that Y is one-dimensional). So, there exists $a \in Z_{M_\theta}(\mathbb{A})$ with a nonzero eigenspace strictly smaller than Y, attached to an eigenvalue $\alpha \neq 0$. This means that $Y_1 := (l(a) - \alpha)Y$ is a proper subspace of Y. Let $F_1 := (l(a) - \alpha)F \in Y_1$. F_1 is obviously nonzero; otherwise $l(b)F$ would be an eigenvector of $l(a)$ for eigenvalue α for every $b \in Z_{M_\theta}(\mathbb{A})$, so that the whole Y is an eigenspace for α; a contradiction. Now, we easily see that the span of the set $\{l(b)F_1 : b \in Z_{M_\theta}(\mathbb{A})\}$ is inside Y_1, which leads to contradiction with our choice of F.

So, we conclude that there exists a character ξ of $Z_M(k) \setminus Z_M(\mathbb{A})$ such that

$$l(z)F(g) = \delta_{P_\theta}^{1/2}(z)\xi(z)F(g) \quad \text{for all } g \in G(\mathbb{A}),\ z \in Z_{M_\theta}(\mathbb{A}). \tag{1}$$

Now, let W_0 denote the $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-subspace of W generated by F. For every vector from this space, (1) holds. Now, since $W = \sum_{a \in Z_{M_\theta}(\mathbb{A})} V_0^a$, where V_0^a are irreducible subspaces, W is also a direct sum of irreducible subspaces (for example, [Lang 2002, Chapter XVII]), and every $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-submodule of W is a direct summand. From this directly follows that W_0 has an irreducible submodule; indeed if $W = \bigoplus_{z \in I} V_0^z$, for some $I \subset Z_{M_\theta}(\mathbb{A})$, then some projection attached to this decomposition $p_z : W \to V_0^z$ is nonzero on W_0. Now $\operatorname{Ker} p_z \cap W_0$ has a direct (invariant) complement W_1 in W, and it is easy to see that $W_1 \cap W_0$ is an irreducible submodule of W_0. This means that we have found an irreducible subspace of W (so necessarily isomorphic to V i.e., to V_0) where the relation (1) holds. This realization of V inside $A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_\xi$ is thus obtained through taking of (maybe translated) constant term along P_θ. From Proposition 2.1 we have

$$A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_\xi = \bigoplus_{\pi \in \Pi_0(M_\theta)_\xi} A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_\pi,$$

and, combining our embedding with an appropriate projection, we have obtained an embedding

$$\Pi \hookrightarrow A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_{\pi_0},$$

for some automorphic (cuspidal) representation π_0 of $M_\theta(\mathbb{A})$. $\square$

Note that the space $A_0(M_\theta(k) \setminus M_\theta(\mathbb{A}))_{\pi_0}$ is semisimple (Gelfand and Piatetski–Shapiro; see [Borel and Jacquet 1979, Section 4]); so there exists an irreducible subspace V_0' of automorphic forms in $A_0(M_\theta(k) \setminus M_\theta(\mathbb{A}))_{\pi_0}$ (thus isomorphic to π_0) such that there is an embedding

$$\Pi \hookrightarrow A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))_{V_0'}$$

(the space on the right-hand side has an obvious meaning). We note that, as a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module, the latter space is isomorphic to the global representation $\operatorname{ind}_{P_\theta(\mathbb{A})}^{G(\mathbb{A})} \pi_0$ (where we use normalized induction and $\mathbf{K}$-finite vectors in this space) [Kim 2004, Section 4.5]. This isomorphism can also be given explicitly by $\phi \mapsto \phi'$, where $\phi'(g) = \phi_g$ and $\phi_g(m) = \delta_{P_\theta}(m)^{-1/2}\phi(mg)$. This is easily checked to be $G(\mathbb{A})$-isomorphism on the space of the smooth (not necessarily $\mathbf{K}$-finite automorphic forms), but then taking $\mathbf{K}$-finite vectors from both spaces, we get the claim (see the second and third lectures in [Cogdell 2004]). This, in turn, proves our main theorem from Section 1.

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