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**A LOWER BOUND FOR EIGENVALUES OF
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We study eigenvalues of the poly-Laplacian of arbitrary order on a bounded domain in an n -dimensional Euclidean space. We obtain a lower bound for these eigenvalues, significantly improving on that of Levine and Protter. In particular, the result of Melas (2003) is subsumed.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ in an n -dimensional Euclidean space \mathbb{R}^n . Let λ_i be the i -th eigenvalue of the Dirichlet eigenvalue problem of the poly-Laplacian with arbitrary order:

$$(1-1) \quad \begin{cases} (-\Delta)^l u = \lambda u & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian in \mathbb{R}^n and ν denotes the outward unit normal vector field of the boundary $\partial\Omega$. It is well known that the spectrum of this eigenvalue problem is real and discrete:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow +\infty,$$

where each λ_i has finite multiplicity and is repeated according to its multiplicity.

Let $V(\Omega)$ denote the volume of Ω and let B_n denote the volume of the unit ball in \mathbb{R}^n . When $l = 1$, the eigenvalue problem (1-1) is called a fixed membrane problem. In this case, one has Weyl's asymptotic formula

$$(1-2) \quad \lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty.$$

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From the above asymptotic formula, one can obtain

$$(1-3) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty.$$

Pólya [1961] proved that

$$(1-4) \quad \lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots,$$

if Ω is a tiling domain in \mathbb{R}^n . Moreover, he proposed the following:

Conjecture of Pólya. If Ω is a bounded domain in \mathbb{R}^n , then the k -th eigenvalue λ_k of the fixed membrane problem satisfies

$$(1-5) \quad \lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots$$

Berezin [1972] and Lieb [1980] gave a partial solution to this conjecture. Li and Yau [1983] proved that

$$(1-6) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots$$

Formula (1-3) shows that (1-6) is sharp in the sense of averages. From (1-6), one can derive

$$(1-7) \quad \lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots,$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$. Melas [2003] has improved the estimate (1-6) to

$$(1-8) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{1}{24(n+2)} \frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \dots,$$

where

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called the *moment of inertia* of Ω .

When $l = 2$, the eigenvalue problem (1-1) is called the clamped plate problem. For the eigenvalues of the clamped plate problem, it follows from [Agmon 1965] and [Pleijel 1950] that

$$(1-9) \quad \lambda_k \sim \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow +\infty.$$

This implies that

$$(1-10) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \sim \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow +\infty.$$

Furthermore, Levine and Protter [1985] proved that the eigenvalues of the clamped plate problem satisfy

$$(1-11) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}.$$

Formula (1-10) shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible constant. Very recently, Cheng and Wei [2011] obtained the following improvement of (1-11):

$$(1-12) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} + c_n \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} \frac{V(\Omega)}{I(\Omega)} k^{\frac{2}{n}} + d_n \left(\frac{V(\Omega)}{I(\Omega)} \right)^2,$$

where c_n and d_n are constants depending only on the dimension n .

When $l \geq 3$, Levine and Protter [1985] proved that

$$(1-13) \quad \frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}}, \quad \text{for } k = 1, 2, \dots$$

From the above formula, one can obtain

$$(1-14) \quad \lambda_k \geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}}, \quad \text{for } k = 1, 2, \dots$$

In this paper we investigate eigenvalues of the Dirichlet eigenvalue problem (1-1) for the Laplacian with any order. We give a significant improvement of (1-13) by adding l lower-order terms than $k^{2l/n}$ to its right-hand side. In fact, we prove:

Theorem. *Let Ω be a bounded domain in an n -dimensional Euclidean space \mathbb{R}^n . Let $\lambda_i, i = 1, 2, \dots$, be the i -th eigenvalue of the eigenvalue problem (1-1). Then*

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^k \lambda_j &\geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}} \\ &+ \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{(24)^p n \dots (n+2p-2)} \frac{(2\pi)^{2(l-p)}}{(B_n V(\Omega))^{\frac{2(l-p)}{n}}} \left(\frac{V(\Omega)}{I(\Omega)} \right)^p k^{\frac{2(l-p)}{n}}. \end{aligned}$$

Remark. If we take $l = 1$, we obtain the inequality (1-8).

2. Proof of the Theorem

Before giving the proof, we introduce some definitions and basic facts about symmetric decreasing rearrangements.

For a bounded domain $\Omega \subset \mathbb{R}^n$, the *moment of inertia* of Ω is defined by

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx.$$

By translating the origin, we may assume that

$$I(\Omega) = \int_{\Omega} |x|^2 dx.$$

Let Ω^* be the symmetric rearrangement of Ω , that is, Ω^* is the open ball centered at the origin with the same volume as Ω . Then

$$\Omega^* = \left\{ x \in \mathbb{R}^n; |x| < \left(\frac{V(\Omega)}{B_n} \right)^{\frac{1}{n}} \right\}.$$

By using the symmetric rearrangement Ω^* of Ω , we have

$$(2-1) \quad I(\Omega) = \int_{\Omega} |x|^2 dx \geq \int_{\Omega^*} |x|^2 dx = \frac{n}{n+2} V(\Omega) \left(\frac{V(\Omega)}{B_n} \right)^{\frac{2}{n}}.$$

Let f be a nonnegative continuous function on Ω . We consider its *distribution function* $\mu_f(t)$ defined by

$$\mu_f(t) = \text{Vol}(\{x \in \Omega; f(x) > t\}).$$

The distribution function can be viewed as a function from $[0, +\infty)$ to $[0, V(\Omega)]$. The *symmetric decreasing rearrangement* f^* of f is defined by

$$f^*(x) = \inf \{ t \geq 0; \mu_f(t) < B_n |x|^n \}, \quad \text{for } x \in \Omega^*.$$

By definition, we know that $f^*(x)$ is a radially symmetric function and

$$\text{Vol}(\{x \in \Omega; f(x) > t\}) = \text{Vol}(\{x \in \Omega^*; f^*(x) > t\}) \quad \text{for all } t > 0.$$

Let $f^*(x) = \phi(|x|)$. Then one gets that $\phi : [0, +\infty) \rightarrow [0, \sup f]$ is a decreasing function of $|x|$. We may assume that ϕ is absolutely continuous. It is well known that

$$(2-2) \quad \int_{\Omega} f(x) dx = \int_{\Omega^*} f^*(x) dx = n B_n \int_0^{+\infty} s^{n-1} \phi(s) ds$$

and

$$(2-3) \quad \int_{\Omega} |x|^{2l} f(x) dx \geq \int_{\Omega^*} |x|^{2l} f^*(x) dx = n B_n \int_0^{+\infty} s^{n+2l-1} \phi(s) ds.$$

Good sources of further information on rearrangements are [Bandle 1980; Pólya and Szegő 1951].

One gets from the coarea formula that

$$\mu_f(t) = \int_t^{\sup f} \int_{\{f=s\}} |\nabla f|^{-1} d\sigma_s ds.$$

Since f^* is radial, we have

$$\begin{aligned} \mu_f(\phi(s)) &= \text{Vol}\{x \in \Omega; f(x) > \phi(s)\} = \text{Vol}\{x \in \Omega^*; f^*(x) > \phi(s)\} \\ &= \text{Vol}\{x \in \Omega^*; \phi(|x|) > \phi(s)\} = B_n s^n. \end{aligned}$$

It follows that

$$n B_n s^{n-1} = \mu'_f(\phi(s))\phi'(s)$$

for almost every s . Putting $\tau := \sup |\nabla f|$, we obtain from the above equations and the isoperimetric inequality that

$$-\mu'_f(\phi(s)) = \int_{\{f=\phi(s)\}} |\nabla f|^{-1} d\sigma_{\phi(s)} \geq \tau^{-1} \text{Vol}_{n-1}(\{f=\phi(s)\}) \geq \tau^{-1} n B_n s^{n-1}.$$

Therefore, one obtains, for almost every s ,

$$(2-4) \quad -\tau \leq \phi'(s) \leq 0.$$

In order to prove our theorem, we need the following lemma.

Lemma. *Let $b \geq 1$ and $\eta, A > 0$, and let $\psi : [0, +\infty) \rightarrow [0, +\infty)$ be a decreasing, absolutely continuous function such that*

$$-\eta \leq \psi'(s) \leq 0, \quad A = \int_0^{+\infty} s^{b-1} \psi(s) ds.$$

For any positive integer l , let

$$A_l := \int_0^{+\infty} s^{b+2l-1} \psi(s) ds.$$

Then, we have

$$A_l \geq \frac{1}{b+2l} \left[(bA)^{\frac{b+2l}{b}} \psi(0)^{-\frac{2l}{b}} + \sum_{p=1}^l \frac{(l+1-p)(bA)^{\frac{b+2(l-p)}{b}} \psi(0)^{\frac{2pb-2(l-p)}{b}}}{6^p b \dots (b+2p-2)\eta^{2p}} \right].$$

Proof. The proof is by induction. Firstly, one can get from the lemma of [Melas 2003] that

$$(2-5) \quad A_1 = \int_0^{+\infty} s^{b+1} \psi(s) ds \geq \frac{1}{b+2} \left[(bA)^{\frac{b+2}{b}} \psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2} \right].$$

To prove the induction step, we assume the statement holds for $l = r$, that is,

$$A_r \geq \frac{1}{b+2r} \left[(bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b \cdots (b+2p-2)\eta^{2p}} \right].$$

Since the formula (2-5) holds for any $b \geq 1$, we have

$$\begin{aligned} & A_{r+1} \\ &= \int_0^{+\infty} s^{b+2r+1} \psi(s) ds \\ &\geq \frac{1}{b+2r+2} \left\{ [(b+2r)A_r]^{\frac{b+2r+2}{b+2r}} \psi(0)^{-\frac{2}{b+2r}} + \frac{A_r \psi(0)^2}{6\eta^2} \right\} \\ &\geq \frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2} \\ &\quad \times \left[(bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b \cdots (b+2p-2)\eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} \\ &\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2(p+1)b-2(r-p)}{b}}}{6^{p+1} b \cdots (b+2p-2)\eta^{2p+2}} \\ &\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2} \\ &= \frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2} \left[(bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} \right]^{\frac{b+2r+2}{b+2r}} \\ &\quad \times \left[1 + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{-2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2)\eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2} \\ &\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^{r+1} \frac{(r+2-p)(bA)^{\frac{b+2r-2p+2}{b}} \psi(0)^{\frac{2pb-2r+2p-2}{b}}}{6^p b \cdots (b+2p-4)\eta^{2p}} \\ &= \frac{(bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}}}{b+2r+2} \left[1 + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{-2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2)\eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} \\ &\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2} + \frac{(bA)\psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2)\eta^{2(r+1)}} \\ &\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4)\eta^{2p}}. \end{aligned}$$

It follows from the Taylor formula that

$$\begin{aligned}
& A_{r+1} \\
& \geq \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
& \quad \times \left[1 + \frac{b+2r+2}{b+2r} \sum_{p=1}^r \frac{(r+1-p)(bA)^{-\frac{2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right] \\
& \quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
& \quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}} \\
& = \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
& \quad + \frac{1}{b+2r} \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \\
& \quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
& \quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}} \\
& = \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
& \quad + \left[\frac{r}{b(b+2r)} + \frac{1}{(b+2r)(b+2r+2)} \right] \frac{1}{6 \eta^2} (bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}} \\
& \quad + \sum_{p=2}^r \left[\frac{r+1-p}{b+2r} + \frac{(r+2-p)(b+2p-2)}{(b+2r)(b+2r+2)} \right] \frac{(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \\
& \quad + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
& \geq \frac{1}{b+2(r+1)} (bA)^{\frac{b+2(r+1)}{b}} \psi(0)^{-\frac{2(r+1)}{b}} \\
& \quad + \frac{1}{b+2(r+1)} \sum_{p=1}^{r+1} \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}}.
\end{aligned}$$

This completes the proof of the lemma. \square

Proof of the Theorem. Let u_j be an orthonormal eigenfunction corresponding to the eigenvalue λ_j , that is, u_j satisfies

$$(2-6) \quad \begin{cases} (-\Delta)^l u_j = \lambda_j u_j, & \text{in } \Omega, \\ u_j = \frac{\partial u_j}{\partial \nu} = \dots = \frac{\partial^{l-1} u_j}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_i u_j = \delta_{ij}, & \text{for any } i, j. \end{cases}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$. We define a function φ_j by

$$(2-7) \quad \varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

The Fourier transform $\widehat{\varphi}_j(z)$ of $\varphi_j(x)$ is then given by

$$(2-8) \quad \widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j(x) e^{i\langle x, z \rangle} dx = (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx.$$

We fix $k \geq 1$ and set

$$f(z) = \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2, \quad \text{for } z \in \mathbb{R}^n.$$

From Bessel's inequality, it follows that

$$(2-9) \quad \begin{aligned} 0 \leq f(z) &= \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2 = (2\pi)^{-n} \sum_{j=1}^k \left| \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 \\ &\leq (2\pi)^{-n} \int_{\Omega} |e^{i\langle x, z \rangle}|^2 dx = (2\pi)^{-n} V(\Omega). \end{aligned}$$

By Parseval's identity, we have

$$(2-10) \quad \begin{aligned} \int_{\mathbb{R}^n} f(z) dz &= \sum_{j=1}^k \int_{\mathbb{R}^n} |\widehat{\varphi}_j(z)|^2 dz = \sum_{j=1}^k \int_{\mathbb{R}^n} \varphi_j^2(x) dx \\ &= \sum_{j=1}^k \int_{\Omega} u_j^2(x) dx = k. \end{aligned}$$

Furthermore, we deduce from integration by parts and Parseval's identity that

$$\begin{aligned}
(2-11) \quad & \int_{\mathbb{R}^n} |z|^{2l} f(z) dz \\
&= \sum_{j=1}^k \int_{\mathbb{R}^n} |z|^{2l} |\widehat{\varphi}_j(z)|^2 dz \\
&= \sum_{j=1}^k \int_{\mathbb{R}^n} |z|^{2l} \left| (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} z_{r_1} \cdots z_{r_l} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} u_j(x) \frac{\partial^l e^{i\langle x, z \rangle}}{\partial x_{r_1} \cdots \partial x_{r_l}} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} \frac{\partial^l u_j(x)}{\partial x_{r_1} \cdots \partial x_{r_l}} e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial^l u_j}{\partial x_{r_1} \cdots \partial x_{r_l}} \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial^l u_j}{\partial x_{r_1} \cdots \partial x_{r_l}} \right)^2 dx \\
&= \sum_{j=1}^k \int_{\Omega} u_j (-\Delta)^l u_j dx = \sum_{j=1}^k \lambda_j.
\end{aligned}$$

Since

$$(2-12) \quad \nabla \widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\Omega} i x u_j(x) e^{i\langle x, z \rangle} dx,$$

we obtain from Bessel's inequality that

$$(2-13) \quad \sum_{j=1}^k |\nabla \widehat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |i x e^{i\langle x, z \rangle}|^2 dx = (2\pi)^{-n} I(\Omega).$$

It follows from (2-9), (2-13) and the Cauchy–Schwarz inequality that, for every $z \in \mathbb{R}^n$,

$$\begin{aligned}
(2-14) \quad |\nabla f(z)| &\leq 2 \left(\sum_{j=1}^k |\widehat{\varphi}_j(z)|^2 \right)^{1/2} \left(\sum_{j=1}^k |\nabla \widehat{\varphi}_j(z)|^2 \right)^{1/2} \\
&\leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}.
\end{aligned}$$

Using the symmetric decreasing rearrangement f^* of f and noting that

$$f^*(x) = \phi(|x|), \quad \tau = \sup |\nabla f| \leq 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)} := \eta,$$

we obtain, from (2-4),

$$(2-15) \quad -\eta \leq -\tau \leq \phi'(s) \leq 0$$

for almost every s . According to (2-2) and (2-10), we infer

$$(2-16) \quad k = \int_{\mathbb{R}^n} f(z) dz = \int_{\mathbb{R}^n} f^*(z) dz = nB_n \int_0^{+\infty} s^{n-1} \phi(s) ds.$$

From (2-3) and (2-11), we obtain

$$(2-17) \quad \begin{aligned} \sum_{j=1}^k \lambda_j &= \int_{\mathbb{R}^n} |z|^{2l} f(z) dz \geq \int_{\mathbb{R}^n} |z|^{2l} f^*(z) dz \\ &= nB_n \int_0^{+\infty} s^{n+2l-1} \phi(s) ds. \end{aligned}$$

Now, we can apply the [Lemma](#) to the function ϕ with

$$(2-18) \quad b = n, \quad A = \frac{k}{nB_n}, \quad \eta = 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)}.$$

We conclude that

$$(2-19) \quad \begin{aligned} \sum_{j=1}^k \lambda_j &\geq \frac{nB_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} \phi(0)^{-\frac{2l}{n}} \\ &\quad + \frac{nB_n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2) \eta^{2p}} \left(\frac{k}{B_n}\right)^{\frac{n+2l-2p}{n}} \phi(0)^{\frac{2pn+2p-2l}{n}}. \end{aligned}$$

Note that $0 < \phi(0) \leq \sup f \leq (2\pi)^{-n} V(\Omega)$. Hence we consider the function F defined by

$$(2-20) \quad \begin{aligned} F(t) &= \frac{nB_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{2l}{n}} \\ &\quad + \frac{nB_n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2) \eta^{2p}} \left(\frac{k}{B_n}\right)^{\frac{n+2l-2p}{n}} t^{\frac{2pn+2p-2l}{n}}, \end{aligned}$$

for $t \in (0, (2\pi)^{-n} V(\Omega)]$. From (2-1), we have

$$(2-21) \quad \eta \geq (2\pi)^{-n} B_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}.$$

By a direct calculation, one gets from $B_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ that

$$(2-22) \quad \frac{B_n^{4/n}}{(2\pi)^2} < \frac{1}{2},$$

where $\Gamma(\frac{n}{2})$ is the gamma function. Thus, it follows from (2-21) and (2-22) that

$$\begin{aligned} F'(t) &= \frac{2B_n t^{-\frac{n+2l}{n}}}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} \left[-l + \sum_{p=1}^l \frac{(l+1-p)(pn+p-l)t^{\frac{2p(n+1)}{n}}}{6^p n \cdots (n+2p-2)\eta^{2p}} \left(\frac{k}{B_n}\right)^{-\frac{2p}{n}} \right] \\ &\leq \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[-l + \sum_{\substack{p > \frac{l}{n+1} \\ p \neq 1}}^l \frac{(l+1-p)(pn+p-l)}{6^p n \cdots (n+2p-2)} \left(\frac{B_n^{\frac{4}{n}}}{(2\pi)^2}\right)^p \right] \\ &< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[-l + \sum_{\substack{p > \frac{l}{n+1} \\ p \neq 1}}^l \frac{(l+1-p)(pn+p-l)}{(12)^p n \cdots (n+2p-2)} \right] \\ &< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[-l + \frac{l(n+1-l)}{12n} + \sum_{\substack{p > \frac{l}{n+1} \\ p \neq 1}}^l \frac{p^2 n(n+1)}{(12)^p n \cdots (n+2p-2)} \right] \\ &< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[-l + \frac{l}{12} + \sum_{\substack{p > \frac{l}{n+1} \\ p \neq 1}}^l \frac{p^2}{(12)^p} \right] \\ &< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[-l + \frac{l}{12} + \frac{1}{12} \right] < 0. \end{aligned}$$

We obtain that $F(t)$ is a decreasing function on $(0, (2\pi)^{-n}V(\Omega)]$. Then we can replace $\phi(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2-19), namely,

$$\begin{aligned} \sum_{j=1}^k \lambda_j &\geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{n+2l}{n}} \\ &\quad + \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2)\eta^{2p}} \frac{(V(\Omega))^{\frac{2pn+2p-2l}{n}}}{(2\pi)^{2pn+2p-2n} B_n^{\frac{2l-2p}{n}}} k^{\frac{n+2l-2p}{n}} \\ &= \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{n+2l}{n}} \\ &\quad + \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{24^p n \cdots (n+2p-2)} \frac{(2\pi)^{2(l-p)}}{(B_n V(\Omega))^{\frac{2(l-p)}{n}}} \left(\frac{V(\Omega)}{I(\Omega)}\right)^p k^{\frac{n+2(l-p)}{n}}. \end{aligned}$$

This completes the proof of the [Theorem](#). □

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On the second K -group of a rational function field	1
KARIM JOHANNES BECHER and MÉLANIE RACZEK	
On existence of a classical solution to a generalized Kelvin–Voigt model	11
MIROSLAV BULÍČEK, PETR KAPLICKÝ and MARK STEINHAUER	
A lower bound for eigenvalues of the poly-Laplacian with arbitrary order	35
QING-MING CHENG, XUERONG QI and GUOXIN WEI	
Quiver algebras, path coalgebras and coreflexivity	49
SORIN DĂSCĂLESCU, MIODRAG C. IOVANOV and CONSTANTIN NĂSTĂSESCU	
A positive density of fundamental discriminants with large regulator	81
ÉTIENNE FOUVRY and FLORENT JOUVE	
On the isentropic compressible Euler equation with adiabatic index $\gamma = 1$	109
DONG LI, CHANGXING MIAO and XIAOYI ZHANG	
Symmetric regularization, reduction and blow-up of the planar three-body problem	129
RICHARD MOECKEL and RICHARD MONTGOMERY	
Canonical classes and the geography of nonminimal Lefschetz fibrations over S^2	191
YOSHIHISA SATO	
Hilbert–Kunz invariants and Euler characteristic polynomials	227
LARRY SMITH	