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Dedicated to our friend Margaret Beattie on the occasion of her retirement

We study the connection between two combinatorial notions associated to a quiver: the quiver algebra and the path coalgebra. We show that the quiver coalgebra can be recovered from the quiver algebra as a certain type of finite dual, and we show precisely when the path coalgebra is the classical finite dual of the quiver algebra, and when all finite-dimensional quiver representations arise as comodules over the path coalgebra. We discuss when the quiver algebra can be recovered as the rational part of the dual of the path coalgebra. Similar results are obtained for incidence (co)algebras. We also study connections to the notion of coreflexive (co)algebras, and give a partial answer to an open problem concerning tensor products of coreflexive coalgebras.

1. Introduction and preliminaries

Let Γ be a quiver, and let K be an arbitrary ground field, which will be fixed throughout the paper. The associated quiver algebra $K[\Gamma]$ is an important object studied extensively in representation theory, and one theme in the field is to relate and understand combinatorial properties of the quiver via the properties of the category of representations of the quiver, and vice versa. Quiver algebras also play a role in general representation theory of algebras; for example, every finite-dimensional pointed algebra is a quiver algebra "with relations". A closely related object is the path coalgebra $K\Gamma$, introduced in [Chin and Montgomery 1997], together with its comodules (quiver corepresentations). Comodules over path coalgebras turn out to form a special kind of representations of the quiver, called locally nilpotent representations in [Chin et al. 2002]. A natural question arises then: what is the precise connection between the two objects $K[\Gamma]$ and $K\Gamma$. We aim to provide such connections, by finding out when one of these objects can be recovered from the

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other one. This is also important from the following viewpoint: one can ask when the finite-dimensional locally nilpotent representations of the quiver (i.e., quiver corepresentations), provide all the finite-dimensional quiver representations. This situation will be exactly the one in which the path coalgebra is recovered from the quiver algebra by a certain natural construction involving representative functions, which we recall below.

Given a coalgebra C, its dual C^* is always an algebra. Given an algebra A, one can associate a certain subspace A^0 of the dual A^*, which has a coalgebra structure. This is called the finite dual of A, and it plays an important role in the representation theory of A, since the category of locally finite left A-modules (i.e., modules which are sums of their finite-dimensional submodules) is isomorphic to the category of right A^0-comodules (see, for example, [Green 1976]). A^0 is sometimes also called the coalgebra of representative functions, and consists of all $f : A \to K$ whose kernel contains a cofinite (i.e., having finite codimension) ideal. We show that the path coalgebra $K\Gamma$ can be reconstructed from the quiver algebra $K[\Gamma]$ as a certain type of "graded" finite dual, that is, $K\Gamma$ embeds in the dual space $K[\Gamma]^*$ as the subspace of linear functions $f : K[\Gamma] \to K$ whose kernel contains a cofinite monomial ideal. This is an "elementwise" answer to the recovery problem; its categorical analogue states that the comodules over the quiver coalgebra are precisely those quiver representations in which the annihilator of every element contains a cofinite monomial ideal. In order to connect these to the classical categorical duality, we first note that in general the quiver algebra does not have identity, but it has enough idempotents. Therefore, we first extend the construction of the finite dual to algebras with enough idempotents (Section 2). To such an algebra A we associate a coalgebra A^0 with counit, and we show that the category of right A^0-comodules is isomorphic to the category of locally finite unital A-modules. In Section 3 we show that the path coalgebra $K\Gamma$ embeds in $K[\Gamma]^0$, and we prove that this embedding is an isomorphism, i.e., the path coalgebra can be recovered as the finite dual of the quiver algebra, if and only if the quiver has no oriented cycles and there are finitely many arrows between any two vertices. On the other hand, $K[\Gamma]$ embeds as an algebra without identity in the dual algebra $(K\Gamma)^*$ of the path coalgebra. We show that the image of this embedding is the rational (left or right) part of $(K\Gamma)^*$, i.e., the quiver algebra can be recovered as the rational part of the dual of the path coalgebra, if and only if for any vertex v of Γ there are finitely many paths starting at v and finitely many paths ending at v. This is also equivalent to the fact that $K\Gamma$ is a left and right semiperfect coalgebra.

In Section 4 we obtain similar results for another class of (co)algebras which are also objects of great combinatorial interest, namely for incidence (co)algebras. See [Joni and Rota 1979], for instance. We show that the incidence coalgebra of a partially ordered set X is always the finite dual of a subalgebra $FIA(X)$ of the incidence algebra which consists of functions of finite support. In this setting, this algebra $FIA(X)$ is the natural analogue of the quiver algebra.

It is also interesting to know when can $K\Gamma$ be recovered from $(K\Gamma)^*$, and how this relates to the results of Section 3. This problem is related to an important notion in coalgebra theory, that of coreflexive coalgebra. A coalgebra C over K is coreflexive if the natural coalgebra embedding $C \to (C^*)^0$ is an isomorphism. In other words, C is coreflexive if it can be completely recovered from its dual. In Section 5 we aim to study this condition for path coalgebras and their subcoalgebras, and give the connection with the results of Section 3. We show that, in fact, a path coalgebra of a quiver with no loops and finitely many arrows between any two vertices is not necessarily coreflexive, and also, that the quivers of coreflexive path coalgebras can contain loops. We then prove a general result stating that under certain conditions a coalgebra C is coreflexive if and only if its coradical is coreflexive. In particular, this result holds for subcoalgebras of a path coalgebra $K\Gamma$ with the property that there are finitely many paths between any two vertices of Γ. The result applies in particular to incidence coalgebras. For both a path coalgebra and an incidence coalgebra the coradical is a grouplike coalgebra (over the set of vertices of the quiver for the first one, or the underlying ordered set for the second one). Thus the coreflexivity of such a coalgebra reduces to the coreflexivity of a grouplike coalgebra $K^{(X)}$. By [Heyneman and Radford 1974, Theorem 3.7.3], if K is an infinite field, then $K^{(X)}$ is coreflexive for most sets in X in set theory and any set of practical use (see Section 5).

We use our results to give a partial answer to a question of E. J. Taft and D. E. Radford asking whether the tensor product of two coreflexive coalgebras is coreflexive. In particular, we show that the tensor product of two coreflexive pointed coalgebras, which embed in path coalgebras of quivers with only finitely many paths between any two vertices, is coreflexive.

Throughout the paper $\Gamma = (\Gamma_0, \Gamma_1)$ will be a quiver. Γ_0 is the set of vertices, and Γ_1 is the set of arrows of Γ. If a is an arrow from the vertex u to the vertex v, we denote $s(a) = u$ and $t(a) = v$. A path in Γ is a finite sequence of arrows $p = a_1a_2 \dots a_n$, where $n \geq 1$, such that $t(a_i) = s(a_{i+1})$ for any $1 \leq i \leq n-1$. We will write $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Also the length of such a p is n. Vertices v in Γ_0 are also considered as paths of length zero, and we write $s(v) = t(v) = v$. If p and q are two paths such that $t(p) = s(q)$, we consider the path pq by taking the arrows of p followed by the arrows of q. We denote by $K\Gamma$ the path coalgebra, which is the vector space with a basis consisting of all paths in Γ, comultiplication Δ defined by $\Delta(p) = \sum_{qr=p} q \otimes r$ for any path p, and counit ϵ defined by $\epsilon(v) = 1$ for any vertex v, and $\epsilon(p) = 0$ for any path of positive length. The underlying space of $K\Gamma$ can be also endowed with a structure of an algebra, not necessarily with identity, with the multiplication defined such that the product of two paths p and

q is pq if $t(p) = s(q)$, and 0 otherwise. We denote this algebra by $K[\Gamma]$; this is known in literature as the quiver algebra or the path algebra of Γ. It has identity if and only if Γ_0 is finite, and in this case the sum of all vertices is the identity.

Besides the above mentioned recovery connections between quiver algebras and path coalgebras, one can also ask whether there is any compatibility between them. More precisely, when do the two structures on the same vector space $K\Gamma$ give rise to a bialgebra structure. This turns out to be only the case for very special quivers. Specifically, consider $K[\Gamma]$ to be the vector space with basis the oriented paths of Γ, and with the quiver algebra and path coalgebra structures. Then $K[\Gamma]$ is a bialgebra (with enough idempotents in general) if and only if in Γ there are no (directed) paths of length ≥ 2 and no multiple edges between vertices (i.e., for any two vertices a, b of Γ there is at most one edge from a to b). Indeed, straightforward computations show that whenever multiple edges $\bullet \overset{x,y}{\Longrightarrow} \bullet$ or paths $\bullet \xrightarrow{x} \bullet \xrightarrow{y} \bullet$ of length at least 2 occur, then $\Delta(xy) \neq \Delta(x)\Delta(y)$. Conversely, a case by case computation for $\Delta(pq)$ with p, q paths of possible length 0 or 1 will show that $\Delta(pq) = \Delta(p)\Delta(q)$.

This shows that the relation between the path coalgebra and quiver algebra is more of a dual nature than an algebraic compatibility. For basic terminology and notation about coalgebras and comodules we refer to [Dăscălescu et al. 2001; Montgomery 1993; Sweedler 1969]. All (co)algebras and (co)modules considered here will be vector spaces over K, and duality $(-)^*$ represents the dual K-vector space.

2. The finite dual of an algebra with enough idempotents

In this section we extend the construction of the finite dual of an algebra with identity to the case where A does not necessarily have a unit, but it has enough idempotents. Throughout this section we consider a K-algebra A, not necessarily having a unit, but having a system $(e_\alpha)_{\alpha\in R}$ of pairwise orthogonal idempotents, such that $A = \bigoplus_{\alpha\in R} Ae_\alpha = \bigoplus_{\alpha\in R} e_\alpha A$. Such an algebra is said to have "enough idempotents", and it is also called an algebra with a complete system of orthogonal idempotents in the literature. Let us note that A has local units, i.e., if $a_1, \ldots, a_n \in A$, then there exists an idempotent $e \in A$ (which can be taken to be the sum of some e_α's) such that $ea_i = a_ie = a_i$ for any $1 \leq i \leq n$. Our aim is to show that there exists a natural structure of a coalgebra (with counit) on the space

$$A^0 = \{f \in A^* \mid \mathrm{Ker}(f) \text{ contains an ideal of } A \text{ of finite codimension}\}.$$

We will call A^0 the finite dual of the algebra A.

Lemma 2.1. *Let I be an ideal of A of finite codimension. Then the set $R' = \{\alpha \in R \mid e_\alpha \notin I\}$ is finite.*

Proof. Denote by $\hat{a}$ the class of an element $a \in A$ in the factor space A/I. We have that $(\widehat{e_\alpha})_{\alpha \in R'}$ is linearly independent in A/I. Indeed, if $\sum_{\alpha \in R'} \lambda_\alpha \widehat{e_\alpha} = 0$, then $\sum_{\alpha \in R'} \lambda_\alpha e_\alpha \in I$. Multiplying by some e_α with $\alpha \in R'$, we find that $\lambda_\alpha e_\alpha \in I$, so then necessarily $\lambda_\alpha = 0$. Since A/I is finite-dimensional, the set R' must be finite. □

Assume now that B is another algebra with enough idempotents, say that $(f_\beta)_{\beta \in S}$ is a system of orthogonal idempotents in B such that $B = \bigoplus_{\beta \in S} B f_\beta = \bigoplus_{\beta \in S} f_\beta B$.

Lemma 2.2. *Let H be an ideal of $A \otimes B$ of finite codimension. Let*

$$I = \{a \in A \mid a \otimes B \subseteq H\} \quad \textit{and} \quad J = \{b \in B \mid A \otimes b \subseteq H\}.$$

Then I is an ideal of A of finite codimension, J is an ideal of B of finite codimension and $I \otimes B + A \otimes J \subseteq H$.

Proof. Let $a \in I$ and $a' \in A$. If $b \in B$ and f is an idempotent in B such that $fb = b$, we have that $a'a \otimes b = a'a \otimes fb = (a' \otimes f)(a \otimes b) \in H$. Thus $a'a \otimes B \subseteq H$, so $a'a \in I$. Similarly $aa' \in I$, showing that I is an ideal of A. Similarly J is an ideal of B.

It is clear that $(e_\alpha \otimes f_\beta)_{\alpha \in R, \beta \in S}$ is a set of orthogonal idempotents in $A \otimes B$ and

$$A \otimes B = \bigoplus_{\substack{\alpha \in R \\ \beta \in S}} (A \otimes B)(e_\alpha \otimes f_\beta) = \bigoplus_{\substack{\alpha \in R \\ \beta \in S}} (e_\alpha \otimes f_\beta)(A \otimes B).$$

By Lemma 2.1, there are finitely many idempotents $e_{\alpha_1} \otimes f_{\beta_1}, \ldots, e_{\alpha_n} \otimes f_{\beta_n}$ which lie outside H. If $\alpha \in R \setminus \{\alpha_1, \ldots, \alpha_n\}$, then for any $\beta \in S$ we have that $e_\alpha \otimes f_\beta \in H$, so $e_\alpha \otimes B f_\beta = (e_\alpha \otimes B f_\beta)(e_\alpha \otimes f_\beta) \subseteq H$. Then $e_\alpha \otimes B \subseteq H$, so $e_\alpha \in I$. Similarly for any $\beta \in S \setminus \{\beta_1, \ldots, \beta_n\}$ we have that $f_\beta \in J$.

For any $\beta \in S$ let $\phi_\beta : A \to A \otimes B$ be the linear map defined by $\phi_\beta(a) = a \otimes f_\beta$. If $a \in A$, then $a \in I$ if and only if for any $\beta \in S$ we have $a \otimes B f_\beta \subseteq H$; because there is a local unit for a, this is further equivalent to $a \otimes f_\beta \in H$ for $\beta \in S$. This condition is obviously satisfied for $\beta \in S \setminus \{\beta_1, \ldots, \beta_n\}$ since $f_\beta \in J$, so we obtain that

$$I = \bigcap_{1 \le i \le n} \phi_{\beta_i}^{-1}(H),$$

a finite intersection of finite codimensional subspaces of A, thus a finite codimensional subspace itself. Similarly J has finite codimension in B. The fact that $I \otimes B + A \otimes J \subseteq H$ is obvious. □

Now we essentially proceed as in [Sweedler 1969, Chapter VI] or [Dăscălescu et al. 2001, Section 1.5], with some arguments adapted to the case of enough idempotents.

Lemma 2.3. *Let A and B be algebras with enough idempotents. The following assertions hold.*

(i) *If* $f : A \to B$ *is a morphism of algebras, then* $f^*(B^0) \subseteq A^0$, *where* f^* *is the dual map of* f.

(ii) *If* $\phi : A^* \otimes B^* \to (A \otimes B)^*$ *is the natural linear injection, then* $\phi(A^0 \otimes B^0) = (A \otimes B)^0$.

(iii) *If* $M : A \otimes A \to A$ *is the multiplication of* A, *and* $\psi : A^* \otimes A^* \to (A \otimes A)^*$ *is the natural injection, then* $M^*(A^0) \subseteq \psi(A^0 \otimes A^0)$.

Proof. It goes as the proof of [Dăscălescu et al. 2001, Lemma 1.5.2], with part of the argument in (ii) replaced by using the construction and the result of Lemma 2.2. □

Lemma 2.3 shows that by restriction and corestriction we can regard the natural linear injection ψ as an isomorphism $\psi : A^0 \otimes A^0 \to (A \otimes A)^0$. We consider the map $\Delta : A^0 \to A^0 \otimes A^0$, $\Delta = \psi^{-1} M^*$. Thus $\Delta(f) = \sum_i u_i \otimes v_i$ is equivalent to $f(xy) = \sum_i u_i(x) v_i(y)$ for any $x, y \in A$. On the other hand, we define a linear map $\varepsilon : A^0 \to K$ as follows. If $f \in A^0$, then $\mathrm{Ker}(f)$ contains a finite codimensional ideal I. By Lemma 2.1, there are finitely many e_α's outside I. Therefore only finitely many e_α's lie outside $\mathrm{Ker}(f)$, so it makes sense to define $\varepsilon(f) = \sum_{\alpha \in R} f(e_\alpha)$ (only finitely many terms are nonzero).

Proposition 2.4. *Let* A *be an algebra with enough idempotents. Then* $(A^0, \Delta, \varepsilon)$ *is a coalgebra with counit.*

Proof. The proof of the coassociativity works exactly as in the case where A has a unit; see [Dăscălescu et al. 2001, Proposition 1.5.3]. To check the property of the counit, let $f \in A^0$ and $\Delta(f) = \sum_i u_i \otimes v_i$. Let $a \in A$ and F a finite subset of R such that $a \in \sum_{\alpha \in F} e_\alpha A$. Then clearly $\left(\sum_{\alpha \in F} e_\alpha\right) a = a$. We have that

$$\begin{aligned}\Big(\sum_i \varepsilon(u_i) v_i\Big)(a) &= \sum_{i,\alpha} u_i(e_\alpha) v_i(a) = \sum_\alpha f(e_\alpha a)\\ &= \sum_{\alpha \in F} f(e_\alpha a) = f\bigg(\Big(\sum_{\alpha \in F} e_\alpha\Big) a\bigg) = f(a),\end{aligned}$$

so $\sum_i \varepsilon(u_i) v_i = f$. Similarly $\sum_i \varepsilon(v_i) u_i = f$, and this ends the proof. □

Let us note that if $f : A \to B$ is a morphism of algebras with enough idempotents, then the map $f^0 : B^0 \to A^0$ induced by f^* is compatible with the comultiplications of A^0 and B^0, but not necessarily with the counits (unless f is compatible in some way to the systems of orthogonal idempotents in A and B).

We denote by $\rightharpoonup$ (respectively $\leftharpoonup$) the usual left (respectively right) actions of A on A^*. As in the unitary case, we have the following characterization of the elements of A^0.

Proposition 2.5. *Let* $f \in A^*$. *With notation as above, the following assertions are equivalent.*

(1) $f \in A^0$.

(2) $M^*(f) \in \psi(A^0 \otimes A^0)$.

(3) $M^*(f) \in \psi(A^* \otimes A^*)$.

(4) $A \rightharpoonup f$ *is finite-dimensional.*

(5) $f \leftharpoonup A$ *is finite-dimensional.*

(6) $A \rightharpoonup f \leftharpoonup A$ *is finite-dimensional.*

(7) $\mathrm{Ker}(f)$ *contains a left ideal of finite codimension.*

(8) $\mathrm{Ker}(f)$ *contains a right ideal of finite codimension.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (6) work exactly as in the case where A has identity; see [Dăscălescu et al. 2001, Proposition 1.5.6]. We adapt the proof of (4) $\Rightarrow$ (1) to the case of enough idempotents. Since $A \rightharpoonup f$ is a left A-submodule of A^*, there is a morphism of algebras (without unit) $\pi : A \to \mathrm{End}(A \rightharpoonup f)$ defined by $\pi(a)(m) = a \rightharpoonup m$ for any $a \in A$, $m \in A \rightharpoonup f$. Since $\mathrm{End}(A \rightharpoonup f)$ has finite dimension, we have that $I = \mathrm{Ker}(\pi)$ is an ideal of A of finite codimension. Let $a \in I$. Then $a \rightharpoonup (b \rightharpoonup f) = (ab) \rightharpoonup f = 0$ for any $b \in A$, so $f(xab) = 0$ for any $x, b \in A$. Let $e \in A$ such that $ea = ae = a$. Then $f(a) = f(eae) = 0$, so $a \in \mathrm{Ker}(f)$. Thus $I \subseteq \mathrm{Ker}(f)$, showing that $f \in A^0$. The equivalence (1) $\Leftrightarrow$ (5) is proved similarly.

(6) $\Rightarrow$ (1) can be adapted from the unital case; see [Montgomery 1993, Lemma 9.1.1], with a small change. Indeed, $R = (A \rightharpoonup f \leftharpoonup A)^{\perp} = \{x \in A \mid g(x) = 0 \text{ for any } g \in A \rightharpoonup f \leftharpoonup A\}$ is an ideal of A of finite codimension, and $R \subseteq \mathrm{Ker}(f)$, since for any $r \in R$ there exists $e \in A$ such that $r = er = re$, so then $f(r) = f(ere) = (e \rightharpoonup f \leftharpoonup e)(r) = 0$.

(1) $\Rightarrow$ (7) is obvious, while (7) $\Rightarrow$ (1) follows from the fact that a left ideal I of finite codimension contains the finite codimensional ideal $J = \{r \in A \mid rA \subseteq I\}$. (1) $\Leftrightarrow$ (8) is similar. □

We end this section with an interpretation of the connection between an algebra A with enough idempotents and its finite dual A^0 from the representation theory point of view. This extends the results presented in [Abe 1980, Chapter 3, Section 1.2] in the case where A has identity. Let M be a left A-module. Then M is called unital if $AM = M$. Also, M is called locally finite if the submodule generated by any element is finite-dimensional. Denote by $\mathrm{LocFin}_A \mathcal{M}$ the full subcategory of the category of left A-modules consisting of all locally finite unital modules. We will also use the notations ${}_A\mathcal{M}, \mathcal{M}_A$ for the categories of left, or right modules over A; similarly, for a coalgebra C, ${}^C\mathcal{M}$ and $\mathcal{M}^C$ will be used to denote the categories of left and respectively right comodules.

Proposition 2.6. *Let A be an algebra with enough idempotents. Then the category $\mathcal{M}^{A^0}$ of right A^0-comodules is isomorphic to the category* $\mathrm{LocFin}_A\,\mathcal{M}$.

Proof. Let M be a right A^0-comodule with comodule structure $m \mapsto \sum m_0 \otimes m_1$. Then M is a left A-module with the action $am = \sum m_1(a)m_0$ for any $a \in A$ and $m \in M$. The counit property $m = \sum \varepsilon(m_1)m_0$, with all m_1's in A^0, shows that $m = \sum_{\alpha \in F} e_\alpha m$ for a finite set F, so M is unital. Since Am is contained in the subspace spanned by all m_0's, we have that M is also locally finite.

Conversely, let $M \in \mathrm{LocFin}_A\,\mathcal{M}$. Let $m \in M$, and let $(m_i)_{i=1,n}$ be a (finite) basis of Am. Define $a_1^*, \dots, a_n^* \in A^*$ such that $am = \sum_{i=1,n} a_i^*(a)m_i$ for any $a \in A$. Since $\bigcap_{i=1,n} \mathrm{ann}_A(m_i) = \mathrm{ann}_A(Am) \subseteq \mathrm{ann}_A(m) = \bigcap_{i=1,n} \mathrm{Ker}\, a_i^*$ and each $\mathrm{ann}_A(m_i)$ has finite codimension, we get that $a_i^* \in A^0$ for any i. Now we define $\rho : M \to M \otimes A^0$ by $\rho(m) = \sum_{i=1,n} m_i \otimes a_i^*$. It is easy to see that the definition of $\rho(m)$ does not depend on the choice of the basis $(m_i)_i$, and that $(\rho \otimes I)\rho = (I \otimes \Delta)\rho$. To show that M is a right A^0-comodule it remains to check the counit property, and this follows from the fact that M is unital.

It is clear that the above correspondences define an isomorphism of categories. □

3. Quiver algebras and path coalgebras

We examine the connection between the quiver algebra $K[\Gamma]$ and the path coalgebra $K\Gamma$ associated to a quiver Γ. The algebra $K[\Gamma]$ has identity if and only if Γ has finitely many vertices. However, $K[\Gamma]$ always has enough idempotents (the set of all vertices). Thus by Section 2 we can consider the finite dual $K[\Gamma]^0$, which is a coalgebra with counit. One has that $K[\Gamma]^0 \supseteq K\Gamma$, i.e., the path coalgebra can be embedded in the finite dual of the quiver algebra. The embedding is given as follows: for each path $p \in \Gamma$, denote by $\theta(p) \in K[\Gamma]^*$ the function $\theta(p)(q) = \delta_{p,q}$. We have that $\theta(p) \in K[\Gamma]^0$ since if we denote by $S(p)$ the set of all subpaths of p, and by P the set of all paths in Γ, the span of $P \setminus S(p)$ is a finite codimensional ideal of $K[\Gamma]$ contained in $\mathrm{Ker}\,\theta(p)$. It is easy to see that $\theta : K\Gamma \hookrightarrow K[\Gamma]^0$ is an embedding of coalgebras. In general, $K[\Gamma] \hookrightarrow (K\Gamma)^*$ is surjective if and only if the quiver Γ is finite. Also, in general, θ is not surjective. To see this, let A be the quiver algebra of a loop Γ, i.e., a quiver with one vertex and one arrow:

so $A = K[X]$, the polynomial algebra in one indeterminate. The finite dual of this algebra is

$$\varinjlim_{\substack{f \text{ irreducible} \\ n \in \mathbb{Z}_{\geq 0}}} (K[X]/(f^n))^* = \bigoplus_{f \text{ irreducible}} \Big[\varinjlim_{n \in \mathbb{Z}_{\geq 0}} (K[X]/(f^n))^* \Big],$$

while the path coalgebra is precisely the divided power coalgebra, which can be written as $\varinjlim_{n\in\mathbb{Z}_{\geq 0}}(K[X]/(X^n))^*$. These two coalgebras are not isomorphic, so the map θ above is not a surjection. Indeed, $K\Gamma$ has just one grouplike element, the vertex of Γ, while the grouplike elements of A^0, which are the algebra morphisms from $A = K[X]$ to K, are in bijection to K.

The embedding of coalgebras $\theta : K\Gamma \hookrightarrow K[\Gamma]^0$ also gives rise to a functor $F^\theta : {}^{K\Gamma}\mathcal{M} \to {}^{K[\Gamma]^0}\mathcal{M}$, associating to a left $K\Gamma$-comodule the left $K[\Gamma]^0$-comodule structure obtained by extension of coscalars via θ. We aim to provide a criterion for when the above map θ is bijective, that is, when the path coalgebra is recovered as the finite dual of the quiver algebra. Even though this is not always the case, we show that it is possible to interpret the quiver algebra as a certain kind of "graded" finite dual. We will think of $K\Gamma$ as embedded into $K[\Gamma]^0$ through θ, and sometimes write $K\Gamma$ instead of $\theta(K\Gamma)$.

Recall that in a quiver algebra $K[\Gamma]$, there is an important class of ideals, those which have a basis of paths; equivalently, the ideals generated by paths. Let us call such an ideal a monomial ideal. When I is a cofinite monomial right ideal, the quotient $K[\Gamma]/I$ produces an interesting type of representation often considered in the representation theory of quivers; we refer to [Villarreal 2001] for the theory monomial algebras and representations. In fact, such a representation also becomes a left $K\Gamma$-comodule, i.e., it is in the "image" of the functor F^θ. To see this, let B be basis of paths for I and let E be the set of paths not belonging to I; then E is finite, and because I is a right ideal, one sees that if $p \in E$ and $p = qr$, then $q \in E$. This shows that KE, the span of E, is a right $K\Gamma$-subcomodule of $K\Gamma$, so it is a rational left $(K\Gamma)^*$-module (for example, by [Dăscălescu et al. 2001, Theorem 2.2.5]). By [Dăscălescu et al. 2001, Lemma 2.2.12], the right $(K\Gamma)^*$-module $(KE)^*$ is rational, and so it has a compatible left $K\Gamma$-comodule structure. Hence $(KE)^*$ is a right $K[\Gamma]$-module via the algebra map $K[\Gamma] \hookrightarrow (K\Gamma)^*$. Now, it is straightforward to see that $K[\Gamma]/I \cong (KE)^*$ as right $K[\Gamma]$-modules, and this proves the claim. Thus, $F^\theta({}^{K\Gamma}((KE)^*)) = {}^{K[\Gamma]^0}(K[\Gamma]/I)$, since every finite-dimensional right $K[\Gamma]$-module is a left $K[\Gamma]^0$-comodule.

We can now state a characterization of the path coalgebra in terms of the quiver algebra, as a certain type of finite dual.

Proposition 3.1. *The coalgebra $\theta(K\Gamma)$ consists of all $f \in K[\Gamma]^*$ such that* $\ker(f)$ *contains a two-sided cofinite monomial ideal.*

Proof. Let P be the set of paths in Γ. If p is a path, and $S(p)$ is the set of subpaths of p, then the cofinite-dimensional vector space with basis $P \setminus S(p)$ is an ideal, and it is obviously contained in $\ker(\theta(p))$. Then clearly $\ker(\theta(z))$ contains a cofinite monomial ideal for any $z \in K\Gamma$.

Let now $f \in K[\Gamma]^*$ such that $\ker(f)$ contains the cofinite monomial ideal I. Let B be a basis of I consisting of paths, and let $E = P \setminus B$, which is finite, since I is cofinite. Then if $q \in B$, we have $f(q) = 0 = \sum_{p\in E} f(p)\theta(p)(q)$, while if $q \in E$, we have $\big(\sum_{p\in E} f(p)\theta(p)\big)(q) = f(q)$. Therefore $f = \sum_{p\in E} f(p)\theta(p)$ lies in $\theta(K\Gamma)$. □

The core of our characterization is the following easy combinatorial condition:

Proposition 3.2. *Let Γ be a quiver. The following conditions are equivalent*:

(i) Γ *has no oriented cycles and between any two vertices there are only finitely many arrows.*

(ii) *For any finite set of vertices $E \subset \Gamma$, there are only finitely many paths passing only through vertices of E.*

We recall that a representation of the quiver Γ is a pair $\mathcal{R} = ((V_u)_{u\in\Gamma_0}, (f_a)_{a\in\Gamma_1})$ consisting of a family of vector spaces and a family of linear maps, such that $f_a : V_u \to V_v$, where $u = s(a)$ and $v = t(a)$ for any $a \in \Gamma_1$. A morphism of representations is a family of linear maps (indexed by Γ_0) between the corresponding V_u's, which are compatible with the corresponding linear morphisms in the two representations. The category Rep Γ of representations of Γ is equivalent to the category $u.\mathcal{M}_{K[\Gamma]}$ of unital right $K[\Gamma]$-modules. The equivalence $H : u.\mathcal{M}_{K[\Gamma]} \to$ Rep Γ works as follows. To a unital right $K[\Gamma]$-module V we associate the representation $H(V) = ((V_u)_{u\in\Gamma_0}, (f_a)_{a\in\Gamma_1})$, where $V_u = Vu$ for any $u \in \Gamma_0$, and for an arrow a from u to v we define $f_a : V_u \to V_v$, $f_a(x) = xa$. An inverse equivalence functor associates to representation $\mathcal{R}$ as above the space $\bigoplus_{u\in\Gamma_0} V_u$ endowed with a right $K[\Gamma]$-action defined by $xp = f_p(x)$ for $p = a_1 \dots a_n$ and $x \in V_u$ such that $s(a_1) = u$. Here we denote $f_p = f_{a_n} \dots f_{a_1}$. If $s(a_1) \neq u$, the action is $xp = 0$.

A representation $\mathcal{R}$ is locally finite if for any $u \in \Gamma_0$ and any $x \in V_u$ the subspace $\langle f_p(x) \mid p \text{ is a path with } s(p) = u\rangle$ of $\bigoplus_{u\in\Gamma_0} V_u$ is finite-dimensional. Denote the subcategory of locally finite representations by LocFinRep Γ. The equivalence H restricts to an equivalence H_1 : LocFin $\mathcal{M}_{K[\Gamma]} \to$ LocFinRep Γ.

Recall from [Chin et al. 2002] that a representation $\mathcal{R}$ is locally nilpotent if for any $u \in \Gamma_0$ and any $x \in V_u$, the set $\{p \mid p \text{ path with } f_p(x) \neq 0\}$ is finite. This is easily seen to be equivalent to each $x \in \bigoplus_{u\in\Gamma_0} V_u$ being annihilated by a monomial ideal of finite codimension. Denote by LocNilpRep Γ the category of locally nilpotent representations, which is clearly a subcategory of LocFinRep Γ.

We have the following diagram:

$$\begin{tikzcd}
{}^{K\Gamma}\mathcal{M} \arrow[r, "F^\theta"] \arrow[d, "H_2"', "\sim"] & {}^{K[\Gamma]^0}\mathcal{M} \arrow[r, "G", "\sim"'] & \mathrm{LocFin}\mathcal{M}_{K[\Gamma]} \arrow[r, "I_1"] \arrow[d, "H_1"', "\sim"] & u\mathcal{M}_{K[\Gamma]} \arrow[d, "H"', "\sim"] \\
\mathrm{LocNilpRep}\ \Gamma \arrow[rr, "I_2"] & & \mathrm{LocFinRep}\ \Gamma \arrow[r, "I_3"] & \mathrm{Rep}\ \Gamma
\end{tikzcd}$$

Here G is the equivalence of categories as in Proposition 2.6 (the version for right modules), and the I_j's are inclusion functors. It is easy to see that the image (on objects) of the functor $H_1 G F^\theta$ lies in $\mathrm{LocNilpRep}\,\Gamma$, so we actually have a functor $H_2 : {}^{K\Gamma}\mathcal{M} \to \mathrm{LocNilpRep}\,\Gamma$, and this is just the equivalence noticed in [Chin et al. 2002, Proposition 6.1]. In this way, at the level of representations, the functor F^θ can be regarded as a functor (embedding) from the locally nilpotent quiver representations to the locally finite quiver representations.

We can now characterize precisely when the path coalgebra can be recovered from the quiver algebra, that is, when the above mentioned embedding θ is an isomorphism.

Theorem 3.3. *Let Γ be a quiver. The following assertions are equivalent*:

(i) *Γ has no oriented cycles and between every two vertices of Γ there are only finitely many arrows.*

(ii) $\theta(K\Gamma) = K[\Gamma]^0$.

(iii) *Every cofinite ideal of $K[\Gamma]$ contains a cofinite monomial ideal.*

(iv) *The functor $F^\theta : {}^{K\Gamma}\mathcal{M} \to {}^{K[\Gamma]^0}\mathcal{M}$ is an equivalence.*

(v) *Every locally finite quiver representation of Γ is locally nilpotent.*

Proof. The equivalence of (ii) and (iv) is a general coalgebra fact: if $C \subseteq D$ is an inclusion of coalgebras, then the corestriction of scalars $F : {}^{C}\mathcal{M} \to {}^{D}\mathcal{M}$ is an equivalence if and only if $C = D$. Indeed, if F is an equivalence, pick an arbitrary $x \in D$ and let $N = xD^* \in {}^{D}\mathcal{M}$ be the finite-dimensional D-subcomodule of D generated by x. Then $N \simeq F(M)$, $M \in {}^{C}\mathcal{M}$, and considering the coalgebras of coefficients C_N and C_M of N and M, we see that $C_N = C_M \subseteq C$ by the definition of F. Since $x \in C_N$, this ends the argument.

The equivalence of (ii) and (iii) follows immediately from Proposition 3.1.

(iv) $\Leftrightarrow$ (v) The previous remarks on F^θ (and the diagram drawn there) show that F^θ is an equivalence functor if and only if so is I_2. On the other hand, the inclusion functor I_2 is an equivalence if and only if every locally finite quiver representation of Γ is locally nilpotent.

(i) $\Rightarrow$ (iii) Let I be an ideal of $K[\Gamma]$ of finite codimension. By Lemma 2.1 applied for the algebra $K[\Gamma]$ and the complete set of orthogonal idempotents Γ_0, we have that the set $S' = \{a \in \Gamma_0 \mid a \notin I\}$ must be finite. Let $S = \{a \in \Gamma_0 \mid a \in I\}$. Note that any path p starting or ending at a vertex in S belongs to I, since $p = s(p)p = pt(p) \in I$ if either $s(p) \in I$ or $t(p) \in I$. Furthermore, this shows that if p contains a vertex in S, then $p \in I$, since in that case $p = qr$ with $x = t(q) = s(r) \in S$. Denote the set of paths containing some vertex in S by M. Let H be the vector space spanned by M and let M' be the set of the rest of the paths in Γ. Obviously, M' consists of the paths whose all vertices belong to S'. Since S' is finite, we see that M' is finite, by the conditions of (i) and Proposition 3.2. Therefore H has finite codimension. Also, since H is spanned by paths passing through some vertex in S, we see that H is an ideal. We conclude that I contains the cofinite monomial ideal H.

(iii) $\Rightarrow$ (i) We show first that there are no oriented cycles in Γ. Assume Γ has a cycle

$$C : v_0 \xrightarrow{x_0} v_1 \xrightarrow{x_1} \cdots \longrightarrow v_{s-1} \xrightarrow{x_{s-1}} v_s = v_0,$$

and consider such a cycle that does not self-intersect. We can consider the vertices $v_0, \ldots, v_{s-1}$ modulo s. Denote by $q_{n,k}$ the path starting at the vertex v_n ($0 \leq n \leq s-1$), winding around the cycle C and of length k. Denote again by P the set of all paths in Γ, and by $X = \{q_{n,k} \mid 0 \leq n \leq s-1, k \geq 0\}$. Since the set X is closed under subpaths, it is easy to see that the vector space H spanned by the set $P \setminus X$ is an ideal of $K[\Gamma]$. Let E be the subspace spanned by $S = \{q_{n,ks+i} - q_{n,i} \mid 0 \leq n \leq s-1, i \geq 0, k \geq 1\}$, and let $I = E + H$. We have

$$\begin{aligned}
(q_{n,ks+i} - q_{n,i})q_{n+i,j} &= q_{n,ks+i+j} - q_{n,i+j} \in S,\\
(q_{n,ks+i} - q_{n,i})q_{m,j} &= 0 \text{ for } m \neq n+i,\\
q_{m,j}(q_{n,ks+i} - q_{n,i}) &= q_{m,ks+i+j} - q_{m,i+j} \in S \text{ if } m+j = n,\\
q_{m,j}(q_{n,ks+i} - q_{n,i}) &= 0 \text{ if } m+j \neq n.
\end{aligned}$$

Here in the notation $q_{n,i}$ the first index is considered modulo s, while the second index is a nonnegative integer. The above equations show that if we multiply an element of S to the left (or right) by an element of X, we obtain either an element of S or 0. Combined with the fact that H is an ideal, this shows that I is an ideal.

It is clear that I has finite codimension, since $S \cup \{q_{n,i} \mid 0 \leq n \leq s-1, 0 \leq i \leq s-1\}$ spans $KC = \langle X \rangle$. Indeed, if $0 \leq n \leq s-1$ and j is a nonnegative integer, write $j = ks + i$ with $k \geq 0$ and $0 \leq i \leq s-1$, and we have that $q_{n,j} = q_{n,ks+i} = (q_{n,ks+i} - q_{n,i}) + q_{n,i}$.

On the other hand, I does not contain a cofinite monomial ideal. Indeed, it is easy to see that an element of the form $q_{m,j}$ cannot be in $\langle S \cup (P \setminus X)\rangle = I$, so any monomial ideal contained in I must have infinite codimension.

Thus, we have found a cofinite ideal I which does not contain a cofinite monomial ideal. This contradicts (iii), and we conclude that Γ cannot contain cycles.

We now show that in Γ there are no pair of vertices with infinitely many arrows between them. Assume such a situation exists between two vertices a, b: $a \xrightarrow{x_n} b$, $n \in \mathbb{Z}_{\geq 0}$. We let $X = \{x_n \mid n \in \mathbb{Z}_{\geq 0}\} \cup \{a, b\}$, H be the span of $P \setminus X$, which is an ideal since X is closed under taking subpaths. Let $S = \{x_n - x_0 \mid n \geq 1\}$ and I be the span of $S \cup (P \setminus X)$. As above, since $x_n - x_0$ multiplied by an element of X gives either $x_n - x_0$ or 0, we have that I is an ideal. I has finite codimension since $\{a, b, x_0\} \cup S \cup (P \setminus X)$ spans $K\Gamma$. Also, I does not contain a monomial ideal of finite codimension since no x_n lies in I. Thus we contradict (iii). In conclusion there are finitely many arrows between any two vertices, and this ends the proof. $\square$

It is clear that a finite quiver Γ (i.e., Γ_0 and Γ_1 are finite) without oriented cycles satisfies condition (i) in Theorem 3.3. In this case $K\Gamma$ is finite-dimensional, and we obviously have $K[\Gamma] = (K\Gamma)^*$ (i.e., the map θ is bijective) and also $K\Gamma = K[\Gamma]^0$. This can also be thought as a trivial case of the above theorem.

We now present a few examples to further illustrate the above theorem.

Example 3.4. Let ${}_\infty\mathbb{A}_\infty$ be the infinite line quiver

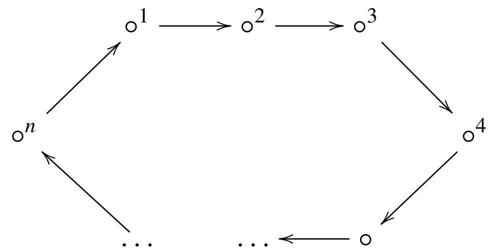
$$\cdots \to \bullet \to \bullet \to \cdots \to \bullet \to \ldots$$

The quiver coalgebra $C = K_\infty\mathbb{A}_\infty$ of this quiver is serial, that is, the injective indecomposable left and right comodules are uniserial, i.e., they have a unique composition series; see [Gómez-Torrecillas and Navarro 2008]. For such a coalgebra, the finite dimensional comodules are easily classified: they are all serial [ibid.]. Moreover, the indecomposable finite-dimensional comodules, i.e., the uniserial ones, correspond to finite paths in ${}_\infty\mathbb{A}_\infty$. Note that this quiver satisfies the conditions of Theorem 3.3, and so the locally nilpotent representations of ${}_\infty\mathbb{A}_\infty$ (i.e., the comodules over $K_\infty\mathbb{A}_\infty$) coincide with the locally finite representations of the quiver algebra $A = K[{}_\infty\mathbb{A}_\infty]$, and also, the finite-dimensional quiver representations of ${}_\infty\mathbb{A}_\infty$ are the comodules over $K_\infty\mathbb{A}_\infty$. Moreover, the coalgebra of representative functions on $K[{}_\infty\mathbb{A}_\infty]$ is isomorphic to $K_\infty\mathbb{A}_\infty$.

Note that in general, it is not easy to describe arbitrary comodules even for a serial coalgebra. By results in [Iovanov 2011], if an infinite dimensional indecomposable injective comodule exists, then there are comodules which do not decompose into indecomposable comodules (and, in particular, are not indecomposable). Moreover, for the left bounded infinite quiver $\mathbb{A}_\infty : \bullet \to \bullet \to \cdots \to \bullet \to \ldots$, it is shown in [Iovanov 2011] that all left comodules over $K\mathbb{A}_\infty$ are serial direct sums of indecomposable uniserial comodules corresponding to finite paths, while in the category of right comodules over $K\mathbb{A}_\infty$ there are objects which do not decompose into direct sums of indecomposables.

Example 3.5. Let $\mathbb{C}_n$ be the following quiver of affine Dynkin type $\widetilde{A}_n$:

$\mathbb{C}_n$



The path coalgebra $K\mathbb{C}_n$ is again serial, and the finite dimensional (left and right) comodules are all direct sum of uniserial objects (corresponding to finite paths). These correspond to finite-dimensional locally nilpotent representations. This quiver does not satisfy the hypothesis of Theorem 3.3. We give an example of a quiver representation which is locally finite (even finite-dimensional) but not locally nilpotent. Let $x_1, \dots, x_n$ denote the arrows of the quiver, with $a_i = s(x_i)$. Let $M = K[\mathbb{C}_n]/I$ where I is the (two sided) ideal generated by elements $p - 1$, where p is a path of length n and with $1 = a_1 + \cdots + a_n$ (M is actually an algebra). One can easily see that M is spanned as a vector space by paths of length less than n. A not too difficult computation shows that I does not contain any monomial ideal of finite codimension, and so M as a representation of $K[\mathbb{C}_n]$ is not locally nilpotent, but it is finite-dimensional. We again note that the infinite dimensional comodules over the coalgebra $K\mathbb{C}_n$ are hard to understand, as there are both left and right comodules which are not direct sum of indecomposable comodules.

An easy particular example of this can be obtained for $n = 1$; in this case, $K[\mathbb{C}_n] \cong K[X]$ — the polynomial algebra. As noted before, the finite dual of this algebra is not the path coalgebra of $\mathbb{C}_1$. Also, the representation $K[X]/(X-1)$ is not locally nilpotent.

Let $\psi : K[\Gamma] \to (K\Gamma)^*$ be the linear map defined by $\psi(p)(q) = \delta_{p,q}$ for any paths p and q. In fact ψ is just θ as a linear map, but we denote it differently since we regard it now as a morphism in the category of algebras not necessarily with identity. Indeed, it is easy to check that ψ is multiplicative. Thus the quiver algebra embeds in the dual of the path coalgebra. Our aim is to show that in certain situations $K[\Gamma]$ can be recovered from $(K\Gamma)^*$ as the rational part. Obviously, this is the case when $K[\Gamma]$ is finite-dimensional, which will also be seen as a consequence of the next result, which characterizes completely these situations. We recall that if C is a coalgebra, the rational part of the left C^*-module C^*, consisting of all elements $f \in C^*$ such that there exist finite families $(c_i)_i$ in C and $(f_i)_i$ in C^* with $c^* f = \sum_i c^*(c_i) f_i$ for any $c^* \in C^*$, is denoted by $C_l^{*\,\mathrm{rat}}$. This is the largest C^*-submodule which is rational, i.e., whose C^*-module structure comes from a right C-comodule structure. Similarly, $C_r^{*\,\mathrm{rat}}$ denotes the rational part of the right C^*-module C^*. A coalgebra C is called right (respectively left) semiperfect if the category of right (respectively left) C-comodules has enough projectives. This is equivalent to the fact that $C_l^{*\,\mathrm{rat}}$ (respectively $C_r^{*\,\mathrm{rat}}$) is dense in C^* in the finite topology, see [Dăscălescu et al. 2001, Section 3.2].

Theorem 3.6. *The following are equivalent.*

(i) $\mathrm{Im}(\psi) = (K\Gamma)_l^{*\,\mathrm{rat}}$.

(ii) $\mathrm{Im}(\psi) = (K\Gamma)_r^{*\,\mathrm{rat}}$.

(iii) *For any vertex v of Γ there are finitely many paths starting at v and finitely many paths ending at v.*

(iv) *The path coalgebra $K\Gamma$ is left and right semiperfect.*

Proof. (iii) $\Rightarrow$ (i) Let p be a path. We show that $p^* = \psi(p) \in \mathrm{Im}(\psi)$. If $c^* \in (K\Gamma)^*$ and q is a path, we have that

$$(c^* p^*)(q) = \sum_{rs=q} c^*(r) p^*(s) = \begin{cases} c^*(r) & \text{if } q = rp \text{ for some path } r, \\ 0 & \text{if } q \text{ does not end with } p. \end{cases}$$

Let $q_1 = r_1 p, \dots, q_n = r_n p$ be all the paths ending with p. By the formula above, $(c^* p^*)(q_i) = c^*(r_i)$ for any $1 \le i \le n$, and $(c^* p^*)(q) = 0$ for any path $q \ne q_1, \dots, q_n$. This shows that $c^* p^* = \sum_{1 \le i \le n} c^*(r_i) q_i^*$, thus $p^* \in (K\Gamma)_l^{*\,\mathrm{rat}}$, and we have that $\mathrm{Im}(\psi) \subseteq (K\Gamma)_l^{*\,\mathrm{rat}}$.

Now let $c^* \in (K\Gamma)_l^{*\,\mathrm{rat}}$, so there exist $(c_i)_{1 \le i \le n}$ in $K\Gamma$ and $(c_i^*)_{1 \le i \le n}$ in $(K\Gamma)^*$ such that $d^* c^* = \sum_{1 \le i \le n} d^*(c_i) c_i^*$ for any $d^* \in (K\Gamma)^*$. Let $p_1, \dots, p_m$ be all the paths that appear with nonzero coefficients in some of the c_i's (represented as a linear combination of paths). Then for any $p \ne p_1, \dots, p_m$ we have that $p^*(c_i) = 0$, so then $p^* c^* = 0$. Let v be a vertex such that no one of $p_1, \dots, p_m$ passes through v. Then for any path p starting at v we have that $0 = (v^* c^*)(p) = v^*(v) c^*(p) = c^*(p)$. Therefore c^* may be nonzero on a path p only if $s(p) \in \{p_1, \dots, p_m\}$. By condition (iii), there are only finitely many such paths p, denote them by $q_1, \dots, q_e$. Then $c^* = \sum_{1 \le i \le e} c^*(q_i) q_i^* \in \mathrm{Im}(\psi)$, and we also have that $(K\Gamma)_l^{*\,\mathrm{rat}} \subseteq \mathrm{Im}(\psi)$.

(i) $\Rightarrow$ (iii) Let v be a vertex. Then $v^* = \psi(v) \in (K\Gamma)_l^{*\,\mathrm{rat}}$, so there exist finite families $(c_i) \subseteq K\Gamma$ and $(c_i^*)_i \subseteq (K\Gamma)^*$ such that $c^* v^* = \sum_i c^*(c_i) c_i^*$ for any $c^* \in (K\Gamma)^*$. Then for any path q,

$$\sum_i c^*(c_i) c_i^*(q) = (c^* v^*)(q) = \begin{cases} c^*(q) & \text{if } q \text{ ends at } v, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If there exist infinitely many paths ending at v, we can find one such path q which does not appear in the representation of any c_i as a linear combination of paths. Then there exists $c^* \in (K\Gamma)^*$ with $c^*(q) \ne 0$ and $c^*(c_i) = 0$ for any i, in contradiction with (1). Thus only finitely many paths can end at v. In particular Γ does not have cycles.

On the other hand, if we assume that there are infinitely many paths $p_1, p_2, \ldots$ starting at v, let $c^* \in (K\Gamma)^*$ which is 1 on each p_i and 0 on any other path. Clearly $c^* \notin \mathrm{Im}(\psi)$. We show that $c^* \in (K\Gamma)_l^{*\,\mathrm{rat}}$, and the obtained contradiction shows that only finitely many paths start at v. Indeed, we have

$$(2)\qquad (d^*c^*)(q) = \begin{cases} d^*(r) & \text{if } q = rp_i \text{ for some } i \geq 1 \text{ and some path } r,\\ 0 & \text{otherwise.}\end{cases}$$

Let $r_1, \ldots, r_m$ be all the paths ending at v (they are finitely many as we proved above). For each $1 \leq j \leq m$ we consider the element $c_j^* \in (K\Gamma)^*$ which is 1 on every path of the form $r_j p_i$, and 0 on any other path. Using (2) and the fact that $r_j p_i \neq r_{j'} p_{i'}$ for $(i, j) \neq (i', j')$ (this follows because $r_j, r_{j'}$ end at v and $p_i, p_{i'}$ start at v, and there are no cycles containing v), we see that $d^*c^* = \sum_{1 \leq j \leq m} d^*(r_j)c_j^*$, and this will guarantee that c^* is a rational element.

(ii) $\Leftrightarrow$ (iii) is similar to (i) $\Leftrightarrow$ (iii).

(iii) $\Leftrightarrow$ (iv) follows from [Chin et al. 2002, Corollary 6.3]. □

4. Incidence coalgebras and incidence algebras

In this section we parallel the results in Section 3 in the framework of incidence (co)algebras. Let $(X, \leq)$ be a partially ordered set which is locally finite, i.e., the set $\{z \mid x \leq z \leq y\}$ is finite for any $x \leq y$ in X. The incidence coalgebra of X, denoted by KX, is the vector space with basis $\{e_{x,y} \mid x, y \in X, x \leq y\}$, and comultiplication and counit defined by $\Delta(e_{x,y}) = \sum_{x \leq z \leq y} e_{x,z} \otimes e_{z,y}$, $\epsilon(e_{x,y}) = \delta_{x,y}$ for any $x, y \in X$ with $x \leq y$. For such a X, we can consider the quiver Γ with vertices the elements of X, and such that there is an arrow from x to y if and only if $x < y$ and there is no element z with $x < z < y$. It was proved in [Dăscălescu et al. ≥ 2013] that the linear map $\phi : KX \to K\Gamma$, defined by

$$\phi(e_{x,y}) = \sum_{\substack{p \text{ path}\\ \text{from } x \text{ to } y}} p$$

for any $x, y \in X, x \leq y$, is an injective coalgebra morphism. We note that this map is surjective if and only if in Γ there is at most one path between any to vertices $x, y \in X$. To see this, let $P(x, y)$ denote the set of paths from x to y. Note that the incidence coalgebra KX is then $KX = \bigoplus_{x,y \in X} \langle P(x, y)\rangle$, and that $\phi(\langle e_{x,y}\rangle) \subset P(x, y)$ for $x \leq y$. Thus, ϕ is surjective if and only if $\dim(P(x, y)) = 1$ for all $x \leq y$, which is equivalent to the above stated condition. In fact, this is also a consequence of the following more general fact.

Proposition 4.1. *A coalgebra C is both an incidence coalgebra and a path coalgebra if and only if it is the path coalgebra of a quiver Γ for which there is at most one path between any two vertices.*

Proof. If the condition is satisfied for a quiver Γ, we can introduce an obvious order on the set X of vertices of Γ setting $x \le y$ if and only if there is a path from x to y. It is easy to check that this is an ordering, and so the above map $\phi : KX \to K\Gamma$ is bijective. Conversely, let $C \cong KX \cong K\Gamma$ for a locally finite partially ordered set X and a quiver Γ. We note that the simple subcoalgebras (and simple left subcomodules, simple right subcomodules) of C are precisely the spaces Kx for $x \in X$ and Kv for v vertex in Γ, and X, respectively Γ correspond to the group-like elements of C. Thus, X must be the set of vertices of Γ. Furthermore, we note that in either an incidence coalgebra or a path coalgebra, the injective hull of a simple left comodule Kx is uniquely determined as follows (note that in general, given an injective module M and a submodule N of M, there is an injective hull of N contained in M but it is not necessarily uniquely determined). For incidence/path coalgebras, the right (left) injective hull $E_r(Kx)$ of Kx (respectively, $E_l(Kx)$) of the right (respectively, left) comodule Kx is the span of all segments/paths starting (respectively, ending) at x (see the proof of [Simson 2009, Proposition 2.5] for incidence coalgebras and [Chin et al. 2002, Corollary 6.2(b)] for path coalgebras). Then, for $x \le y$, from the incidence coalgebra results we get $E_r(Kx) \cap E_l(Ky) = \langle e_{x,y} \rangle$ and from the path coalgebra we get $E_r(Kx) \cap E_l(Ky) = \langle P(x, y) \rangle$. This shows that $\langle P(x, y) \rangle$ is one dimensional, and the proof is finished. □

Apart from the incidence coalgebra KX, there is another associated algebraic object with a combinatorial relevance. This is the incidence algebra $IA(X)$, which is the space of all functions $f : \{(x, y) \mid x, y \in X, x \le y\} \to K$ (functions on the set of closed intervals of X), with multiplication given by convolution:

$$(fg)(x, y) = \sum_{x \le z \le y} f(x, z) g(z, y)$$

for any $f, g \in IA(X)$ and any $x, y \in X,\ x \le y$. See [Spiegel and O'Donnell 1997] for details on the combinatorial relevance of the incidence algebra. It is clear that $IA(X)$ is isomorphic to the dual algebra of KX, if we identify a map $f \in IA(X)$ with the element of $(KX)^*$ which takes $e_{x,y}$ to $f(x, y)$ for any $x \le y$. For simplicity, we will identify $IA(X)$ with $(KX)^*$.

Comparing to path coalgebras and quiver algebras, the situation is different, since the incidence algebra always has identity. However, we can consider the subspace $FIA(X)$ of $IA(X)$ spanned by all the elements $E_{x,y}$ with $x \le y$, where $E_{x,y}(e_{u,v}) = \delta_{x,u}\delta_{y,v}$ for any $u \le v$. Equivalently, $FIA(X)$ consists of all functions on $\{(x, y) \mid x, y \in X, x \le y\}$ that have finite support. Then $FIA(X)$ is a subalgebra of $IA(X)$ which does not have an identity when X is infinite, but it has enough idempotents, the set of all $E_{x,x}$. The algebra $FIA(X)$ plays the role of the quiver algebra in this new framework.

The subspace $FIA(X)$ is dense in $IA(X)$ in the finite topology, since it is easy to see that $FIA(X)^{\perp} = 0$ (see [Dăscălescu et al. 2001, Corollary 1.2.9]). We have a coalgebra morphism $\theta : KX \to FIA(X)^0$, defined by $\theta(c)(c^*) = c^*(c)$ for any $c \in KX$ and any $c^* \in FIA(X)$. We note that $\theta(c)$ indeed lies in $FIA(X)^0$, since $\mathrm{Ker}(\theta(c)) = \langle c\rangle^{\perp} \cap FIA(X) \supseteq D^{\perp} \cap FIA(X)$, where D is the (finite dimensional) subcoalgebra generated by c in KX. Then $D^{\perp}$ is an ideal of $IA(X)$ of finite codimension, and then $D^{\perp} \cap FIA(X)$ is an ideal of $FIA(X)$ of finite codimension. Since $FIA(X)$ is dense in $IA(X)$, θ is injective. The next result shows that we can recover the incidence coalgebra KX as the finite dual of the algebra with enough idempotents $FIA(X)$. The result parallels Theorem 3.3; note that the conditions analogous to the ones in (i) in Theorem 3.3 are always satisfied in incidence algebras.

Theorem 4.2. *For any locally finite partially ordered set X, the map*

$$\theta : KX \to FIA(X)^0$$

is an isomorphism of coalgebras.

Proof. It is enough to show that θ is surjective. Let $F \in FIA(X)^0$, so F maps $FIA(X)$ to K and $\mathrm{Ker}(F)$ contains an ideal I of $FIA(X)$ of finite codimension. Then the set $X_0 = \{x \in X \mid E_{x,x} \notin I\}$ is finite by Lemma 2.1.

If $x \in X \setminus X_0$, then $E_{x,y} = E_{x,x}E_{x,y} \in I$ for any $x \le y$. Similarly $E_{x,y} \in I$ for any $y \in X \setminus X_0$ and $x \le y$. Thus in order to have $E_{x,y} \notin I$, both x and y must lie in X_0. This shows that only finitely many $E_{x,y}$'s lie outside I. Let $\mathcal{F}$ be the set of all pairs (x, y) such that $E_{x,y} \notin I$. Then we have that $F = \sum_{(x,y)\in\mathcal{F}} F(E_{x,y})\theta(e_{x,y})$. Indeed, when evaluated at $E_{u,v}$, both sides are 0 if $(u, v) \notin \mathcal{F}$, or $F(E_{u,v})$ if $(u, v) \in \mathcal{F}$. Thus $F \in \mathrm{Im}(\theta)$. □

The next result and its proof parallel Theorem 3.6.

Theorem 4.3. *Let $C = KX$. The following assertions are equivalent.*

(i) $FIA(X) = C_l^{*\,\mathrm{rat}}$.

(ii) $FIA(X) = C_r^{*\,\mathrm{rat}}$.

(iii) *For any $x \in X$ there are finitely many elements $u \in X$ such that $u \le x$, and finitely many elements $y \in X$ such that $x \le y$.*

(iv) *KX is a left and right semiperfect coalgebra.*

Proof. (i) ⇒ (iii) Since $E_{x,x} \in C_l^{*\,\mathrm{rat}}$, there exist finite families $(c_i)_i$ in C and $(c_i^*)_i$ in C^* such that $c^*E_{x,x} = \sum_i c^*(c_i)c_i^*$ for any $c^* \in C^*$. If there are infinitely many elements u in X such that $u \le x$, then we can choose such an element u_0 for which $e_{u_0,x}$ does not show up in the representation of any c_i (as a linear combination of the standard basis of C). Since $E_{u_0,x}(e_{p,q}) = \delta_{u_0,p}\delta_{x,q}$, we get $E_{u_0,x}(c_i) = 0$ for any i, so $\sum_i E_{u_0,x}(c_i)c_i^* = 0$, while $(E_{u_0,x}E_{x,x})(e_{u_0,x}) = 1$, a contradiction.

Assume now that for some $x \in X$ the set of all elements y with $x \leq y$, say $(y_i)_i$, is infinite. Let $c^* \in C^*$ which is 1 on each e_{x,y_i} and 0 on any other $e_{p,q}$. Then it is easy to see that

$$(d^*c^*)(e_{u,v}) = \begin{cases} d^*(e_{u,x}) & \text{if } u \leq x \leq v, \text{ and } v \in \{y_i \mid i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(u_j)_j$ be the family of all elements u with $u \leq x$. As we proved above, this family is finite. For each j, let $c_j^* \in C^*$ equal 1 on every e_{u_j,y_i}, and 0 on any other $e_{p,q}$. We have that $d^*c^* = \sum_j d^*(e_{u_j,x})c_j^*$ for any $d^* \in C^*$. Indeed, using the formula above we see that both sides equal $d^*(e_{u_{j_0},x})$ when evaluated at some $e_{u_{j_0},y_i}$, and 0 when evaluated at any other $e_{p,q}$.

Therefore $c^* \in C_l^{*\,\mathrm{rat}}$, but obviously $c^* \notin FIA(X)$, since it is nonzero on infinitely many $e_{p,q}$'s.

(iii) $\Rightarrow$ (i) Choose some x, y with $x \leq y$. Then for any $c^* \in C^*$ we have that

$$(c^*E_{x,y})(e_{u,v}) = \begin{cases} c^*(e_{u,x}) & \text{if } u \leq x \leq y = v, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that if $(u_j)_j$ is the finite family of all elements u with $u \leq x$, then $c^*E_{x,y} = \sum_j c^*(e_{u_j,x})E_{u_j,y}$, so $E_{x,y}$ lies in $C_l^{*\,\mathrm{rat}}$.

Now let $c^* \in C_l^{*\,\mathrm{rat}}$, so

$$d^*c^* = \sum_i d^*(c_i)c_i^*$$

for some finite families $(c_i)_i$ in C and $(c_i^*)_i$ in C^*. If $x \in X$ such that $e_{x,x}$ does not appear in any c_i (with nonzero coefficient), then $E_{x,x}c^* = 0$. In particular $0 = (E_{x,x}c^*)(e_{x,y}) = c^*(e_{x,y})$ for any $x \leq y$. Since only finitely many $e_{u,u}$ appear in the representations of the c_i's, and for any such u there are finitely many v with $u \leq v$, we obtain that $c^*(e_{u,v})$ is nonzero for only finitely many $e_{u,v}$. So c^* lies in the span of all $E_{x,y}$'s, which is $FIA(X)$.

(ii) $\Leftrightarrow$ (iii) is similar.

(iii) $\Leftrightarrow$ (iv) follows from [Simson 2009, Lemma 5.1]. □

5. Coreflexivity for path subcoalgebras and subcoalgebras of incidence coalgebras

We recall from [Radford 1973; Taft 1972] that a coalgebra C is called coreflexive if any finite-dimensional left (or equivalently, any finite-dimensional right) C^*-module is rational. This is also equivalent to asking that the natural embedding of C into the finite dual of C^*, $C \to (C^*)^0$ is surjective (so an isomorphism), or that any left (equivalently, any right) cofinite ideal is closed in the finite topology. See [Radford 1974; 1973; Taft 1972; 1977] for further equivalent characterizations.

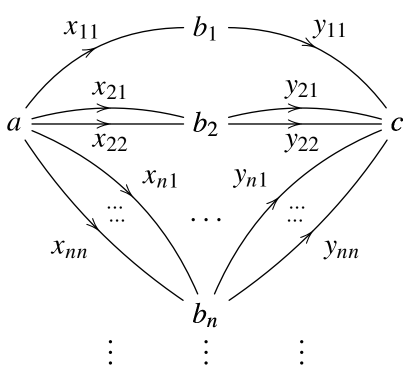
Given the definition of coreflexivity and the characterizations of the previous section, it is natural to ask what is the connection between the situation when the path coalgebra can be recovered as the finite dual of the quiver algebra, and the coreflexivity of the path coalgebra. We note that these two are closely related. We have an embedding $\iota : K\Gamma \hookrightarrow (K\Gamma)^{*0}$; at the same time, we note that the embedding of algebras (without identity) $\psi : K[\Gamma] \hookrightarrow (K\Gamma)^*$ which is dense in the finite topology of $(K\Gamma)^*$, produces a comultiplicative morphism $\varphi : (K\Gamma)^{*0} \to K[\Gamma]^0$. Note that φ is not necessarily a morphism of coalgebras, since it may not respect the counits. It is easy to see that these canonical morphisms are compatible with θ, i.e., they satisfy $\theta = \varphi \circ \iota$:

$$\begin{tikzcd} K\Gamma \arrow[r, hook, "\iota"] \arrow[dr, "\theta"'] & (K\Gamma)^{*0} \arrow[d, "\varphi"] \\ & K[\Gamma]^0 \end{tikzcd}$$

It is then natural to ask what is the connection between the bijectivity of θ, and coreflexivity of $K\Gamma$, i.e., bijectivity of ι. In fact, we notice that if C is coreflexive (equivalently, ι is surjective), then φ is necessarily injective.

The following two examples will show that, in fact, C can be coreflexive and θ not an isomorphism, and also that θ can be an isomorphism without C being coreflexive.

Example 5.1. Consider the path coalgebra of the following quiver Γ:



Here there are n arrows from vertex a to vertex b_n and n arrows from b_n to c for each positive integer n. We note that the one-dimensional vector space I spanned by $a - c$ is a coideal, since $a - c$ is an (a, c)-skew-primitive element. It is not difficult to observe that the quotient coalgebra C/I is isomorphic to the coalgebra from [Radford 1974, Example 3.4], and so C/I is not coreflexive, as shown in [Radford 1974]. By [Heyneman and Radford 1974, 3.1.4], we know that if I is a finite-dimensional coideal of a coalgebra C then C is coreflexive if and only if

C/I is coreflexive. Therefore, C is not coreflexive. However, it is obvious that C satisfies the quiver conditions of Theorem 3.3, and therefore, $K\Gamma = K[\Gamma]^0$.

Hence, a path coalgebra of a quiver with no cycles and finitely many arrows between any two vertices is not necessarily coreflexive. Conversely, we note that in a coreflexive path coalgebra there are only finitely many arrows between any two vertices. This is true since a coreflexive coalgebra is locally finite by [Heyneman and Radford 1974, 3.2.4], which means that the wedge $X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y)$ of any two finite-dimensional vector subspaces X, Y of C is finite-dimensional (one applies this for $X = Ka$ and $Y = Kb$). However, if a path coalgebra $K\Gamma$ is coreflexive, Γ may contain cycles: consider the path coalgebra C of a loop (a graph with one vertex and one arrow); C is then the divided power coalgebra, $C^* = K[[X]]$, the ring of formal power series, and its only ideals are (X^n), which are closed in the finite topology of C^*. Thus, every finite dimensional C^*-module is rational and C is coreflexive.

We will prove coreflexivity of an interesting class of path coalgebras, whose quiver satisfy a slightly stronger condition than that required by Theorem 3.3 (so in particular, they will satisfy $K\Gamma = K[\Gamma]^0$). We first prove a general coreflexivity criterion.

Theorem 5.2. *Let C be a coalgebra with the property that for any finite dimensional subcoalgebra V there exists a finite-dimensional subcoalgebra W such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. Then C is coreflexive if and only if its coradical C_0 is coreflexive.*

Proof. If C is coreflexive, then so is C_0, since a subcoalgebra of a coreflexive coalgebra is coreflexive (see [Heyneman and Radford 1974, Proposition 3.1.4]). Conversely, let C_0 be coreflexive. We prove that any finite-dimensional left C^*-module M is rational, by induction on the length $l(M)$ of M. If $l(M) = 1$, i.e., M is simple, then M is also a left $C^*/J(C^*)$-module, where $J(C^*)$ is the Jacobson radical of C^*. Since $C^*/J(C^*) \simeq C_0^*$ and C_0 is coreflexive, we have that M is a rational C_0^*-module, so then it is a rational C^*-module, too.

Assume now that the statement is true for length $< n$, where $n > 1$, and let M be a left C^*-module of length n. Let M' be a simple submodule of M, and consider the associated exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

By the induction hypothesis M' and M'' are rational. By [Dăscălescu et al. 2001, Theorem 2.2.14] we have that $\mathrm{ann}_{C^*}(M')$ and $\mathrm{ann}_{C^*}(M'')$ are finite codimensional closed two-sided ideals in C^*. Using [Dăscălescu et al. 2001, Corollary 1.2.8 and Proposition 1.5.23], $\mathrm{ann}_{C^*}(M') = U_1^\perp$ and $\mathrm{ann}_{C^*}(M'') = U_2^\perp$ for some finite-dimensional subcoalgebras of C. Using the hypothesis for $V = U_1 + U_2$, there is a finite dimensional subcoalgebra W of C such that $U_1 \subseteq W$, $U_2 \subseteq W$ and $W^\perp W^\perp = W^\perp$. Then, by [Dăscălescu et al. 2001, Proposition 1.5.23],

$$W^\perp = W^\perp W^\perp \subseteq U_1^\perp U_2^\perp = \operatorname{ann}_{C^*}(M') \operatorname{ann}_{C^*}(M'') \subseteq \operatorname{ann}_{C^*}(M)$$

is a two-sided closed ideal of C^*, of finite codimension. Therefore, M is a rational C^*-module by using again [Dăscălescu et al. 2001, Theorem 2.2.14]. □

Proposition 5.3. *Let C be the path coalgebra $K\Gamma$, where Γ is a quiver such that there are finitely many paths between any two vertices. Then for any finite-dimensional subcoalgebra V of C there exists a finite-dimensional subcoalgebra W such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. As a consequence, C is coreflexive if and only if the coradical C_0 (which is the grouplike coalgebra over the set of vertices of Γ) is coreflexive.*

Proof. Let V be a finite-dimensional subcoalgebra of $C = K\Gamma$. An element $c \in V$ is of the form

$$c = \sum_{i=1}^{n} \alpha_i p_i, \quad \alpha_i \neq 0,$$

a linear combination of paths $p_1, \dots, p_n$. Consider the set of all vertices at least one of these paths passes through, and let S_0 be the union of all these sets of vertices when c runs through the elements of V. Since V is finite-dimensional, we have that S_0 is finite (in fact, one can see that S_0 consists of all vertices in Γ which belong to V, so that KS_0 is the socle of V). Let P be the set of all paths p such that $s(p), t(p) \in S_0$. We consider the set S of all vertices at least one path of P passes through. It is clear that P is finite, and then so is S. We note that if $v_1, v_2 \in S$ and p is a path from v_1 to v_2, then any vertex on p lies in S. Indeed, v_1 is on a path from u_1 to u_1' (vertices in S_0), and let p_1 be its subpath from u_1 to v_1. Similarly, v_2 is on a path from u_2 to u_2' (in S_0), and let p_2 be the subpath from v_2 to u_2'. Then $p_1 p p_2 \in P$, so any vertex of p is in S. Let W be the subspace spanned by all paths starting and ending at vertices in S. It is clear that any subpath of a path in W is also in W, so then W is a finite-dimensional subcoalgebra containing V (since S_0 is contained in S).

We show that $W^\perp W^\perp = W^\perp$. For this, given $\eta \in W^\perp$, we construct elements $f_1, f_2, g_1, g_2 \in W^\perp$ such that $\eta = f_1 g_1 + f_2 g_2$. We define $f_i(p)$ and $g_i(p)$, $i = 1, 2$, on all paths p by induction on the length of p. For paths p of length zero, i.e., if p is a vertex v, we define $f_i(v) = g_i(v) = 0$, $i = 1, 2$, for any $v \in S$, while for $v \notin S$, we set $f_1(v) = g_2(v) = 1$, and $f_2(v)$ and $g_2(v)$ are such that $g_1(v) + f_2(v) = \eta(v)$. Then clearly $\eta = f_1 g_1 + f_2 g_2$ on paths of length zero. For the induction step, assume that we have defined f_i and g_i, $i = 1, 2$, on all paths of length $< l$, and that $\eta = f_1 g_1 + f_2 g_2$ on any such path. Let now p be a path of length l, starting at u and ending at v. If $u, v \in S$, then we define $f_i(p) = g_i(p) = 0$, $i = 1, 2$, and clearly $\eta(p) = \sum_{i=1,2} \sum_{qr=p} f_i(q)g_i(r)$, since both sides are zero. If either $u \notin S$ or $v \notin S$, we need the following equality to hold:

$$(3) \quad f_1(u)g_1(p) + f_1(p)g_1(v) + f_2(u)g_2(p) + f_2(p)g_2(v) \\ = \eta(p) - \sum_{i=1,2} \sum_{\substack{qr=p \\ q\neq p, r\neq p}} f_i(q)g_i(r).$$

We note that the terms of the right-hand side of the equality (3) have already been defined, because when $p = qr$ and $q \neq p$, $r \neq p$, the length of the paths q and r is strictly less than the length of p. We define $f_1(p)$ and $g_2(p)$ to be any elements of K, and then since either $f_1(u) = 1$ or $g_2(v) = 1$ (since either $u \notin S$ or $v \notin S$), we can choose suitable $g_1(p)$ and $f_2(p)$ such that (3) holds true.

The fact that C is coreflexive if and only if so is C_0 follows now follows directly from Theorem 5.2. □

Moreover, we can extend the result in the previous proposition to subcoalgebras of path coalgebras.

Proposition 5.4. *Let C be a subcoalgebra of a path coalgebra $K\Gamma$, such that there are only finitely many paths between any two vertices in Γ. Then C is coreflexive if and only if C_0 is coreflexive.*

Proof. Let Γ' be the subquiver of Γ whose vertices are all the vertices v of Γ such that there is an element $c = \sum_i \alpha_i p_i \in C$, where the α_i's are nonzero scalars and the p_i's are pairwise distinct paths, and at least one p_i passes through v. The arrows of Γ' are all the arrows of Γ between vertices of Γ'. Clearly, there are only finitely many paths between any two vertices in Γ'. Then we have that C is a subcoalgebra of $K\Gamma'$ and $C_0 = (K\Gamma')_0$. Obviously, $C_0 \subset (K\Gamma')_0$; for the converse, let us consider a vertex u in Γ', so there is $c \in C$ such that $c = \sum_i \alpha_i p_i$, with $\alpha_i \neq 0$ and distinct p_i's, and some p_k passes through u. Let us write then $p_k = qr$ such that q ends at u and r begins at u. Since C is a subcoalgebra of $K\Gamma'$ it is also a sub-bicomodule, so then $r^*cq^* \in C$, where $q^*, r^* \in (K\Gamma')^*$ are equal to 1 on q, r respectively and 0 on all other paths of $K\Gamma'$. Now

$$r^* p_i q^* = \sum_{p_i = stw} q^*(s)tr^*(w)$$

and the only nonzero terms can occur if $p_i = qt_i r$, where t_i is a path starting and ending at u (loop at u). Let J be the set of these indices. In this situation $r^* p_i q^* = t_i$. Note that since the p_i's are distinct, the t_j's, $j \in J$ are distinct too. Also, since $p_k = qr$, there is at least such a j. We have

$$r^*cq^* = \sum_j \alpha_j t_j,$$

with all t_j beginning and ending at u, and $t_k = u$. Let $l \in J$ be an index such that t_l has maximum length among the t_j's, $j \in J$. We note then that $t_l^* t_j = 0$ if $j \neq l$, since for any decomposition $t_j = st$, we have $t \neq t_l$ because of the maximality of t_l and of the fact that $t_j \neq t_l$. However, $t_l^* t_l = u$. Therefore, $t_l^* c = \alpha_l u \in C$, so $u \in C$ since $\alpha_l \neq 0$.

Thus if C_0 is coreflexive, we have that $(K\Gamma')_0$ is coreflexive, and then by Proposition 5.3, we have that $K\Gamma'$ is coreflexive. Then C is coreflexive, as a subcoalgebra of $K\Gamma'$. Conversely, if C is coreflexive, clearly C_0 is coreflexive. □

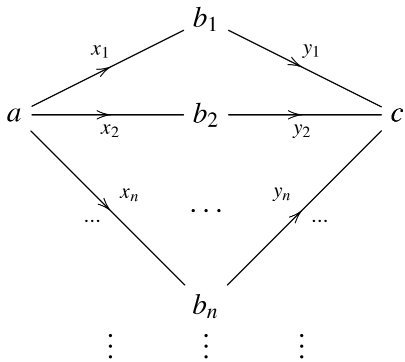
Corollary 5.5. *Let C be a subcoalgebra of an incidence coalgebra KX. Then C is coreflexive if and only if C_0 is coreflexive.*

Proof. As explained in Section 4, KX can be embedded in a path coalgebra $K\Gamma$, where Γ is a quiver for which there are finitely many paths between any two vertices. Then C is isomorphic to a subcoalgebra of $K\Gamma$ and we apply Proposition 5.4. □

Recall that for a path coalgebra or incidence coalgebra C, $C_0 \sim K^{(X)}$, where X is the set of grouplike elements in C. At this point, we believe it is worth mentioning that by [Heyneman and Radford 1974, Theorem 3.7.3], $K^{(X)}$ is coreflexive whenever X is a nonmeasurable cardinal. More precisely, an ultrafilter $\mathcal{F}$ on a set X is called an *Ulam* ultrafilter if $\mathcal{F}$ is closed under countable intersection. X is called *nonmeasurable* (or *reasonable* in the language of [Heyneman and Radford 1974]) if every Ulam ultrafilter is principal (i.e., it equals the collection of all subsets of X containing some fixed $x \in X$). The class of nonmeasurable sets contains the countable sets and is closed under usual set-theoretic constructions, such as the power set, subsets, products, and unions. If a nonreasonable (i.e., measurable) set exists, its cardinality has to be "very large" (inaccessible in the sense of set theory).

We now give an example to show that it is possible to have a coalgebra which is both coreflexive, and satisfies the path coalgebra "recovery" conditions of Theorem 3.3; however, in its quiver, some vertices are joined by infinitely many paths. Thus, in general, the coreflexivity question for path coalgebras is more complicated.

Example 5.6. Consider the path coalgebra C of the following quiver Γ:



Here there are infinitely many vertices b_n, one for each positive integer n. Let W_n be the finite-dimensional subcoalgebra of C with basis

$$B = \{a, c, b_1, \ldots, b_n, x_1, \ldots, x_n, y_1, \ldots, y_n, x_1y_1, \ldots, x_ny_n\}.$$

We show that $W_n^\perp = W_n^\perp \cdot W_n^\perp$. Let $f \in W_n^\perp$. We show that we can find elements $g_1, g_2, h_1, h_2 \in W_n^\perp$ such that $f = g_1h_1 + g_2h_2$. This condition is already true on elements of B if we set g_1, g_2, h_1, h_2 to be zero on W_n. For $k > n$ we define g_1, g_2, h_1, h_2 on x_k, y_k and x_ky_k such that

$$\begin{aligned}
f(x_ky_k) &= \sum_{i=1,2} (g_i(a)h_i(x_ky_k) + g_i(x_k)h_i(y_k) + g_i(x_ky_k)h_i(c)),\\
f(x_k) &= \sum_{i=1,2} (g_i(a)h_i(x_k) + g_i(x_k)h_i(b_k)),\\
f(y_k) &= \sum_{i=1,2} (g_i(b_k)h_i(y_k) + g_i(y_k)h_i(c)),\\
f(b_k) &= \sum_{i=1,2} g_i(b_k)h_i(b_k).
\end{aligned}$$

and since $g_i(a) = h_i(a) = g_i(c) = h_i(c) = 0$ this is equivalent to the matrix equality

$$\begin{pmatrix} f(b_k) & f(y_k) \\ f(x_k) & f(x_ky_k) \end{pmatrix} = \begin{pmatrix} g_1(b_k) \\ g_1(x_k) \end{pmatrix} \cdot \begin{pmatrix} h_1(b_k) & h_1(y_k) \end{pmatrix} + \begin{pmatrix} g_2(b_k) \\ g_2(x_k) \end{pmatrix} \cdot \begin{pmatrix} h_2(b_k) & h_2(y_k) \end{pmatrix}.$$

But it is a standard linear algebra fact that any arbitrary 2×2 matrix can be written this way as a sum of two matrices of rank 1, and thus the claim is proved. Since every finite-dimensional subcoalgebra V of C is contained in some W_n with $W_n^\perp = W_n^\perp \cdot W_n^\perp$ and $C_0 \cong K^{\mathbb{Z}_{>0}}$ is coreflexive, by Theorem 5.2 we obtain that C is coreflexive.

Reflexivity for quiver and incidence algebras. Recall from [Taft 1972] that an algebra is called reflexive if the natural (evaluation) map from A to A^{0*} is an isomorphism. Using our construction in Section 2, we can extend this to algebras with enough idempotents, and call such an algebra *reflexive* if the map $\Phi : a \longmapsto (f \mapsto f(a)) \in A^{0*}$ is an isomorphism. We note that in general the coalgebra A^0 is a coalgebra with counit, and therefore, A^{0*} is an algebra with unit. Hence, a reflexive algebra must be unital. Parallel to algebras with unit we call an algebra *proper* if the map Φ is injective and we call A *weakly reflexive* if Φ is surjective. It is not difficult to see that an algebra is proper if and only if the intersection of all cofinite ideals is 0.

Theorem 5.7. *Let Γ be a quiver.*

(i) *The quiver algebra $K[\Gamma]$ is proper.*

(ii) *$K[\Gamma]$ is reflexive (equivalently, weakly reflexive) if and only if it is finite-dimensional, equivalently, Γ has finitely many vertices and arrows, and has no oriented cycles.*

Proof. (i) follows since $K[\Gamma]$ embeds in $(K\Gamma)^*$ which is proper by Proposition 3.1 of [Taft 1972], and one can easily see that Proposition 3.4 of the same reference, stating that a subalgebra of a proper algebra is proper can be extended to algebras with enough idempotents. Alternatively, one can see that the intersection of cofinite ideals of $K[\Gamma]$ is always 0.

(ii) Assume $K[\Gamma]$ is weakly reflexive, so $K[\Gamma] \to K[\Gamma]^{0*}$ is surjective. The inclusion $K\Gamma \subseteq K[\Gamma]^0$ yields a surjective morphism of algebras

$$K[\Gamma]^{0*} \to (K\Gamma)^*.$$

This shows that the natural map $\psi : K[\Gamma] \hookrightarrow (K\Gamma)^*$ is surjective (and, in fact, bijective). Consider the "gamma function" on $K[\Gamma]$, i.e., the function $\gamma \in K[\Gamma]$ equal to 1 on all paths. Then γ is in the image of ψ, and since every function in the image of ψ has finite support as a function on the set of paths of Γ, it follows that there are only finitely many paths in Γ. Therefore, $K[\Gamma]$ is finite-dimensional. The converse is obvious (as noticed before). $\square$

In the case of incidence algebras, using [Taft 1972, Proposition 6.1] which states that a coalgebra C is coreflexive if and only if C^* is reflexive, and using also Corollary 5.5, we immediately get this:

Theorem 5.8. *Let X be a locally finite partially ordered set. The following assertions are equivalent:*

(i) *The incidence algebra $IA(X)$ of X over K is reflexive.*

(ii) *The incidence coalgebra KX is coreflexive.*

(iii) *The coalgebra* $(KX)_0 = KX_0$ (*the grouplike coalgebra on the elements of* X) *is coreflexive.*

(iv) *The algebra* K^X *of functions on* X *is reflexive.*

These yield as a corollary the algebra analogue of Proposition 4.1.

Corollary 5.9. *Let* A *be an algebra of a nonmeasurable cardinality. Then* A *is isomorphic both to a quiver algebra and to an incidence algebra if and only if and only if it is the quiver algebra of a finite quiver with no oriented cycles, equivalently, it is elementary, finite dimensional and hereditary.*

Proof. If $A \cong K[\Gamma] \cong IA(X)$ for a quiver Γ and a locally finite partially ordered set X, then $K^{(X)}$ is coreflexive by [Heyneman and Radford 1974] since X is also nonmeasurable. Now $A \cong IA(X)$ is reflexive since $K^X \cong (K^{(X)})^*$ is reflexive by [Taft 1972, Proposition 6.1]. By Theorem 5.7, $A \cong K[\Gamma]$ must be finite dimensional since it is reflexive. The final statements follow from the well known characterizations of finite-dimensional quiver algebras. $\square$

An application. We give now an application of our considerations on coreflexive coalgebras. If Γ, Γ' are quivers, then we consider the quiver $\Gamma \times \Gamma'$ defined as follows. The vertices are all pairs (a, a') for a, a' vertices in Γ and Γ' respectively. The arrows are the pairs (a, x'), which is an arrow from (a, a'_1) to (a, a'_2), where a is a vertex in Γ and x' is an arrow from a'_1 to a'_2 in Γ', and the pairs (x, a'), which is an arrow from (a_1, a') to (a_2, a'), where x is an arrow from a_1 to a_2 in Γ, and a' is a vertex in Γ'. Let $p = x_1 x_2 \dots x_n$ be a path in Γ going (in order) through the vertices $a_0, a_1, \dots, a_n$ and $q = y_1 y_2 \dots y_k$ be a path in Γ' going through vertices $b_0, b_1, \dots, b_k$ (some vertices may repeat). We consider the 2 dimensional lattice $L = \{0, \dots, n\} \times \{0, \dots, k\}$. A lattice walk is a sequence of elements of L starting with $(0, 0)$ and ending with (n, k), and always going either one step to the right or one step upwards in L, i.e., (i, j) is followed either by $(i+1, j)$ or by $(i, j+1)$. There are $\binom{n+k}{k}$ such walks.

To p, q and a lattice walk $(0, 0) = (i_0, j_0), (i_1, j_1), \dots, (i_{n+k}, j_{n+k}) = (n, k)$ in L we associate a path of length $n + k$ in $\Gamma \times \Gamma'$, starting at (a_0, b_0) and ending at (a_n, b_k) such that the r-th arrow of the path, from $(a_{i_{r-1}}, b_{j_{r-1}})$ to (a_{i_r}, b_{j_r}) is $(x_{r-1}, b_{j_{r-1}})$ if $i_r = i_{r-1} + 1$, and $(a_{i_{r-1}}, y_{r-1})$ if $j_r = j_{r-1} + 1$.

Conversely, if γ is a path in $\Gamma \times \Gamma'$, there are (uniquely determined) paths p in Γ and q in Γ', and a lattice walk such that γ is associated to p, q and that lattice walk as above. Indeed, we take p to be the path in Γ formed by considering the arrows x such that there are arrows of the form (x, a') in γ, taken in the order they appear in γ. Similarly, q is formed by considering the arrows of the form (a, y) in γ. The lattice walk is defined according to the succession of arrows in γ.

For two such paths p, q let us denote $W(p,q)$ the set of all paths in $\Gamma \times \Gamma'$ associated to p and q via lattice walks.

Functoriality, (co)products of quivers and recovery problems. We note that if Γ and Γ' satisfy condition (i) in Theorem 3.3 (i.e., if their path coalgebras can be recovered as finite duals of the corresponding quiver algebras), then $\Gamma \times \Gamma'$ satisfies this condition, too. Indeed, the description of the arrows in $\Gamma \times \Gamma'$ shows that there are finitely many arrows between any two vertices. Also, if an oriented cycle existed in $\Gamma \times \Gamma'$, then it would produce an oriented cycle in each of Γ and Γ'.

Also, if Γ and Γ' satisfy condition (iii) in Theorem 3.6 (i.e., if their quiver algebras can be recovered as the rational part of the dual of the corresponding path coalgebras), then $\Gamma \times \Gamma'$ satisfies this condition, too. Indeed, a path in $\Gamma \times \Gamma'$ starting at the vertex (a, a') is determined by a path in Γ starting at a, a path in Γ' starting at a' (and there are finitely many such paths in both cases), and a lattice walk (chosen from a finite family). These can be extended to finite products of quivers in the obvious way.

Given a family of quivers $(\Gamma_i)_i$, one can consider the coproduct quiver $\Gamma = \coprod_i \Gamma_i$. The path coalgebra functor commutes with coproducts and one has

$$K\Gamma = \bigoplus_i K\Gamma_i.$$

Also, the quiver algebra functor from the category of quivers to the category of algebras with enough idempotents has the property that

$$K\left[\coprod_i \Gamma_i\right] = \bigoplus_i K[\Gamma_i].$$

It is clear that $\coprod_i \Gamma_i$ satisfies the conditions of Theorem 3.3 (i) if and only if each Γ_i satisfies the same condition, so each $K\Gamma_i$ can be recovered from $K[\Gamma_i]$ if and only if $K\Gamma$ is recoverable from $K[\Gamma]$. Also, each of the quivers $(\Gamma_i)_i$ satisfies condition (iii) in Theorem 3.6, if and only if so does their disjoint union $\coprod_i \Gamma_i$. In coalgebra terms, this is justified by the fact that a direct sum $\bigoplus_i C_i$ of coalgebras is semiperfect if and only if each C_i is semiperfect.

Returning to coreflexivity problems, we need the following.

Lemma 5.10. *The linear map $\alpha : K\Gamma \otimes K\Gamma' \hookrightarrow K(\Gamma \times \Gamma')$ defined by $\alpha(p \otimes q) = \sum_{w \in W(p,q)} w$, where $p \in \Gamma$ and $q \in \Gamma'$ are paths, is an injective morphism of K-coalgebras.*

Proof. We keep the notations above. Denote δ and Δ the comultiplications of $K\Gamma \otimes K\Gamma'$ and $K(\Gamma \times \Gamma')$. We have

$$\delta\alpha(p \otimes q) = \sum_{w \in W(p,q)} \sum_{w'w''=w} w' \otimes w'',$$

$$(\alpha \otimes \alpha)\Delta(p \otimes q) = \sum_{p'p''=p} \sum_{q'q''=q} \sum_{\substack{u \in W(p',q') \\ v \in W(p'',q'')}} u \otimes v.$$

On the one hand, if $p = p'p''$, $q = q'q''$, $u \in W(p', q')$ and $v \in W(p'', q'')$, we have $uv \in W(p, q)$. On the other hand, if $w \in W(p, q)$ and $w = w'w''$, then there exist $p'p''$ in Γ and q', q'' in Γ' such that $p = p'p''$, $q = q'q''$, $w' \in W(p', q')$ and $w'' \in W(p'', q'')$. These show that $\delta\alpha(p \otimes q) = (\alpha \otimes \alpha)\Delta(p \otimes q)$, i.e., α is a morphism of coalgebras (the compatibility with counits is easily verified).

To prove injectivity, if $p = x_1x_2 \ldots x_n$ is a path in Γ starting at a_0 and ending at a_n, and $q = y_1y_2 \ldots y_k$ is a path in Γ' starting at b_0 and ending at b_k, we denote by (p^*, q^*) the linear map on $K(\Gamma \times \Gamma')$ which equals 1 on the path $(x_1, b_0), \ldots, (x_n, b_0), (a_n, y_1), \ldots, (a_n, y_k)$ (for simplicity we also denote this path by $(p, b_0); (a_n, q))$ and 0 on the rest of the paths. Let $\sum_i \lambda_i p_i \otimes q_i \in \mathrm{Ker}(\alpha)$. Then we have

$$\sum_i \sum_{w \in W(p_i,q_i)} \lambda_i w = 0. \tag{4}$$

Fix some j. Say that p_j ends at a_n and q_j starts at b_0. We have that

$$(p_j^*, q_j^*)(w) = \begin{cases} 0 & \text{if } w \in W(p_i, q_i), i \neq j, \\ 0 & \text{if } w \in W(p_j, q_j) \text{ and } w \neq (p_j, b_0), (a_n, q_j), \\ 1 & \text{if } w = (p_j, b_0), (a_n, q_j). \end{cases}$$

Note that we used the fact that $W(p, q) \cap W(p', q') = \varnothing$ for $(p, q) \neq (p', q')$. Now applying (p_j^*, q_j^*) to (4) we see that $\lambda_j = 0$. We conclude that α is injective. □

Combining the above, we derive a result about tensor products of certain coreflexive coalgebras. It is known that a tensor product of a coreflexive and a strongly coreflexive coalgebra is coreflexive (see [Radford 1973]; see also [Taft 1977]). It is not known whether the tensor product of coreflexive coalgebras is necessarily coreflexive. We have the following consequences.

Proposition 5.11. *Let C, D be coreflexive subcoalgebras of path coalgebras $K\Gamma$ and $K\Gamma'$ respectively such that between any two vertices in Γ and Γ' respectively there are only finitely many paths. Then $C \otimes D$ is coreflexive.*

Proof. Without any loss of generality we may assume that $C_0 = (K\Gamma)_0 = K^{(\Gamma_0)}$ and $D_0 = (K\Gamma')_0 = K^{(\Gamma_0)}$ (otherwise we replace Γ and Γ' by appropriate subquivers), where $K^{(\Gamma_0)}$ denotes the grouplike coalgebra with basis the set Γ_0 of vertices of Γ.

Now $C \otimes D$ is a subcoalgebra of $K\Gamma \otimes K\Gamma'$, so by Lemma 5.10, it also embeds in $K(\Gamma \times \Gamma')$. Since the coradical of $K(\Gamma \times \Gamma')$ is $K^{(\Gamma_0 \times \Gamma'_0)}$, and

$$K^{(\Gamma_0 \times \Gamma'_0)} \simeq K^{(\Gamma_0)} \otimes K^{(\Gamma'_0)} = C_0 \otimes D_0 \subseteq C \otimes D,$$

we must have that $(C \otimes D)_0 = K^{(\Gamma_0 \times \Gamma'_0)}$. We claim that $K^{(\Gamma_0 \times \Gamma'_0)}$ is coreflexive. Indeed, this is obvious if Γ_0 and Γ'_0 are both finite. Otherwise, $\mathrm{card}(\Gamma_0 \times \Gamma'_0) = \max\{\mathrm{card}(\Gamma_0), \mathrm{card}(\Gamma'_0)\}$, hence $K^{(\Gamma_0 \times \Gamma'_0)}$ is isomorphic either to $K^{(\Gamma_0)}$ or to $K^{(\Gamma'_0)}$, both of which are coreflexive by Proposition 5.4. Since it is clear that in $\Gamma \times \Gamma'$ there are also finitely many paths between any two vertices, we can use Proposition 5.4 to show that $C \otimes D$ is coreflexive. □

Corollary 5.12. *If C, D are coreflexive subcoalgebras of incidence coalgebras, then $C \otimes D$ is coreflexive.*

Proof. It follows immediately from the embedding of C and D in path coalgebras verifying the hypothesis of Proposition 5.11. □

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