Pacific Journal of Mathematics

ON THE ISENTROPIC COMPRESSIBLE EULER EQUATION WITH ADIABATIC INDEX $\gamma = 1$

DONG LI, CHANGXING MIAO AND XIAOYI ZHANG

Volume 262 No. 1 March 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 262, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.262.109

ON THE ISENTROPIC COMPRESSIBLE EULER EQUATION WITH ADIABATIC INDEX $\gamma = 1$

DONG LI, CHANGXING MIAO AND XIAOYI ZHANG

We consider the isentropic compressible Euler equations with polytropic gamma law $P(\rho) = \rho^\gamma$ in dimensions $d \le 3$. We address the borderline case when adiabatic index $\gamma = 1$ and establish local theory in the Sobolev space $C_t^0 L_x^p \cap C_t^0 \dot{H}_x^k$ for $d < p \le 4$. This covers a class of physical solutions which can decay to vacuum at spatial infinity and are not compact perturbations of steady states. We construct a blowup scenario where initially the fluid is quiet in a neighborhood of the origin but is supersonic near the spatial infinity. For this special class of noncompact initial data, we prove the formation of singularities in finite time.

1. Introduction and main results

We consider the Cauchy problem for the d-dimensional, $d \le 3$, isentropic compressible Euler equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) + \nabla P = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d. \\ (\rho, \boldsymbol{v})(0, x) = (\rho_0, \boldsymbol{v}_0)(x). \end{cases} \tag{1-1}$$

Here, $\rho = \rho(t, x)$ is a scalar function representing density, $\boldsymbol{v} = \boldsymbol{v}(t, x)$ is a vector-valued function representing velocity. P is the pressure, satisfying the polytropic gamma law

$$P(\rho) = A\rho^\gamma, \quad \gamma \ge 1,$$

where $A > 0$ is a constant and γ is so-called adiabatic index. In this paper, we will mainly consider the borderline case $\gamma = 1$. For simplicity we shall set $A = 1$.

There is an extensive one-dimensional theory on the singularity formation of solutions to the compressible Euler equation and related equations (see [John 1974; Klainerman and Majda 1980; Lax 1964; Liu 1979]). The proofs are usually

Li is supported in part by an NSERC Discovery grant and by the NSF under agreement DMS-1128155. Miao is supported by the NSF of China (11171033 and 11231006). Zhang is supported by an Alfred P. Sloan fellowship.

MSC2010: primary 35Q35; secondary 76N10.

Keywords: compressible Euler equation, blowup solutions.

based on method of characteristics which is not robust enough to treat dimensions $d \geq 2$ (see, however, [Chae and Ha 2009] for a blowup result in 3D using method of characteristics). Sideris [1985] considered the following three-dimensional compressible Euler system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) + \nabla P = 0, \\ \partial_t S + \boldsymbol{v} \cdot \nabla S = 0, \\ (\rho, \boldsymbol{v}, S)(0, x) = (\rho_0, \boldsymbol{v}_0, S_0)(x). \end{cases} \tag{1-2}$$

Here $S = S(t, x)$ denotes the specific entropy and the pressure law is given by

$$P(\rho, S) = A\rho^\gamma e^S, \quad A > 0,\ \gamma > 1. \tag{1-3}$$

If we set $S(t, x) \equiv \bar{S} = \text{const}$, the system (3-1) reduces to (1-1) (hence the name "isentropic"). The following set of initial data was considered in [Sideris 1985], where $R > 0$ is fixed:

$$\begin{aligned} &\rho_0(x) > 0 && \text{for all } x, \\ &\rho_0(x) = \bar{\rho}, \quad \boldsymbol{v}_0(x) = 0, \quad S_0(x) = \bar{S} && \text{if } |x| \geq R. \end{aligned} \tag{1-4}$$

Such initial data can be viewed as compact perturbations of the steady state $(\rho, \boldsymbol{v}, S) \equiv (\bar{\rho}, 0, \bar{S})$. By a change of variable $c = \text{const} \cdot \rho^{(\gamma-1)/2}$, one can rewrite (3-1) as a symmetric positive hyperbolic system in terms of $(c, \boldsymbol{v}, S)$. For initial data of the form (1-4), local wellposedness of $(c, \boldsymbol{v}, S)$ in $C_t^0 H_x^{(5/2)+}$ then follows easily; see [Kato 1975]. The speed of sound σ is defined by

$$\sigma = \left(\frac{\partial P(\rho, S)}{\partial \rho} \right)^{1/2} \Bigg|_{(\rho, S) = (\bar{\rho}, \bar{S})} = (A\gamma \bar{\rho}^{\gamma-1} e^{\bar{S}})^{1/2}. \tag{1-5}$$

A result of [Sideris 1985], roughly speaking, is that for a set of initial data (1-4) which is supersonic in a neighborhood of the origin, the corresponding C^1 solution to (1-2)–(1-3) must have finite lifespan. This was extended to the two-dimensional case by Rammaha [1989]. There are also more precise results on the estimate of lifespan of blowup solutions which are small perturbations of steady states. For the 3D compressible Euler equation (1-1) with irrotational (i.e., $\nabla \times \boldsymbol{v} = 0$) initial data $(\rho_0, \boldsymbol{v}_0) = (\epsilon \tilde{\rho}_0 + \bar{\rho}, \epsilon \tilde{\mathbf{v}}_0)$, where $\tilde{\rho}_0 \in \mathcal{S}(\mathbb{R}^3)$, $\tilde{\mathbf{v}}_0 \in \mathcal{S}(\mathbb{R}^3)^3$ ($\mathcal{S}(\mathbb{R}^3)$ is the usual Schwartz space), Sideris [1991] proved that the lifespan of the classical solution $T_\epsilon > \exp(C/\epsilon)$. For the upper bound it follows from [Sideris 1985] that $T_\epsilon < \exp(C/\epsilon^2)$ under some mild conditions on the initial data. For initial data which is spherically symmetric and is smooth compact ϵ-perturbation of the constant state, Godin [2005] obtained by using a suitable approximation solution the precise asymptotic of the lifespan T_ϵ as

$$\lim_{\epsilon\to 0} \epsilon \log T_\epsilon = T^*,$$

where T^* is a constant. These results rely crucially on the observation that after some simple manipulations, the compressible Euler equation in rescaled variables is given by a vectorial nonlinear wave equation with pure quadratic nonlinearities. This fact together with the positivity of fundamental solutions of the wave operator were also exploited in [Sideris 1985; Rammaha 1989] to establish a different set of blowup results which are analogs of corresponding results on nonlinear wave equations.

In this paper we will be concerned with the d-dimensional isentropic compressible Euler system (1-1) with adiabatic index $\gamma = 1$. This is the borderline case, since previous results in the literature are mainly for the case $\gamma > 1$. We discuss first the local theory. In the case $\gamma > 1$, all the results mentioned before essentially deal with initial data which contain no vacuum states and are compact perturbations of steady states, cf. (1-4). Local wellposedness to (1-1) in $C_t^0 H_x^s$ for some regularity index $s > d/2 + 1$ then follow easily from [Kato 1975] after some change of variables transforming to a symmetric positive hyperbolic system. In principle one can essentially repeat this kind of analysis in the case $\gamma = 1$ and obtain local wellposedness for initial data which are compact perturbations of steady states. However we shall not discuss this simple case and will focus instead on the more interesting case where the initial data can be essentially *noncompact*. A useful example is where the initial density $\rho_0(x)$ decays as $(1+|x|^2)^{-\beta}$ for some large exponent β as $|x| \to \infty$; in other words, we allow the density to decay to vacuum at spatial infinity. As it turns out, even the local theory for such initial data requires a bit of work, since the standard H_x^k spaces which fit so well with the usual symmetric hyperbolic systems are not suitable for closing the estimates due to problems at low frequencies. Instead, we will establish the local existence in $L^p(\mathbb{R}^d) \cap \dot{H}^k(\mathbb{R}^d)$:

Theorem 1.1 (local existence). *Let the dimension be $d \le 3$. Let $k \ge 10d$ be a large integer and take p such that $d < p \le 4$. Assume the initial data satisfy*

$$\begin{aligned} &\rho_0 > 0, && {\rho_0}^{-1}\nabla\rho_0 \in L^p(\mathbb{R}^d) \cap \dot{H}^{k-1}(\mathbb{R}^d), \\ &\rho_0 \in L^p(\mathbb{R}^d) \cap \dot{H}^{k-1}(\mathbb{R}^d), && \boldsymbol{v}_0 \in L^p(\mathbb{R}^d) \cap \dot{H}^{k}(\mathbb{R}^d). \end{aligned} \tag{1-6}$$

Then there exists $T > 0$ such that the Cauchy problem (1-1) admits a unique solution

$$\rho \in C([0,T]; L^p(\mathbb{R}^d) \cap \dot{H}^{k-1}(\mathbb{R}^d)), \quad \boldsymbol{v} \in C([0,T]; L^p(\mathbb{R}^d) \cap \dot{H}^{k}(\mathbb{R}^d)),$$

with $\rho > 0$. Moreover, $\rho^{-1}\nabla\rho \in C([0,T], L^p(\mathbb{R}^d) \cap \dot{H}^{k-1}(\mathbb{R}^d))$. If in addition $\rho_0 \in L^1(\mathbb{R}^d)$, we have mass conservation:

$$\int \rho(t,x)\,dx = \int \rho_0(x)\,dx.$$

Remark 1.2. In Theorem 1.1, the restriction $p > d$ comes from the physical assumptions we put on the initial data ρ_0. Since we allow ρ_0 to be essentially noncompact, in particular we can take $\rho_0 \sim (1+|x|^2)^{-\beta}$ for $|x| \gg 1$. It is not difficult to check that in this case $\rho_0^{-1}\nabla\rho_0 \in L^p(\mathbb{R}^d)$ *only for* $p > d$. On the other hand the upper bound $p \le 4$ comes from bounding certain L^2-norm of products in the nonlinear estimates. For example (see also (2-2)), if we have two functions f, g with frequencies supported on the ball $|\xi| \le 1$, that is, $f \sim P_{\le 1} f$, $g \sim P_{\le 1} g$ (here $P_{\le 1}$ is the usual Littlewood–Paley projector, see Section 2), and we only know that f and g are bounded in L^p, then

$$\|fg\|_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|_{L_x^{2p/(p-2)}} \|g\|_p \lesssim \|f\|_p \|g\|_p,$$

where in the last inequality we have to use the Bernstein inequality for which the constraint $2p/(p-2) \ge p$ or $p \le 4$ is deduced. By the constraint $d < p \le 4$ we deduce $d \le 3$ and this is the main reason for the restriction of the dimension.

The next result is on the formation of singularities in finite time. We will show that the local solutions constructed in Theorem 1.1 have finite life spans. As was mentioned before, the class of data that leads to blowups is a not a compact perturbation of the constant state. More precisely we have the following

Theorem 1.3 (blowup from spatial infinity). *Let ρ_0, $\boldsymbol{v}_0$ satisfy the conditions in (1-6) and $\rho_0 \in L^1(\mathbb{R}^d)$. For $d = 2, 3$, we also assume $\boldsymbol{v}_0$ is irrotational*: $\operatorname{curl}(\boldsymbol{v}_0) = 0$. *Let $\rho_0(x) = 1$, $\boldsymbol{v}_0(x) = 0$, for all $|x| \le 10$. Let $\phi(x)$ be a Schwartz function such that $\nabla^2\phi(x)$ is positive definite on $|x| > 1$. Set*

$$N := \int \rho_0 \boldsymbol{v}_0 \cdot \nabla\phi(x)\, dx. \tag{1-7}$$

Then there exist a constant $C = C(\|\rho_0\|_1) > 0$ such that whenever $N > C$, the corresponding solution constructed in Theorem 1.1 blows up at some time $T^* < 1$.

Remark 1.4. The blowup constructed in Theorem 1.3 is different from the usual case where the initial data is concentrated near the origin. In our scenario, the bulk of the initial data is concentrated near spatial infinity and the quantity N defined in (1-7) measures this concentration. The intuitive picture is that initially the fluid is quiet in an $O(1)$-neighborhood of the origin but is supersonic near the spatial infinity. After an $O(1)$-finite time the fluid develops singularities in the transient region away from the origin.

2. Preliminaries

We will often use the notation $X \lesssim Y$ whenever there exists some constant C such that $X \le CY$. For any two operators A, B, we use the notation $[A, B] := AB - BA$ to denote the commutator.

We will also need to use the Littlewood–Paley theory. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \le 2$ and equal to one on the ball $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood–Paley operators

$$\widehat{P_{\le N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi), \quad \widehat{P_{>N} f}(\xi) := [1-\varphi(\xi/N)]\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi).$$

Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\le N}$ for $P_{\le N} f$ and similarly for the other operators. We recall the following standard Bernstein- and Sobolev-type inequalities:

Lemma 2.1. *For any* $1 \le p \le q \le \infty$ *and* $s > 0$, *we have*

$$\|P_{\ge N} f\|_{L^p_x} \lesssim N^{-s} \||\nabla|^s P_{\ge N} f\|_{L^p_x},$$
$$\||\nabla|^s P_{\le N} f\|_{L^p_x} \lesssim N^s \|P_{\le N} f\|_{L^p_x},$$
$$\||\nabla|^{\pm s} P_N f\|_{L^p_x} \sim N^{\pm s} \|P_N f\|_{L^p_x},$$
$$\|P_{\le N} f\|_{L^q_x} \lesssim N^{d/p-d/q} \|P_{\le N} f\|_{L^p_x},$$
$$\|P_N f\|_{L^q_x} \lesssim N^{d/p-d/q} \|P_N f\|_{L^p_x}.$$

We will use the following simple estimate frequently:

$$\|f\|_\infty \lesssim \|P_{\le 1} f\|_p + \sum_{\substack{N>1\\ N\in 2^{\mathbb{Z}}}} N^{d/2} \|P_N f\|_2 \lesssim \|P_{\le 1} f\|_p + \|P_{>1} f\|_{\dot{H}^{d/2+1}}. \tag{2-1}$$

We prove below some commutator estimates which will be useful in controlling the nonlinear terms. To simple notations we shall assume that the functions are scalar-valued. The extension to vector-valued functions is rather trivial. In order not to be burdened with notations, we will sometimes use the same notations for vector-valued functions as in the scalar-valued case. For example if $\boldsymbol{v} = (v_1, \cdots, v_d)$ and $v_j \in L^2_x(\mathbb{R}^d)$, we shall simply write $\boldsymbol{v} \in L^2_x(\mathbb{R}^d)$ in place of $\boldsymbol{v} \in L^2_x(\mathbb{R}^d)^d$.

Lemma 2.2. *Let* $f, g \in \mathcal{S}(\mathbb{R}^d)$. *Let* ∂ *denote any partial derivative. Let* $2 \le p \le 4$ *and* $k > d+2$. *Then*

$$\|[\partial^k, f\partial]g\|_2 \lesssim \|f\|_{\dot{H}^k \cap L^p} \|g\|_{\dot{H}^k \cap L^p}, \qquad \|[\partial^k, f]g\|_2 \lesssim \|f\|_{\dot{H}^k \cap L^p} \|g\|_{\dot{H}^{k-1} \cap L^p},$$
$$\|[\partial^{k-1}, f]\partial g\|_2 \lesssim \|f\|_{\dot{H}^k \cap L^p} \|g\|_{\dot{H}^{k-1} \cap L^p}.$$

Proof. We only prove the first one. By the chain rule and the triangle inequality, we have the bound

$$\|[\partial^k, f\partial]g\|_2 \lesssim \sum_{1\le j\le k} \|\partial^j f \partial^{k+1-j} g\|_2.$$

In the case $1 \le j \le k/2$, we split g into low and high frequencies. For the low-frequency piece, we use the fact $p \le 4$ and Bernstein to get

$$\begin{aligned}\|\partial^j f \partial^{k+1-j} P_{\le 1} g\|_2 &\le \|\partial^j f\|_{2p/(p-2)} \|\partial^{k+1-j} P_{\le 1} g\|_p \\ &\lesssim (\|\partial^j P_{\le 1} f\|_{2p/(p-2)} + \|\partial^j P_{>1} f\|_{2p/(p-2)}) \|g\|_p \\ &\lesssim \|f\|_{\dot{H}^k \cap L^p} \|g\|_p.\end{aligned}$$

In the last estimate, we used a similar estimate as in (2-1). For the high-frequency piece, we use Sobolev embedding and Bernstein to get

$$\begin{aligned}\|\partial^j f \partial^{k+1-j} P_{>1} g\|_2 &\le \|\partial^j f\|_\infty \|\partial^{k+1-j} P_{>1} g\|_2 \\ &\lesssim \|f\|_{\dot{H}^k \cap L^p} \|g\|_{\dot{H}^k}.\end{aligned} \tag{2-2}$$

Again we invoke (2-1) in the last step. In the case $k/2 < j \le k$, we can instead split f into low and high frequencies. Then the estimate just follows by symmetry. □

We need to use the following space which will be useful for proving some contraction estimates in Section 3. For any positive integer k, define

$$X_k = \{f, \|f\|_{X_k} := \|f\|_p + \|P_{>1} f\|_{\dot{H}^k} < \infty\}. \tag{2-3}$$

It is not difficult to check that for $k > d/2$ the space X_k forms an algebra. This fact together with some useful commutator estimates and product estimates are stated in the next

Lemma 2.3. *Under the same conditions as in Lemma 2.2, we have*:

$$\begin{aligned}\|[\partial^{k-1}, f\partial] P_{>1} g\|_2 &\lesssim \|f\|_{X_{k-1}} \|P_{>1} g\|_{\dot{H}^{k-1}}, \\ \|\partial^{k-1}(f P_{\lesssim 1} g)\|_2 &\lesssim \|f\|_{X_{k-1}} \|g\|_p, \\ \|[\partial^{k-1}, f] P_{>1} g\|_2 &\lesssim \|f\|_{X_{k-1}} \|g\|_{X_{k-2}}, \\ \|\partial^{k-1}(f \partial g)\|_2 &\lesssim \|f\|_{X_{k-1}} \|g\|_{X_k}, \\ \|\partial^{k-1}(fg)\|_2 &\lesssim \|f\|_{X_{k-1}} \|g\|_{X_{k-1}}.\end{aligned}$$

Proof. The proof proceeds in a similar way as in Lemma 2.2. One has to split both f and g into high- and low-frequency pieces and discuss several cases. We omit the details. □

3. Proof of Theorem 1.1

To construct the local solution, we will use the usual Picard iteration but in a slightly nonstandard space and exploiting in an essential way the structure of the system. Due to the singular nature of the problem, we need both the hyperbolic formulation of the equation and the original formulation. The tricky part of the analysis is to define a good iteration scheme.

To this end, we define

$$f = \log \rho,$$

and rewrite the Cauchy problem (1-1) in terms of $(f, \boldsymbol{v})$ as

$$\begin{cases} \partial_t f + \boldsymbol{v} \cdot \nabla f + \nabla \cdot \boldsymbol{v} = 0, \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla f = 0, \\ f(0, x) = \log \rho_0(x), \quad \boldsymbol{v}(0, x) = \boldsymbol{v}_0(x). \end{cases} \tag{3-1}$$

By bringing in the f-function, we have obtained the hyperbolic formulation (3-1) for the original system.

Remark 3.1. It is tempting to invoke the usual wellposedness theory in H^k, $k > d/2$ spaces and conclude that the system (3-1) admits local solutions in $C_t^0 H_x^k$. However there is a serious problem with this due to the physical assumptions we put on the initial data. Namely $f = \log \rho$ does not lie in L_x^2 in general. To see it one can consider the sample case $\rho(x) = \langle x \rangle^{-C}$ which immediately yields $f \sim \log\langle x \rangle \notin L_x^2$. In fact it is not difficult to check $f \notin \dot{H}_x^k(\mathbb{R}^d)$ for any $k \le d/2$.

By Remark 3.1, we shall proceed differently from the usual fashion and work with an enlarged (and redundant) system of equations which includes both the hyperbolic formulation and the original system. The advantage is that with a little bit of work we can obtain regularity of all functions at one stroke.

We start with the zeroth iterate, defined as

$$\rho^{(0)}(t, x) = \rho_0(x), \quad \boldsymbol{v}^{(0)}(t, x) = \boldsymbol{v}_0(x), \quad f^{(0)}(t, x) = \log \rho_0(x).$$

For any integer $n \ge 0$, we inductively define $(\rho^{(n+1)}, \boldsymbol{v}^{(n+1)}, f^{(n+1)})$ as solutions to the linear system

$$\begin{cases} \partial_t \rho^{(n+1)} + \nabla \cdot (\rho^{(n+1)} \boldsymbol{v}^{(n)}) = 0, \\ \partial_t f^{(n+1)} + \boldsymbol{v}^{(n)} \cdot \nabla f^{(n+1)} + \nabla \cdot \boldsymbol{v}^{(n+1)} = 0, \\ \partial_t \boldsymbol{v}^{(n+1)} + (\boldsymbol{v}^{(n)} \cdot \nabla)\boldsymbol{v}^{(n+1)} + \nabla f^{(n+1)} = 0, \\ \rho^{(n+1)}(0, x) = \rho_0(x), \quad f^{(n+1)}(0, x) = \log \rho_0(x), \quad \boldsymbol{v}^{(n+1)}(0, x) = \boldsymbol{v}_0(x). \end{cases} \tag{3-2}$$

Remark 3.2. Strictly speaking, instead of $f^{(n+1)}$, we should be working with $\boldsymbol{g}^{(n+1)} = \nabla f^{(n+1)}$ and write the second equation in (3-2) as

$$\partial_t \boldsymbol{g}^{(n+1)} + \nabla(\boldsymbol{v}^{(n)} \cdot \boldsymbol{g}^{(n+1)}) + \nabla(\nabla \cdot \boldsymbol{v}^{(n+1)}) = 0,$$

with initial data $\boldsymbol{g}^{(n+1)} = \nabla \rho_0 / \rho_0$. Correspondingly in the third equation of (3-2) we should replace $\nabla f^{(n+1)}$ by $\boldsymbol{g}^{(n+1)}$. In this way we do not need to prove any regularity or solvability estimates of $f^{(n+1)}$ themselves in the iteration system.

We first show that the sequence of functions $\boldsymbol{v}^{(n)}$ are uniformly bounded in the space $L_t^\infty([0,T];\dot H^k\cap L^p)$, $(\rho^{(n)},\nabla f^{(n)})$ are uniformly bounded in the space $L_t^\infty([0,T];\dot H^{k-1}\cap L^p)$ for some suitably small T.

Step 1: The L^p boundedness of the iterates $(\rho^{(n+1)},\boldsymbol{v}^{(n+1)},\nabla f^{(n+1)})$. Multiplying the first equation in (3-2) by $|\rho^{(n+1)}|^{p-2}\rho^{(n+1)}$ and integrating by parts, we get

$$\frac{1}{p}\frac{d}{dt}\|\rho^{(n+1)}(t)\|_p^p+\frac{p-1}{p}\int(\rho^{(n+1)})^p\nabla\cdot\boldsymbol{v}^{(n)}\,dx=0.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\|\rho^{(n+1)}(t)\|_p&\le\|\nabla\cdot\boldsymbol{v}^{(n)}(t)\|_\infty\|\rho^{(n+1)}(t)\|_p\\&\lesssim\|\boldsymbol{v}^{(n)}(t)\|_{\dot H^k\cap L^p}\|\rho^{(n+1)}(t)\|_p.\end{aligned}\tag{3-3}$$

Next we take the inner product with $|\boldsymbol{v}^{(n+1)}|^{p-2}\boldsymbol{v}^{(n+1)}$ on both sides of the third equation in (3-2). After integrating on $\mathbb{R}^d$, we get

$$\frac{1}{p}\frac{d}{dt}\|\boldsymbol{v}^{(n+1)}(t)\|_p^p-\frac{1}{p}\int\nabla\cdot\boldsymbol{v}^{(n)}|\boldsymbol{v}^{(n+1)}|^p\,dx+\int|\boldsymbol{v}^{(n+1)}|^{p-2}\nabla f^{(n+1)}\cdot\boldsymbol{v}^{(n+1)}\,dx=0.$$

Hölder's inequality yields

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{v}^{(n+1)}(t)\|_p&\lesssim\|\nabla\cdot\boldsymbol{v}^{(n)}(t)\|_\infty\|\boldsymbol{v}^{(n+1)}(t)\|_p+\|\nabla f^{(n+1)}(t)\|_p\\&\lesssim\|\boldsymbol{v}^{(n)}(t)\|_{\dot H^k\cap L^p}\|\boldsymbol{v}^{(n+1)}(t)\|_p+\|\nabla f^{(n+1)}(t)\|_p.\end{aligned}\tag{3-4}$$

To close the estimate, we need to estimate $\|\nabla f^{(n+1)}\|_p$. Differentiating the second equation in (3-2) once, we have the equation for $\partial_i f^{(n+1)}$:

$$\partial_t\partial_i f^{(n+1)}+\partial_i(\boldsymbol{v}^{(n)}\cdot\nabla f^{(n+1)})+\nabla\cdot\partial_i\boldsymbol{v}^{(n+1)}=0.$$

Multiplying both sides by $|\partial_i f^{(n+1)}|^{p-2}\partial_i f^{(n+1)}$ and integrating by parts, we get

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\partial_i f^{(n+1)}(t)\|_p^p+\int\partial_i\boldsymbol{v}^{(n)}\cdot\nabla f^{(n+1)}|\partial_i f^{(n+1)}|^{p-2}\partial_i f^{(n+1)}\,dx\\&\quad-\frac{1}{p}\int\nabla\cdot\boldsymbol{v}^{(n)}|\partial_i f^{(n+1)}|^p\,dx+\int\nabla\cdot\partial_i\boldsymbol{v}^{(n+1)}|\partial_i f^{(n+1)}|^{p-2}\partial_i f^{(n+1)}\,dx=0.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\partial_i f^{(n+1)}(t)\|_p^p&\le\|\partial_i\boldsymbol{v}^{(n)}(t)\|_\infty\|\nabla f^{(n+1)}(t)\|_p\|\partial_i f^{(n+1)}(t)\|_p^{p-1}\\&\quad+\frac{1}{p}\|\nabla\cdot\boldsymbol{v}^{(n)}(t)\|_\infty\|\partial_i f^{(n+1)}(t)\|_p^p+\|\partial_i\nabla\cdot\boldsymbol{v}^{(n+1)}(t)\|_p\|\partial_i f^{(n+1)}(t)\|_p^{p-1}.\end{aligned}$$

Summing in $i = 1, \dots, d$ gives

$$
\begin{aligned}
(3\text{-}5)\quad & \frac{d}{dt}\|\nabla f^{(n+1)}(t)\|_p \\
&\lesssim \sum_{i=1}^{d} \|\partial_i \boldsymbol{v}^{(n)}(t)\|_\infty \|\nabla f^{(n+1)}(t)\|_p + \sum_{j,i=1}^{d} \|\partial_j \partial_i \boldsymbol{v}^{(n+1)}(t)\|_p \\
&\lesssim \|\boldsymbol{v}^{(n)}(t)\|_{\dot H^k \cap L^p} \|\nabla f^{(n+1)}(t)\|_p + \|\boldsymbol{v}^{(n+1)}(t)\|_{\dot H^k \cap L^p}.
\end{aligned}
$$

This ends the L^p-estimate. Next we turn to high-order energy estimates.

Step 2: $\dot H^k$-estimates. Let ∂^k denote a differential operator of order k, we compute

$$
\begin{aligned}
(3\text{-}6)\quad & \frac{d}{dt}\int |\partial^k \boldsymbol{v}^{(n+1)}|^2\,dx \\
&= 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot \partial^k \partial_t \boldsymbol{v}^{(n+1)}\,dx \\
&= -2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot \partial^k [(\boldsymbol{v}^{(n)} \cdot \nabla)\boldsymbol{v}^{(n+1)}]dx - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot \partial^k \nabla f^{(n+1)}\,dx \\
&= -2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot [(\boldsymbol{v}^{(n)} \cdot \nabla)\partial^k \boldsymbol{v}^{(n+1)}] - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot [\partial^k, (\boldsymbol{v}^{(n)} \cdot \nabla)]\boldsymbol{v}^{(n+1)}\,dx \\
&\qquad - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot \partial^k \nabla f^{(n+1)}\,dx \\
&= \int \nabla \cdot \boldsymbol{v}^{(n)} |\partial^k \boldsymbol{v}^{(n+1)}|^2 dx - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot [\partial^k, (\boldsymbol{v}^{(n)} \cdot \nabla)]\boldsymbol{v}^{(n+1)}\,dx \\
&\qquad - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot \nabla \partial^k f^{(n+1)}\,dx.
\end{aligned}
$$

Similarly for $f^{(n+1)}$ we have

$$
\begin{aligned}
(3\text{-}7)\quad \frac{d}{dt}\int |\partial^k f^{(n+1)}|^2\,dx &= \int \nabla \cdot \boldsymbol{v}^{(n)} |\partial^k f^{(n+1)}|^2 \\
&\quad - 2\int \partial^k f^{(n+1)} [\partial^k, \boldsymbol{v}^{(n)}] \cdot \nabla f^{(n+1)} - 2\int \partial^k f^{(n+1)} \partial^k \nabla \cdot \boldsymbol{v}^{(n+1)}\,dx.
\end{aligned}
$$

Adding (3-6), (3-7) together, we have

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\partial^k \boldsymbol{v}^{(n+1)}(t)\|_2^2 + \|\partial^k f^{(n+1)}(t)\|_2^2\big) \\
&\quad = \int \nabla \cdot \boldsymbol{v}^{(n)} |\partial^k \boldsymbol{v}^{(n+1)}|^2 dx - 2\int \partial^k \boldsymbol{v}^{(n+1)} \cdot [\partial^k, (\boldsymbol{v}^{(n)} \cdot \nabla)]\boldsymbol{v}^{(n+1)}\,dx \\
&\qquad + \int \nabla \cdot \boldsymbol{v}^{(n)} |\partial^k f^{(n+1)}|^2 dx - 2\int \partial^k f^{(n+1)} [\partial^k, \boldsymbol{v}^{(n)}] \cdot \nabla f^{(n+1)}\,dx.
\end{aligned}
$$

By Hölder's inequality and Lemma 2.2, we have

$$\frac{d}{dt}\big(\|\partial^k \boldsymbol{v}^{(n+1)}(t)\|_2^2+\|\partial^k f^{(n+1)}(t)\|_2^2\big)$$
$$\lesssim \|\nabla\cdot \boldsymbol{v}^{(n)}\|_\infty\big(\|\partial^k \boldsymbol{v}^{(n+1)}\|_2^2+\|\partial^k f^{(n+1)}\|_2^2\big)$$
$$+\|\boldsymbol{v}^{(n)}\|_{\dot H^k\cap L^p}\big(\|\partial^k \boldsymbol{v}^{(n+1)}\|_2\|\boldsymbol{v}^{(n+1)}\|_{\dot H^k\cap L^p}+\|\partial^k f^{(n+1)}\|_2\|\nabla f^{(n+1)}\|_{\dot H^{k-1}\cap L^p}\big).$$

Since $\|\nabla\cdot \boldsymbol{v}^{(n)}\|_\infty\lesssim\|\boldsymbol{v}^{(n)}\|_{\dot H^k\cap L^p}$, we then obtain

(3-8) $$\frac{d}{dt}\big(\|\boldsymbol{v}^{(n+1)}(t)\|_{\dot H^k}^2+\|f^{(n+1)}(t)\|_{\dot H^k}^2\big)$$
$$\lesssim \|\boldsymbol{v}^{(n)}(t)\|_{\dot H^k\cap L^p}\big(\|\boldsymbol{v}^{(n+1)}(t)\|_{\dot H^k\cap L^p}^2+\|\nabla f^{(n+1)}(t)\|_{\dot H^{k-1}\cap L^p}^2\big).$$

The estimates are now complete. However, to prove the contraction estimates, we still need the high-order energy estimate of $\rho^{(n+1)}$: the $\dot H^{k-1}$-norm. By using integration by parts, we compute

$$\begin{aligned}\frac{d}{dt}\int|\partial^{k-1}\rho^{(n+1)}|^2\,dx=&-\int\partial^{k-1}\rho^{(n+1)}\nabla\partial^{k-1}\rho^{(n+1)}\cdot\boldsymbol{v}^{(n)}\,dx\\&-\int\partial^{k-1}\rho^{(n+1)}[\partial^{k-1},\boldsymbol{v}^{(n)}]\cdot\nabla\rho^{(n+1)}\,dx\\&-\int\partial^{k-1}\rho^{(n+1)}\partial^{k-1}(\rho^{(n+1)}\nabla\cdot\boldsymbol{v}^{(n)})\,dx.\end{aligned}$$

By Hölder and using again Lemma 2.2, we obtain

(3-9) $$\frac{d}{dt}\|\rho^{(n+1)}(t)\|_{\dot H^{k-1}}\lesssim\|\boldsymbol{v}^{(n)}(t)\|_{\dot H^k\cap L^p}\|\rho^{(n+1)}(t)\|_{\dot H^{k-1}\cap L^p}.$$

Set

$$M^{(n+1)}(t):=\|\rho^{(n+1)}(t)\|_{\dot H^{k-1}\cap L^p}^2+\|\boldsymbol{v}^{(n+1)}\|_{\dot H^k\cap L^p}^2+\|\nabla f^{(n+1)}\|_{\dot H^{k-1}\cap L^p}^2.$$

Collecting the estimates (3-3), (3-4), (3-5), (3-8) and (3-9), we have

$$\begin{cases}\dfrac{d}{dt}M^{(n+1)}(t)\le CM^{(n+1)}(t)(1+M^{(n)}(t)),\\ M^{(n+1)}(0)=\|\rho_0\|_{\dot H^{k-1}\cap L^p}^2+\|\boldsymbol{v}_0\|_{\dot H^k\cap L^p}^2+\|\nabla f_0\|_{\dot H^{k-1}\cap L^p}^2:=M_0.\end{cases}$$

Here the constant depends only on p, d. Applying Gronwall's inequality, we obtain

(3-10) $$M^{(n+1)}(t)\le M_0\exp\Big\{C\int_0^t(1+M^{(n)}(s))\,ds\Big\}.$$

It suffices to take T small enough such that

(3-11) $$8CT(1+M_0)\le\tfrac{1}{100}.$$

Then the sequence $M^{(n)}(t)$ are uniformly bounded as

$$\|M^{(n)}\|_{L_t^\infty([0,T])} \le 2M_0. \tag{3-12}$$

Therefore, for the chosen T, the sequence $\{\rho^{(n)}, \nabla f^{(n)}\}$ are bounded in

$$L_t^\infty\big([0,T]; (\dot H^{k-1} \cap L^p)\big),$$

and $\{\boldsymbol{v}^{(n)}\}$ are bounded in $L_t^\infty\big([0,T]; (\dot H^k \cap L^p)\big)$. In the next step, we shall show that they are Cauchy in an intermediate topology.

Step 3: Contraction estimates. It is easy to check that the differences $\rho^{(n+1)} - \rho^{(n)}$, $\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}$, and $f^{(n+1)} - f^{(n)}$ satisfy the system of equations

$$\begin{aligned}
\partial_t(\rho^{(n+1)} - \rho^{(n)}) + \nabla\cdot((\rho^{(n+1)} - \rho^{(n)})\boldsymbol{v}^{(n)}) + \nabla\cdot(\rho^{(n)}(\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)})) &= 0,\\
\partial_t(\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}) + (\boldsymbol{v}^{(n)}\cdot\nabla)(\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)})&\\
+ [(\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)})\cdot\nabla]\boldsymbol{v}^{(n)} + \nabla(f^{(n+1)} - f^{(n)}) &= 0,\\
\partial_t(f^{(n+1)} - f^{(n)}) + \boldsymbol{v}^{(n)}\cdot(\nabla f^{(n+1)} - \nabla f^{(n)})&\\
+ (\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)})\cdot\nabla f^{(n)} + \nabla\cdot(\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}) &= 0.
\end{aligned}$$

We shall prove that the sequence $\boldsymbol{v}^{(n)}$ is Cauchy in X_{k-1} and $(\rho^{(n)}, f^{(n)})$ is Cauchy in X_{k-2}. Here the space X_j is defined in (2-3). We first estimate the L^p norm as

$$\begin{aligned}
\frac{d}{dt}\|\rho^{(n+1)} - \rho^{(n)}(t)\|_p^p \lesssim{}& \|\nabla\cdot\boldsymbol{v}^{(n)}\|_\infty \|\rho^{(n+1)} - \rho^{(n)}\|_p^p\\
&+ \|\nabla\rho^{(n)}\|_\infty \|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}\|_p \|\rho^{(n+1)} - \rho^{(n)}\|_p^{p-1}\\
&+ \|\rho^{(n)}\|_\infty \|\nabla\cdot(\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)})\|_p \|\rho^{(n+1)} - \rho^{(n)}\|_p^{p-1}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|\rho^{(n)}\|_\infty + \|\nabla\rho^{(n)}\|_\infty &\lesssim \|\rho^{(n)}\|_{X_{k-1}},\\
\|\nabla\cdot\boldsymbol{v}^{(n-1)}\|_\infty &\lesssim \|\boldsymbol{v}^{(n-1)}\|_{X_k},\\
\|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}\|_p &\lesssim \|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}\|_{X_{k-1}},\\
\|\nabla\cdot(\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n+1)})\|_p &\lesssim \|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}\|_{X_{k-1}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\|\rho^{(n+1)} - \rho^{(n)}\|_p&\\
\lesssim \|\boldsymbol{v}^{(n)}\|_{X_k}\|\rho^{(n+1)} - \rho^{(n)}\|_p &+ \|\rho^{(n)}\|_{X_{k-1}}\|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}\|_{X_{k-1}}.
\end{aligned} \tag{3-13}$$

Similarly we also have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{v}^{(n+1)}(t)-\boldsymbol{v}^{(n)}(t)\|_p^p \\ \lesssim \|\nabla\cdot\boldsymbol{v}^{(n)}(t)\|_\infty\|(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})(t)\|_p^p \\ +\|\nabla\boldsymbol{v}^{(n)}(t)\|_\infty\|(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})(t)\|_p\|(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})(t)\|_p^{p-1} \\ +\|(\nabla f^{(n+1)}-\nabla f^{(n)})(t)\|_p\|(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})(t)\|_p^{p-1}.\end{aligned}$$

Using the fact $\|\nabla f^{(n+1)}-\nabla f^{(n)}\|_p\lesssim\|\nabla f^{(n+1)}-\nabla f^{(n)}\|_{X_{k-2}}$, we arrive at

$$\frac{d}{dt}\|(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})(t)\|_p\lesssim\|\boldsymbol{v}^{(n)}(t)\|_{X_k}\|(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})(t)\|_{X_{k-1}} +\|(\nabla f^{(n+1)}-\nabla f^{(n)})(t)\|_{X_{k-2}}. \tag{3-14}$$

For the L^p estimate of $\nabla f^{(n+1)}-\nabla f^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\|\partial f^{(n+1)}(t)-\partial f^{(n)}(t)\|_p^p \\ \lesssim\|\nabla\boldsymbol{v}^{(n)}\|_\infty\|\nabla(f^{(n+1)}-f^{(n)})\|_p\|\partial f^{(n+1)}-\partial f^{(n)}\|_p^{p-1} \\ +\|\nabla\cdot\boldsymbol{v}^{(n)}\|_\infty\|\partial f^{(n+1)}-\partial f^{(n)}\|_p^p \\ +\|\nabla f^{(n)}\|_\infty\|\partial(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})\|_p\|\partial f^{(n+1)}-\partial f^{(n)}\|_p^{p-1} \\ +\|\partial\nabla f^{(n)}\|_\infty\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_p\|\partial f^{(n+1)}-\partial f^{(n)}\|_p^{p-1} \\ +\|\partial\nabla\cdot(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_p\|\partial f^{(n+1)}-\partial f^{(n)}\|_p^{p-1}.\end{aligned}$$

Or, simplifying a bit,

$$\begin{aligned}\frac{d}{dt}\|(\nabla f^{(n+1)}-\nabla f^{(n)})(t)\|_p \\ \lesssim\|\boldsymbol{v}^{(n)}(t)\|_{X_k}\|(\nabla f^{(n+1)}-\nabla f^{(n)})(t)\|_{X_{k-2}} \\ +\|\nabla f^{(n)}(t)\|_{X_{k-1}}\|(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})(t)\|_{X_{k-1}}+\|(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})(t)\|_{X_{k-1}}.\end{aligned} \tag{3-15}$$

We now turn to the $\dot{H}^{k-1}$ estimates of the high-frequency part of the iterate differences. From direct computation, we have

$$\frac{d}{dt}\int|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})|^2\,dx=I_1+I_2+I_3, \tag{3-16}$$

where we have set

$$\begin{aligned}I_1&=-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}\cdot\nabla)(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})]\,dx,\\ I_2&=-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})\cdot\nabla\boldsymbol{v}^{(n)}]\,dx,\\ I_3&=-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}\nabla(f^{(n+1)}-f^{(n)})\,dx.\end{aligned}$$

We can write

$$\begin{aligned}
I_1 =&-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}\cdot\nabla)P_{>1}^2(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})]\,dx\\
&-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}\cdot\nabla)(I-P_{>1}^2)(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})]\,dx\\
=&-2\int P_{>1}^2\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot(\boldsymbol{v}^{(n)}\cdot\nabla)\partial^{k-1}P_{>1}^2(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\,dx\\
&-2\int P_{>1}^2\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot[\partial^{k-1},(\boldsymbol{v}^{(n)}\cdot\nabla)]P_{>1}^2(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\,dx\\
&-2\int P_{>1}^2\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot\partial^{k-1}[(\boldsymbol{v}^{(n)}\cdot\nabla)(I-P_{>1}^2)(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})]\,dx.
\end{aligned}$$

Integrating by parts and using Hölder's inequality together with Lemma 2.3 (the first two), we obtain the estimate

$$\begin{aligned}
I_1 \lesssim&\ \|\nabla\cdot\boldsymbol{v}^{(n)}\|_\infty\|\partial^{k-1}P_{>1}^2(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2^2\\
&+\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|[\partial^{k-1},(\boldsymbol{v}^{(n)}\cdot\nabla)]P_{>1}^2(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\\
&+\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|\partial^{k-1}((\boldsymbol{v}^{(n)}\cdot\nabla)(I-P_{>1}^2)(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}))\|_2\\
\lesssim&\ \|\boldsymbol{v}^{(n)}\|_{X_k}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2^2\\
&+\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|\boldsymbol{v}^{(n)}\|_{X_k}\|\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}\|_{X_{k-1}}\\
\lesssim&\ \|\boldsymbol{v}^{(n)}\|_{X_k}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}\|_{X_{k-1}}.
\end{aligned}$$

For the next term, we use Lemma 2.3 to write

$$\begin{aligned}
I_2 &\lesssim \|\partial^{k-1}P_{>1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|P_{>1}\partial^{k-1}([(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})\cdot\nabla]\boldsymbol{v}^{(n)})\|_2\\
&\lesssim \|\boldsymbol{v}^{(n)}\|_{X_k}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}.
\end{aligned}\tag{3-17}$$

Collecting the estimates above, we have

$$\begin{aligned}
&\frac{d}{dt}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2^2\\
&\quad\le C\|\boldsymbol{v}^{(n)}\|_{X_k}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2\\
&\qquad\qquad\times\big[\|\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}\|_{X_{k-1}}+\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\big]\\
&\qquad\qquad-2\int P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\cdot P_{>1}\partial^{k-1}\nabla(f^{(n+1)}-f^{(n)})\,dx.
\end{aligned}\tag{3-18}$$

The estimate for $f^{(n+1)}-f^{(n)}$ follows similarly. We compute

$$
\begin{aligned}
&\frac{d}{dt}\int |P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})|^2 dx\\
&= -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})P_{>1}\partial^{k-1}[\boldsymbol{v}^{(n)}\cdot(\nabla f^{(n+1)}-\nabla f^{(n)})]\,dx\\
&\qquad -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})\cdot\nabla f^{(n)}]\,dx\\
&\qquad -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\nabla\cdot P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\,dx\\
&= -2\int P_{>1}^2\partial^{k-1}(f^{(n+1)}-f^{(n)})\boldsymbol{v}^{(n)}\cdot\partial^{k-1}P_{>1}^2(\nabla f^{(n+1)}-\nabla f^{(n)})\,dx\\
&\qquad -2\int P_{>1}^2\partial^{k-1}(f^{(n+1)}-f^{(n)})[\partial^{k-1},\boldsymbol{v}^{(n)}]\cdot P_{>1}^2(\nabla f^{(n+1)}-\nabla f^{(n)})\,dx\\
&\qquad -2\int P_{>1}^2\partial^{k-1}(f^{(n+1)}-f^{(n)})\partial^{k-1}(\boldsymbol{v}^{(n)}\cdot(I-P_{>1}^2)(\nabla f^{(n+1)}-\nabla f^{(n)}))\,dx\\
&\qquad -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})P_{>1}\partial^{k-1}[(\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)})\cdot\nabla f^{(n)}]\,dx\\
&\qquad -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\nabla\cdot P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\,dx.
\end{aligned}
$$

Applying Lemma 2.3 (the last two estimates), we get

$$
\begin{aligned}
\text{(3-19)}\quad &\frac{d}{dt}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2^2\\
&\le C\|\boldsymbol{v}^{(n)}\|_{X_k}\|\nabla f^{(n+1)}-\nabla f^{(n)}\|_{X_{k-2}}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2\\
&\quad + C\|\nabla f^{(n)}\|_{X_{k-1}}\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2\\
&\quad -2\int P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\nabla\cdot P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\,dx.
\end{aligned}
$$

Adding together (3-18) and (3-19), we have

$$
\begin{aligned}
&\frac{d}{dt}\|P_{>1}\partial^{k-1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_2^2+\frac{d}{dt}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2^2\\
&\quad\le C\|\boldsymbol{v}^{(n)}\|_{X_k}\|P_{>1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|_{\dot H^{k-1}}\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\\
&\qquad + C\|\boldsymbol{v}^{(n)}\|_{X_k}\|\nabla f^{(n+1)}-\nabla f^{(n)}\|_{X_{k-2}}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2\\
&\qquad + C\|\nabla f^{(n)}\|_{X_{k-1}}\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\|P_{>1}\partial^{k-1}(f^{(n+1)}-f^{(n)})\|_2.
\end{aligned}
$$

Summing over all the partial derivatives and using Cauchy–Schwartz, we have

$$\begin{aligned}(3\text{-}20)\quad &\frac{d}{dt}\big(\|P_{>1}(\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)})\|^2_{\dot H^{k-1}}+\|P_{>1}\nabla(f^{(n+1)}-f^{(n)})\|^2_{\dot H^{k-2}}\big)\\ &\lesssim \|\boldsymbol{v}^{(n)}\|_{X_k}\cdot\big(\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|^2_{X_{k-1}}+\|\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}\|^2_{X_{k-1}}\big)\\ &\quad+\|\boldsymbol{v}^{(n)}\|_{X_k}\cdot\|\nabla(f^{(n+1)}-f^{(n)})\|^2_{X_{k-2}}\\ &\quad\quad+\|\nabla f^{(n)}\|_{X_{k-1}}\cdot\big(\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|^2_{X_{k-1}}+\|\nabla(f^{(n+1)}-f^{(n)})\|^2_{X_{k-2}}\big).\end{aligned}$$

Similarly, we get the estimate for ρ as follows:

$$\begin{aligned}(3\text{-}21)\quad \frac{d}{dt}\|P_{>1}(\rho^{(n+1)}-\rho^{(n)})\|_{\dot H^{k-2}} &\lesssim \|\rho^{(n)}\|_{X_{k-1}}\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\\ &\quad+\|\boldsymbol{v}^{(n)}\|_{X_k}\|\rho^{(n+1)}-\rho^{(n)}\|_{X_{k-2}}.\end{aligned}$$

Let

$$N^n(t)=\|\boldsymbol{v}^{(n+1)}-\boldsymbol{v}^{(n)}\|^2_{X_{k-1}}+\|\nabla f^{(n+1)}-\nabla f^{(n)}\|^2_{X_{k-2}}+\|\rho^{(n+1)}-\rho^{(n)}\|^2_{X_{k-2}}.$$

Collecting estimates (3-13)–(3-15), (3-20), (3-21), and integrating in t, we have

$$N^{(n+1)}(t)\le C\int_0^t(1+M^{(n)}(\tau))N^{(n)}(\tau)\,d\tau+C\int_0^t(1+M^{(n)}(\tau))N^{(n+1)}(\tau)\,d\tau.$$

Using Gronwall's inequality we get

$$\|N^{(n+1)}\|_{L^\infty_t([0,T])}\le CT\|M^{(n)}N^{(n)}\|_{L^\infty_t([0,T])}\exp\big\{CT(1+\|M^{(n)}\|_{L^\infty_t([0,T])})\big\}.$$

From (3-12) and the choice of T (3-11), we have

$$\begin{aligned}(3\text{-}22)\quad \|N^{(n+1)}\|_{L^\infty_t([0,T])}&\le 2CM_0T\|N^{(n)}\|_{L^\infty_t([0,T])}\exp\{2CT(1+M_0)\}\\ &\le \tfrac12\|N^{(n)}\|_{L^\infty_t([0,T])}.\end{aligned}$$

Step 4: Limiting system and regularity of solutions. The estimate (3-22) easily implies that

$$\begin{aligned}\{\rho^{(n)}\}_{n=1}^\infty &\text{ is Cauchy in } L^\infty_t([0,T],X_{k-2}),\\ \{\boldsymbol{v}^{(n)}\}_{n=1}^\infty &\text{ is Cauchy in } L^\infty_t([0,T],X_{k-1}),\\ \{\nabla f^{(n)}\}_{n=1}^\infty &\text{ is Cauchy in } L^\infty_t([0,T],X_{k-2}).\end{aligned}$$

From the condition $k\ge 10d$, and using the embedding $X_{k-2}\subset W^{[k/2],p}$, we know that all sequences $\{\rho^{(n)},\boldsymbol{v}^{(n)},\nabla f^{(n)}\}_{n=1}^\infty$ are Cauchy in $L^\infty_t([0,T];W^{[k/2],p})$. Using the iteration system (3-2) and noting $W^{[k/5],p}$ is an algebra, we can upgrade the regularity in time and obtain that $\{\rho^{(n)},\boldsymbol{v}^{(n)},\nabla f^{(n)}\}_{n=1}^\infty$ are Cauchy in

$W_t^{[k/5],\infty}([0,T]; W^{[k/5],p})$. Therefore there exist

$$\rho \in L_t^\infty([0,T], \dot{H}^{k-1} \cap L^p) \cap W_t^{[k/5],\infty}([0,T]; W^{[k/5],p}),$$
$$\boldsymbol{g} \in L_t^\infty([0,T], \dot{H}^{k-1} \cap L^p) \cap W_t^{[k/5],\infty}([0,T]; W^{[k/5],p}),$$
$$v \in L_t^\infty([0,T], \dot{H}^{k} \cap L^p) \cap W_t^{[k/5],\infty}([0,T]; W^{[k/5],p}).$$

such that the following equations hold true in the classical sense:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \boldsymbol{g} = 0, \\ \partial_t \boldsymbol{g} + \nabla(\boldsymbol{v} \cdot \boldsymbol{g}) + \nabla(\nabla \cdot \boldsymbol{v}) = 0. \end{cases} \tag{3-23}$$

Step 5: Continuity in highest norm. Since $(\rho, \boldsymbol{v}, \boldsymbol{g}) \in C([0,T], L_x^p)$, we only need to show $(\rho, \boldsymbol{g}) \in C([0,T], \dot{H}_x^{k-1})$, $\boldsymbol{v} \in C([0,T], \dot{H}^k)$. We shall only prove it for ρ as the others are similar. Fix any $t_0 \in [0,T]$, we compute

$$\begin{aligned} &\|\partial^{k-1}(\rho(t) - \rho(t_0))\|_2^2 \\ &\quad = \|\partial^{k-1}\rho(t)\|_2^2 - \|\partial^{k-1}\rho(t_0)\|_2^2 + 2\langle \partial^{k-1}\rho(t_0) - \partial^{k-1}\rho(t), \partial^{k-1}\rho(t_0)\rangle, \end{aligned} \tag{3-24}$$

where $\langle\, ,\, \rangle$ is the usual L^2-pairing. By a simple density argument and the fact that $\rho \in C([0,T], L_x^p)$, we have[1]

$$\lim_{t \to t_0} |(3\text{-}24)| = 0.$$

Therefore we only need to check the norm continuity, that is:

$$\lim_{t \to t_0} \|\partial^{k-1}\rho(t)\|_2^2 = \|\partial^{k-1}\rho(t_0)\|_2^2.$$

But this follows from a simple Gronwall estimate, which was essentially done in Step 1. We omit the details.

Finally to recover the equation in (1-1) we still need to show $\rho > 0$ and $\boldsymbol{g} = \nabla\rho/\rho$. Since the initial data ρ_0 is positive, the positivity of ρ follows easily from the method of characteristics and the fact that $\boldsymbol{v} \in C^2$. We leave the proof that $\boldsymbol{g} = \nabla\rho/\rho$ to the next step.

Step 6: Identification of $\boldsymbol{g}$ with $\nabla\rho/\rho$. We first show that

$$\frac{\nabla\rho}{\rho} \in C([0,T], L_x^p). \tag{3-25}$$

[1]If $t_0 = 0$, then the left continuity can be obtained by the simple fact that our solution actually belongs to $C([-T_1, T], L_x^p)$ for some small T_1 since our system is inviscid.

From Step 4 and using the positivity of $\rho^{(n)}$ and ρ, it is not difficult to check that up to a subsequence,

$$\frac{\nabla\rho^{(n)}}{\rho^{(n)}}(t,x)\to\frac{\nabla\rho}{\rho}(t,x),\quad a.e.\ (t,x)\in[0,T]\times\mathbb{R}^d.$$

Thus (3-25) can be proved if the sequence $\nabla\rho^{(n)}/\rho^{(n)}$ is Cauchy in $C_t^0L_x^p$. To this end, we set $\boldsymbol{g}_1^{(n+1)}=\nabla\rho^{(n+1)}/\rho^{(n+1)}$. By the ρ-equation in (3-2) we have

$$\partial_t\boldsymbol{g}_1^{(n+1)}+\nabla(\nabla\cdot\boldsymbol{v}^{(n)})+\nabla(\boldsymbol{v}^{(n)}\cdot\boldsymbol{g}_1^{(n+1)})=0.$$

Using integration by parts (note that $\boldsymbol{g}_1^{(n+1)}$ is gradient-like), we obtain

$$\frac{d}{dt}\|\boldsymbol{g}_1^{(n+1)}(t)\|_p\lesssim\|\boldsymbol{v}^{(n)}(t)\|_{X_k}(1+\|\boldsymbol{g}_1^{(n+1)}(t)\|_p).$$

From Gronwall's inequality and the choice of T (shrinking T if necessary), we obtain

$$\|\boldsymbol{g}_1^{(n+1)}\|_{L_t^\infty([0,T];L_x^p)}\le 2M_0.$$

Similarly, we have

$$\|\partial\boldsymbol{g}_1^{(n+1)}\|_{L_t^\infty([0,T];L_x^p)}\le 2M_0.$$

Summing over all partial derivatives we see $\nabla\boldsymbol{g}_1^{(n+1)}$ is bounded in L^p.

For the L^p norm of the difference, we have

$$\frac{d}{dt}\|\boldsymbol{g}_1^{(n+1)}-\boldsymbol{g}_1^{(n)}\|_p\lesssim\|\boldsymbol{v}^{(n)}\|_{X_k}\|\boldsymbol{g}_1^{(n+1)}-\boldsymbol{g}_1^{(n)}\|_p+\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{X_{k-1}}\|\boldsymbol{g}_1^{(n)}\|_{W^{1,p}}.$$

Using the boundedness of $\boldsymbol{g}_1^{(n)}$ in $W^{1,p}$ and Gronwall, we have

$$\|\boldsymbol{g}_1^{(n+1)}-\boldsymbol{g}_1^{(n)}\|_{L_t^\infty([0,T];L^p)}\le C\|\boldsymbol{v}^{(n)}-\boldsymbol{v}^{(n-1)}\|_{L_t^\infty([0,T];X_{k-1})}.$$

Therefore $\boldsymbol{g}_1^{(n)}$ is Cauchy in $C_t^0([0,T];L_x^p)$. This completes the proof of (3-25).

We are now ready to show $\boldsymbol{g}=\nabla\rho/\rho$. Indeed, from the first equation in (3-23), we see $\nabla\rho/\rho$ satisfies in the classical sense

$$\partial_t\left(\frac{\nabla\rho}{\rho}\right)+\nabla\left(\boldsymbol{v}\cdot\frac{\nabla\rho}{\rho}\right)+\nabla(\nabla\cdot\boldsymbol{v})=0.$$

This equation has exactly the same form as the g-equation in (3-23). The identification of $\boldsymbol{g}$ with $\nabla\rho/\rho$ then follows from the uniqueness of the solutions in the L^p class, to the following vector equation

$$\partial_t\boldsymbol{h}+\nabla(\boldsymbol{v}\cdot\boldsymbol{h})=0,\qquad \boldsymbol{h}(0)\in L^p.$$

The uniqueness in L^p follows from a simple energy estimate which is omitted here. We note that if $\rho_0\in L^1$, then the mass conservation follows from a standard truncation argument. We omit the details.

4. Proof of Theorem 1.3

By Theorem 1.1, for any chosen ρ_0, $\boldsymbol{v}_0$, there exists a time $T > 0$ such that (1-1) admits a unique solution $(\rho(t, x), \boldsymbol{v}(t, x))$ on $[0, T]$. In particular, the local solution is at least C^2 and satisfies the equation in the classical sense. Since $\operatorname{curl}(\boldsymbol{v}_0) = 0$, it is easy to check that $\operatorname{curl}(\boldsymbol{v}(t)) = 0$ for any t. We first observe the property of finite propagation speed. Indeed, set

$$f = \log \rho.$$

Then the Euler equation (1-1) can be written as

$$\begin{cases} \partial_t f + \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla f = 0, \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla f = 0. \end{cases}$$

Taking one more derivative in t for both equations and using the irrotational condition $\operatorname{curl} \boldsymbol{v} = 0$, we have

$$\begin{cases} \partial_{tt} f - \Delta f = \frac{1}{2}\Delta(|\boldsymbol{v}|^2) + \frac{1}{2}\nabla f \cdot \nabla(|\boldsymbol{v}|^2) + |\nabla f|^2 + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla(\boldsymbol{v} \cdot \nabla f), \\ \partial_{tt} \boldsymbol{v} - \Delta \boldsymbol{v} = \nabla(\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) + 2\nabla(\boldsymbol{v} \cdot \nabla f). \end{cases}$$

This is a standard quasilinear wave equation. The standard arguments (compare [Sogge 1995]), yields the finite propagation speed. In particular, we have $\rho(t, x) = 1$, $\boldsymbol{v}(t, x) = 0$ for all t, x such that $|x| \le 10 - t$ and $t \le T$.

We claim that the corresponding local solution $\rho(t, x)$, $\boldsymbol{v}(t, x)$ must blow up before $t = 1$. We argue by contradiction. Suppose ρ, $\boldsymbol{v}$ exist on $[0, 1]$, then we have

$$\frac{d}{dt} \int \rho \phi \, dx = \int \rho \boldsymbol{v} \cdot \nabla \phi \, dx.$$

Taking one more derivative in t, we get

$$\begin{aligned} \frac{d^2}{dt^2} \int \rho \phi \, dx &= \frac{d}{dt} \int \rho \boldsymbol{v} \cdot \nabla \phi \, dx \\ &= \int \rho \partial_t \boldsymbol{v} \cdot \nabla \phi + \int \partial_t \rho \boldsymbol{v} \cdot \nabla \phi \, dx \\ &= -\int (\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) \cdot \nabla \phi \, dx - \int \nabla \cdot (\rho \boldsymbol{v})\boldsymbol{v} \cdot \nabla \phi \, dx - \int \nabla \rho \cdot \nabla \phi \, dx \\ &= \int \rho \boldsymbol{v}_j \boldsymbol{v}_k \partial_{jk} \phi(x) \, dx + \int \rho \Delta \phi \, dx. \end{aligned}$$

Note $\boldsymbol{v}(t, x)$ vanishes on $|x| \le 1$ for all $t \in [0, 1]$. For $|x| > 1$, we use the fact that $\nabla^2 \phi$ is positive definite and the boundedness of $\Delta \phi$ to get

$$\frac{d^2}{dt^2} \int \rho \phi \, dx > -C,$$

for some C depending on $\|\rho_0\|_1$. Therefore from the condition (1-7), we have

$$\frac{d}{dt}\int \rho\phi\,dx \geq N - C \quad \text{for } t \in [0, 1].$$

This implies

$$\int \rho(1, x)\phi(x)\,dx \geq \int \rho_0(x)\phi(x) + N - C,$$

which, for N large enough, contradicts the fact that

$$\int \rho(1, x)\phi(x)\,dx \leq \|\rho(1)\|_1 \|\phi\|_\infty.$$

This completes the proof of Theorem 1.3.

References

[Chae and Ha 2009] D. Chae and S.-Y. Ha, "On the formation of shocks to the compressible Euler equations", *Commun. Math. Sci.* **7**:3 (2009), 627–634. MR 2011a:35331 Zbl 1183.35225

[Godin 2005] P. Godin, "The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions", *Arch. Ration. Mech. Anal.* **177**:3 (2005), 479–511. MR 2006i:76092 Zbl 1075.76052

[John 1974] F. John, "Formation of singularities in one-dimensional nonlinear wave propagation", *Comm. Pure Appl. Math.* **27** (1974), 377–405. MR 51 #6163 Zbl 0302.35064

[Kato 1975] T. Kato, "The Cauchy problem for quasi-linear symmetric hyperbolic systems", *Arch. Rational Mech. Anal.* **58**:3 (1975), 181–205. MR 52 #11341 Zbl 0343.35056

[Klainerman and Majda 1980] S. Klainerman and A. Majda, "Formation of singularities for wave equations including the nonlinear vibrating string", *Comm. Pure Appl. Math.* **33**:3 (1980), 241–263. MR 81f:35080 Zbl 0443.35040

[Lax 1964] P. D. Lax, "Development of singularities of solutions of nonlinear hyperbolic partial differential equations", *J. Mathematical Phys.* **5** (1964), 611–613. MR 29 #2532 Zbl 0135.15101

[Liu 1979] T. P. Liu, "Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations", *J. Differential Equations* **33**:1 (1979), 92–111. MR 80g:35075 Zbl 0379.35048

[Rammaha 1989] M. A. Rammaha, "Formation of singularities in compressible fluids in two-space dimensions", *Proc. Amer. Math. Soc.* **107**:3 (1989), 705–714. MR 90f:35127 Zbl 0692.35015

[Sideris 1985] T. C. Sideris, "Formation of singularities in three-dimensional compressible fluids", *Comm. Math. Phys.* **101**:4 (1985), 475–485. MR 87d:35127 Zbl 0606.76088

[Sideris 1991] T. C. Sideris, "The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit", *Indiana Univ. Math. J.* **40**:2 (1991), 535–550. MR 92f:35119 Zbl 0736.35087

[Sogge 1995] C. D. Sogge, *Lectures on nonlinear wave equations*, International Press, Boston, MA, 1995. MR 2000g:35153 Zbl 1089.35500

Received January 8, 2012. Revised August 1, 2012.

DONG LI
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540
UNITED STATES

and

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC V6T 1Z2
CANADA

mpdongli@gmail.com

CHANGXING MIAO
INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS
BEIJING 100088
CHINA

miao_changxing@iapcm.ac.cn

XIAOYI ZHANG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
14 MACLEAN HALL
IOWA CITY, IA 52242
UNITED STATES

and

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCES
BEIJING 100080
CHINA

zh.xiaoyi@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 262 No. 1 March 2013