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We study the Hilbert–Kunz invariants of homogeneous ideals in graded polynomial algebras and develop a homological formula for the Hilbert–Kunz multiplicity resembling the formula of J.-P. Serre using Koszul homology for the ordinary multiplicity of an ideal. We apply this in the special case of maximal primary irreducible ideals to obtain several new results, among which is a reciprocity formula for the Hilbert–Kunz invariants of directly linked ideals in a graded polynomial algebra.

The Hilbert–Kunz invariants grew out of the paper of E. Kunz [1969] characterizing regular local rings in characteristic $p \neq 0$ and they were put into their present form by P. Monsky [1983]. These invariants are defined by analogy with the Hilbert function and its associated multiplicity, but instead of using the ordinary powers of an ideal to do so, one uses its Frobenius powers instead. Specifically, fix a field $\mathbb{F}$ of characteristic $p \neq 0$ and a commutative graded connected[1] $\mathbb{F}$-algebra A. Recall that for an ideal $I \subset A$ the *Frobenius power* $I^{[p]}$ of I is the ideal generated by the p-th powers of elements of I. If A is Noetherian and M is a finitely generated A-module, so M is of finite type,[2] one defines the *Hilbert–Kunz function* $\mathsf{HK}_{(I,M)} : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ of a maximal primary ideal $I \subset A$ on M by $\mathsf{HK}_{(I,M)}(e) = \dim_{\mathbb{F}}(M/I^{[p^e]} \cdot M)$ for $e \in \mathbb{N}_0$. The *Hilbert–Kunz multiplicity* of I on M is defined to be the real number

$$e_{\mathsf{HK}}(I, M) = \lim_{e\to\infty}\left\{\frac{\dim_{\mathbb{F}}\big(M/I^{[p^e]} \cdot M\big)}{p^{e\cdot\dim(A)}}\right\} = \lim_{e\to\infty}\left\{\frac{\mathsf{HK}_{(I,M)}(e)}{p^{e\cdot\dim(A)}}\right\}.$$

The fact that the numbers $\{\dim_{\mathbb{F}}(M/I^{[p^e]} \cdot M)/p^{e\cdot\dim(A)}\}|_{e\in\mathbb{N}_0}$ form a bounded Cauchy sequence, so that the preceding limit makes sense, was proved by P. Monsky

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[1] By a *connected* algebra over $\mathbb{F}$ is meant a nonnegatively graded algebra R whose degree 0 component is $\mathbb{F} \cdot 1$ where $1 \in R$ is the unit of the algebra. The terminology comes from algebraic topology: a (nonpathological) topological space X is connected if and only if its cohomology algebra is connected.

[2] A graded vector space V is of *finite type* if all its homogeneous components V_i are finite-dimensional.

(see, e.g., [Monsky 1983]). If the ideal I is the maximal ideal[3] then one speaks of the Hilbert–Kunz function of M, and writes it as $\mathsf{HK}_M(\text{—})$, and the Hilbert–Kunz multiplicity of M, and denotes it by $e_{\mathsf{HK}}(M)$.

The colength formula of [Watanabe and Yoshida 2000] for $e_{\mathsf{HK}}(I)$ provided our starting point. Using it we obtain a homological formula, Proposition 3.1, for $e_{\mathsf{HK}}(I)$ based on work of W. Smoke [1972] going back to D. Hilbert [1890]. The colength formula yields a relation between the Hilbert–Kunz multiplicity of bundle and base ideals in the context of the projective bundle theorem (see [Smith and Stong 2011] and Section 2), as well as the Hilbert–Kunz multiplicity of Gorenstein ideals with socle degree 2 or 3 in polynomial algebras (see Section 2 and Section 4). Proposition 3.1 also leads to a reciprocity relation for the Hilbert–Kunz multiplicity of a pair of directly linked ideals (see Section 5) in polynomial algebras.

We pay particular attention here to setting things up in a graded context. Being of the J. C. Moore school,[4] a $\mathbb{Z}$-graded object X is a *collection* $\{X_i \mid i \in \mathbb{Z}\}$, not a direct sum, and only homogeneous elements are considered. If the direct sum makes sense we write $\mathrm{Tot}(X)$ for the direct sum $\bigoplus X_i$ to distinguish it from the graded object X. Unless specifically mentioned to the contrary all graded vector spaces are nonnegatively graded, i.e., $X_i = 0$ for all $i < 0$.

1. Background from homological algebra

In this section we collect results from homological algebra needed for the proofs in the later sections. These consist of a brief review of [Smoke 1972] which formulates some fundamental ideas of D. Hilbert, in particular the syzygy theorem and its application to computing Poincaré series (see, e.g., [Hilbert 1890]) in the language of homological algebra.

Fix a ground field $\mathbb{F}$ which for the present may be arbitrary. Let R denote a commutative graded connected algebra over $\mathbb{F}$. Unless otherwise stated to the contrary the algebra R should be assumed Noetherian, i.e., finitely generated over $\mathbb{F}$.

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[3]In the graded context there is only one maximal ideal in A; to wit, the *augmentation ideal*, which sometimes is referred to as the *irrelevant ideal*, denoted by $\overline{A}$ and consisting of all the homogeneous elements of strictly positive degree and a zero in all other degrees, i.e., $\overline{A}$ is the kernel of the *augmentation map* $\eta : A \longrightarrow \mathbb{F}$ defined by

$$\eta(a) = \begin{cases} a & \text{if } \deg(a) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $A = \mathbb{F}[x_1, \dots, x_n]$ is a polynomial algebra, then, since the notation $\overline{\mathbb{F}[x_1, \dots, x_n]}$ is quite ugly we prefer to write $\mathfrak{m}$ for the maximal ideal in this case and also to use the expression $\mathfrak{m}$-*primary* for a maximal primary ideal in $\mathbb{F}[x_1, \dots, x_n]$. More generally, we write $\mathfrak{m}_A$ for the maximal ideal of A if A is a complicated symbol such as $\mathbb{F}[V]^G$.

[4]Though I myself am a Massey product.

Definition. A function ξ from isomorphism classes of R-modules of finite type to an abelian group A is called an *Euler characteristic with values in* A, or said to have the *Euler characteristic property*, if for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules of finite type one has

$$\xi(M') + \xi(M'') = \xi(M).$$

Example 1. The Poincaré series,[5] to wit,

$$P(M, t) = \sum_{i \in \mathbb{N}_0} \dim_{\mathbb{F}}(M_i) t^i,$$

for a finite type R-module M, defines an Euler characteristic with values in the abelian group $\mathbb{Z}[\![t]\!]$ of formal power series with integral coefficients.

The following general nonsense result should at least be recorded. A proof is not really necessary (but if you insist on seeing one, consult, e.g., [Fraser 1969]).

Lemma 1.1. *Let R be a commutative graded connected algebra over $\mathbb{F}$ and denote by $K(R)$ the Grothendieck group of the category of finite type R-modules. Then an Euler characteristic ξ with values in the abelian group A is nothing but a homomorphism of abelian groups $K(R) \longrightarrow A$.*

In other words, the map $[-]$ that assigns to an R-module of finite type its equivalence class in the Grothendieck group $K(R)$ is a universal function with the Euler characteristic property. This means that given any Euler characteristic ξ with values in the abelian group A, there is a unique group homomorphism $\Xi : K(R) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} \mathbf{MOD}_R \arrow[r, "{[\,]}"] \arrow[d, "\xi"'] & K(R) \arrow[dl, "\Xi"] \\ A & \end{tikzcd}$$

commutes, where $\mathbf{MOD}_R$ denotes the set (sic!) of isomorphism classes of R-modules of finite type.

Definition. A resolution of an R-module M of finite type

$$\mathcal{F} \qquad \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

[5] We prefer to work with Poincaré series in place of the Hilbert function.

is said to be *weakly minimal* if

$$\cdots > \mathrm{cx}(F_i) > \mathrm{cx}(F_{i-1}) > \cdots > \mathrm{cx}(F_0),$$

where $\mathrm{cx}(-)$ denotes the connectivity[6] of the module $-$. If in addition all the induced maps $Q(F_i) \longrightarrow Q(F_{i-1})$ of the vector spaces of indecomposables[7] are zero for $i \in \mathbb{N}$ then we say that $\mathcal{F}$ is *minimal.*

The notion of a weakly minimal resolution is a bit ad hoc, but it is the condition that was employed in [Broer et al. 2011] to prove the following lemmas culminating in the formula of Proposition 1.5 below, which is the nonequivariant version of the starting point for [Broer et al. 2011]. Also, by working with weakly minimal resolutions we can choose one resolution with some special algebraic structure to be put on homology modules, and another resolution to prove finiteness results such as in the next lemma. This kind of strategy was used in [Broer et al. 2011], especially Section 2, to incorporate a group action and character series. For another example of this kind see the discussion following Proposition 3.1 to follow where such a special structure is imposed for example by choosing a resolution as in [Buchsbaum and Eisenbud 1977], though one could alternatively invoke [Avramov and Golod 1971].

Lemma 1.2. *If $\mathcal{F}$ is a weakly minimal resolution of an R-module of finite type with each term F_i of $\mathcal{F}$ of finite type for $i \in \mathbb{N}_0$ then the alternating sum of their Poincaré series*

$$\sum_{I \in \mathbb{N}_0} (-1)^i P(F_i, t) \in \mathbb{Z}[\![t]\!]$$

makes sense as a formal power series.

Proof. For any integer j there are only finitely many F_i for $i \in \mathbb{N}_0$ with $(F_i)_j \neq 0$, so for any i and j there are only finitely many $P(F_i, t)$ in which t^j has a nonzero coefficient. □

The next lemma says that the formal power series in Lemma 1.2 does not depend on the choice of the weakly minimal free resolution and provides a value for it.

Lemma 1.3. *If $\mathcal{F}$ is a weakly minimal free resolution of an R-module M of finite type and each term F_i ($i \in \mathbb{N}_0$) of $\mathcal{F}$ is of finite type, then*

$$P(R, t) \cdot \sum_{i \in \mathbb{N}_0} (-1)^i P(V_i, t) = \sum_{i \in \mathbb{N}_0} (-1)^i P(F_i, t) = P(M, t) \in \mathbb{Z}[\![t]\!],$$

where $V_i = \mathbb{F} \otimes_R F_i$ is the indecomposable module of F_i for $i \in \mathbb{N}_0$.

[6] The *connectivity* $\mathrm{cx}(M)$ of a graded vector space M is the largest integer such that $M_i = 0$ for $i \leq \mathrm{cx}(M)$.

[7] If M is an R-module its vector space of *indecomposable elements* is $\mathbb{F} \otimes_R M$. It is often denoted by $Q(M)$.

Proof. This follows from the Euler characteristic property of the function $P(—, t)$ and the fact that $P(F_i, t) = P(R, t) \cdot P(V_i, t)$. $\square$

So Lemma 1.3 tells us for an R-module of finite type that the alternating sum

$$\sum_{I \in \mathbb{N}_0} (-1)^i P(F_i, t) \in \mathbb{Z}[\![t]\!] \tag{1-1}$$

associated with a weakly minimal resolution $\mathcal{F}$ of finite type[8] is independent of the resolution $\mathcal{F}$. To evaluate it we are free to pick $\mathcal{F}$ in a particular way; for example to be a minimal resolution. For a minimal resolution $\mathcal{F}$ of a finite type module M one has[9]

$$F_i \cong R \otimes \operatorname{Tor}_i^R(M, \mathbb{F}) \quad \text{for } i \in \mathbb{N}_0,$$

so we obtain a second way to evaluate the alternating sum (1-1). To wit:

Lemma 1.4. *If $\mathcal{F}$ is a weakly minimal resolution of an R-module of finite type with each term F_i of $\mathcal{F}$ of finite type for $i \in \mathbb{N}_0$ then*

$$\sum_{I \in \mathbb{N}_0} (-1)^i P(F_i, t) = P(R, t) \cdot \sum_{i \in \mathbb{N}_0} (-1)^i P\big(\operatorname{Tor}_i^R(M, \mathbb{F}), t\big).$$

To summarize this part of the discussion we have proven the following result going back in spirit to [Hilbert 1890].

Proposition 1.5. *Let M be an R-module of finite type. Then*

$$P(M, t) = P(R, t) \cdot \sum_{i \in \mathbb{N}_0} (-1)^i \cdot P\big(\operatorname{Tor}_i^R(M, \mathbb{F}), t\big).$$

2. Background on Hilbert–Kunz invariants and first applications

The definition of the Hilbert–Kunz multiplicity in general requires the asymptotics introduced by P. Monsky to prove its existence. However in the special case of ideals in a polynomial algebra this is unnecessary. The existence is a more or less a direct consequence of the formula for the Hilbert–Kunz function of an ideal in terms of the Frobenius functor and the exactness of that functor for polynomial algebras (see, e.g., [Huneke and Yao 2002, Remark 2.4]). For the sake of simplicity we assume that all algebras in this section have a *standard grading*, i.e., are generated as algebras by their homogeneous component of degree 1.

8 We say that a resolution $\mathcal{F}$ is of *finite type* if the modules F_i in it are finitely generated for all $i \in \mathbb{N}_0$.

9 Minimal resolutions are unique up to isomorphism, hence *the* minimal resolution: though one should not overdo it here: the isomorphism is not unique nor functorial. The isomorphism $F_i \cong R \otimes \operatorname{Tor}_i^R(\mathbb{F}, M)$ is a result of the fact that per definition of minimal resolution the differentials in the complex $\mathbb{F} \otimes_R \mathcal{F}$ are identically zero and of course F_i is a free R-module for $i \in \mathbb{N}_0$.

Proposition 2.1. *Let $I \subset \mathbb{F}[x_1, \dots, x_n]$ be a maximal primary ideal in the standard graded polynomial algebra $S = \mathbb{F}[x_1, \dots, x_n]$ over the field $\mathbb{F}$ of characteristic $p \neq 0$ and set $H = \mathbb{F}[x_1, \dots, x_n]/I$. Then the Hilbert–Kunz function $\mathsf{HK}_{(I,S)}(\text{—})$ is given by*

$$\mathsf{HK}_{(I,S)}(e) = p^{e \cdot n} \cdot \dim_{\mathbb{F}}(H) \quad \textit{for } e \in \mathbb{N}_0,$$

and the Hilbert–Kunz multiplicity by $e_{\mathsf{HK}}(I, S) = \dim_{\mathbb{F}}(H)$, i.e., the colength of I.

The following simple example illustrates this; additional examples may be found further on in this section as well as Section 4. It is due to F. S. Macaulay [1916, Section 71] and has served ever since to demonstrate that irreducible ideals need not be generated by a regular sequence.

Example 2 (F. S. Macaulay). Let $\mathbb{F}$ be a field and consider the five quadratic forms

$$z^2 - x^2,\ z^2 - y^2,\ xy,\ xz,\ yz \in \mathbb{F}[x, y, z]$$

and the ideal $M \subset \mathbb{F}[x, y, z]$ that they generate. The quotient algebra is easily seen to be a Poincaré duality algebra, so by [Meyer and Smith 2005, Proposition I.1.4] the ideal M is irreducible. In fact, the quotient algebra is the $\mathbb{F}$-cohomology with the grading halved of the complex surface[10] $\mathbb{CP}(2) \# \mathbb{CP}(2) \# \mathbb{CP}(2)$, which is the connected sum of three copies of the complex projective plane $\mathbb{CP}(2)$. The preceding formula tells us that for any field of characteristic $p \neq 0$ the Hilbert–Kunz multiplicity is $e_{\mathsf{HK}}(M, \mathbb{F}[x, y, z]) = 5$.

In [Smith and Stong 2011] we studied the algebra associated with the projective bundle theorem of algebraic topology (see, e.g., [Stong 1968, p. 62]). For such ideals Proposition 2.1 yields a formula for the Hilbert–Kunz multiplicity of the bundle ideal in terms of the base ideal and the bundle dimension. Recall that for $I \subset \mathbb{F}[V, X]$ an $\mathfrak{m}$-primary ideal and $J = I \cap \mathbb{F}[V]$ we call I a *projective bundle ideal* with *base ideal* J if $\mathbb{F}[V, X]/I$ is a free $\mathbb{F}[V]/J$-module with respect to the module structure defined by the canonical inclusion $\mathbb{F}[V]/J \hookrightarrow \mathbb{F}[V, X]/I$. For such an ideal there is a coexact sequence[11]

$$\mathbb{F} \longleftarrow \mathbb{F}[X]/(X^{k+1}) \longleftarrow \mathbb{F}[V, X]/I \longleftarrow \mathbb{F}[V]/J \longleftarrow \mathbb{F} \tag{2-1}$$

[10] Or of $\mathbb{CP}(2) \# (S^2 \times S^2)$, which is the connected sum of a projective plain and a torus (see, e.g., the proof of Lemma 1.3 in [Smith and Stong 2010]).

[11] If $A'' \xleftarrow{f''} A \xleftarrow{f'} A'$ are maps between commutative graded connected algebras, the sequence is called *coexact* if $\ker f''$ is the ideal $f'(\overline{A'}) \cdot A$ of A generated by the image of the augmentation ideal $\overline{A'}$ of A'. Equivalently, $f''(A) \cong \mathbb{F} \otimes_{A'} A$. The category $\mathcal{CCA}_*$ of commutative graded connected algebras over a field has categorical images and cokernels, the image of a map $f : A' \longrightarrow A''$ being the monomorphism $\iota_f : f(A') \hookrightarrow A''$ and the cokernel the epimorphism $\eta_f : A'' \longrightarrow \mathbb{F} \otimes'_A A''$. To say that the sequence is coexact is equivalent to requiring that the natural map of the categorical cokernel of f' to categorical image of f'' is an isomorphism. The categorical cokernel of a map $f : R \longrightarrow S$ in $\mathcal{CCA}_*$ is often denoted by $R//f$ or $R//S$. So coexact is the categorical concept dual to exact.

of algebras. This sequence is an analogue of the coexact sequence of cohomology algebras

$$\mathbb{F} \longleftarrow H^*(\mathbb{CP}(k);\mathbb{F}) \longleftarrow H^*(\mathbb{P}(\xi \downarrow B);\mathbb{F}) \longleftarrow H^*(B;\mathbb{F}) \longleftarrow \mathbb{F}$$

associated to a complex vector bundle $\xi \downarrow B$ of dimension $k+1$ over the base space B, where $\mathbb{P}(\xi \downarrow B)$ is the associated projective space bundle (see, e.g., [Stong 1968, loc. cit.]). For this reason we use the following terminology in connection with the coexact sequence (2-1) of a projective bundle ideal. The integer $k+1$ is called the *bundle dimension*, $\mathbb{F}[V, X]/I$ the *bundle algebra*, $\mathbb{F}[V]/J$ the *base algebra*, and $\mathbb{F}[X]/(X^{k+1})$ the *fiber algebra*. A detailed example of a projective bundle ideal follows Proposition 2.2 which relates the Hilbert–Kunz multiplicity of the three algebras in the coexact sequence (2-1).

The proof of Lemma 2.2 in [Smith and Stong 2011] yields the formula

$$P(\mathbb{F}[V, X]/I, t) = P(\mathbb{F}[V]/J, t) \cdot P(\mathbb{F}[X]/(X^{k+1}), t) \tag{2-2}$$

relating the Poincaré series of the terms of the coexact sequence (2-1). Therefore one has the following relation for the Hilbert–Kunz multiplicities (compare [Huneke and Yao 2002, Lemma 2.1]).

Proposition 2.2. *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal with base ideal $J \subset \mathbb{F}[V]$ and bundle dimension $k+1$. Then*

$$e_{\mathsf{HK}}(I, \mathbb{F}[V, X]) = (k+1) \cdot e_{\mathsf{HK}}(J, \mathbb{F}[V]).$$

Proof. Evaluate both sides of formula (2-2) at $t=1$ and use Proposition 2.1. □

Here is an example to illustrate Proposition 2.2. The choice of $\mathbb{F}_2$ as ground field is merely a convenience in relating this example to its topological origins.

Example 3. Let $t, r \in \mathbb{N}_0$ be nonnegative integers and $V = \mathbb{F}_2^{2t+r}$. Denote by $x_1, \dots, x_t, y_1, \dots, y_t, u_1, \dots, u_r$ a basis for the space V^* of linear forms on V. Choose a linear form $w_1 \in \mathbb{F}[V]_1$ and a quadratic form $w_2 \in \mathbb{F}[V]_2$. In the polynomial algebra $\mathbb{F}_2[V, X]$ consider the ideal I generated by the following $t^2 + \binom{t}{2} + 2tr + 1$ forms:

$$\begin{gathered}
x_1^2, \dots, x_t^2, y_1^2, \dots, y_t^2, \\
x_i \cdot y_j \quad \text{for } 1 \le i \ne j \le t, \\
x_i \cdot u_j \quad \text{for } 1 \le i \le t \text{ and } 1 \le j \le r, \\
y_i \cdot u_j \quad \text{for } 1 \le i \le t \text{ and } 1 \le j \le r, \\
X^2 + w_1 \cdot X + w_2.
\end{gathered}$$

This is a projective bundle ideal with bundle dimension 2 and base ideal $J \subset \mathbb{F}_2[V]$ generated by all the previous forms except for $X^2 + w_1 \cdot X + w_2$. The quotient of

$\mathbb{F}_2[V]$ by the base ideal is the cohomology with $\mathbb{F}_2$ coefficients of the closed surface

$$F = \underbrace{(S^1 \times S^1)\,\#\cdots\#\,(S^1 \times S^1)}_{t} \;\#\; \underbrace{\mathbb{RP}(2)\,\#\cdots\#\,\mathbb{RP}(2)}_{r},$$

where # denotes the connected sum of closed manifolds. So

$$e_{\mathsf{HK}}(\mathbb{F}[V]/J) = \dim_{\mathbb{F}_2}\big(\mathrm{Tot}(\mathbb{F}[V]/J)\big) = 4t + 2r,$$

and since the bundle dimension is 2,

$$e_{\mathsf{HK}}(\mathbb{F}[V, X]/I) = 2 \cdot (4t + 2r).$$

The corresponding Poincaré duality quotient algebra $\mathbb{F}_2[V, X]/I$ is isomorphic to the $\mathbb{F}_2$-cohomology of the projective space bundle of a 2-plane bundle ξ over the closed surface F whose Stiefel–Whitney classes are w_1 and w_2.

Ever since the publication of [Cartan and Eilenberg 1956], change of rings phenomena have played an important role in algebra. An essential such result for Hilbert–Kunz multiplicity was proven in [Watanabe and Yoshida 2000]. Here it is in the graded form.

Theorem 2.3 (K. Watanabe and K. Yoshida). *Let $A \hookrightarrow B$ be a finite extension of graded connected commutative Noetherian integral domains over the field $\mathbb{F}$ of characteristic $p \neq 0$ and $I \subset A$ a maximal primary ideal. Then*

$$e_{\mathsf{HK}}(I \cdot B, B) = r \cdot e_{\mathsf{HK}}(I, A),$$

where r is equal to the degree $|\mathbb{L} : \mathbb{K}|$ of the field extension $\mathbb{K} \subseteq \mathbb{L}$, and we have written $\mathbb{K}$ for the field of fractions[12] *$\mathbf{FF}(A)$ of A and $\mathbb{L}$ for the field of fractions $\mathbf{FF}(B)$ of B.*

The following example shows that an analogous formula for the Hilbert–Kunz function does not hold.

Example 4. Let A be the subalgebra of $B = \mathbb{F}[x, y]$ that is generated by x^2, xy, y^2. Since $x^2, y^2 \in A$ is a system of parameters for B the extension $A \hookrightarrow B$ is finite. If $\mathbb{K}$ is the field of fractions of A and $\mathbb{L}$ the field of fractions of B then the field extension $\mathbb{K} \subset \mathbb{L}$ has degree $r = |\mathbb{L} : \mathbb{K}| = 2$. One way[13] to see this is to let $\mathbb{E}$ be

[12] The terminology *quotient field* or *field of fractions* of A, where A is a domain, is unfortunately not so clear as it might be. What is meant is the field consisting of all the fractions of the form a/b where $a, b \in A$ and $b \neq 0$; not the quotient of A by its maximal ideal. If A is graded then only homogeneous elements would be allowed and the resulting graded object is $\mathbb{Z}$-graded and a graded field. We employ the notation $\mathbf{FF}(A)$ for the field of fractions of an integral domain A, graded or not. In the special case of the field of fractions of $\mathbb{F}[V]$, we denote it by $\mathbb{F}(V)$, or $\mathbb{F}(z_1, \ldots, z_n)$ if $z_1, \ldots, z_n$ is a basis for the linear forms.

[13] Alternatively, for $\mathbb{F}$ of characteristic different from 2 one has $A \cong \mathbb{F}[x, y]^{\mathbb{Z}/2}$ where $\mathbb{Z}/2 < \mathrm{GL}(2, \mathbb{F})$ is generated by $-\mathrm{Id} \in \mathrm{GL}(2, \mathbb{F})$. Galois theory would then tell us the degree of the extension is 2.

the field of fractions of $\mathbb{F}[x^2, y^2]$ so $\mathbb{E} \subsetneqq \mathbb{K} \subsetneqq \mathbb{L}$. Since $|\mathbb{L} : \mathbb{E}|$ has degree 4 the only possible value for $|\mathbb{L} : \mathbb{K}|$ is 2 since it must be a proper nontrivial divisor of 4.

We consider the augmentation ideal $\overline{A}$ of A and note that $\dim_{\mathbb{F}}(A/\overline{A}) = 1 = \mathsf{HK}_{(\overline{A},A)}(0)$. Note that $\overline{A} \cdot B = (x^2, xy, y^2) \subset \mathbb{F}[x, y] = B$ so $\mathsf{HK}_{(\overline{A}\cdot B,B)}(0) = \dim(B/(\overline{A} \cdot B)) = 4$. Therefore

$$\mathsf{HK}_{(\overline{A}\cdot B,B)}(0) = 4 \neq 2 \cdot 1 = r \cdot \mathsf{HK}_{(\overline{A},A)}(0).$$

A similar computation would apply to any $e \in \mathbb{N}$ by Proposition 2.1.

As an illustration of Theorem 2.3 let us return to Example 2 and instead of considering the ideal of $\mathbb{F}[x, y, z]$ generated by the five quadratic forms listed there the subalgebra they generate.

Example 5. Let $A \subset \mathbb{F}[x, y, z]$ be the subalgebra generated by the five forms

$$z^2 - x^2,\ z^2 - y^2,\ xy,\ xz,\ yz \in \mathbb{F}[x,\ y,\ z].$$

Then Theorem 2.3 tells us that we can compute the Hilbert–Kunz multiplicity of A over a field of nonzero characteristic from the ideal in $\mathbb{F}[x, y, z]$ the five forms generate with the formula

$$e_{\mathsf{HK}}(A) = r \cdot e_{\mathsf{HK}}\big(\overline{A} \cdot \mathbb{F}[x, y, z], \mathbb{F}[x, y, z]\big),$$

where r is the degree of the field extension $\mathit{FF}(A) \subset \mathbb{F}(x, y, z)$. That the degree of this field extension is 4 may be seen by enlarging $\mathbb{F}$ to contain a square root $\boldsymbol{i}$ of -1. This does not change the degree of the resulting field extension. Then apply the automorphism α of $\mathbb{F}[x, y, z]$ given by sending z to $\boldsymbol{i} \cdot z$ and leaving x and y fixed. The algebra A gets mapped to $\alpha(A)$ which is generated by $z^2 + x^2$, $z^2 + y^2$, xy, xz, $yz \in \mathbb{F}[x, y, z]$. The element $z \in \mathbb{F}(x, y, z)$ is integral over $\alpha(A)$ with minimal polynomial

$$t^4 + (x^2 + y^2)t^2 + (xy)^2 \in \alpha(A)[t].$$

If we adjoin z to $\mathit{FF}(\alpha(A))$ then the resulting field extension also contains $y = yz/z$ and $x = xz/z$ so coincides with $\mathbb{F}(x, y, z)$. Hence $r = 4$ and therefore $e_{\mathsf{HK}}(A) = e_{\mathsf{HK}}(\alpha(A)) = 5/4$.

If A is a commutative graded connected Noetherian algebra of Krull dimension $n = \dim(A)$ over the field $\mathbb{F}$ then its Poincaré series has integral coefficients and a pole of order n at $t = 1$. Therefore the rational number

$$(1 - t)^n \cdot P(A, t)\big|_{t=1} = \deg(A)$$

is well defined; it is called the *degree* of A (see [Smith 1995, Section 5.5] for a discussion of this invariant and its occurrence in invariant theory in particular). For a finite extension $A \hookrightarrow B$ of commutative graded connected Noetherian integral domains, the ratio of their degrees is the degree[14] of the corresponding field extension of their respective fields of fractions. Specifically, if $\mathbb{K}$ is the field of fractions of A and $\mathbb{L}$ the field of fractions of B then $\deg(B) = |\mathbb{L} : \mathbb{K}| \cdot \deg(A)$ (see, e.g., [Smith 1995, Proposition 5.5.2]). So using the notion of the degree of an algebra allows us to rephrase the change of rings theorem for integral domains in a more symmetric form.

Corollary 2.4. *Let $A \hookrightarrow B$ be a finite extension of Noetherian integral domains over the field $\mathbb{F}$ of characteristic $p \neq 0$ and $I \subset A$ a maximal primary ideal. Then*

$$e_{\mathsf{HK}}(I \cdot B, B) \cdot \deg(A) = e_{\mathsf{HK}}(I, A) \cdot \deg(B).$$

In this next example it is easier to compute the degree of the subalgebra of $\mathbb{F}[V]$ being investigated rather than the degree of the field extension.

Example 6. Consider the subalgebra A in the polynomial algebra $\mathbb{F}[x, y, z]$ generated by the four forms x^2, xy, y^2, z^4. It is not hard to see that x^2, y^2, z^4 is a system of parameters for A and that A is Cohen–Macaulay with basis $1, xy$ as an $\mathbb{F}[x^2, y^2, z^4]$-module. Hence the Poincaré series of A is

$$P(A, t) = \frac{1+t^2}{(1-t^2)^2} \cdot \frac{1}{1-t^4} = \frac{1}{(1-t^2)^3}$$

so for the degree we have $\deg(A) = 1/8$. Since the quotient of $\mathbb{F}[x, y, z]$ by the ideal I has dimension 12 we obtain from Corollary 2.4 that $e_{\mathsf{HK}}(A) = 12/8 = 3/2$. So, although the Poincaré series of A looks like that of a polynomial algebra on three elements of degree 2 the Hilbert–Kunz invariant tells it is not (see, e.g., [Kunz 1969]). This example is well known from invariant theory (see, e.g., [Stanley 1979]).

3. An Euler characteristic formula for the Hilbert–Kunz multiplicity

In a famous paper, D. Hilbert [1890] proved not only the finiteness of the number of generators of the ring of invariants of certain classical groups, but also of the number of relations between invariants, and relations between relations, etc. In modern terms (we follow the notations and terminology of [Smith 1995]), and formulated for finite groups, what he did was to choose a minimal resolution[15]

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{F}[V]_G \longrightarrow 0 \tag{3-1}$$

of the ring[16] of coinvariants $\mathbb{F}[V]_G$ of a representation $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$, of a finite group G over the field $\mathbb{F}$, regarded as an $\mathbb{F}[V]$-module. Then, by the Euler

[14] This multiple use of *degree* hopefully will cause no confusion.

[15] Which he first had to prove existed!

[16] By definition the coinvariant algebra $\mathbb{F}[V]_G$ is $\mathbb{F} \otimes_{\mathbb{F}[V]^G} \mathbb{F}[V]$.

characteristic property of the Poincaré series one has, as in Section 1,

$$P(\mathbb{F}[V]_G,t)=\sum_{i=0}^{n}(-1)^i P(F_i,t). \tag{3-2}$$

From the definition of a minimal resolution one finds (loc. cit.)

$$F_i \cong \mathbb{F}[V]\otimes \mathrm{Tor}_i^{\mathbb{F}[V]}(\mathbb{F}[V]_G,\mathbb{F}) \quad \text{for } i=0,\dots,n, \tag{3-3}$$

so putting (3-2) and (3-3) together leads to the formula

$$\begin{aligned} P(\mathbb{F}[V]_G,t) &= P(\mathbb{F}[V],t)\cdot\sum_{i=0}^{n}(-1)^i P(\mathrm{Tor}_i^{\mathbb{F}[V]}(\mathbb{F}[V]_G,\mathbb{F}),t)\\ &= \frac{1}{(1-t)^n}\cdot\sum_{i=0}^{n}(-1)^i P(\mathrm{Tor}_i^{\mathbb{F}[V]}(\mathbb{F}[V]_G,\mathbb{F}),t). \end{aligned} \tag{3-4}$$

The discussion in Section 1 allows us to reformulate this in the following more general terms for use in computing Hilbert–Kunz multiplicities. It is an analog for Hilbert–Kunz multiplicity of the formula of J.-P. Serre (see, e.g., [Serre 1965, Part V]) for the ordinary multiplicity.

Proposition 3.1. *Let $S=\mathbb{F}[x_1,\dots,x_n]$ be a graded polynomial algebra on generators with degrees $\deg(x_i)=d_i$ for $i=1,\dots,n$ over the field $\mathbb{F}$ of characteristic $p\neq 0$, $I\subset S$ an $\bar{S}$-primary ideal,[17] and $S/I=R$. Then*

$$e_{\mathsf{HK}}(I,S)=\left[\frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})}\cdot\sum_{i=0}^{n}(-1)^i P(\mathrm{Tor}_i^S(R,\mathbb{F}),t)\right]\Bigg|_{t=1}.$$

Proof. This follows Proposition 1.5, after accounting for the degrees of the variables, and Proposition 2.1. □

Although this formula seems pretty useless on the surface (after all, how is one to *compute* the Poincaré series of the various syzygy modules without really having so firm a grip on R that one knows its Poincaré series already?), there are several answers to this objection in the case of irreducible ideals $I\subset S=\mathbb{F}[x_1,\dots,x_n]$, because, in this case, the algebra $\mathrm{Tor}_*^S(R,\mathbb{F})_*$ carries the additional structure of a Poincaré duality algebra (see, e.g., [Meyer and Smith 2005, Part I] for a discussion of the relation between Poincaré duality quotients of graded Gorenstein algebras and irreducible ideals). Specifically, the modules (see [Avramov and Golod 1971] for the local case) $\mathrm{Tor}_*^S(R,\mathbb{F})_*$ form a bigraded algebra, which, if we regrade it by total degree, are, apart from the cosmetic difference[18] of being graded commutative

[17] There is no loss of generality in assuming that I is $\bar{S}$-primary since $e_{\mathsf{HK}}(I,S)=0$ if it is not.

[18] For an algebraic topologist in fact this is not a difference: it is with these commutation rules that Poincaré duality algebras arise as the cohomology of manifolds.

instead of commutative, a Poincaré duality algebra. Moreover, R itself is a Poincaré duality algebra (loc. cit.), and if R has formal dimension d (which means the socle of R is in homogeneous degree d) then the formal dimension of the singly graded algebra $\mathrm{Tor}^S(R, \mathbb{F})$ is $n+d$, where $n = \dim_{\mathbb{F}}(V)$. Therefore the ordinary Poincaré series of this singly graded torsion algebra, to wit, the formal series

$$\sum_{i=0}^{n} P\big(\mathrm{Tor}_i^S(R, \mathbb{F}), t\big),$$

must be a palindromic polynomial of degree $n+d$, i.e., if

$$\sum_{i=0}^{n} P\big(\mathrm{Tor}_i^S(R, \mathbb{F}), t\big) = a_0 + a_1 t + \cdots + a_{n+d} t^{n+d}, \quad a_0, \ldots, a_{n+d} \in \mathbb{N}_0,$$

then $a_i = a_{n+d-i}$ for all $i = 0, \ldots, \lfloor (n+d)/2 \rfloor$. This means that in case $n = 3$ we can actually write down a closed formula for the Hilbert–Kunz multiplicity of a maximal primary irreducible ideal $I \subset \mathbb{F}[x, y, z]$ knowing only the degrees of the generators of I and the socle degree of the quotient $R = \mathbb{F}[x, y, z]/I$.

Proposition 3.2. *Let $\mathbb{F}$ be a field of characteristic $p \neq 0$, $I \subset \mathbb{F}[x, y, z]$ be a maximal primary irreducible ideal in the polynomial algebra on three generators x, y, z of degrees a, b, c, and set d equal to the degree of the socle of the quotient algebra $R = \mathbb{F}[x, y, z]/I$. Let the degrees of a minimal set of ideal generators for I be $d_1, \ldots, d_k$. Then $e_{\mathsf{HK}}(I, \mathbb{F}[x, y, z])$ is equal to*

$$\left[\frac{1}{(1-t^a)\cdot(1-t^b)\cdot(1-t^c)}\cdot\Big[1-(t^{d_1}+\cdots+t^{d_k})+(t^{3+d-d_1}+\cdots+t^{3+d-d_k})-t^{3+d}\Big]\right]\Bigg|_{t=1}.$$

Proof. Write S for $\mathbb{F}[x, y, z]$. Let $(\mathcal{K}, \partial)$ be the Koszul resolution for $\mathbb{F}$ regarded as an S-module which has

$$\mathcal{K} = S \otimes E(u, v, w), \quad \begin{cases} \partial(f \otimes 0) = 0 & \text{for } f \in S \quad \text{and} \\ \partial(1 \otimes u) = x, & \partial(1 \otimes v) = y, \quad \partial(1 \otimes w) = z. \end{cases}$$

So there are no boundaries of homological degree 3 and $f \otimes u \cdot v \cdot w$ is a cycle if and only if

$$0 = \partial(f \otimes u \cdot v \cdot w) = f \cdot x \otimes v \cdot w + f \cdot y \otimes u \cdot w + f \cdot w \otimes u \cdot v.$$

Since the elements vw, uw, uv are linearly independent in $E(u, v, w)$ this is the case if and only if $f \cdot x = f \cdot y = f \cdot w = 0$, and therefore $f \in \mathrm{soc}(R)$. Hence

$$\mathrm{Tor}_3^S(R, \mathbb{F}) = \mathrm{soc}(R) \otimes uvw$$

and is 1-dimensional concentrated in degree $3 + \deg(\mathrm{soc}(R))$ just as it should be.

There is a short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow R \longrightarrow 0$ of S-modules which leads to the long exact sequence of torsion modules

$$0 = \mathrm{Tor}_1^S(S, \mathbb{F}) \longrightarrow \mathrm{Tor}_1^S(R, \mathbb{F}) \xrightarrow{\partial} I \otimes_S \mathbb{F} \longrightarrow S \otimes_S \mathbb{F} \xrightarrow{\pi} R \otimes_S \mathbb{F} \longrightarrow 0.$$

Since $S \otimes_S \mathbb{F} \cong \mathbb{F} \cong R \otimes_S \mathbb{F}$ the map π is an isomorphism and hence so is ∂. This tells us that

$$P(\mathrm{Tor}_1^S(R, \mathbb{F}), t) = t^{d_1} + \cdots + t^{d_k}$$

and therefore the Euler characteristic polynomial for the torsion product is

$$P(\mathrm{Tor}^S(R, \mathbb{F}), t) = 1 - (t^{d_1} + \cdots + t^{d_k}) + (t^{3+d-d_1} + \cdots + t^{3+d-d_k}) - t^{3+d},$$

as follows from the preceding discussion. The final formula is then a consequence of Proposition 3.1. □

A maximal primary irreducible ideal I in a polynomial algebra $\mathbb{F}[V]$ would more often than not be specified by giving its Macaulay dual μ_I in the sense of [Macaulay 1916, Part IV] (see also [Meyer and Smith 2005, Parts I and VI]). The element μ_I may be viewed in several different ways: first, as an element of the divided polynomial algebra $\Gamma(V)$ with degree s equal to the formal dimension of the quotient algebra $\mathbb{F}[V]/I$; equivalently, as a form of degree $-s$ in the inverse polynomial algebra associated with $\mathbb{F}[V]$ and a basis for the space of linear forms V^*; or, as an element in the local cohomology module $H^n_{\mathfrak{m}}(\mathbb{F}[V])_{-s-n}$, where $n = \dim_{\mathbb{F}}(V)$ (see, e.g., [Greenlees and Smith 2008; Smith 2013]). So the degree of the socle would be a priori known. In the case of $\mathbb{F}_2[x, y, z]$ and socle degree 3 for the quotient algebra there are up to automorphism twenty-one possible choices for the Macaulay dual, and the corresponding ideals and quotient algebras have been classified and listed in [Smith and Stong 2010]. For all of these Proposition 3.2 gives the Hilbert–Kunz multiplicity.[19] Here is an example.

Example 7 [Smith and Stong 2010, Section 5, Orbit 10]. Consider the inverse ternary cubic form[20]

$$\theta_{10} = x^{-3} + y^{-3} + x^{-1}y^{-2} + x^{-1}y^{-1}z^{-1} \in \mathbb{F}_2\left[x^{-1}, y^{-1}, z^{-1}\right],$$

which defines a maximal primary ideal $I(\theta_{10}) \subset \mathbb{F}[x, y, z]$. Using the method of catalecticant matrices due to J. J. Sylvester (see, e.g., [Meyer and Smith 2005,

[19] It would be interesting to know how to express the Hilbert–Kunz multiplicity of these examples in terms of the invariants from [Smith and Stong 2010, Section 6] used to separate them.

[20] This is the classical terminology for a form in three variables (cubic) of degree three. Since we are dealing with variables of degree -1 (inverse) this means that θ_{10} is a form in three inverse variables, here x^{-1}, y^{-1}, and z^{-1}, and has degree -3.

Section 6.2]), one finds that this ideal is generated by the three forms x^2+xz+y^2, x^2+yz, z^2 as an ideal of $S=\mathbb{F}[x, y, z]$. If one examines the catalecticant matrix

$\boldsymbol{cat}_{\theta_{10}}(1,2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	1
y	0	1	0	1	1	0
z	0	0	0	1	0	0

representing this orbit, one can see that $z^2=0$, and that the algebra $H(\theta_{10})=\mathbb{F}[x, y, z]/I(\theta_{10})$ corresponding to this matrix can be visualized as pictured in Diagram 1. As in [Meyer and Smith 2005] the entries on a given horizontal line in the diagram are a basis for the homogeneous component of H of degree equal to the number of lines above the unit 1 of the algebra. So one reads off that the dimension of H is 8 and hence $e_{\mathbf{HK}}(I(\theta_{10}), S)=8$.

$$\begin{array}{ccc} & \overset{\bullet}{xyz} & \\ \overset{\bullet}{xy} & \overset{\bullet}{xz} & \overset{\bullet}{yz} \\ \overset{\bullet}{x} & \overset{\bullet}{y} & \overset{\bullet}{z} \\ & \overset{\bullet}{1} & \end{array}$$

Diagram 1. The algebra $H(\theta_{10})$.

From Diagram 1 one finds the relations

$$x^2=yz, \quad y^2=xz+yz.$$

This shows that $H(\theta_{10})$ is a free module over the subalgebra $\mathbb{F}[z]/(z^2)\subset H(\theta_{10})$ with basis the four elements $1, x, y, xy$. So $H(\theta_{10})$ looks like the $\mathbb{F}_2$-cohomology of the total space M^3 of a fibering $S^1\times S^1 \hookrightarrow M^3 \downarrow S^1$ which is totally nonhomologous to zero. Such a fibered manifold is constructed in [Smith and Stong 2010, Section 7].

The situation for surface algebras $H=\mathbb{F}[x, y, z]/I$, where I is a maximal primary irreducible ideal and the socle degree of H is 2 is somewhat simpler. Here is how this goes.

Example 8. Consider a nonzero inverse quadratic form μ in three inverse variables in $\mathbb{F}[x^{-1}, y^{-1}, z^{-1}]$ which defines a maximal primary ideal $I(\mu)\subset\mathbb{F}[x, y, z]=S$ with quotient algebra $R(\mu)=S/I(\mu)$ a Poincaré duality algebra of formal dimension 2; a surface algebra in the language of [Smith and Stong 2010]. Making use of Proposition 3.2 and [Eisenbud 1995, Exercise 21.6] allows us to construct the following table:

rank(μ)	$P(\mathrm{Tor}^S(R(\mu), t)$	$e_{\mathsf{HK}}(I(\mu))$
1	$1-2t+t^2-t^3+2t^4-t^5$	3
2	$1-t-t^2+t^3+t^4-t^5$	4
3	$1-5t+5t^2-t^5$	5

Table 1. Hilbert–Kunz multiplicity of ternary surface algebras.

In [Smith and Stong 2010, Section 2] we showed that any surface algebra over $\mathbb{F}_2$ can be written as a connected sum of the two basic examples: $\mathbb{F}_2[x, y]/(x^2, y^2)$, with Macaulay dual form of rank 2, and $\mathbb{F}_2[z]/(z^3)$, with Macaulay dual form of rank 1, so with the aide of the above table and Proposition 4.1 one has a formula for the Hilbert–Kunz multiplicity of any surface algebra at least over $\mathbb{F}_2$. See the discussion of connected sums in Section 4 and Examples 10, 11 there.

In fact, already the two-variable case of Proposition 3.1 is interesting, as we explain next. We use its proof to provide a short and simple proof of the result of F. S. Macaulay that an irreducible ideal in a polynomial algebra in two variables is generated by a regular sequence (for a different modern proof, see, e.g., [Vasconcelos 1967]).

Theorem 3.3 [Macaulay 1904]. *Let $\mathbb{F}$ be a field and $I \subset \mathbb{F}[x, y] = S$ an ideal such that $R = S/I$ is a Poincaré duality algebra. Then I is generated by a regular sequence.*

Proof. To evaluate the formula in Proposition 3.1 in this case we recycle the proof of Proposition 3.2 to compute $\mathrm{Tor}_i^S(R, \mathbb{F})$ for $i = 1$ and 2. This results in the formula

$$P(R, t) = \frac{1}{(1-t^a)\cdot(1-t^b)}\cdot\left[1-(t_1^k+\cdots+t^{k_r})+t^{2+d}\right],$$

where $\deg(x) = a$, $\deg(y) = b$, $d = a+b$, $k_1, \ldots, k_r$ are the degrees of a minimal set of generators for I, and $d = \text{f–dim}(R)$, i.e., the socle degree of R. The left hand side of this equality is a polynomial so the right hand side must be one also. This says that

$$(1-t^a)\cdot(1-t^b) = (1-t)^2\cdot(1+t+\cdots+t^{a-1})\cdot(1+t+\cdots+t^{b-1})$$

must divide

$$p(t) = 1-(t^{k_1}+\cdots+t^{k_r})+t^{2+d} \tag{3-5}$$

so $p(1) = 0$. Evaluating $p(1)$ from the formula (3-5) and equating the result to zero gives $0 = p(1) = 2-r$, so $r = 2$ and I is generated by two elements f, h which must then be a system of parameters since R is totally finite. Since S is Cohen–Macaulay it follows that $f, h \in S = \mathbb{F}[x, y]$ is a regular sequence. □

We can change viewpoint and replace the ideal I in the statement of Proposition 3.1 with a subalgebra A such as an algebra of invariants. The reformulated result takes the following form. An illustrative example is given in Example 9.

Proposition 3.4. *Let $S = \mathbb{F}[x_1, \dots, x_n]$ be a graded polynomial algebra on generators with degrees $\deg(x_i) = d_i$ for $i = 1, \dots, n$ over the field $\mathbb{F}$ of characteristic $p \neq 0$, $A \subset S$ a subalgebra making S into a finitely generated A-module, and set*[21] *$R = S /\!/ A \cong S \otimes_A \mathbb{F}$. Then*

$$e_{\mathsf{HK}}(A) = \deg(A) \cdot \left[\frac{1}{(1-t^{d_1}) \cdots (1-t^{d_n})} \cdot \sum_{i=0}^{n} (-1)^i P(\mathrm{Tor}_i^S(R, \mathbb{F}), t) \right] \Bigg|_{t=1}.$$

Proof. This follows from Theorem 2.3, Proposition 1.5 after accounting for the degrees of the variables, and Proposition 2.1. □

4. Further applications and examples

In this section we collect some examples of computations of Hilbert–Kunz invariants to illustrate the behavior of these in special circumstances. We begin with the possibility that there is an integral form of the algebra being studied. Then one can ask if, and if so how, these invariants change with the characteristic. Rings of invariants of permutation groups are natural candidates in this context. The following example provides such a case where there seems to be a connection with F-rationality (see, e.g., [Glassbrenner 1995; Singh 1998; Smith 2004]).

Example 9. Consider the ring of invariants $\mathbb{F}[z_1, \dots, z_n]^{A_n}$ of the alternating group A_n acting by means of its tautological permutation representation on the variables. Denote by $e_1, \dots, e_n \in \mathbb{F}[z_1, \dots, z_n]$ the elementary symmetric polynomials in $z_1, \dots, z_n$. These are invariants of the full symmetric group Σ_n and hence also of its alternating subgroup, so they belong to $\mathbb{F}[z_1, \dots, z_n]^{A_n}$. If the characteristic of $\mathbb{F}$ is not 2 and we restrict the permutation representation of Σ_n to the alternating subgroup A_n, then, as is also well known, the ring of invariants $\mathbb{F}[z_1, \dots, z_n]^{A_n}$ is a complete intersection generated by $e_1, \dots, e_n$ and the discriminant

$$\Delta_n = \prod_{1 \le i < j \le n} (z_i - z_j) = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \cdot z_{\sigma(1)}^0 z_{\sigma(2)}^1 \cdots z_{\sigma(n)}^{n-1},$$

the square Δ_n^2 being a polynomial in $e_1, \dots, e_n$ given by the resultant of φ_n and φ_n' (see, e.g., [Smith 1995, Section 1.3; Glassbrenner 1995, Section 12]), where

$$\varphi_n(t) = e_n + e_{n-1} \cdot t + \cdots + e_1 \cdot t^{n-1} + t^n = \prod_{i=1}^{n} (t + z_i) \in \mathbb{F}[z_1, \dots, z_n][t].$$

[21] Recall that $S /\!/ A$ is defined to be $S \otimes_A \mathbb{F}$ and is the categorical cokernel of the map including A into S in the category of commutative graded connected algebras over the field $\mathbb{F}$.

Less well known[22] would appear to be the invariants in characteristic 2. If we set

$$\mathfrak{S}_n = \sum_{\sigma \in A_n} z^0_{\sigma(1)} z^1_{\sigma(2)} \cdots z^{n-1}_{\sigma(n)},$$

then *regardless of the characteristic*, $\mathbb{F}[z_1, \ldots, z_n]^{A_n}$ is a hypersurface algebra (hence Gorenstein) generated by $e_1, \ldots, e_n$ and $\mathfrak{S}_n$, the square $\mathfrak{S}_n^2$ being a polynomial in $e_1, \ldots, e_n$ (see, e.g., [Smith 1995, Theorem 1.3.5]).

D. Glassbrenner [1995] discovered that the Hilbert ideals[23] $\mathfrak{h}(A_n)$ and $\mathfrak{h}(\Sigma_n)$ coincide if the characteristic p of the field $\mathbb{F}$ divides $\binom{n}{2}$. This was extended to all odd $p \leq n$ in [Singh 1998] and all $p \leq n$ in [Smith 2004]. Specifically, one has

$$\mathfrak{h}(A_n) = (e_1, \ldots, e_n) = \mathfrak{h}(\Sigma_n) \quad \Longleftrightarrow \quad p \leq n.$$

This being the case, we get from Proposition 3.4 the following formulae for the Hilbert–Kunz multiplicity of the algebra $\mathbb{F}[z_1, \ldots, z_n]^{A_n}$ as a function of the characteristic of the ground field $\mathbb{F}$ (see also [Brenner 2010] for a more complete discussion of Hilbert–Kunz multiplicities of algebras of invariants):

$$e_{\mathsf{HK}}(\mathbb{F}[z_1, \ldots, z_n]^{A_n}) = \begin{cases} \dfrac{n!}{(1/2)\cdot n!} = 2 & \text{if } p \leq n, \\[2ex] \dfrac{n!-1}{(1/2)\cdot n!} = 2 - \dfrac{2}{n!} & \text{otherwise.} \end{cases}$$

This follows from the discussion of this example in [Smith 2004], in particular the computation of a Macaulay dual for the Hilbert ideal $\mathfrak{h}(A_n)$, which shows that $\mathbb{F}[z_1, \ldots, z_n]_{A_n}$ is the algebra $\mathbb{F}[z_1, \ldots, z_n]_{\Sigma_n}$ with the socle removed if $p > n$, and that the degree of the algebra $\mathbb{F}[z_1, \ldots, z_n]^{A_n}$ is $(1/2) \cdot n!$ independent of the characteristic of $\mathbb{F}$ (see, e.g., [Smith 1995, Theorem 5.5]).

Remark. If one lets $n \uparrow \infty$ in these formulae one gets 2 in all cases, i.e., independent of p. Does this have any significance? Can it be explained by some integral analog for integral alternating invariants of the ring of integral symmetric polynomials in infinitely many variables?

A standard way to study ideals, or even to define special properties for them, is to examine the corresponding quotient algebra. In [Smith and Stong 2010; 2011] we studied a natural construction coming from algebraic topology on Poincaré duality algebras called the *connected sum*.[24] If R' and R'' are Poincaré duality algebras over the field $\mathbb{F}$ of the same formal dimension d then their connected sum $R' \# R''$ is

[22]This fact gets *rediscovered* every couple of years and published circa once a decade.

[23]If $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a group over the field $\mathbb{F}$ then the *Hilbert ideal* is the ideal in $\mathbb{F}[z_1, \ldots, z_n]$ generated by all G-invariant forms of strictly positive degree.

[24]This construction seems much more natural on the quotients than on the ideals defining them.

defined by identifying in their direct sum the two units and fundamental classes,[25] so by the requirements

$$(R' \# R'')_i = \begin{cases} \mathbb{F} \cdot 1 & \text{if } i = 0, \\ R'_i \oplus R''_i & \text{if } 0 < i < d, \text{ and} \\ \mathbb{F} \cdot [R' \# R''] & \text{if } i = d, \end{cases}$$

where $[R'] \in R'$ and $[R''] \in R''$ are chosen fundamental classes. Put another way, if $\mathbb{S}(d)$ denotes the Poincaré duality algebra[26] with $\mathbb{F} \cdot 1$ in degree 0 and $[\mathbb{S}] \cdot \mathbb{F}$ in degree d with all other homogeneous degrees being 0, then for any Poincaré duality algebra R of formal dimension d with fundamental class $[R]$ there is a natural map $\tau : \mathbb{S}(d) \longrightarrow R$ sending unit to unit and fundamental class to fundamental class. The connected sum is defined by requiring that

$$\begin{tikzcd} \mathbb{S}(d) \arrow[r, "\tau"] \arrow[d] \arrow[dr, phantom, "\ulcorner\bullet"] & R' \arrow[d] \\ R'' \arrow[r, "\tau"] & R' \# R'' \end{tikzcd}$$

be a pushout diagram.

If $R' = S'/I'$ and $R'' = S''/I''$ where S' and S'' are standard graded polynomial algebras so $I' \subset S'$ and $I'' \subset S''$ are maximal primary irreducible ideals, then $R' \# R''$ is of the form $(S' \otimes S'')/I$ for a maximal primary irreducible ideal $I' \# I'' = I \subset S = S' \otimes S''$ in the standard graded polynomial algebra S. From the colength formula in Proposition 2.1 we then get the following formula for the Hilbert–Kunz multiplicity.

Proposition 4.1. *Let $S' = \mathbb{F}[x'_1, \ldots, x'_{n'}]$ and $S'' = \mathbb{F}[x''_1, \ldots, x''_{n''}]$ be standard graded polynomial algebras over the field $\mathbb{F}$ of characteristic $p \neq 0$, and let $I' \subset S'$ and $I'' \subset S''$ be maximal primary ideals with Poincaré duality quotients $R' = S'/I'$ and $R'' = S''/I''$ of the same formal dimension $d > 0$. If $I \subset S' \otimes S'' = \mathbb{F}[x_1, \ldots, x_{n'}, x''_1, \ldots, x''_{n''}]$ defines the Poincaré duality quotient algebra $R' \# R''$ as a quotient of S then $e_{\mathsf{HK}}(I, S) = e_{\mathsf{HK}}(I', S) + e_{\mathsf{HK}}(I'', S'') - 2$.*

The case $d = 0$ of the previous result is trivial because in this case $R' = \mathbb{F} = R''$ so I is the maximal ideal and $e_{\mathsf{HK}}(I, S) = 1$. The result for more than two parts in the connected sum is easily extended by induction to yield the formula

$$e_{\mathsf{HK}}(\underbrace{I(1) \# \cdots \# I(k)}_{\longleftarrow k \longrightarrow}, S) = e_{\mathsf{HK}}(I(1), (S(1)) + \cdots + e_{\mathsf{HK}}(I(k), S(k)) - 2 \cdot (k-1)$$

for the Hilbert–Kunz multiplicity of the ideal defining the connected sum of k irreducible ideals in k standard graded polynomial algebras.

[25] A *fundamental class* is a nonzero element of the socle.

[26] This is nothing but $H^*(S^d; \mathbb{F})$.

Corollary 4.2. *Let $I \subset \mathbb{F}_2[z_1, \dots, z_n] = S$ be a maximal primary irreducible ideal with quotient algebra $H = S/I$ a surface algebra, i.e., the socle degree of H is 2. If $q_I \in \mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$ is an inverse quadratic form that is a Macaulay dual for I then $e_{\mathsf{HK}}(I, S) = 2 + \operatorname{rank}(q_I)$.*

Proof. By [Smith and Stong 2010, Corollary 2.6] any surface algebra over $\mathbb{F}_2$ is a connected sum of the algebras with ranks 1 or 2 listed in Table 1. The result then follows from Proposition 4.1 by induction on the number of terms in the connected sum. □

As an example of Proposition 4.1 we return to Example 2 from Section 2.

Example 10. The connected sum

$$\big(\mathbb{F}[x]/(x^3)\big) \# \big(\mathbb{F}[y]/(y^3)\big) \# \big(\mathbb{F}[z]/(z^3)\big)$$

is a Poincaré duality quotient of $\mathbb{F}[x, y, z]$ with formal dimension 3. Its defining ideal is the ideal M of Example 2. Therefore we find from Proposition 4.1 for its Hilbert–Kunz multiplicity $e_{\mathsf{HK}}(M, \mathbb{F}[x, y, z]) = 3+3+3-2\cdot 2 = 9-4 = 5$, just as computed previously.

In [Smith and Stong 2010] we provided several criteria to check if a Poincaré duality algebra is in fact a connected sum, one of which we use in the next example.

Example 11. Consider the ideal $I \subset \mathbb{F}[x, y, z] = S$ generated by the five quadratic forms

$$x^2,\ y^2,\ xz,\ yz,\ z^2 - xy.$$

It is not hard to see that Lemma 1.1 of [Smith and Stong 2010] applies to the quotient algebra $R = S/I$ with $H'_1 = \operatorname{Span}_{\mathbb{F}}\{x, y\}$ and $H''_1 = \operatorname{Span}_{\mathbb{F}}\{z\}$ so[27] $R = \mathbb{F}[x, y]/(x^2, y^2) \# \mathbb{F}[z]/(z^3)$. Therefore Proposition 4.1 tells us that $e_{\mathsf{HK}}(I, S) = 4+3-2 = 5$.

Remark. Let $S' = \mathbb{F}[x'_1, \dots, x'_{n'}]$ and $S'' = \mathbb{F}[x''_1, \dots, x''_{n''}]$ be standard graded polynomial algebras over the field $\mathbb{F}$ of characteristic $p \neq 0$ and $I' \subset S'$, $I'' \subset S''$ maximal primary ideals with Poincaré duality quotients $R' = S'/I'$ and $R'' = S''/I''$ of the same formal dimension $d > 0$. If $I \subset S' \otimes S'' = \mathbb{F}[x_1, \dots, x_{n'}, x''_1, \dots, x''_{n''}]$ defines the Poincaré duality algebra $R = R' \# R''$ as a quotient of S, then the theorem of [Avramov and Golod 1971] tells us that the three torsion products

$$\operatorname{Tor}^{S'}(R', \mathbb{F}), \quad \operatorname{Tor}^{S''}(R'', \mathbb{F}), \quad \text{and} \quad \operatorname{Tor}^{S}(R, \mathbb{F})$$

[27]Topologists should recognize this as $H^*((S^2 \times S^2)\#\mathbb{CP}(2); \mathbb{F})$ after halving the grading degrees; algebraists as the ideal with Macaulay dual $z^{-2} + x^{-1}y^{-1} \in \mathbb{F}[x^{-1}, y^{-1}, z^{-1}]$ (see, e.g., [Eisenbud 1995, Example 21.7]). In characteristic 2 the algebras in this and the previous example are isomorphic; see, e.g., [Smith and Stong 2010, Lemma 2.4].

are Poincaré duality algebras, the first of dimension $d+n'$, the second of dimension $d+n''$ and the third of dimension $d+n$, where $n=n'+n''$. So all three of the algebras

$$\mathrm{Tor}^{S'}(R', \mathbb{F}) \otimes E'', \quad \mathrm{Tor}^{S''}(R'', \mathbb{F}) \otimes E', \quad \text{and} \quad \mathrm{Tor}^{S}(R, \mathbb{F})$$

are Poincaré duality algebras of formal dimension $d+n$, where E'', E' are exterior the algebras $E'' = \mathrm{Tor}^{S''}(\mathbb{F}, \mathbb{F}) = E(u''_1, \dots, u''_{n''})$ and $E' = \mathrm{Tor}^{S'}(\mathbb{F}, \mathbb{F}) = E(u'_1, \dots, u'_{n'})$. If we regard R' and R'' as quotients of $S = S' \otimes S''$ by means of the maps

$$R' \cong R' \otimes \mathbb{F} \xleftarrow{\pi' \otimes \epsilon''} S' \otimes S'' \xrightarrow{\epsilon' \otimes \pi''} \mathbb{F} \otimes R'' \cong R'',$$

where ϵ', ϵ'' are the augmentation maps of S' and S'' respectively, and π' and π'' the quotient maps from S' and S'' onto R' and R'' respectively, then

$$\mathrm{Tor}^{S'}(R', \mathbb{F}) \otimes E'' \cong \mathrm{Tor}^{S}(R', \mathbb{F}),$$
$$\mathrm{Tor}^{S''}(R'', \mathbb{F}) \otimes E' \cong \mathrm{Tor}^{S}(R'', \mathbb{F}),$$

so all three of the torsion products

$$\mathrm{Tor}^{S}(R', \mathbb{F}), \quad \mathrm{Tor}^{S}(R'', \mathbb{F}), \quad \text{and} \quad \mathrm{Tor}^{S}(R, \mathbb{F})$$

become Poincaré duality algebras of formal dimension $d+n$. Moreover there is a map

$$\eta : \mathrm{Tor}^{S}(R', \mathbb{F}) \# \mathrm{Tor}^{S}(R'', \mathbb{F}) \longrightarrow \mathrm{Tor}^{S}(R' \# R'', \mathbb{F})$$

of degree one basically induced by forming the connected sum of the two maps

$$\mathrm{Tor}^{S}(R', \mathbb{F}) \longrightarrow \mathrm{Tor}^{S}(R' \# R'', \mathbb{F}) \longleftarrow \mathrm{Tor}^{S}(R'', \mathbb{F}).$$

The map η being of degree one must be a monomorphism (see, e.g., the proof, not the statement, of Lemma I.3.1 in [Meyer and Smith 2005]). It does not seem to be an isomorphism for the case of the connected sum $\mathbb{F}[x]/(x^3) \# \mathbb{F}[y]/(y^3)$: so what can we say about it?

5. Reciprocity formulae for linked ideals

Recall that two ideals I, $J \subset A$ in a commutative graded connected algebra A over the field $\mathbb{F}$ are said to be *directly linked* if there is a regular sequence $f_1, \dots, f_m \in \overline{A}$ such that

$$I = \big((f_1, \dots, f_m) \underset{A}{:} J\big) \quad \text{and} \quad J = \big((f_1, \dots, f_m) \underset{A}{:} I\big).$$

In this case one also says that I and J are linked over the complete intersection ideal $\mathfrak{f} = (f_1, \dots, f_m)$ in A. If A is a Gorenstein algebra, then an ideal generated by a regular sequence of maximal length is irreducible (see, e.g., [Meyer and Smith 2005, Proposition I.1.4 and Lemma I.1.3]) and hence the Noether involution theorem [loc. cit., Theorem I.2.1] assures us that either one of these conditions implies the other.

The purpose of this section is to prove the following reciprocity formula for the Hilbert–Kunz multiplicity of a pair of directly linked maximal primary ideals in a polynomial algebra:

$$e_{\mathsf{HK}}(I, S) + e_{\mathsf{HK}}(J, S) = e_{\mathsf{HK}}(\mathfrak{f}, S). \tag{5-1}$$

Here $S = \mathbb{F}[x_1, \dots, x_n]$ is a polynomial algebra over the field $\mathbb{F}$, $\mathfrak{f} = (f_1, \dots, f_n) \subset S$ is an ideal generated by a regular sequence $f_1, \dots, f_n \in \overline{S}$ of maximal length, and $I \subset S$ is a maximal primary ideal containing $\mathfrak{f}$ with $J = (\mathfrak{f} :_A I)$ the directly linked ideal. Note that the right hand side of (5-1) may be evaluated by means of the colength formula to yield

$$e_{\mathsf{HK}}(\mathfrak{f}, S) = \dim_{\mathbb{F}}(S/\mathfrak{f}) = \frac{\prod_{i=1}^{n} \deg(f_i)}{\prod_{i=1}^{n} \deg(x_i)},$$

since S is a free module over the subalgebra $\mathbb{F}[f_1, \dots, f_n]$.

The plan for the proof of formula (5-1) is to use Proposition 2.1 and first prove the reciprocity formula

$$\dim_{\mathbb{F}}(S/I) + \dim_{\mathbb{F}}(S/J) = \dim_{\mathbb{F}}(S/\mathfrak{f}) \tag{5-2}$$

for the dimensions of the corresponding quotient algebras.[28] To do this we make use of some elementary homological *tic-toc-toe*. We begin with the following basic fact.[29]

Lemma 5.1. *Let A be a commutative graded connected algebra over the field $\mathbb{F}$, $f_1, \dots, f_n \in \overline{A}$, and M an A-module. If $f_1, \dots, f_n$ form a regular sequence on M, then on the category of $A/(f_1, \dots, f_n)$-modules there are natural equivalences of functors*

$$\operatorname{Ext}^i_A(-, M) \cong \begin{cases} \operatorname{Hom}_A(-, M/(f_1, \dots, f_n) \cdot M) & \text{if } i = n, \\ 0 & \text{for } i < n. \end{cases}$$

The proof of this lemma rests on the following observation.

Lemma 5.2. *Let A be a commutative graded connected algebra over the field $\mathbb{F}$ and M, N a pair of A-modules. If $\operatorname{Ann}_A(N)$ contains a regular element on M then* $\operatorname{Hom}_A(N, M) = 0$.

[28] Here, and throughout this section, we abuse notation and write $\dim_{\mathbb{F}}(X)$, where X is a totally finite graded vector space for the more correct $\dim_{\mathbb{F}}(\operatorname{Tot}(X))$.

[29] Versions of these lemmas go back at least to [Serre 1965] and can be found in [Bass 1963, Proposition 2.9] as well as [Bruns and Herzog 1993, Lemmas 1.2.3 and 1.2.4].

Proof. Let $\varphi : N \longrightarrow M$ be a homomorphism of A-modules and $u \in \mathrm{Ann}_A(N)$ a regular element on M. If $w \in N$ then $u \cdot \varphi(w) = \varphi(u \cdot w) = \varphi(0) = 0$ implies that $\varphi(w) = 0$ since u is a regular element on M; hence $\varphi = 0$ since w was arbitrary. □

Proof of Lemma 5.1. By induction on n. For $n = 0$ there is nothing to prove, so suppose that $n > 0$ and the result is established for $n-1$. Let N be an $A/(f_1, \dots, f_n)$-module. By the induction hypothesis,

$$\mathrm{Ext}_A^{n-1}(N, M) \cong \mathrm{Hom}_A(N, M/(f_1, \dots, f_{n-1}) \cdot M).$$

Since $f_n \in \mathrm{Ann}_A(N)$ is a regular element on the A-module $M/(f_1, \dots, f_{n-1}) \cdot M$, Lemma 5.2 tells us that $\mathrm{Hom}_A\big(N, M/(f_1, \dots, f_{n-1}) \cdot M\big) = 0$. Therefore of course $\mathrm{Ext}_A^{n-1}(N, M) = 0$ as well.

The element $f_n \in \overline{A}$ being regular on M means one has a short exact sequence of A-modules

$$0 \longrightarrow M \xrightarrow{\cdot f_n} M \longrightarrow M/f_n \cdot M \longrightarrow 0.$$

The long exact sequence for $\mathrm{Ext}^\bullet(N, —)$ associated to it yields[30]

$$0 = \mathrm{Ext}_A^{n-1}(N, M) \longrightarrow \mathrm{Ext}_A^{n-1}(N, M/f_n \cdot M) \\ \xrightarrow{\delta} \mathrm{Ext}_A^{n-1}(N, M) \xrightarrow{\cdot f_n} \mathrm{Ext}_A^{n}(N, M) \longrightarrow \cdots.$$

The map $\cdot f_n$ is induced by multiplication with f_n on M, but, $\mathrm{Ext}_A^\bullet(—, —)$ is a balanced functor so it is equally well induced by multiplication with f_n on N which is the zero map. Therefore $\delta : \mathrm{Ext}_A^{n-1}(N, M) \longrightarrow \mathrm{Ext}_A^{n}(N, M)$ is an isomorphism. The $n-1$ elements $f_1, \dots, f_{n-1}$ form a regular sequence on $M/f_n \cdot M$, so the induction hypothesis yields an isomorphism

$$\mathrm{Ext}_A^{n-1}(N, M/f_n \cdot M) \cong \mathrm{Hom}_A\big(N, M/(f_1, \dots, f_{n-1}, f_n) \cdot M\big),$$

completing the inductive proof that $\mathrm{Ext}_A^{n}(N, M) \cong \mathrm{Hom}_A(N, M/(f_1, \dots, f_n) \cdot M)$. To complete the inductive step note that for $k > 0$ we have $\mathrm{Ext}_A^{n-k}(N, M) \cong \mathrm{Hom}(N, M/(f_1, \dots, f_{n-k}) \cdot M)$ and f_n is a regular element on the quotient module $M/(f_1, \dots, f_{n-k}) \cdot M$. Since f_n annihilates N, Lemma 5.2 tells us that $\mathrm{Hom}(N, M/(f_1, \dots, f_{n-k}) \cdot M) = 0$, and hence $\mathrm{Ext}_A^{n-k}(N, M) = 0$ as well. □

There are a number of special cases of these lemmas that are relevant to the notion of linkage. We need to record these, but before we do so, note that, if $f_1, \dots, f_n \in \overline{A}$ is a regular sequence in the commutative graded connected algebra A over the field $\mathbb{F}$ and the ideal $\mathfrak{f} = (f_1, \dots, f_n)$ is irreducible, then the Noether involution theorem (see, e.g., [Meyer and Smith 2005, Theorem I.2.1]) implies that $J = (\mathfrak{f} :_A I)$ if and only if $I = (\mathfrak{f} :_A J)$. This is the case if $n = \dim(A)$ and A is Gorenstein. It will

[30] We will use a $\bullet$ to denote the indexing of derived functors rather than a $*$ to distinguish it from the internal grading index on these functors.

allow us under these circumstances to interchange the roles of I and J in the next result.

Lemma 5.3. *Let A be a commutative graded connected algebra over the field $\mathbb{F}$, $f_1, \dots, f_n \in \overline{A}$ a regular sequence, and $I \subset S$ a maximal primary ideal. Set $J = ((f_1, \dots, f_n) :_A I)$. Then $\operatorname{Ext}^n_A(S/I, A) \cong J/(f_1, \dots, f_n)$.*

Proof. In Lemma 5.1, put $— = A/I$ and $M = A$. The result is an isomorphism

$$\operatorname{Ext}^n_A(A/I, A) \cong \operatorname{Hom}_A(A/I, A/(f_1, \dots, f_n)).$$

Any element $\varphi \in \operatorname{Hom}_A(A/I, A/(f_1, \dots, f_n))$ is determined by $\varphi(1)$ from the requirement that it be an A-module homomorphism, viz., $\varphi(a) = \varphi(a \cdot 1) = a \cdot \varphi(1)$. In order that this formula define a map $A/I \longrightarrow A/(f_1, \dots, f_n)$ it is necessary and sufficient that $\varphi(1)$ annihilate the image of I in $A/(f_1, \dots, f_n)$. Note that

$$\begin{aligned}\varphi(1) \in \operatorname{Ann}_{A/(f_1,\dots,f_n)}\big(I/(f_1, \dots, f_n)\big) &= \big(0 \underset{A/(f_1,\dots,f_n)}{:} I/(f_1, \dots, f_n)\big)\\ &\cong \big((f_1, \dots, f_n) \underset{A}{:} I\big)/(f_1, \dots, f_n)\\ &\cong J/(f_1, \dots, f_n).\end{aligned}$$

Hence the map $\operatorname{Hom}_A(A/I, A/(f_1, \dots, f_n)) \longrightarrow J/(f_1, \dots, f_n)$ defined by sending an element $\varphi \in \operatorname{Hom}_A(A/I, A/(f_1, \dots, f_n))$ to $\varphi(1) \in J/(f_1, \dots, f_n)$ is an isomorphism, which combined with the isomorphism of Lemma 5.1, $\operatorname{Ext}^n_A(A/I, A) \cong \operatorname{Hom}_A(A/I, A/(f_1, \dots, f_n))$, yields the desired conclusion. □

Remark. As a special case of Lemma 5.3 we can put $I = (f_1, \dots, f_n)$ and conclude

$$\operatorname{Ext}^n_A(A/(f_1, \dots, f_n), A) \cong A/(f_1, \dots, f_n).$$

This will prove useful in the sequel.

In Lemma 5.3 the Noether involution theorem tells us that if the ideal $(f_1, \dots, f_n) \subset A$ is maximal primary and irreducible then we can interchange the roles of I and J. What is somewhat surprising is that we can also interchange the roles of A/I and $J/(f_1, \dots, f_n)$ if A is a polynomial algebra;[31] to wit:

Lemma 5.4. *Let $S = \mathbb{F}[x_1, \dots, x_n]$ be a graded polynomial algebra over the field $\mathbb{F}$ and $f_1, \dots, f_n \in \overline{S}$ a regular sequence (so the ideal $(f_1, \dots, f_n) \subset S$ is maximal primary and irreducible). Let $I \subset S$ be an ideal containing $f_1, \dots, f_n$ and $J = ((f_1, \dots, f_n) :_S I)$ the directly linked ideal. Then $\operatorname{Ext}^n_S(J/(f_1, \dots, f_n), S) \cong S/I$.*

Proof. Consider the short exact sequence of S-modules

$$0 \longrightarrow J/(f_1, \dots, f_n) \longrightarrow S/(f_1, \dots, f_n) \longrightarrow S/J \longrightarrow 0.$$

[31] Careful study of the proof shows it would be enough to suppose that the ideal $(f_1, \dots, f_n)$ is maximal primary and irreducible as well as $\operatorname{Ext}^{n+1}_S(S/J, S) = 0$.

Apply the functor $\mathrm{Ext}_S^n(-, S)$ to it. One gets a long exact sequence

$$\text{(5-3)}\quad \cdots \longleftarrow \mathrm{Ext}_S^n(J/(f_1, \ldots, f_n), S) \longleftarrow \mathrm{Ext}_S^n(S/(f_1, \ldots, f_n), S) \\ \longleftarrow \mathrm{Ext}_S^n(S/J, S) \longleftarrow \cdots.$$

By Lemma 5.3 and the Remark following it we find

$$\mathrm{Ext}_S^n(S/J, S) \cong I/(f_1, \ldots, f_n)$$

and

$$\mathrm{Ext}_S^n(S/(f_1, \ldots, f_n), S) \cong S/(f_1, \ldots, f_n).$$

If we put this into (5-3) we obtain the exact sequence

$$\cdots \longleftarrow \mathrm{Ext}_S^n(J/(f_1, \ldots, f_n), S) \longleftarrow S/(f_1, \ldots, f_n) \longleftarrow I/(f_1, \ldots, f_n) \longleftarrow \cdots.$$

The map $I/(f_1, \ldots, f_n) \longrightarrow S/(f_1, \ldots, f_n)$ is monic, and in addition the map $\mathrm{Ext}_S^n(S/(f_1, \ldots, f_n), S) \longrightarrow \mathrm{Ext}_S^n(J/(f_1, \ldots, f_n), S)$ in the exact sequence (5-3) is epic since its cokernel lies in $\mathrm{Ext}_S^{n+1}(S/J, S)$, which is zero because S has global dimension n. Therefore we have a short exact sequence

$$0 \longleftarrow \mathrm{Ext}_S^n(J/(f_1, \ldots, f_n), S) \longleftarrow S/(f_1, \ldots, f_n) \longleftarrow I/(f_1, \ldots, f_n) \longleftarrow 0$$

so $\mathrm{Ext}_S^n(J/(f_1, \ldots, f_n), S) \cong S/I$ as required. □

Again, Noether's involution theorem tells us we can interchange the roles of I and J in this lemma. In the remainder of this section we will use the isomorphism

$$S/J \cong \mathrm{Ext}_S^n(I/(f_1, \ldots, f_n), S)$$

to prove the formula (5-2), from which the formula (5-1) follows by Proposition 2.1. To do this we will construct a weakly minimal free resolution of $I/(f_1, \ldots, f_n)$, use (see Lemma 5.1) that $\mathrm{Ext}_S^i(I/(f_1, \ldots, f_n), S) = 0$ for $i \neq n$, and the Euler characteristic of an exact sequence is zero, as well as Lemma 1.3. We begin with a review of the mapping cone construction from homological algebra (see, e.g., [MacLane 1963, pp. 46–47]).

Recollection. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a map of chain complexes the *mapping cone* $\mathcal{C}(\varphi) = \mathcal{C}$ is the chain complex with chains $\mathcal{C}_i = \mathcal{B}_i \oplus \mathcal{A}_{i-1}$ for $i \in \mathbb{Z}$ and boundary ∂ maps defined by $\partial(b, a) = (\partial_B(b) + \varphi(a), \partial_A(a))$ where ∂_B, ∂_A are the boundary maps of the complexes $\mathcal{B}$ and $\mathcal{A}$, respectively.

Note that the mapping cone $\mathcal{C}$ of a chain map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ fits into a short exact sequence of complexes

$$0 \longrightarrow \mathcal{B} \xrightarrow{\iota_\varphi} \mathcal{C} \xrightarrow{\pi} \Sigma(\mathcal{A}) \longrightarrow 0,$$

where the map ι_φ is defined by $\iota_\varphi(b) = (b, 0)$. In the resulting long exact sequence in homology the boundary map $\partial : H_i(\Sigma(\mathcal{C})) \longrightarrow H_{i-1}(\mathcal{B})$ may be identified up to sign with the induced map $\varphi_* : H_{i-1}(\mathcal{C}) \longrightarrow H_{i-1}(\mathcal{B})$ (loc. cit.).

Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial algebra over the field $\mathbb{F}$, $f_1, \ldots, f_n \in \bar{S}$ a regular sequence, and $I \subset S$ a maximal primary ideal containing $f_1, \ldots, f_n$ with directly linked ideal $J = ((f_1, \ldots, f_n) :_S I)$. We next describe[32] how to construct (see, e.g., [Peskine and Szpiro 1974, Proposition 2.6; Martsinkovsky and Strooker 2004, Proposition 10]) a (weakly minimal) free resolution of $I/(f_1, \ldots, f_n)$ as S-module. Choose (weakly minimal) free resolutions of finite type, $\mathcal{F}$ of S/I and $\mathcal{K}$ of $S/(f_1, \ldots, f_n)$ (e.g., $\mathcal{K}$ could be the Koszul complex for $f_1, \ldots, f_n \in \bar{S}$) as S-modules. Let $\mathcal{C}$ be the mapping cone of a map of complexes $\varphi : \mathcal{K} \longrightarrow \mathcal{F}$ lifting the natural quotient map $S/(f_1, \ldots, f_n) \longrightarrow S/I$. Then $\mathcal{C}$ is a (weakly minimal) complex of free S-modules of finite type. We claim that apart from a degree shift it is a resolution of $I/(f_1, \ldots, f_n)$. To see this we examine the long exact homology sequence associated with the exact sequence of complexes

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \longrightarrow \Sigma(\mathcal{K}) \longrightarrow 0.$$

Since $\mathcal{F}$ and $\mathcal{K}$ are acyclic the only portion of this long exact sequence with nonzero terms looks as follows:

$$\begin{tikzcd} 0 \arrow[r] & H_1(\mathcal{C}) \arrow[r] & H_1(\Sigma(\mathcal{K})) \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"] & H_0(\mathcal{F}) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & H_0(\mathcal{C}) \arrow[r] & 0 \\ & & S/(f_1, \ldots, f_n) \arrow[r, "\pi"] & S/I, & & \end{tikzcd}$$

where π is the natural quotient map. Hence $H_0(\mathcal{C}) = 0$ and $H_1(\mathcal{C}) \cong I/(f_1, \ldots, f_n)$. Therefore we have proven the following result (loc. cit.).

Lemma 5.5. *Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial algebra over the field $\mathbb{F}$, $f_1, \ldots, f_n \in \bar{S}$ a regular sequence, and $I \subset S$ a maximal primary ideal containing $f_1, \ldots, f_n$ with directly linked ideal $J = ((f_1, \ldots, f_n) :_S I)$. Let $\mathcal{F}$ be a (weakly minimal) free resolution of S/I and $\mathcal{K}$ of $S/(f_1, \ldots, f_n)$. If $\mathcal{C}$ is the mapping cone of a map of complexes lifting the natural quotient map $S/(f_1, \ldots, f_n) \longrightarrow S/I$ then $\Sigma^{-1}(\mathcal{C})$ is a (weakly minimal) free resolution of $I/(f_1, \ldots, f_n)$, where $\Sigma^{-1}(\mathcal{C})$ denotes the shifted[33] complex $\Sigma^{-1}(\mathcal{C})_i = \mathcal{C}_{i+1}$ for $i \in \mathbb{Z}$.*

Continuing with the notations preceding Lemma 5.5 we note that the cocomplex $\mathcal{H} = \mathrm{Hom}_S(\Sigma^{-1}(\mathcal{C}), S)$ has as cohomology $H^\bullet(\mathcal{H}) = \mathrm{Ext}_S^\bullet(I/(f_1, \ldots, f_n), S)$ and that by Lemma 5.1, $\mathrm{Ext}_S^i(I/(f_1, \ldots, f_n), S) = 0$ for $i \neq n$. We augment the

[32]The geometric version of this construction would appear to be due to D. Ferrand (see [Peskine and Szpiro 1974, Section 2]).

[33]A topologist would say *desuspended*.

cocomplex $\mathcal{H}$ with $\operatorname{Ext}_S^n(I/(f_1, \ldots, f_n), S)$ to obtain an exact sequence, viz.,

$$0 \longleftarrow \operatorname{Ext}_S^n(I/(f_1, \ldots, f_n), S) \longleftarrow \mathcal{H}^n \longleftarrow \cdots \longleftarrow \mathcal{H}^0 \longleftarrow \mathcal{H}^{-1} \longleftarrow 0.$$

The Euler characteristic of an exact sequence is zero, so after rearranging things we obtain the following equality for Euler characteristic polynomials:

$$P\big(\operatorname{Ext}_S^n(I/(f_1, \ldots, f_n), S), t\big) = \sum (-1)^i P(\mathcal{H}_i, t).$$

At this point we require an elementary, but necessary, lemma.

Lemma 5.6. *Let A be a commutative graded connected algebra over the field $\mathbb{F}$ and L a finitely generated free A-module. Then as graded vector spaces* $\operatorname{Hom}_A(L, A) \cong Q(L)^* \otimes A$ *where* $Q(L)^* = \operatorname{Hom}_{\mathbb{F}}(L \otimes_A \mathbb{F}, \mathbb{F})$, *where* $Q(L) = L \otimes_A \mathbb{F}$.

Proof. Set $Q(L) = L \otimes_A \mathbb{F}$. We have isomorphisms of graded vector spaces

$$\begin{aligned}\operatorname{Hom}_A(L, A) &\cong \operatorname{Hom}_A(A \otimes Q(L), A) \cong \operatorname{Hom}_{\mathbb{F}}(Q(L), \mathbb{F}) \\ &\cong \operatorname{Hom}_{\mathbb{F}}\big(\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}, Q(L)), A\big) \cong \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}, Q(L)^* \otimes A) \cong Q(L)^* \otimes A,\end{aligned}$$

where the next to the last isomorphism results from the Hom — $\otimes$ interchange. □

Returning to the discussion preceding the lemma, write $\mathcal{F} = S \otimes \mathcal{U}$ and $\mathcal{K} = S \otimes \mathcal{V}$ as bigraded vector spaces. Unravel the definition of the cocomplex $\mathcal{H}$ and use Lemma 5.6 to write

$$\begin{aligned}\mathcal{H} &= \operatorname{Hom}_S(\Sigma^{-1}(\mathcal{C}), S) = \Sigma^{-1}\big(\operatorname{Hom}_S(\mathcal{F} \oplus \Sigma(\mathcal{K}), S)\big) \\ &= \Sigma^{-1}\big(\operatorname{Hom}_S(\mathcal{F}, S) \oplus \operatorname{Hom}_S(\Sigma(\mathcal{K}), S)\big) \\ &= \Sigma^{-1}\big(\operatorname{Hom}_S(S \otimes \mathcal{U}, S) \oplus \operatorname{Hom}_S(S \otimes \Sigma(\mathcal{V}), S)\big) \\ &= \big[\Sigma^{-1}(S \otimes \operatorname{Hom}_{\mathbb{F}}(\mathcal{U}, \mathbb{F}))\big] \oplus \big[S \otimes \operatorname{Hom}_{\mathbb{F}}(\Sigma(\mathcal{V}), \mathbb{F})\big]\end{aligned}$$

as graded vector spaces. By taking Euler characteristic polynomials and applying Lemma 1.3 we obtain

$$\begin{aligned}&P\big(\operatorname{Ext}_S^n(I/(f_1, \ldots, f_n), S), t\big) \\ &\qquad = P(S, t) \cdot \sum (-1)^i P(\mathcal{U}_i, t) - P(S, t) \cdot \sum (-1)^i P(\mathcal{V}_i, t) \\ &\qquad = P\big(S/(f_1, \ldots, f_n), t\big) - P(S/I, t),\end{aligned} \tag{5-4}$$

and with this we can prove the formula (5-2); to wit:

Theorem 5.7. *Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial algebra over the field $\mathbb{F}$, $f_1, \ldots, f_n \in \overline{S}$ a regular sequence, and $I \subset S$ a maximal primary ideal containing $f_1, \ldots, f_n$ with directly linked ideal $J = ((f_1, \ldots, f_n) :_S I)$. Then*

$$\dim_{\mathbb{F}}(S/I) + \dim_{\mathbb{F}}(S/J) = \dim_{\mathbb{F}}(S/(f_1, \ldots, f_n)).$$

Proof. By Lemma 5.1 we have an isomorphism

$$S/J \cong \mathrm{Ext}^n_S(I/(f_1, \ldots, f_n), S). \tag{5-5}$$

By (5-4) we have an equality of Poincaré series

$$P\big(\mathrm{Ext}^n_S(I/(f_1, \ldots, f_n), S), t\big) = P\big(S/(f_1, \ldots, f_n), t\big) - P(S/I, t),$$

so putting $t = 1$ into this equality yields an equality of dimensions

$$\dim_{\mathbb{F}}\big(\mathrm{Ext}^n_S(I/(f_1, \ldots, f_n), S)\big) = \dim_{\mathbb{F}}\big(S/(f_1, \ldots, f_n)\big) - \dim_{\mathbb{F}}(S/I) \tag{5-6}$$

for the corresponding vector spaces. Combining the two equalities (5-5) and (5-6) completes the proof. □

Corollary 5.8. *Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial algebra over the field $\mathbb{F}$, $f_1, \ldots, f_n \in \bar{S}$ a regular sequence, and $I \subset S$ a maximal primary ideal containing $f_1, \ldots, f_n$ with directly linked ideal $J = ((f_1, \ldots, f_n) :_S I)$. Then*

$$e_{\mathsf{HK}}(S/I) + e_{\mathsf{HK}}(S/J) = e_{\mathsf{HK}}(S/(f_1, \ldots, f_n)).$$

Proof. This follows from Theorem 5.7 and Proposition 2.1. □

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