Pacific Journal of Mathematics

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Volume 262 No. 2 April 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 262, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.262.397

KLEIN FOUR-SUBGROUPS OF LIE ALGEBRA AUTOMORPHISMS

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We classify the Klein four-subgroups Γ of $\mathrm{Aut}(\mathfrak{u}_0)$ for each compact simple Lie algebra $\mathfrak{u}_0$ up to conjugation, by calculating the symmetric subgroups $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ and their involution classes. This leads to a new approach to the classification of semisimple symmetric pairs and $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetric spaces. We also determine the fixed point subgroups $\mathrm{Aut}(\mathfrak{u}_0)^\Gamma$.

1. Introduction

Riemannian symmetric pairs were classified by Élie Cartan (see [Carter 1993], for example) and the more general semisimple symmetric pairs were classified by Marcel Berger [1957]. The algebraic structure of semisimple symmetric spaces is even more interesting for geometric and analytic reasons. Some of the recent works are Ōshima and Sekiguchi's classification [1984] of reduced root systems and Helminck's classification [1988] for algebraic groups. Most recently some new approaches to the classification and the parametrization of semisimple symmetric pairs were given in [Huang 2002] by using admissible quadruplets and in [Chuah and Huang 2010] by using double Vogan diagrams.

In this paper we study semisimple symmetric spaces from a different point of view — by determining the Klein four-subgroups in Lie algebra automorphisms. Let $\mathfrak{u}_0$ be a compact simple Lie algebra and $\mathfrak{g}$ be its complexification. Denote by $\mathrm{Aut}(\mathfrak{u}_0)$ the automorphism group of $\mathfrak{u}_0$. For any involution θ in $\mathrm{Aut}(\mathfrak{u}_0)$, we first determine the centralizer $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ of θ, which is a symmetric subgroup. By understanding the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta$, we proceed to classify Klein four-subgroups Γ of $\mathrm{Aut}(\mathfrak{u}_0)$ up to conjugation. This gives a new approach to the classification of commuting pairs of involutive automorphisms of $\mathfrak{u}_0$ or $\mathfrak{g}$. We note that the ordered commuting pairs of involutions correspond to

The research work described in this paper was partially supported by a Research Grant from Research Grant Council of HKSAR, China; the second author's current work is supported by a grant from SNF (Schweizerischer Nationalfonds). Finally, we would like to thank the anonymous referee for his/her careful reading and helpful comments.

MSC2010: primary 20E45; secondary 53C35.

Keywords: automorphism group, involution, symmetric subgroup, Klein four-group, involution type.

Berger's classification of semisimple symmetric pairs.

If Γ is a finite abelian subgroup of the automorphism group of a Lie group G, then the homogeneous space G/H is called a Γ-symmetric space provided that $(G^\Gamma)_0 \subseteq H \subseteq G^\Gamma$; see [Lutz 1981]. In the case of $\Gamma = \mathbb{Z}_2$ this is a symmetric space and in the case of $\Gamma = \mathbb{Z}_k$ it is the k-symmetric space studied in [Wolf and Gray 1968]. In the case of $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ it is the Klein four-group; $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces were studied in [Bahturin and Goze 2008; Kollross 2009]. This paper contains a complete list of all $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric pairs and our method is very different from theirs. Finally, we determine the fixed point subgroups $\mathrm{Aut}(\mathfrak{u}_0)^\Gamma$.

2. Preliminaries

2A. ***Complex semisimple Lie algebras and Dynkin diagrams.*** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Then $\mathfrak{g}$ has a root-space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \Big),$$

where $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ is the root system of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space of the root $\alpha \in \Delta$. Let B be the Killing form on $\mathfrak{g}$. It is a nondegenerate symmetric form. The restriction of B to $\mathfrak{h}$ is also nondegenerate. For any $\lambda \in \mathfrak{h}^*$, let $H_\lambda \in \mathfrak{h}$ be determined by

$$B(H_\lambda, H) = \lambda(H) \quad \text{for all } \ H \in \mathfrak{h}.$$

For any $\lambda, \mu \in \mathfrak{h}^*$, define $\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu)$.

For any root α, we have

$$H_\alpha \in \mathfrak{h}. \tag{1}$$

Define

$$H'_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha, \tag{2}$$

which is called a coroot; let

$$0 \neq X_\alpha \in \mathfrak{g}_\alpha \tag{3}$$

be any nonzero vector (recall that $\dim \mathfrak{g}_\alpha = 1$), which is called a root vector of the root α. The notation H_α, H'_α, X_α will be used frequently in this paper.

Note that, for any $\alpha, \beta \in \Delta$,

$$\langle \alpha, \beta \rangle = B(H_\alpha, H_\beta) = \beta(H_\alpha) = \alpha(H_\beta) \in \mathbb{R},$$
$$\langle \alpha, \alpha \rangle = B(H_\alpha, H_\alpha) = \alpha(H_\alpha) \neq 0,$$

and $2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle \in \mathbb{Z}$. We also note that $\mathrm{span}_{\mathbb{R}}\{\alpha \mid \alpha \in \Delta\} \subset \mathfrak{h}^*$ is a real vector space of dimension equal to $r = \mathrm{rank}\, \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{h}$; see [Knapp 2002, pp. 140–162].

We set $A_{\alpha,\beta} = 2\langle\alpha, \beta\rangle/\langle\beta, \beta\rangle = \alpha(H'_\beta)$. Then

$$[H'_\alpha, X_\beta] = \beta(H'_\alpha)X_\beta = \frac{2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} X_\beta = A_{\beta,\alpha}X_\beta.$$

Choose a lexicography order of $\operatorname{span}_{\mathbb{R}}\{\alpha \mid \alpha \in \Delta\}$ to get a positive system Δ^+ and a simple system Π. Let

$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}. \tag{4}$$

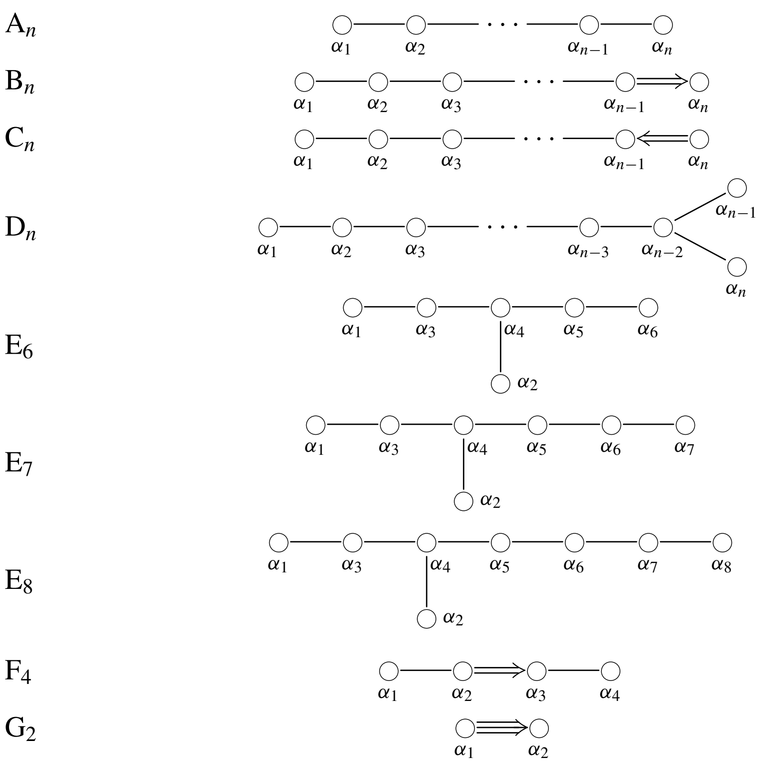
For brevity, we write

$$H_i,\ H'_i \tag{5}$$

instead of H_{α_i}, H'_{α_i} for a simple root α_i.

Draw $A_{\alpha,\beta}A_{\beta,\alpha}$ edges to connect any two distinct simple roots α and β, and draw an arrow from α to β if $\langle\alpha, \alpha\rangle > \langle\beta, \beta\rangle$; this gives us a graph. This graph is connected if and only if $\mathfrak{g}$ is a simple Lie algebra; in this case it is called the Dynkin diagram of $\mathfrak{g}$. In this paper, we always follow Bourbaki numbering to order the simple roots; see [Bourbaki 2002, pp. 265–300]. The following are all the possible (connected) Dynkin diagrams.[1]

A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2



[1] These diagrams are drawn by using a Latex package of Professor Jiu-Kang Yu. We are grateful to him for the kind permission to use this package.

Let $\mathrm{Aut}(\mathfrak{g})$ be the group of all complex linear automorphisms of $\mathfrak{g}$ and $\mathrm{Int}(\mathfrak{g})$ be the subgroup of inner automorphisms. We define

$$\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\,\mathrm{Int}(\mathfrak{g}).$$

The exponential map $\exp : \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$ is given by

$$\exp(X) = \exp(\mathrm{ad}(X)) \quad \text{for all } X \in \mathfrak{g} = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})).$$

2B. ***A compact real form.*** One can normalize the root vectors $\{X_\alpha, X_{-\alpha}\}$ so that $B(X_\alpha, X_{-\alpha}) = 2/\alpha(H_\alpha)$. Then $[X_\alpha, X_{-\alpha}] = H'_\alpha$. Moreover, one can normalize $\{X_\alpha\}$ appropriately, such that

$$\mathfrak{u}_0 = \mathrm{span}_{\mathbb{R}}\{X_\alpha - X_{-\alpha},\ i(X_\alpha + X_{-\alpha}),\ iH_\alpha : \alpha \in \Delta^+\} \tag{6}$$

is a compact real form of $\mathfrak{g}$ [Knapp 2002, pp. 348–354]. Define

$$\theta(X + iY) := X - iY \quad \text{for all } X, Y \in \mathfrak{u}_0.$$

Then θ is a Cartan involution of $\mathfrak{g}$ (as a real semisimple Lie algebra) and $\mathfrak{u}_0 = \mathfrak{g}^\theta$ is a maximal compact subalgebra of $\mathfrak{g}$. Any other compact real form of $\mathfrak{g}$ is conjugate to $\mathfrak{u}_0$. Below, whenever we discuss a compact real form of $\mathfrak{g}$, we always use this compact real form $\mathfrak{u}_0$ in (6).

Let $\mathrm{Aut}(\mathfrak{u}_0)$ be the group of automorphisms of $\mathfrak{u}_0$ and $\mathrm{Int}(\mathfrak{u}_0)$ be the subgroup of inner automorphisms. Any automorphism of $\mathfrak{u}_0$ extends uniquely to a holomorphic automorphism of $\mathfrak{g}$, so $\mathrm{Aut}(\mathfrak{u}_0) \subset \mathrm{Aut}(\mathfrak{g})$. Similarly, $\mathrm{Int}(\mathfrak{u}_0) \subset \mathrm{Int}(\mathfrak{g})$. Define

$$\Theta(f) := \theta f \theta^{-1} \quad \text{for all } f \in \mathrm{Aut}(\mathfrak{g}).$$

Then it is a Cartan involution of $\mathrm{Aut}(\mathfrak{g})$ with differential θ. It follows that $\mathrm{Aut}(\mathfrak{u}_0) = \mathrm{Aut}(\mathfrak{g})^\Theta$ and $\mathrm{Int}(\mathfrak{u}_0) = \mathrm{Int}(\mathfrak{g})^\Theta$ are maximal compact subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$, respectively. We also have

$$\mathrm{Out}(\mathfrak{u}_0) := \mathrm{Aut}(\mathfrak{u}_0)/\,\mathrm{Int}(\mathfrak{u}_0) \cong \mathrm{Out}(\mathfrak{g}) \cong \mathrm{Aut}(\Pi),$$

where $\mathrm{Aut}(\Pi)$ is the symmetry group of the graph Π consisting of permutations of vertices preserving the multiples of edges and directions of arrows.

2C. ***Notation.*** We denote by $\mathfrak{e}_6$ the compact simple Lie algebra of type $\mathbf{E_6}$. Let E_6 be the connected and simply connected Lie group with Lie algebra $\mathfrak{e}_6$. Let $\mathfrak{e}_6(\mathbb{C})$ and $\mathrm{E}_6(\mathbb{C})$ denote their complexifications. Similar notation will be used for other types.

Let $Z(G)$ and $\mathfrak{z}(\mathfrak{g})$ denote the center of a group G and a Lie algebra $\mathfrak{g}$, respectively, and G_0 denote the connected component of G containing identity element. For Lie groups $H \subset G$, let $Z_G(H)$ denote the centralizer of H in G, and for Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, let $Z_{\mathfrak{g}}(\mathfrak{h})$ denote the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Let $N_G(H)$ denote the normalizer of H in G. For any two elements $x, y \in G$, we write $x \sim y$ to mean x, y are conjugate in G, that is, $y = gxg^{-1}$ for some $g \in G$ and $x \sim_H y$ to mean $y = gxg^{-1}$ for some $g \in H$.

In the case of $G = \mathrm{E}_6$ or E_7, let c denote a nontrivial element in $Z(G)$.

In the case of $\mathfrak{u}_0 = \mathfrak{e}_7$, let

$$H_0' = \frac{H_2' + H_5' + H_7'}{2} \in i\mathfrak{e}_7 \subset \mathfrak{e}_7(\mathbb{C}).$$

Let Pin(n) (Spin(n)) be the Pin (Spin) group in degree n. Write

$$c = e_1 e_2 \cdots e_n \in \mathrm{Pin}(n).$$

Then c is in Spin(n) if and only if n is even; in this case $c \in Z(\mathrm{Spin}(n))$. If n is odd, then Spin(n) has a spinor module M of dimension $2^{(n-1)/2}$. If n is even, then Spin(n) has two spinor modules M_+, M_- of dimension $2^{(n-2)/2}$. We distinguish M_+ and M_- by requiring that c acts on M_+ as the identity when $4 \mid n$ and as multiplication by $-i$ when $4 \mid n-2$ (and thus c acts on M_- as multiplication by -1 and i, respectively, in the same two cases).

We define the matrices

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix},$$

$$I'_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}, \quad J_{p,q} = \begin{pmatrix} 0 & I_p & 0 & 0 \\ -I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix},$$

$$K_p = \begin{pmatrix} 0 & 0 & 0 & I_p \\ 0 & 0 & -I_p & 0 \\ 0 & I_p & 0 & 0 \\ -I_p & 0 & 0 & 0 \end{pmatrix}.$$

and the groups

$$Z_m = \{\lambda I_m \mid \lambda^m = 1\},$$

$$Z' = \{(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \mid \epsilon_i = \pm 1, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1\},$$

$$\Gamma_{p,q,r,s} = \left\langle \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_s \end{pmatrix}, \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ 0 & 0 & 0 & I_s \end{pmatrix} \right\rangle.$$

3. Involutions

The classical compact simple Lie algebras are as follows. For $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, let $M_n(F)$ be the set of $n \times n$ matrices with entries in F, and

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\},$$
$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X + X^* = 0, \operatorname{tr} X = 0\},$$
$$\mathfrak{sp}(n) = \{X \in M_n(\mathbb{H}) \mid X + X^* = 0\}.$$

Then $\{\mathfrak{su}(n) : n \geq 3\}$, $\{\mathfrak{so}(2n+1) : n \geq 1\}$, $\{\mathfrak{sp}(n) : n \geq 3\}$, $\{\mathfrak{so}(2n) : n \geq 4\}$ represent all isomorphism classes of compact classical simple Lie algebras.

Let $\mathfrak{u}_0$ be a compact simple Lie algebra and $\mathfrak{g} = (\mathfrak{u}_0) \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Note that the conjugacy classes of involutions in $\operatorname{Aut}(\mathfrak{u}_0)$ are in one-to-one correspondence with isomorphism classes of noncompact real forms of $\mathfrak{g}$, and are also in one-to-one correspondence with isomorphism classes of irreducible Riemannian symmetric pairs $(\mathfrak{u}_0, \mathfrak{k}_0)$ of compact type or $(\mathfrak{g}_0, \mathfrak{k}_0)$ of noncompact type; see [Huang 2002; Helminck 1988] and references therein. One direction of this correspondence is as follows: let θ be an involutive automorphism of a compact real simple Lie algebra $\mathfrak{u}_0$, and extend it to a holomorphic automorphism of $\mathfrak{g}$. Let $\mathfrak{k}_0 \subset \mathfrak{u}_0$ and $i\mathfrak{p}_0 \subset \mathfrak{u}_0$ (so $\mathfrak{p}_0 \subset i\mathfrak{u}_0$) be the $+1$, -1 eigenspaces of θ on $\mathfrak{u}_0$, respectively. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

(this is also the Cartan decomposition of $\mathfrak{g}_0$). Then $\mathfrak{g}_0$ is a real simple Lie algebra (that is, a real form of $\mathfrak{g}$), $(\mathfrak{u}_0, \mathfrak{k}_0)$ is a Riemannian symmetric pair of compact type and $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a Riemannian symmetric pair of noncompact type. The other direction of this correspondence needs a sophisticated argument.

These objects were classified by Élie Cartan in 1926. We list this classification here. Our presentation below is mainly from [Knapp 2002, pp. 408–426; Helgason 2001, pp. 515–518]. In each case, we also define a specific involution in each conjugacy class of involutions in $\operatorname{Aut}(\mathfrak{u}_0)$, which corresponds to a real simple Lie algebra or symmetric space. In the exceptional simple Lie algebras case, these involutions are labeled as σ_1, σ_2, σ_3, σ_4, σ and $\tau = \sigma_3$ (this is used only in the E_6 case). We will use this notation for involutions frequently in the rest of this paper.

The notation **AI**–**G** is Cartan notation and the notation $\mathfrak{e}_{6,-2}$, etc., is Helgason notation (with a little difference). For a real simple Lie algebra $\mathfrak{g}_0$ with a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and whose complexified Lie algebra $\mathfrak{g}$ is an exceptional simple Lie algebra, Helgason [2001, pp. 517–518] made an interesting observation: the isomorphism type of $\mathfrak{g}_0$ is distinguished by the type of $\mathfrak{g}$ (or its compact real form $\mathfrak{u}_0$) and the integer $\dim \mathfrak{k}_0 - \dim \mathfrak{p}_0$. For example, the notation $\mathfrak{e}_{6,-2}$ (written by Helgason as $\mathfrak{e}_{6(2)}$, as he used the integer $\dim \mathfrak{p}_0 - \dim \mathfrak{k}_0$ instead) means the compact real form of the complexified Lie algebra has type $\mathfrak{e}_6$ and $\dim \mathfrak{k}_0 - \dim \mathfrak{p}_0 = -2$.

The elements (coroots) H'_i are defined in (2) and (5).

i) Type **A**. For $\mathfrak{u}_0 = \mathfrak{su}(n)$, $n \geq 3$, $\{\mathrm{Ad}(I_{p,n-p}) \mid 1 \leq p \leq n/2\}$ (type **AIII**), $\{\tau = \text{complex conjugation}\}$ (type **AI**), $\{\tau \circ \mathrm{Ad}(J_{n/2})\}\}$ (type **AII**) represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$. The corresponding real forms are $\mathfrak{su}(p, n-p)$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sl}(\frac{n}{2}, \mathbb{H})$.

ii) Type **B**. For $\mathfrak{u}_0 = \mathfrak{so}(2n+1)$, $n \geq 1$, $\{\mathrm{Ad}(I_{p,2n+1-p}) \mid 1 \leq p \leq n\}$ (type **BI**) represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$. The corresponding real forms are $\mathfrak{so}(p, 2n+1-p)$.

iii) Type **C**. For $\mathfrak{u}_0 = \mathfrak{sp}(n)$, $n \geq 3$, $\{\mathrm{Ad}(I_{p,n-p}) \mid 1 \leq p \leq n/2\}$ (type **CII**) and $\{\mathrm{Ad}(\mathbf{i}I)\}$ (type **CI**) represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$. The corresponding real forms are $\mathfrak{sp}(p, n-p)$, $\mathfrak{sp}(n, \mathbb{R})$.

iv) Type **D**. For $\mathfrak{u}_0 = \mathfrak{so}(2n)$, $n \geq 4$, $\{\mathrm{Ad}(I_{p,2n-p}) \mid 1 \leq p \leq n\}$ (type **DI**) and $\{\mathrm{Ad}(J_n)\}$ (type **DIII**) represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$. The corresponding real forms are $\mathfrak{so}(p, 2n-p)$, $\mathfrak{so}^*(2n, \mathbb{R})$.[2]

v) Type $\mathbf{E_6}$. For $\mathfrak{u}_0 = \mathfrak{e}_6$, let τ be a specific diagram involution defined by

$$\begin{aligned}
\tau(H_{\alpha_1}) &= H_{\alpha_6}, & \tau(H_{\alpha_6}) &= H_{\alpha_1}, & \tau(H_{\alpha_3}) &= H_{\alpha_5},\\
\tau(H_{\alpha_5}) &= H_{\alpha_3}, & \tau(H_{\alpha_2}) &= H_{\alpha_2}, & \tau(H_{\alpha_4}) &= H_{\alpha_4},\\
\tau(X_{\pm\alpha_1}) &= X_{\pm\alpha_6}, & \tau(X_{\pm\alpha_6}) &= X_{\pm\alpha_1}, & \tau(X_{\pm\alpha_3}) &= X_{\pm\alpha_5},\\
\tau(X_{\pm\alpha_5}) &= X_{\pm\alpha_3}, & \tau(X_{\pm\alpha_2}) &= X_{\pm\alpha_2}, & \tau(X_{\pm\alpha_4}) &= X_{\pm\alpha_4}.
\end{aligned}$$

Let $\sigma_1 = \exp(\pi i H'_2)$, $\sigma_2 = \exp(\pi i (H'_1 + H'_6))$, $\sigma_3 = \tau$, $\sigma_4 = \tau \exp(\pi i H'_2)$. Then σ_1, σ_2, σ_3, σ_4 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric pairs of type **EII**, **EIII**, **EIV**, **EI** and the corresponding real forms are $\mathfrak{e}_{6,-2}$, $\mathfrak{e}_{6,14}$, $\mathfrak{e}_{6,26}$, $\mathfrak{e}_{6,-6}$. Also, σ_1, σ_2 are inner automorphisms and σ_3, σ_4 are outer automorphisms.

vi) Type $\mathbf{E_7}$. For $\mathfrak{u}_0 = \mathfrak{e}_7$, let

$$\begin{aligned}
\sigma_1 &= \exp(\pi i H'_2),\\
\sigma_2 &= \exp\left(\pi i \frac{H'_2 + H'_5 + H'_7}{2}\right),\\
\sigma_3 &= \exp\left(\pi i \frac{H'_2 + H'_5 + H'_7 + 2H'_1}{2}\right).
\end{aligned}$$

Then σ_1, σ_2, σ_3 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric pairs of type **EVI**, **EVII**, **EV** and the corresponding real forms are $\mathfrak{e}_{7,5}$, $\mathfrak{e}_{7,25}$, $\mathfrak{e}_{7,-7}$.

vii) Type $\mathbf{E_8}$. For $\mathfrak{u}_0 = \mathfrak{e}_8$, let

$$\sigma_1 = \exp(\pi i H'_2), \quad \sigma_2 = \exp(\pi i (H'_2 + H'_1)).$$

[2] When $n = 4$, we have $\mathrm{Ad}(I_{2,6}) \sim \mathrm{Ad}(J_4)$, and $\mathfrak{so}(2, 6) \cong \mathfrak{so}^*(8)$.

Then σ_1, σ_2 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric pairs of type **EIX**, **EVIII** and the corresponding real forms are $\mathfrak{e}_{8,24}$, $\mathfrak{e}_{8,-8}$.

viii) Type $\mathbf{F_4}$. For $\mathfrak{u}_0 = \mathfrak{f}_4$, let

$$\sigma_1 = \exp(\pi i H_1'), \quad \sigma_2 = \exp(\pi i H_4').$$

Then σ_1, σ_2 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric pairs of type **FI**, **FII** and the corresponding real forms are $\mathfrak{f}_{4,-4}$, $\mathfrak{f}_{4,20}$.

ix) Type $\mathbf{G_2}$. For $\mathfrak{u}_0 = \mathfrak{g}_2$, let $\sigma = \exp(\pi H_1')$, which represents the unique conjugacy class of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$ and corresponds to a Riemannian symmetric pair of type **G** and the corresponding real form is $\mathfrak{g}_{2,-2}$.

4. Centralizer of an automorphism

In this section we prove a property of the centralizer G^x of an element x in a complex or compact Lie group G. First, we recall a theorem of Steinberg [Carter 1993, pp. 93–95].

Proposition 4.1 (Steinberg). *Let G be a connected and simply connected semisimple complex (or compact) Lie group. Then the centralizer G^x for any $x \in G$ is connected.*

For an element x in a group, we write $o(x)$ for the order of x. The notation

$$\mathrm{Int}(\mathfrak{g})_0^\theta \tag{7}$$

in this paper always means $(\mathrm{Int}(\mathfrak{g})^\theta)_0$, not $(\mathrm{Int}(\mathfrak{g})_0)^\theta$. Similarly for

$$\mathrm{Int}(\mathfrak{u}_0)_0^\theta,\ \mathrm{Aut}(\mathfrak{u}_0)_0^\theta,\ \mathrm{Aut}(\mathfrak{g})_0^\theta. \tag{8}$$

Proposition 4.2. *Let $\mathfrak{g}$ be a complex simple Lie algebra. Suppose that the order of an element $\theta \in \mathrm{Aut}(\mathfrak{g})$ is equal to the order of the coset element $\theta\,\mathrm{Int}(\mathfrak{g})$ in $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$, that is, $o(\theta) = o(\theta\,\mathrm{Int}(\mathfrak{g}))$. Then $Z_{\mathrm{Int}(\mathfrak{g})}(\mathrm{Int}(\mathfrak{g})_0^\theta) = 1$.*

Proof. By the assumption, θ is a diagram automorphism; this means there exists a Cartan subalgebra $\mathfrak{t}$ which is stable under θ and θ maps Δ^+ to itself, where $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ and Δ^+ is a positive system. For any $\alpha \in \Delta$, let $\theta(X_\alpha) = a_\alpha X_{\theta\alpha}$ with $a_\alpha \neq 0$.

Let $k = o(\theta) = o(\theta\,\mathrm{Int}(\mathfrak{g}))$. Then, for any $\alpha \in \Delta$,

$$X_\alpha = \theta^k(X_\alpha) = \Bigl(\prod_{0 \le j \le k-1} a_{\theta^j\alpha}\Bigr) X_{\theta^k\alpha}.$$

It follows that

$$\prod_{0\le j\le k-1} a_{\theta^j\alpha}=1.$$

Let $L=\mathrm{Int}(\mathfrak{g})_0^\theta$, $\mathfrak{s}=\mathfrak{t}^\theta$, $T=\exp(\mathrm{ad}\,\mathfrak{t})$ and $S=\exp(\mathrm{ad}\,\mathfrak{s})$. It is clear that $S\subset L$.

We first show that $Z_{\mathrm{Int}(\mathfrak{g})}(S)=T$. It is clear that $\mathfrak{t}\subset Z_{\mathfrak{g}}(\mathfrak{s})$. Suppose that $X_\alpha\in Z_{\mathfrak{g}}(\mathfrak{s})$ for some $\alpha\in\Delta^+$. Since $\theta^k=1$, we have $\sum_{0\le j\le k-1}\theta^j(H)\in\mathfrak{t}^\theta=\mathfrak{s}$ for any $H\in\mathfrak{t}$. Then $\left[\sum_{0\le j\le k-1}\theta^j(H),X_\alpha\right]=0$.

For any j, we have

$$\begin{aligned}[\theta^j H,X_\alpha]=\theta^j([H,\theta^{k-j}X_\alpha])&=\theta^j\Bigg(\Bigg(\prod_{0\le i\le k-j-1}a_{\theta^i\alpha}\Bigg)\cdot((\theta^{k-j}\alpha)H)\cdot X_{\theta^{k-j}\alpha}\Bigg)\\&=\Bigg(\prod_{0\le i\le k-j-1}a_{\theta^i\alpha}\Bigg)\cdot((\theta^{k-j}\alpha)H)\cdot\Bigg(\prod_{0\le i\le j-1}a_{\theta^{k-j+i}\alpha}\Bigg)X_\alpha\\&=\Bigg(\prod_{0\le i\le k-1}a_{\theta^i\alpha}\Bigg)\cdot((\theta^{k-j}\alpha)H)\cdot X_\alpha=((\theta^{k-j}\alpha)H)\cdot X_\alpha.\end{aligned}$$

Hence $0=\left[\sum_{0\le j\le k-1}\theta^j(H),X_\alpha\right]=\left(\left(\sum_{0\le j\le k-1}\theta^{k-j}\alpha\right)H\right)\cdot X_\alpha$. This implies

$$\sum_{0\le j\le k-1}\theta^j\alpha=0,$$

which contradicts that all $\theta^j\alpha$ are positive roots. So $Z_{\mathfrak{g}}(\mathfrak{s})=\mathfrak{t}$. Since $Z_{\mathrm{Int}(\mathfrak{g})}(S)$ is connected (by Corollary 4.51 of [Knapp 2002, p. 260], which also applies to complex semisimple groups), $Z_{\mathrm{Int}(\mathfrak{g})}(S)=T$.

Now we show that $Z_{\mathrm{Int}(\mathfrak{g})}(L)=1$. Suppose that $1\ne\tau\in Z_{\mathrm{Int}(\mathfrak{g})}(L)$. By the above, we have $Z_{\mathrm{Int}(\mathfrak{g})}(L)\subset Z_{\mathrm{Int}(\mathfrak{g})}(S)=T$, then $\tau=\exp(\mathrm{ad}\,H)$ for some $H\in\mathfrak{t}$. For any $\alpha\in\Delta$, $\sum_{0\le j\le k-1}\theta^j(X_\alpha)\in\mathfrak{g}^\theta$ (since $\theta^k=1$), so

$$\sum_{0\le j\le k-1}\theta^j(X_\alpha)=\tau\Bigg(\sum_{0\le j\le k-1}\theta^j(X_\alpha)\Bigg)=\sum_{0\le j\le k-1}\tau(\theta^j(X_\alpha))=\sum_{0\le j\le k-1}e^{(\theta^j\alpha)H}\theta^j(X_\alpha).$$

Since each $\theta^j(X_\alpha)$ is of the form $\theta^j(X_\alpha)=b_jX_{\theta^j\alpha}$ for some $b_j\ne 1$, the last equality implies $\tau(X_\alpha)=X_\alpha$ if $\{\theta^j\alpha,0\le j\le k-1\}$ are distinct.

Claim 4.3. *Those $\alpha\in\Delta$ with roots in $\{\theta^j\alpha,0\le j\le k-1\}$ pairwise different generate Δ (as a root system).*

Since $\tau(X_\alpha)=X_\alpha$ when the elements $\theta^j\alpha$ are distinct for $0\le j\le k-1$, by Claim 4.3, we have $\tau(X_\alpha)=X_\alpha$ for any $\alpha\in\Delta$. Hence $\tau=1$, which is to say, $Z_{\mathrm{Int}(\mathfrak{g})}(\mathrm{Int}(\mathfrak{g})_0^\theta)=1$. □

Proof of Claim 4.3. Note that θ maps Δ^+ to itself, so it maps the simple system $\Pi = \{\alpha_1, \dots, \alpha_r\}$ to itself. We have four cases to consider, that is, $\Delta = \mathrm{A}_n$ $(n \geq 2)$, D_n $(n \geq 4)$, E_6 and θ is an automorphism of order 2, or $\Delta = \mathrm{D}_4$ and θ is an automorphism of order 3. We give the proof when $\Delta = \mathrm{A}_{2n}$ $(n \geq 1)$ and $o(\theta) = 2$. The proof for other cases is similar.

When $\Delta = \mathrm{A}_{2n}$ $(n \geq 1)$ and $o(\theta) = 2$, we have $\theta(\alpha_i) = \alpha_{2n+1-i}$ and $\theta(\alpha_{2n+1-i}) = \alpha_i$ for any i, $1 \leq i \leq n$. For $1 \leq i \leq n$, let

$$\beta_i = \sum_{1 \leq j \leq i} \alpha_j \quad \text{and} \quad \beta_i' = \sum_{1 \leq j \leq i} \alpha_{2n+1-j}.$$

Then $\theta(\pm\beta_i) \neq \pm\beta_i$, $\theta(\pm\beta_i') \neq \pm\beta_i'$ and $\{\pm\beta_i, \pm\beta_i' : 1 \leq i \leq n\}$ generate Δ. □

Corollary 4.4. *Let* $\mathfrak{u}_0$ *be a compact simple Lie algebra. If* $\theta \in \mathrm{Aut}(\mathfrak{u}_0)$ *satisfies the condition* $o(\theta) = o(\theta \, \mathrm{Int}(\mathfrak{u}_0))$, *then* $Z_{\mathrm{Int}(\mathfrak{u}_0)}(\mathrm{Int}(\mathfrak{u}_0)_0^{\theta}) = 1$.

Corollary 4.4 indicates that if G is a compact (simple) Lie group of adjoint type and x is of minimal possible order among all elements in the connected component containing it, then $(G^x)_0$ is also of adjoint type and the conjugation action of any element $y \in G^x - (G^x)_0$ on $(G^x)_0$ is an outer automorphism.

5. Symmetric subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$

Let $\mathfrak{u}_0$ be a compact simple Lie algebra. For each conjugacy class of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, we choose a representative θ as in Section 3 and determine the symmetric subgroup $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$.

When $\mathfrak{u}_0$ is a classical simple Lie algebra nonisomorphic to $\mathfrak{so}(8)$ or $\mathfrak{u}_0 = \mathfrak{so}(8)$ but $\theta \not\sim \mathrm{Ad}(I_{4,4})$, we can use matrices to represent involutions θ and calculate the corresponding $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$. In the case of $\theta = \mathrm{Ad}(I_{4,4}) \in \mathrm{Aut}(\mathfrak{so}(8))$, we have $\theta \sim \exp(\pi i H_2')$. Then

$$\mathrm{Int}(\mathfrak{so}(8))^{\theta} = (\mathrm{Sp}(1)^4/Z') \rtimes D,$$

where $Z' = \{(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \mid \epsilon_i = \pm 1, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1\}$, and $D \subset S_4$ is the (unique) normal order four subgroup of S_4 with conjugation action on $(\mathrm{Sp}(1)^4)/Z'$ by permutations. Then we observe that there exists a subgroup of $\mathrm{Aut}(\mathfrak{so}(8))$ that projects isomorphically to $\mathrm{Aut}(\mathfrak{so}(8))/\mathrm{Int}(\mathfrak{so}(8)) \cong S_3$ and is contained in $\mathrm{Aut}(\mathfrak{so}(8))^{\theta}$. A little more argument shows

$$\mathrm{Aut}(\mathfrak{so}(8))^{\theta} = (\mathrm{Sp}(1)^4/Z') \rtimes S_4.$$

When $\mathfrak{u}_0$ is an exceptional simple Lie algebra, we first determine the symmetric subalgebra $\mathfrak{k}_0 = \mathfrak{u}_0^{\theta}$ and the highest weights of the isotropic space $\mathfrak{p}_0 = \mathfrak{u}_0^{-\theta}$ as a $\mathfrak{k}_0$-module. The results are summarized in Table 1. The coroots H_i' are defined in (2) and (5) and the involutions are defined in Section 3.

	θ	$\mathfrak{k}_0$	$\mathfrak{p}$
EI	$\sigma_4 = \tau \exp(\pi i H'_2)$	$\mathfrak{sp}(4)$	V_{ω_4}
EII	$\sigma_1 = \exp(\pi i H'_2)$	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1)$	$\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2$
EIII	$\sigma_2 = \exp(\pi i (H'_1 + H'_6))$	$\mathfrak{so}(10) \oplus i\mathbb{R}$	$(M_+ \otimes 1) \oplus (M_- \otimes \overline{1})$
EIV	$\sigma_3 = \tau$	$\mathfrak{f}_4$	V_{ω_4}
EV	$\sigma_3 = \exp(\pi i (H'_1 + H'_0))$	$\mathfrak{su}(8)$	$\bigwedge^4 \mathbb{C}^8$
EVI	$\sigma_1 = \exp(\pi i H'_2)$	$\mathfrak{so}(12) \oplus \mathfrak{sp}(1)$	$M_+ \otimes \mathbb{C}^2$
EVII	$\sigma_2 = \exp(\pi i H'_0)$	$\mathfrak{e}_6 \oplus i\mathbb{R}$	$(V_{\omega_1} \otimes 1) \oplus (V_{\omega_6} \otimes \overline{1})$
EVIII	$\sigma_2 = \exp(\pi i (H'_1 + H'_2))$	$\mathfrak{so}(16)$	M_+
EIX	$\sigma_1 = \exp(\pi i H'_1)$	$\mathfrak{e}_7 \oplus \mathfrak{sp}(1)$	$V_{\omega_7} \otimes \mathbb{C}^2$
FI	$\sigma_1 = \exp(\pi i H'_1)$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	$V_{\omega_3} \otimes \mathbb{C}^2$
FII	$\sigma_2 = \exp(\pi i H'_4)$	$\mathfrak{so}(9)$	M
G	$\sigma = \exp(\pi i H'_1)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\mathrm{Sym}^3 \mathbb{C}^2 \otimes \mathbb{C}^2$

Table 1. Symmetric pairs and isotropic modules (exceptional Lie algebras case).

Since any element of $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ which acts trivially on both $\mathfrak{k}_0$ and $\mathfrak{p}_0$ must be trivial, the isomorphism type of $\mathfrak{k}_0$ and its isotropic module $\mathfrak{p}$ determine $\mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ completely. We may get $\mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ in the following way. Start with a compact connected Lie group H of the form $H = A \times H_s$ with $A = Z(\mathrm{Aut}(\mathfrak{u}_0)^\theta_0)_0$ a connected torus ($A \cong \mathrm{U}(1)^s$ with $s = \dim \mathfrak{z}(\mathfrak{k}_0)$) and H_s a connected and simply connected compact Lie group with $\mathrm{Lie}\, H_s = [\mathfrak{k}_0, \mathfrak{k}_0]$ (then $\mathrm{Lie}\, H = \mathfrak{k}_0 = \mathfrak{u}_0^\theta$). Then we have a surjective homomorphism

$$\pi : H \to \mathrm{Aut}(\mathfrak{u}_0)$$

determined by $\mathfrak{g}$ as a $\mathfrak{k}_0$-module. With this construction, it is clear that $\mathrm{Im}(\pi) = \mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ and $\ker \pi$ is determined by $\mathfrak{k}_0$ and its module $\mathfrak{p}$ (as described in Table 1). By Proposition 4.1 and Corollary 4.4, we can also determine the number of connected components of $\mathrm{Aut}(\mathfrak{u}_0)^\theta$. Then we could find elements outside $\mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ to generate $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ together with $\mathrm{Aut}(\mathfrak{u}_0)^\theta_0$. We show the detailed argument in most cases below. The results about the symmetric subgroups $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ are given in the last column of Table 2. The information about the first three columns of Table 2 is contained in [Knapp 2002, pp. 408–426]. The fourth column is from Section 3.

5A. ***Type* $\mathbf{E_6}$.** Now $\mathfrak{u}_0 = \mathfrak{e}_6$. Consider an outer automorphism $\theta = \sigma_3$ or σ_4. By Corollary 4.4, any element in $\mathrm{Int}(\mathfrak{u}_0)^\theta - \mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ acts on $\mathfrak{u}_0^\theta$ as an outer automorphism. Note that $\mathfrak{u}_0^\theta \cong \mathfrak{sp}(4)$ or $\mathfrak{f}_4$, so it has no outer automorphisms. By Corollary 4.4, it follows that $\mathrm{Int}(\mathfrak{u}_0)^\theta = \mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ and $\mathrm{Aut}(\mathfrak{u}_0)^\theta = \mathrm{Aut}(\mathfrak{u}_0)^\theta_0 \times \langle\theta\rangle$. Moreover, $\mathrm{Aut}(\mathfrak{u}_0)^\theta_0$ is of adjoint type by Corollary 4.4.

Type	$(\mathfrak{u}_0,\mathfrak{k}_0)$	rank	θ	symmetric subgroup $\mathrm{Aut}(\mathfrak{u}_0)^\theta$
AI	$(\mathfrak{su}(n),\mathfrak{so}(n))$	$n-1$	$\overline{X}$	$(O(n)/\langle -I\rangle)\times\langle\theta\rangle$
AII	$(\mathfrak{su}(2n),\mathfrak{sp}(n))$	$n-1$	$J_n\overline{X}J_n^{-1}$	$(\mathrm{Sp}(n)/\langle -I\rangle)\times\langle\theta\rangle$
AIII $p<q$	$(\mathfrak{su}(p+q),\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q)))$	p	$I_{p,q}XI_{p,q}$	$(S(U(p)\times U(q))/Z_{p+q})\rtimes\langle\tau\rangle$ $\mathrm{Ad}(\tau)=$ complex conjugation
AIII $p=q$	$(\mathfrak{su}(2p),\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(p)))$	p	$I_{p,p}XI_{p,p}$	$(S(U(p)\times U(p))/Z_{2p})\rtimes\langle\tau,J_p\rangle$ $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$
BDI $p<q$	$(\mathfrak{so}(p+q),\mathfrak{so}(p)+\mathfrak{so}(q))$	p	$I_{p,q}XI_{p,q}$	$(O(p)\times O(q))/\langle(-I_p,-I_q)\rangle$
DI $p>4$	$(\mathfrak{so}(2p),\mathfrak{so}(p)+\mathfrak{so}(p))$	p	$I_{p,p}XI_{p,p}$	$((O(p)\times O(p))/\langle(-I_p,-I_p)\rangle)\rtimes\langle J_p\rangle$ $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$
DI $p=4$	$(\mathfrak{so}(8),\mathfrak{so}(4)+\mathfrak{so}(4))$	4	$I_{4,4}XI_{4,4}$	$((\mathrm{Sp}(1)^4)/Z')\rtimes S_4$ S_4 acts by permutations
DIII	$(\mathfrak{so}(2n),\mathfrak{u}(n))$	n	$J_nXJ_n^{-1}$	$(U(n)/\{\pm I\})\rtimes\langle I_{n,n}\rangle$ $\mathrm{Ad}(I_{n,n})=$ complex conjugation
CI	$(\mathfrak{sp}(n),\mathfrak{u}(n))$	n	$(\mathbf{i}I)X(\mathbf{i}I)^{-1}$	$(U(n)/\{\pm I\})\rtimes\langle\mathbf{j}I\rangle$ $\mathrm{Ad}(\mathbf{j}I)=$ complex conjugation
CII $p<q$	$(\mathfrak{sp}(p+q),\mathfrak{sp}(p)+\mathfrak{sp}(q))$	p	$I_{p,q}XI_{p,q}$	$(\mathrm{Sp}(p)\times\mathrm{Sp}(q))/\langle(-I_p,-I_q)\rangle$
CII $p=q$	$(\mathfrak{sp}(2p),\mathfrak{sp}(p)+\mathfrak{sp}(p))$	p	$I_{p,p}XI_{p,p}$	$((\mathrm{Sp}(p)\times\mathrm{Sp}(p)/\langle(-I_p,-I_p)\rangle)\rtimes\langle J_p\rangle$ $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$
EI	$(\mathfrak{e}_6,\ \mathfrak{sp}(4))$	6	σ_4	$(\mathrm{Sp}(4)/\langle -1\rangle)\times\langle\theta\rangle$
EII	$(\mathfrak{e}_6,\ \mathfrak{su}(6)+\mathfrak{sp}(1))$	4	σ_1	$(SU(6)\times\mathrm{Sp}(1)/\langle(e^{\frac{2\pi i}{3}}I,1),(-I,-1)\rangle)\rtimes\langle\tau\rangle$ $\mathfrak{k}_0^\tau=\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$
EIII	$(\mathfrak{e}_6,\ \mathfrak{so}(10)+i\mathbb{R})$	2	σ_2	$(\mathrm{Spin}(10)\times U(1)/\langle(c,i)\rangle)\rtimes\langle\tau\rangle$ $\mathfrak{k}_0^\tau=\mathfrak{so}(9)$
EIV	$(\mathfrak{e}_6,\ \mathfrak{f}_4)$	2	σ_3	$F_4\times\langle\theta\rangle$
EV	$(\mathfrak{e}_7,\ \mathfrak{su}(8))$	7	σ_3	$(SU(8)/\langle iI\rangle)\rtimes\langle\omega\rangle$ $\mathfrak{k}_0^\omega=\mathfrak{sp}(4)$
EVI	$(\mathfrak{e}_7,\ \mathfrak{so}(12)+\mathfrak{sp}(1))$	4	σ_1	$(\mathrm{Spin}(12)\times\mathrm{Sp}(1))/\langle(c,1),(-1,-1)\rangle$
EVII	$(\mathfrak{e}_7,\ \mathfrak{e}_6+i\mathbb{R})$	3	σ_2	$((E_6\times U(1))/\langle(c,e^{\frac{2\pi i}{3}})\rangle)\rtimes\langle\omega\rangle$ $\mathfrak{k}_0^\omega=\mathfrak{f}_4$
EVIII	$(\mathfrak{e}_8,\ \mathfrak{so}(16))$	8	σ_2	$\mathrm{Spin}(16)/\langle c\rangle$
EIX	$(\mathfrak{e}_8,\ \mathfrak{e}_7+\mathfrak{sp}(1))$	4	σ_1	$E_7\times\mathrm{Sp}(1)/\langle(c,-1)\rangle$
FI	$(\mathfrak{f}_4,\ \mathfrak{sp}(3)+\mathfrak{sp}(1))$	4	σ_1	$(\mathrm{Sp}(3)\times\mathrm{Sp}(1))/\langle(-I,-1)\rangle$
FII	$(\mathfrak{f}_4,\ \mathfrak{so}(9))$	1	σ_2	$\mathrm{Spin}(9)$
G	$(\mathfrak{g}_2,\ \mathfrak{sp}(1)+\mathfrak{sp}(1))$	2	σ	$(\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\langle(-1,-1)\rangle$

Table 2. Symmetric pairs and symmetric subgroups. (When $n=4$, DIII is identical to BDI when $p=2$ and $q=6$.)

Consider an inner automorphism $\theta = \sigma_1$ or σ_2. Let $\theta' \in \mathrm{E}_6$ be an involution which maps to θ under the covering $\pi : \mathrm{E}_6 \to \mathrm{Int}(\mathfrak{e}_6)$. We have

$$\mathrm{Int}(\mathfrak{e}_6)^\theta = \{g \in \mathrm{E}_6 \mid \theta' g \theta'^{-1} g^{-1} \in Z(\mathrm{E}_6)\}/Z(\mathrm{E}_6),$$
$$\mathrm{Int}(\mathfrak{e}_6)^\theta_0 = \{g \in \mathrm{E}_6 \mid \theta' g \theta'^{-1} g^{-1} = 1\}/Z(\mathrm{E}_6),$$

(use Proposition 4.1 here). If $\{g \in \mathrm{E}_6 \mid \theta' g \theta'^{-1} g^{-1} \in Z(\mathrm{E}_6)\} \neq \mathrm{E}_6^\theta$, then there exists $g \in \mathrm{E}_6$ such that $\theta' g \theta'^{-1} g^{-1} = c \in Z(E_6)$. Then $g\theta' g^{-1} = \theta' c^{-1}$. But $o(\theta') = 2 \neq 6 = o(\theta' c^{-1})$. So $g\theta' g^{-1} \neq \theta' c^{-1}$. Then $\{g \in \mathrm{E}_6 \mid \theta(g) g^{-1} \in Z(\mathrm{E}_6)\} = \mathrm{E}_6^\theta$ and so $\mathrm{Int}(\mathfrak{e}_6)^\theta = \mathrm{Int}(\mathfrak{e}_6)^\theta_0$. Since σ_1, σ_2 commutes with τ,

$$\mathrm{Aut}(\mathfrak{e}_6)^\theta = \mathrm{Int}(\mathfrak{e}_6)^\theta_0 \rtimes \langle \tau \rangle.$$

The conjugation action of τ on $\mathrm{Int}(\mathfrak{e}_6)^\theta_0$ is determined by its action on $\mathfrak{k}_0 = \mathfrak{u}_0^\theta$, and

$$(\mathfrak{e}_6^{\sigma_1})^\tau = \mathfrak{sp}(3) \oplus \mathfrak{sp}(1), \quad (\mathfrak{e}_6^{\sigma_2})^\tau = \mathfrak{so}(9).$$

5B. ***Type*** **E₇.** Now $\mathfrak{u}_0 = \mathfrak{e}_7$ and $\mathrm{Aut}(\mathfrak{e}_7) = \mathrm{Int}(\mathfrak{e}_7)$ is connected. Let $\pi : \mathrm{E}_7 \to \mathrm{Aut}(\mathfrak{e}_7)$ be the adjoint homomorphism, which is a 2-fold covering. Let

$$\sigma_1' = \exp(\pi i H_2') \in \mathrm{E}_7,$$
$$\sigma_2' = \exp\left(\pi i \frac{H_2' + H_5' + H_7'}{2}\right) \in \mathrm{E}_7,$$
$$\sigma_3' = \exp\left(\pi i \frac{2H_1' + H_2' + H_5' + H_7'}{2}\right) \in \mathrm{E}_7 .$$

Then $\pi(\sigma_i') = \sigma_i$, $o(\sigma_1') = 2$, $o(\sigma_2') = 4$ and $o(\sigma_3') = 4$. One has

$$\mathrm{Aut}(\mathfrak{e}_7)^{\sigma_i} \cong \{g \in \mathrm{E}_7 \mid g\sigma_i' g^{-1} \sigma_i'^{-1} \in Z(\mathrm{E}_7)\}/Z(\mathrm{E}_7),$$
$$\mathrm{Aut}(\mathfrak{e}_7)^{\sigma_i}_0 \cong \{g \in \mathrm{E}_7 \mid g\sigma_i' g^{-1} \sigma_i'^{-1} = 1\}/Z(\mathrm{E}_7)$$

(use Proposition 4.1 here), where $Z(\mathrm{E}_7) = \langle \exp(\pi i(H_2' + H_5' + H_7'))\rangle \cong \mathbb{Z}/2\mathbb{Z}$ is the center of E_7.

For $\theta = \sigma_1$, suppose that there exists $g \in \mathrm{E}_7$ such that

$$g\sigma_1' g^{-1} (\sigma_1')^{-1} = \exp(\pi i(H_2' + H_5' + H_7')).$$

Then $g \exp(\pi i H_2') g^{-1} = \exp(\pi i(H_5' + H_7'))$. Then there exists $w \in W$ such that $w(\exp(\pi i H_2')) = \exp(\pi i(H_5' + H_7'))$. Since $w(\exp(\pi i H'_{\alpha_2})) = \exp(\pi i H'_{w(\alpha_2)})$, we get $\exp(\pi i H'_{w(\alpha_2)}) = \exp(\pi i(H_5' + H_7'))$. Then

$$w(\alpha_2) \in (\alpha_5 + \alpha_7) + 2\,\mathrm{span}_{\mathbb{Z}}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}.$$

There are no roots in $(\alpha_5 + \alpha_7) + 2\,\mathrm{span}_{\mathbb{Z}}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, so there are no $g \in \mathrm{E}_7$ such that $(g\sigma_1' g^{-1})\sigma_1'^{-1} = \exp(\pi i(H_2' + H_5' + H_7'))$. Then

$$\{g \in \mathrm{E}_7 \mid (g\sigma_1' g^{-1})\sigma_1'^{-1} \in Z(\mathrm{E}_7)\} = \mathrm{E}_7^{\sigma_1'} .$$

So $\mathrm{Aut}(\mathfrak{e}_7)^{\sigma_1} = \mathrm{Aut}(\mathfrak{e}_7)_0^{\sigma_1}$.

For $\theta = \sigma_2$ or σ_3, let

$$\omega = \exp\left(\frac{\pi(X_{\alpha_2} - X_{-\alpha_2})}{2}\right)\exp\left(\frac{\pi(X_{\alpha_5} - X_{-\alpha_5})}{2}\right)\exp\left(\frac{\pi(X_{\alpha_7} - X_{-\alpha_7})}{2}\right).$$

Then

$$\omega\sigma_2'\omega^{-1} = \sigma_2'^{-1} = \sigma_2' \exp(\pi i(H_2' + H_5' + H_7')),$$
$$\omega\sigma_3'\omega^{-1} = \sigma_3'^{-1} = \sigma_3' \exp(\pi i(H_2' + H_5' + H_7')),$$

and $\omega^2 = 1$. Then $\mathrm{Aut}(\mathfrak{e}_7)^{\theta} = \mathrm{Aut}(\mathfrak{e}_7)_0^{\theta} \rtimes \langle\omega\rangle$. The conjugation action of ω on $\mathrm{Aut}(\mathfrak{e}_7)_0^{\theta}$ is determined by its action on $\mathfrak{k}_0 = \mathfrak{u}_0^{\theta}$, and we have

$$(\mathfrak{e}_7^{\sigma_2})^{\omega} = \mathfrak{f}_4, \qquad (\mathfrak{e}_7^{\sigma_3})^{\omega} = \mathfrak{sp}(4).$$

Further, ω acts on $\mathfrak{h}$ as $s_{\alpha_2}s_{\alpha_5}s_{\alpha_7}$, where s_α in the Weyl group is the reflection corresponding to the root α.

5C. ***Types*** $\mathbf{E_8}$**,** $\mathbf{F_4}$**,** $\mathbf{G_2}$**.** If $\mathfrak{u}_0 = \mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$, then $\mathrm{Aut}(\mathfrak{u}_0)$ is connected and simply connected. By Proposition 4.1, $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$ is connected. Then they are determined by $\mathfrak{u}_0^{\theta}$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$.

6. Klein four-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$

In this section, we classify Klein four-subgroups Γ (called simply Klein subgroups) in $\mathrm{Aut}(\mathfrak{u}_0)$ up to conjugation. We also determine the fixed-point subgroups $\mathrm{Aut}(\mathfrak{u}_0)^{\Gamma}$. Note that such a Γ is equal to $\{1, \theta, \sigma, \theta\sigma\}$ for two commuting involutions $\theta \neq \sigma$. Fix an involution θ; the conjugacy class of Γ is determined by the conjugacy classes of the involution $\sigma(\neq \theta)$ in $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$.

6A. ***Ordered commuting pairs of involutions and semisimple symmetric pairs.*** For a compact simple Lie algebra $\mathfrak{u}_0$ and its complexification $\mathfrak{g}$, the isomorphism classes of semisimple symmetric pairs $(\mathfrak{g}_0, \mathfrak{h}_0)$ with $\mathfrak{g}_0$ a real form of $\mathfrak{g}$ and $\mathfrak{h}_0(\neq \mathfrak{g}_0)$ noncompact are in one-to-one correspondence with the conjugacy classes of ordered commuting pairs of involutions (θ, σ) in $\mathrm{Aut}(\mathfrak{u}_0)$ with $\theta \neq \sigma$. One direction of this correspondence is as follows: let $\mathfrak{u}_{i,j}$ $(i,\ j = 0$ or $1)$ be the joint eigenspace of θ and σ where θ acts on it as $(-1)^i$ and σ acts on it as $(-1)^j$. Then we have a decomposition

$$\mathfrak{u}_0 = \mathfrak{u}_{0,0} \oplus \mathfrak{u}_{0,1} \oplus \mathfrak{u}_{1,0} \oplus \mathfrak{u}_{1,1}.$$

Then $\mathfrak{k}_0 = \mathfrak{u}_0^{\theta} = \mathfrak{u}_{0,0} \oplus \mathfrak{u}_{0,1}$ and $i\mathfrak{p}_0 = \mathfrak{u}_0^{-\theta} = \mathfrak{u}_{1,0} \oplus \mathfrak{u}_{1,1}$. Extend θ, σ to holomorphic automorphisms of $\mathfrak{g}$ and let

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{u}_{0,0} + \mathfrak{u}_{0,1} + i(\mathfrak{u}_{1,0} + \mathfrak{u}_{1,1}) \quad \text{and} \quad \mathfrak{h}_0 = \mathfrak{g}_0^{\sigma} = \mathfrak{u}_{0,0} + i\mathfrak{u}_{1,0}.$$

$\mathfrak{u}_0$	Γ_i	$\mathfrak{l}_0=\mathfrak{u}_0^{\Gamma_i}$	Type
$\mathfrak{su}(p+q)$	$\Gamma_{p,q}=\langle\tau,I_{p,q}\rangle$	$\mathfrak{so}(p)+\mathfrak{so}(q)$	**AI-AI-AIII**, S
$\mathfrak{su}(2p)$	$\Gamma_p=\langle\tau,J_p\rangle$	$\mathfrak{u}(p)$	**AI-AII-AIII**, N
$\mathfrak{su}(2p+2q)$	$\Gamma'_{p,q}=\langle\tau J_{p+q},I'_{p,q}\rangle$	$\mathfrak{sp}(p)+\mathfrak{sp}(q)$	**AII-AII-AIII**, S
$\mathfrak{su}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q)+\mathfrak{u}(r)+\mathfrak{u}(s))$	**AIII-AIII-AIII**, NSV
$\mathfrak{su}(2p)$	$\Gamma_p=\langle I_{p,p},J_p\rangle$	$\mathfrak{su}(p)$	**AIII-AIII-AIII**, V
$\mathfrak{so}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$\mathfrak{so}(p)+\mathfrak{so}(q)+\mathfrak{so}(r)+\mathfrak{so}(s))$	**BDI-BDI-BDI**, NSV
$\mathfrak{so}(2p)$	$\Gamma_p=\langle J_p,I_{p,p}\rangle$	$\mathfrak{so}(p)$	**DI-DI-DIII**, S
$\mathfrak{so}(2p+2q)$	$\Gamma_{p,q}=\langle J_{p+q},I'_{p,q}\rangle$	$\mathfrak{u}(p)+\mathfrak{u}(q)$	**DI-DIII-DIII**, S
$\mathfrak{so}(4p)$	$\Gamma'_p=\langle J_{2p},K_p\rangle$	$\mathfrak{sp}(p)$	**DIII-DIII-DIII**, V
$\mathfrak{sp}(p)$	$\Gamma_p=\langle\mathbf{i}I,\mathbf{j}I\rangle$	$\mathfrak{so}(p)$	**CI-CI-CI**, V
$\mathfrak{sp}(p+q)$	$\Gamma_{p,q}=\langle\mathbf{i}I,I_{p,q}\rangle$	$\mathfrak{u}(p)+\mathfrak{u}(q)$	**CI-CI-CII**, S
$\mathfrak{sp}(2p)$	$\Gamma'_p=\langle\mathbf{i}I,\mathbf{j}J_p\rangle$	$\mathfrak{sp}(p)$	**CI-CII-CII**, S
$\mathfrak{sp}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$\mathfrak{sp}(p)+\mathfrak{sp}(q)+\mathfrak{sp}(r)+\mathfrak{sp}(s)$	**CII-CII-CII**, NSV

Table 3. Klein subgroups in $\mathrm{Aut}(\mathfrak{u}_0)$ for the classical cases. (When $p=1$, $q=3$, $\Gamma_{1,3}$ is very special since $\mathrm{Ad}(I_{2,6})\sim\mathrm{Ad}(J_4)$.)

Then $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and $(\mathfrak{g}_0,\mathfrak{h}_0)$ is a semisimple symmetric pair with $\mathfrak{h}_0\neq\mathfrak{g}_0$ and noncompact. The other direction of this correspondence needs a more sophisticated argument.

When θ is fixed, the conjugacy classes of the pairs (θ,σ) in $\mathrm{Aut}(\mathfrak{u}_0)$ are in one-to-one correspondence with the $\mathrm{Aut}(\mathfrak{u}_0)^\theta$-conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$.

For an exceptional compact simple Lie algebra $\mathfrak{u}_0$ and any representative θ of involution classes in Section 3, we give the representatives of classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$ and identify their classes in $\mathrm{Aut}(\mathfrak{u}_0)$. For any classical compact simple Lie algebra $\mathfrak{u}_0$ and a representative θ of an involution class, we have a similar classification of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$; we omit it here but remark that the representatives can be constructed from Table 3. This gives a new proof to Berger's classification of semisimple symmetric pairs.

In most cases the symmetric subgroup $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ is a product of classical groups with some twisting, for which we can classify their involution classes by matrix calculations. In the remaining cases, $\mathfrak{u}_0^\theta=\mathfrak{s}_0\oplus\mathfrak{z}$ for an exceptional simple Lie algebra $\mathfrak{s}_0$ and an algebra $\mathfrak{z}=0$, $i\mathbb{R}$ or $\mathfrak{sp}(1)$. We have a homomorphism

$$p:\mathrm{Aut}(\mathfrak{u}_0)^\theta\to\mathrm{Aut}(\mathfrak{s}_0).$$

Then what we need to do is to classify involutions in $p^{-1}(\sigma)$ for $\sigma\in\mathrm{Aut}(\mathfrak{s}_0)$ an involution or the identity element, which is not hard in general.

For an exceptional compact simple Lie algebra $\mathfrak{u}_0$, the conjugacy class of an involution $\sigma\in\mathrm{Aut}(\mathfrak{u}_0)$ is determined by $\dim\mathfrak{g}^\sigma$. (This is an accidental phenomenon observed by Helgason [2001, pp. 517–518].) For any involution $\sigma \in \mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$, the class of σ in $\mathrm{Aut}(\mathfrak{u}_0)$ is determined by $\dim \mathfrak{g}^\sigma = \dim \mathfrak{k}^\sigma + \dim \mathfrak{p}^\sigma$ and the dimensions $\dim \mathfrak{k}^\sigma$, $\dim \mathfrak{p}^\sigma$ can be calculated from the class of σ in $\mathrm{Aut}(\mathfrak{u}_0)^\theta$. The coroots H_i' are defined in (2) and (5) and the involutions σ_i, σ, τ are defined in Section 3.

Type $\mathbf{E_6}$. Now $\mathfrak{u}_0 = \mathfrak{e}_6$. For $\theta = \sigma_1 = \exp(\pi i H_2')$, one has

$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_1} = (\mathrm{SU}(6) \times \mathrm{Sp}(1)/\langle (e^{2\pi i/3} I, 1), (-I, -1)\rangle) \rtimes \langle \tau \rangle,$$

$\sigma_1 = (I, -1) = (-I, 1)$, where $\mathrm{Ad}(\tau)(X, Y) = (J_3 \overline{X} J_3^{-1}, Y)$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$\left(\begin{pmatrix} -I_4 & 0 \\ 0 & I_2 \end{pmatrix}, 1\right) \sim \sigma_2, \qquad \left(\begin{pmatrix} -I_2 & 0 \\ 0 & I_4 \end{pmatrix}, 1\right) \sim \sigma_1,$$

$$\left(\begin{pmatrix} iI_5 & 0 \\ 0 & -iI_1 \end{pmatrix}, \mathbf{i}\right) \sim \sigma_2, \qquad \left(\begin{pmatrix} iI_3 & 0 \\ 0 & -iI_3 \end{pmatrix}, \mathbf{i}\right) \sim \sigma_1,$$

$$\tau \sim \sigma_3, \qquad \tau\sigma_1 \sim \sigma_4, \qquad \tau(J_3, \mathbf{i}) \sim \sigma_4.$$

These elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

For $\theta = \sigma_2 = \exp(\pi i (H_1' + H_6'))$, one has

$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_2} = \big((\mathrm{Spin}(10) \times \mathrm{U}(1))/\langle (c, i)\rangle\big) \rtimes \langle \tau \rangle, \quad \sigma_2 = (-1, 1) = (1, -1),$$

where $c = e_1 e_2 \cdots e_{10}$ and $\mathrm{Ad}(\tau)(x, z) = ((e_1 e_2 \cdots e_9) x (e_1 e_2 \cdots e_9)^{-1}, z^{-1})$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$(e_1 e_2 e_3 e_4, 1) \sim \sigma_1, \quad (e_1 e_2 \cdots e_8, 1) \sim \sigma_2,$$

$$\left(\delta, \frac{1+i}{\sqrt{2}}\right) \sim \sigma_2, \quad \left(-\delta, \frac{1+i}{\sqrt{2}}\right) \sim \sigma_1,$$

$$\tau \sim \sigma_3, \quad \tau(e_1 e_2 e_3 e_4, 1) \sim \sigma_4,$$

where

$$\delta = \frac{1 + e_1 e_2}{\sqrt{2}} \, \frac{1 + e_3 e_4}{\sqrt{2}} \cdots \frac{1 + e_9 e_{10}}{\sqrt{2}}.$$

These elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

For $\theta = \sigma_3 = \tau$, one has $\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_3} = \mathrm{F}_4 \times \langle \tau \rangle$. Let τ_1, τ_2 be involutions in F_4 with

$$\mathfrak{f}_4^{\tau_1} \cong \mathfrak{sp}(3) \oplus \mathfrak{sp}(1), \quad \mathfrak{f}_4^{\tau_2} \cong \mathfrak{so}(9).$$

Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$\tau_1 \sim \sigma_1, \quad \tau_2 \sim \sigma_2,$$

$$\sigma_3 \tau_1 \sim \sigma_4, \quad \sigma_3 \tau_2 \sim \sigma_3,$$

these elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

For $\theta=\sigma_4=\tau\exp(\pi i H'_2)$, one has $\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_4}=(\mathrm{Sp}(4)/\langle -I\rangle)\times\langle\sigma_4\rangle$. Let

$$\tau_1=\mathbf{i}I,\quad \tau_2=\begin{pmatrix}-I_2 & 0\\ 0 & I_2\end{pmatrix},\quad \tau_3=\begin{pmatrix}-1 & 0\\ 0 & I_3\end{pmatrix}.$$

Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$\tau_1\sim\sigma_1,\qquad \tau_2\sim\sigma_2,\qquad \tau_3\sim\sigma_1,$$
$$\sigma_4\tau_1\sim\sigma_4,\quad \sigma_4\tau_2\sim\sigma_4,\quad \sigma_4\tau_3\sim\sigma_3.$$

These elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}-\{\theta\}$.

Type $\mathbf{E}_7$. Now $\mathfrak{u}_0=\mathfrak{e}_7$. For $\theta=\sigma_1=\exp(\pi i H'_2)$, one has

$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_1}=(\mathrm{Spin}(12)\times\mathrm{Sp}(1))/\langle(c,1),(-1,-1)\rangle,$$

where $\sigma_1=(-1,1)=(1,-1)$, $c=e_1e_2\cdots e_{12}$. Let

$$\delta=\frac{1+e_1e_2}{\sqrt{2}}\,\frac{1+e_3e_4}{\sqrt{2}}\cdots\frac{1+e_{11}e_{12}}{\sqrt{2}}.$$

Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$(e_1e_2e_3e_4,1)\sim\sigma_1,\quad (e_1e_2,\mathbf{i})\sim\sigma_2,\quad (e_1e_2\cdots e_6,\mathbf{i})\sim\sigma_3,$$
$$(\delta,1)\sim\sigma_2,\quad (-\delta,1)\sim\sigma_3,\qquad (e_1\delta e_1,\mathbf{i})\sim\sigma_1.$$

These elements represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}-\{\theta\}$. Moreover,

$$\langle\sigma_1,(e_1e_2e_3e_4,1)\rangle\sim F_2,\quad \langle\sigma_1,(e_1\delta e_1,\mathbf{i})\rangle\sim F_1.$$

For $\theta=\sigma_2=\tau=\exp\left(\pi i\dfrac{H'_2+H'_5+H'_7}{2}\right)$, one has

$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_2}_0=((\mathrm{E}_6\times\mathrm{U}(1))/\langle(c,e^{\frac{2\pi i}{3}})\rangle)\rtimes\langle\omega\rangle,$$

where c is a nontrivial central element of E_6 with $o(c)=3$, $\sigma_2=(1,-1)$ and $(\mathfrak{e}_6\oplus i\mathbb{R})^{\omega}=\mathfrak{f}_4\oplus 0$. Let τ_1,τ_2 be involutions in E_6 with

$$\mathfrak{e}_6^{\tau_1}\cong\mathfrak{su}(6)\oplus\mathfrak{sp}(1),\quad \mathfrak{e}_6^{\tau_2}\cong\mathfrak{so}(10)\oplus i\mathbb{R}.$$

Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$\tau_1\sim\sigma_1,\qquad \tau_2\sim\sigma_1,$$
$$\tau_1\sigma_2\sim\sigma_3,\quad \tau_2\sigma_2\sim\sigma_2,$$
$$\omega\sim\sigma_2,\qquad \omega\eta\sim\sigma_3,$$

where $\eta\in\mathrm{F}_4=\mathrm{E}_6^{\omega}$ is an involution with $(\mathfrak{f}_4)^{\eta}\cong\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$. These elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}-\{\theta\}$.

For
$$\theta=\sigma_3=\exp\left(\pi i\frac{H_2'+H_5'+H_7'+2H_1'}{2}\right),$$
one has
$$\mathrm{Aut}(\mathfrak{u}_0)_0^{\sigma_3}=(\mathrm{SU}(8)/\langle iI\rangle)\rtimes\langle\omega\rangle,\quad \sigma_3=\frac{1+i}{\sqrt{2}}I,$$
where $\mathrm{Ad}(\omega)X=J_4\overline{X}J_4^{-1}$. Let $\tau_1=\begin{pmatrix}-I_2 & \\ & I_6\end{pmatrix}$, $\tau_2=\begin{pmatrix}-I_4 & \\ & I_4\end{pmatrix}$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,
$$\tau_1\sim\sigma_1,\quad \tau_2\sim\sigma_1,\quad \tau_1\sigma_3\sim\sigma_2,\quad \tau_2\sigma_3\sim\sigma_3,$$
$$\omega\sim\sigma_2,\quad \omega\sigma_3\sim\sigma_3,\quad \omega J_4\sim\sigma_3.$$
These elements represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$.

Type $\mathbf{E_8}$. Now $\mathfrak{u}_0=\mathfrak{e}_8$. For $\theta=\sigma_1=\exp(\pi i H_2')$, one has
$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_1}\cong(\mathrm{E}_7\times\mathrm{Sp}(1))/\langle(c,-1)\rangle,$$
where $\sigma_1=(1,-1)=(c,1)$. Let τ_1, τ_2 denote the elements in E_7 with $\tau_1^2=\tau_2^2=c$ and $\mathfrak{e}_7^{\tau_1}\cong\mathfrak{e}_6\oplus i\mathbb{R}$, $\mathfrak{e}_7^{\tau_2}\cong\mathfrak{su}(8)$. Let τ_3, τ_4 be involutions in E_7 such that there exist Klein subgroups $\Gamma,\Gamma'\subset E_7$ with three nonidentity elements in Γ all conjugate to τ_3, three nonidentity elements in Γ' all conjugate to τ_4, and $\mathfrak{e}_7^\Gamma\cong\mathfrak{su}(6)\oplus(i\mathbb{R})^2$, $\mathfrak{e}_7^{\Gamma'}\cong\mathfrak{so}(8)\oplus(\mathfrak{sp}(1))^3$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,
$$(\tau_1,\mathbf{i})\sim\sigma_1,\quad(\tau_2,\mathbf{i})\sim\sigma_2,\quad(\tau_3,1)\sim\sigma_1,\quad(\tau_4,1)\sim\sigma_2.$$
These elements represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$.

For $\theta=\sigma_2=\exp(\pi i(H_2'+H_1'))$, one has $\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_2}\cong\mathrm{Spin}(16)/\langle c\rangle$, where $\sigma_2=-1$, $c=e_1e_2\cdots e_{16}$. Let
$$\delta=\frac{1+e_1e_2}{\sqrt{2}}\,\frac{1+e_3e_4}{\sqrt{2}}\cdots\frac{1+e_{15}e_{16}}{\sqrt{2}},$$
$$\tau_1=e_1e_2e_3e_4,\quad \tau_2=e_1e_2e_3\cdots e_8,\quad \tau_3=\delta,\quad \tau_4=-\delta.$$
Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,
$$\tau_1\sim\sigma_1,\quad \tau_2\sim\sigma_2,\quad \tau_3\sim\sigma_1,\quad \tau_4\sim\sigma_2.$$
These elements represent all the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta-\{\theta\}$.

Type $\mathbf{F_4}$. When $\mathfrak{u}_0=\mathfrak{f}_4$, for $\theta=\sigma_1=\exp(\pi i H_1')$,
$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_1}\cong\mathrm{Sp}(3)\times\mathrm{Sp}(1)/\langle(-I,-1)\rangle,$$
where $\sigma_1=(-I,1)=(I,-1)$. Let
$$\tau_1=\left(\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},1\right),\quad \tau_2=\left(\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},1\right),\quad \tau_3=(\mathbf{i}I,\mathbf{i}).$$

Then, in $\mathrm{Aut}(\mathfrak{u}_0)$,

$$\tau_1 \sim \sigma_1, \quad \tau_2 \sim \sigma_2, \quad \tau_3 \sim \sigma_1.$$

These elements represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

For $\theta = \sigma_2 = \exp(\pi i H_4')$, one has $\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_2} \cong \mathrm{Spin}(9)$, $\sigma_2 = -1$. Let $\tau_1 = e_1e_2e_3e_4$, $\tau_2 = e_1e_2e_3\cdots e_8$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$, we have $\tau_1 \sim \sigma_1$ and $\tau_2 \sim \sigma_2$. These elements represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

Type $\mathbf{G_2}$. When $\mathfrak{u}_0 = \mathfrak{g}_2$ and $\theta = \sigma = \exp(\pi i H_1')$, one has

$$\mathrm{Aut}(\mathfrak{u}_0)^{\sigma_1} \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\langle(-1, -1)\rangle,$$

where $\sigma_1 = (-1, 1) = (1, -1)$. Denote $\tau = (\mathbf{i}, \mathbf{i})$. Then, in $\mathrm{Aut}(\mathfrak{u}_0)$, we have $\tau \sim \sigma$, and τ represents the unique conjugacy class of involutions in $\mathrm{Aut}(\mathfrak{u}_0)^\theta - \{\theta\}$.

By the above, we have reproved Berger's classification of semisimple symmetric pairs. The next proposition is an immediate consequence of this classification.

Proposition 6.1. *There are* 23, 19, 8, 5, *and* 1 *isomorphism classes of nontrivial* (*that is*, $\mathfrak{h}_0 \neq \mathfrak{g}_0$) *semisimple symmetric pairs* $(\mathfrak{g}_0, \mathfrak{h}_0)$ *with* $\mathfrak{g}_0$ *noncompact and* $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ *a complex simple Lie algebra of types* $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$, *and* $\mathbf{G_2}$, *respectively.*

6B. ***Klein subgroups, speciality, regularity and centralizers.*** For a Klein group $\Gamma \subset \mathrm{Aut}(\mathfrak{u}_0)$, we call the conjugacy classes of the involutions in Γ the *involution type* of Γ, and the classes of Riemannian symmetric pairs corresponding to the involutions in Γ the *symmetric space type* of Γ. Since there is a one-to-one correspondence between these two types, we simply say *type* of Γ for either involution type or symmetric space type.

For a compact simple Lie algebra $\mathfrak{u}_0$, a Klein subgroup Γ of $\mathrm{Aut}(\mathfrak{u}_0)$ is called *regular* if any two distinct conjugate (in $\mathrm{Aut}(\mathfrak{u}_0)$) elements $\sigma, \theta \in \Gamma$ are conjugate by an element $g \in \mathrm{Aut}(\mathfrak{u}_0)$ commuting with $\theta\sigma$ (that is, $g \in \mathrm{Aut}(\mathfrak{u}_0)^{\theta\sigma}$).

A Klein subgroup $\Gamma \subset \mathrm{Aut}(\mathfrak{u}_0)$ is called *special* if there are two (distinct) elements of Γ which are conjugate in $\mathrm{Aut}(\mathfrak{u}_0)$. It is called *very special* if three involutions of Γ are pairwise conjugate in $\mathrm{Aut}(\mathfrak{u}_0)$. Otherwise it is called nonspecial. The definition of special is due to [Ōshima and Sekiguchi 1984].

In Tables 3 and 4, we list some Klein subgroups $\Gamma_i \subset \mathrm{Aut}(\mathfrak{u}_0)$ for each compact simple Lie algebra $\mathfrak{u}_0$ together with their symmetric space types (when $\mathfrak{u}_0$ is classical) or involution types (when $\mathfrak{u}_0$ is exceptional). These subgroups are not conjugate to each other since their fixed point subalgebras $\mathfrak{u}_0^{\Gamma_i}$ are nonisomorphic. In the last column we also indicate whether they are special or not. For brevity, we write N to mean nonspecial, S to mean special but not very special, V to mean very special. The speciality of the subgroups $\Gamma_{p,q,r,s}$ depends on the parameters. In general they can be nonspecial, special or very special; in this case we use NSV to denote their

$\mathfrak{u}_0$	Γ_i	$\mathfrak{l}_0=\mathfrak{u}_0^{\Gamma_i}$	Type
$\mathfrak{e}_6$	$\Gamma_1=\langle\exp(\pi i H_2'),\exp(\pi i H_4')\rangle$	$(\mathfrak{su}(3))^2\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_1,\sigma_1)$, V
$\mathfrak{e}_6$	$\Gamma_2=\langle\exp(\pi i H_4'),\exp(\pi i(H_3'+H_4'+H_5'))\rangle$	$\mathfrak{su}(4)\oplus(\mathfrak{sp}(1))^2\oplus i\mathbb{R}$	$(\sigma_1,\sigma_1,\sigma_2)$, S
$\mathfrak{e}_6$	$\Gamma_3=\langle\exp(\pi i(H_2'+H_1')),\exp(\pi i(H_4'+H_1'))\rangle$	$\mathfrak{su}(5)\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_2,\sigma_2)$, S
$\mathfrak{e}_6$	$\Gamma_4=\langle\exp(\pi i(H_1'+H_6')),\exp(\pi i(H_3'+H_5'))\rangle$	$\mathfrak{so}(8)\oplus(i\mathbb{R})^2$	$(\sigma_2,\sigma_2,\sigma_2)$, V
$\mathfrak{e}_6$	$\Gamma_5=\langle\exp(\pi i H_2'),\tau\rangle$	$\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$	$(\sigma_1,\sigma_3,\sigma_4)$, N
$\mathfrak{e}_6$	$\Gamma_6=\langle\exp(\pi i H_2'),\tau\exp(\pi i H_4')\rangle$	$\mathfrak{so}(6)\oplus i\mathbb{R}$	$(\sigma_1,\sigma_4,\sigma_4)$, S
$\mathfrak{e}_6$	$\Gamma_7=\langle\exp(\pi i(H_1'+H_6'))),\tau\rangle$	$\mathfrak{so}(9)$	$(\sigma_2,\sigma_3,\sigma_3)$, S
$\mathfrak{e}_6$	$\Gamma_8=\langle\exp(\pi i(H_1'+H_6')),\tau\exp(\pi i H_2')\rangle$	$\mathfrak{so}(5)\oplus\mathfrak{so}(5)$	$(\sigma_2,\sigma_4,\sigma_4)$, S
$\mathfrak{e}_7$	$\Gamma_1=\langle\exp(\pi i H_2'),\exp(\pi i H_4')\rangle$	$\mathfrak{su}(6)\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_1,\sigma_1)$, V
$\mathfrak{e}_7$	$\Gamma_2=\langle\exp(\pi i H_2'),\exp(\pi i H_3')\rangle$	$\mathfrak{so}(8)\oplus(\mathfrak{sp}(1))^3$	$(\sigma_1,\sigma_1,\sigma_1)$, V
$\mathfrak{e}_7$	$\Gamma_3=\langle\exp(\pi i H_2'),\tau\rangle$	$\mathfrak{so}(10)\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_2,\sigma_2)$, S
$\mathfrak{e}_7$	$\Gamma_4=\langle\exp(\pi i H_1'),\tau\rangle$	$\mathfrak{su}(6)\oplus\mathfrak{sp}(1)\oplus i\mathbb{R}$	$(\sigma_1,\sigma_2,\sigma_3)$, N
$\mathfrak{e}_7$	$\Gamma_5=\langle\exp(\pi i H_2'),\tau\exp(\pi i H_1')\rangle$	$\mathfrak{su}(4)\oplus\mathfrak{su}(4)\oplus i\mathbb{R}$	$(\sigma_1,\sigma_3,\sigma_3)$, S
$\mathfrak{e}_7$	$\Gamma_6=\langle\tau,\omega\rangle$	$\mathfrak{f}_4$	$(\sigma_2,\sigma_2,\sigma_2)$, V
$\mathfrak{e}_7$	$\Gamma_7=\langle\tau,\omega\exp(\pi i H_1')\rangle$	$\mathfrak{sp}(4)$	$(\sigma_2,\sigma_3,\sigma_3)$, S
$\mathfrak{e}_7$	$\Gamma_8=\langle\tau\exp(\pi i H_1'),\omega\exp(\pi i H_3')\rangle$	$\mathfrak{so}(8)$	$(\sigma_3,\sigma_3,\sigma_3)$, V
$\mathfrak{e}_8$	$\Gamma_1=\langle\exp(\pi i H_2'),\exp(\pi i H_4')\rangle$	$\mathfrak{e}_6\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_1,\sigma_1)$, V
$\mathfrak{e}_8$	$\Gamma_2=\langle\exp(\pi i H_2'),\exp(\pi i H_1')\rangle$	$\mathfrak{so}(12)\oplus(\mathfrak{sp}(1))^2$	$(\sigma_1,\sigma_1,\sigma_2)$, S
$\mathfrak{e}_8$	$\Gamma_3=\langle\exp(\pi i H_2'),\exp(\pi i(H_1'+H_4'))\rangle$	$\mathfrak{su}(8)\oplus i\mathbb{R}$	$(\sigma_1,\sigma_2,\sigma_2)$, S
$\mathfrak{e}_8$	$\Gamma_4=\langle\exp(\pi i(H_2'+H_1')),\exp(\pi i(H_5'+H_1'))\rangle$	$\mathfrak{so}(8)\oplus\mathfrak{so}(8)$	$(\sigma_2,\sigma_2,\sigma_2)$, V
$\mathfrak{f}_4$	$\Gamma_1=\langle\exp(\pi i H_2'),\exp(\pi i H_1')\rangle$	$\mathfrak{su}(3)\oplus(i\mathbb{R})^2$	$(\sigma_1,\sigma_1,\sigma_1)$, V
$\mathfrak{f}_4$	$\Gamma_2=\langle\exp(\pi i H_3'),\exp(\pi i H_2')\rangle$	$\mathfrak{so}(5)\oplus(\mathfrak{sp}(1))^2$	$(\sigma_1,\sigma_1,\sigma_2)$, S
$\mathfrak{f}_4$	$\Gamma_3=\langle\exp(\pi i H_4'),\exp(\pi i H_3')\rangle$	$\mathfrak{so}(8)$	$(\sigma_2,\sigma_2,\sigma_2)$, V
$\mathfrak{g}_2$	$\Gamma=\langle\exp(\pi i H_1'),\exp(\pi i H_2')\rangle$	$(i\mathbb{R})^2$	(σ,σ,σ), V

Table 4. Klein four-subgroups in $\mathrm{Aut}(\mathfrak{u}_0)$ for the exceptional cases.

speciality. The reader can determine for which parameters they are nonspecial, special or very special. The notation $I_{p,q}$, J_p, etc. is defined in Section 2C.

Theorem 6.2. *For a compact simple Lie algebra* $\mathfrak{u}_0$, *any Klein subgroup* $\Gamma\subset\mathrm{Aut}(\mathfrak{u}_0)$ *is conjugate to one in Table 3 or Table 4 and they are all regular.*

Proof. When $\mathfrak{u}_0$ is a classical compact simple Lie algebra, we can do matrix calculation to show Table 3 is complete and any Klein subgroup is regular. When $\mathfrak{u}_0$ is an exceptional compact simple Lie algebra, from Klein subgroups we get nonconjugate commuting pairs of involutions (θ_1,θ_2) distinguished by the isomorphism type of $\mathfrak{u}_0^{\langle\theta_1,\theta_2\rangle}$ or the distribution of the classes of the (ordered) tuples $(\theta_1,\theta_2,\theta_3)$. When $\mathfrak{u}_0$ is of type $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$, or $\mathbf{G_2}$, we get (at least) 23, 19, 8, 5, or 1 nonconjugate commuting pairs, respectively. By Proposition 6.1, they represent all conjugacy classes of commuting pairs of involutions. So Table 4 is complete.

For an exceptional simple Lie algebra $\mathfrak{u}_0$, suppose that some Klein subgroup fails to be regular. Then we can construct nonconjugate commuting pairs (θ_1, θ_2) and (θ_1', θ_2') $(= (\theta_2, \theta_1))$ with $\langle\theta_1, \theta_2\rangle = \langle\theta_1', \theta_2'\rangle$, $\theta_1 \sim \theta_1'$, $\theta_2 \sim \theta_2'$, $\theta_1\theta_2 \sim \theta_1'\theta_2'$. Then there should exist more isomorphism classes of semisimple symmetric pairs. But it is not the case, and it follows that any Klein subgroup is regular. □

Another way of proving all Klein subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ are regular is as follows. First we just need to check for any commuting pair of involutions $\theta_1, \theta_2 \in \mathrm{Aut}(\mathfrak{u}_0)$ with $\theta_1 \sim \theta_2$ (in $\mathrm{Aut}(\mathfrak{u}_0)$), θ_1, θ_2 are conjugate in $\mathrm{Aut}(\mathfrak{u}_0)^\theta$, where $\theta = \theta_1\theta_2$. Fix θ as a representative in Section 3, when $\mathfrak{u}_0$ is an exceptional simple Lie algebra. This was already checked in the last subsection; when $\mathfrak{u}_0$ is a classical simple Lie algebra, we can check this from the data in Table 3 (list of Klein groups with symmetric space type) and Table 2 (symmetric subgroups).

A statement equivalent to the regularity of all Klein subgroups (Theorem 6.2) is that two commuting pairs of involutions (θ, σ) and (θ', σ') are conjugate in $\mathrm{Aut}(\mathfrak{u}_0)$ if and only if

$$\theta \sim \theta', \quad \sigma \sim \sigma', \quad \theta\sigma \sim \theta'\sigma'$$

and the Klein subgroups $\langle\theta, \sigma\rangle$, $\langle\theta', \sigma'\rangle$ are conjugate. This statement clearly implies the second statement in Theorem 6.2. To derive this statement from Theorem 6.2, give two pairs (θ, σ) and (θ', σ') with $\theta \sim \theta'$, $\sigma \sim \sigma'$, $\theta\sigma \sim \theta'\sigma'$ and $\langle\theta, \sigma\rangle \sim \langle\theta', \sigma'\rangle$. After replacing (θ', σ') by a pair conjugate to it, we may assume $\langle\theta, \sigma\rangle = \langle\theta', \sigma'\rangle$, that is, (θ, σ) and (θ', σ') generate the same Klein subgroup. By Theorem 6.2, $\langle\theta, \sigma\rangle$ is regular, so (θ, σ) and (θ', σ') are conjugate. Since any Klein subgroup of $\mathrm{Aut}(\mathfrak{u}_0)$ is regular, a conjugacy class of Klein subgroups gives 6, 3, or 1 isomorphism types of semisimple symmetric pairs when it is nonspecial, special but not very special, or very special, respectively.

The fact that all Klein subgroups in $\mathrm{Aut}(\mathfrak{u}_0)$ are regular is an interesting phenomenon. The property of regularity can be generalized to closed subgroups of any Lie group; a vast array of examples of nonregular subgroups is given in [Larsen 1994].

From Tables 1 and 4, we can abstract the following facts.

Proposition 6.3. *When* $\mathfrak{u}_0$ *is an exceptional compact simple Lie algebra, any two classes of involutions have commuting representatives*; *for any Klein group* $\Gamma \subset \mathrm{Aut}(\mathfrak{u}_0)$ *the centralizer* $\mathrm{Aut}(\mathfrak{u}_0)^\Gamma$ *intersects of* $\mathrm{Aut}(\mathfrak{u}_0)$.

For classical compact simple Lie algebras, both statements of the above proposition fail in general. For example, in $\mathrm{Aut}(\mathfrak{su}(2n))$ and for an odd p with $1 \leq p \leq n-1$, $\tau \circ \mathrm{Ad}(I_{n,n})$ ($\tau =$ complex conjugation) doesn't commute with any involution conjugate to $\mathrm{Ad}(I_{p,2n-p})$; in $\mathrm{Aut}(\mathfrak{so}(4n))$, $\mathrm{Aut}(\mathfrak{so}(4n))^{\Gamma_n} \subset \mathrm{Int}(\mathfrak{so}(4n))$ (see Table 3 for the definition of Γ_n).

For each Klein subgroup Γ listed in Table 3 or 4 with two generators $\theta, \sigma \in \mathrm{Aut}(\mathfrak{u}_0)$, we get the centralizer $\mathrm{Aut}(\mathfrak{u}_0)^\Gamma$ by calculating $(\mathrm{Aut}(\mathfrak{u}_0)^\theta)^\sigma$. The results about $\mathrm{Aut}(\mathfrak{u}_0)^\Gamma$ are listed in Table 5 for classical compact simple Lie algebras and in Table 6 for exceptional compact simple Lie algebras.

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Received January 17, 2012. Revised July 15, 2012.

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$\mathfrak{u}_0$	Γ_i	$L = \mathrm{Aut}(\mathfrak{u}_0)^{\Gamma_i}$
$\mathfrak{su}(p+q), p \neq q$	$\Gamma_{p,q}$	$((O(p)\times O(q))/\langle(-I_p,-I_q)\rangle)\times\langle\tau\rangle$
$\mathfrak{su}(2p)$	$\Gamma_{p,p}$	$((O(p)\times O(p))/\langle(-I_p,-I_p)\rangle)\rtimes\langle\tau,J_p\rangle$, $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$, $\mathrm{Ad}(\tau)=1$
$\mathfrak{su}(2p)$	Γ'_p	$(U(p)/\langle -I_p\rangle)\rtimes\langle\tau,z\rangle$, $\mathrm{Ad}(z)=1$
$\mathfrak{su}(2p+2q), p \neq q$	$\Gamma'_{p,q}$	$((\mathrm{Sp}(p)\times \mathrm{Sp}(q))/\langle(-I_p,-I_q)\rangle)\times\langle\tau J_{p+q}\rangle$
$\mathfrak{su}(4p)$	$\Gamma'_{p,p}$	$((\mathrm{Sp}(p)\times \mathrm{Sp}(p))/\langle(-I_p,-I_p)\rangle)\rtimes\langle\tau J_{2p},J_p\rangle$, $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$, $\mathrm{Ad}(\tau J_{2p})=1$
$\mathfrak{su}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$((S(U(p)\times U(q)\times U_r\times U_s)/\langle Z_{p+q+r+s}\rangle)\rtimes\langle\tau\rangle$ $\mathrm{Ad}(\tau)=$ complex conjugation
$\mathfrak{su}(2p+2r), p\neq r$	$\Gamma_{p,p,r,r}$	$((S(U(p)\times U(p)\times U_r\times U_r)/\langle Z_{2p+2r}\rangle)\rtimes\langle\tau,J_{p,r}\rangle$ $\mathrm{Ad}(J_{p,r})(X_1,X_2,X_3,X_4)=(X_2,X_1,X_4,X_3)$
$\mathfrak{su}(4p)$	$\Gamma_{p,p,p,p}$	$((S(U(p)\times U(p)\times U(p)\times U(p))/\langle Z_{4p}\rangle)\rtimes\langle\tau,J_{2p},J_{p,p}\rangle$ $\mathrm{Ad}(J_{2p})(X_1,X_2,X_3,X_4)=(X_3,X_4,X_1,X_2)$
$\mathfrak{su}(2p)$	Γ_p	$PSU(p)\rtimes\langle F_p,\tau\rangle$ $\mathrm{Ad}(\tau)=$ complex conjugation, $\mathrm{Ad}(F_p)=1$
$\mathfrak{so}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$(O(p)\times O(q)\times O(r)\times O(s))/\langle -I_{p+q+r+s}\rangle$
$\mathfrak{so}(2p+2r), p\neq r$	$\Gamma_{p,p,r,r}$	$((O(p)\times O(p)\times O(r)\times O(r))/\langle -I_{2p+2r}\rangle))\rtimes\langle J_{p,r}\rangle$ $\mathrm{Ad}(J_{p,r})(X_1,X_2,X_3,X_4)=(X_2,X_1,X_4,X_3)$
$\mathfrak{so}(4p)$, $p\neq 2$	$\Gamma_{p,p,p,p}$	$((O(p))^4/\langle -I_{4p}\rangle)\rtimes\langle J_{2p},J_{p,p}\rangle$ $\mathrm{Ad}(J_{2p})(X_1,X_2,X_3,X_4)=(X_3,X_4,X_1,X_2)$
$\mathfrak{so}(8)$	$\Gamma_{2,2,2,2}$	$(U(1)^4/Z')\rtimes\langle\epsilon_{1,2},\epsilon_{1,3},\epsilon_{1,4},S_4\rangle$ $\mathrm{Ad}(\epsilon_{1,2})(X_1,X_2,X_3,X_4)=(-X_1,-X_2,X_3,X_4)$,etc S_4 acts by permutations
$\mathfrak{so}(2p)$	Γ_p	$(O(p)/\langle -I_p\rangle)\times F_p$
$\mathfrak{so}(2p+2q), p\neq q$	$\Gamma_{p,q}$	$((U(p)\times U(q))/\langle(-I_p,-I_q)\rangle)\rtimes\langle\tau\rangle$ $\mathrm{Ad}(\tau)=$ complex conjugation
$\mathfrak{so}(4p)$	$\Gamma_{p,p}$	$((U(p)\times U(p))/\langle(-I_p,-I_p)\rangle)\rtimes\langle\tau,J_p\rangle$, $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$
$\mathfrak{so}(4p)$	Γ'_p	$(\mathrm{Sp}(p)/\langle -I_p\rangle)\times F'_p$
$\mathfrak{sp}(n)$	Γ_p	$(O(n)/\langle -I_n\rangle)\times F_p$
$\mathfrak{sp}(p+q), p\neq q$	$\Gamma_{p,q}$	$((U(p)\times U(q))/\langle(-I_p,-I_q)\rangle)\times\langle\tau\rangle$
$\mathfrak{sp}(2p)$	$\Gamma_{p,p}$	$((U(p)\times U(p))/\langle(-I_p,-I_p)\rangle)\rtimes\langle\tau,J_p\rangle$, $\mathrm{Ad}(\tau)=$ complex conjugation, $\mathrm{Ad}(J_p)(X,Y)=(Y,X)$
$\mathfrak{sp}(2p)$	Γ'_p	$(\mathrm{Sp}(p)/\langle -I_p\rangle)\times F'_p$
$\mathfrak{sp}(p+q+r+s)$	$\Gamma_{p,q,r,s}$	$(\mathrm{Sp}(p)\times\mathrm{Sp}(q)\times\mathrm{Sp}(r)\times\mathrm{Sp}(s))/\langle -I_{p+q+r+s}\rangle$
$\mathfrak{sp}(2p+2r), p\neq r$	$\Gamma_{p,p,r,r}$	$((\mathrm{Sp}(p)\times\mathrm{Sp}(p)\times\mathrm{Sp}(r)\times\mathrm{Sp}(r))/\langle -I_{2p+2r}\rangle)\rtimes\langle J_{p,r}\rangle$ $\mathrm{Ad}(J_{p,r})(X_1,X_2,X_3,X_4)=(X_2,X_1,X_4,X_3)$
$\mathfrak{sp}(4p)$	$\Gamma_{p,p,p,p}$	$((\mathrm{Sp}(p))^4/\langle -I_{4p}\rangle)\rtimes\langle J_{2p},J_{p,p}\rangle$ $\mathrm{Ad}(J_{2p})(X_1,X_2,X_3,X_4)=(X_3,X_4,X_1,X_2)$

Table 5. Fixed point subgroups of Klein four-subgroups: classical cases.

$\mathfrak{u}_0$	Γ_i	$L=\mathrm{Aut}(\mathfrak{u}_0)^{\Gamma_i}$
$\mathfrak{e}_6$	Γ_1	$((SU(3)\times SU(3)\times U(1)\times U(1))/\langle(e^{\frac{2\pi i}{3}}I,I,e^{\frac{2\pi i}{3}},1),(I,e^{\frac{2\pi i}{3}}I,e^{\frac{-2\pi i}{3}},1)\rangle)\rtimes\langle z,\tau\rangle$, $\mathrm{Ad}(\tau)(X,Y,\lambda,\mu)=(\overline{Y},\overline{X},\lambda,\mu)$, $\mathrm{Ad}(z)(X,Y,\lambda,\mu)=(Y,X,\lambda^{-1},\mu^{-1})$
$\mathfrak{e}_6$	Γ_2	$(SU(4)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times U(1))/\langle(iI,-1,1,i),(I,-1,-1,-1)\rangle)\rtimes\langle\tau\rangle$, $\mathrm{Ad}(\tau)(X,y,z,\lambda)=(J_2\overline{X}(J_2)^{-1},y,z,\lambda^{-1})$
$\mathfrak{e}_6$	Γ_3	$(SU(5)\times U(1)\times U(1))\rtimes\langle\tau'\rangle$, $\mathrm{Ad}(\tau')(X,\lambda,\mu)=(\overline{X},\lambda^{-1},\mu^{-1})$
$\mathfrak{e}_6$	Γ_4	$((\mathrm{Spin}(8)\times U(1)\times U(1))/\langle(-1,-1,1),(c,1,-1)\rangle)\rtimes\langle\tau\rangle$, $\mathrm{Ad}(\tau)(x,\lambda,\mu)=(x,\lambda^{-1},\mu^{-1})$
$\mathfrak{e}_6$	Γ_5	$((\mathrm{Sp}(3)\times\mathrm{Sp}(1))/\langle(-I,-1)\rangle)\times\langle\tau\rangle$
$\mathfrak{e}_6$	Γ_6	$((SO(6)\times U(1))/\langle(-I,-1)\rangle)\rtimes\langle\tau',z\rangle$, $\mathrm{Ad}(z)(X,\lambda)=(I_{3,3}XI_{3,3},\lambda^{-1})$, $\mathrm{Ad}(\tau')=1$
$\mathfrak{e}_6$	Γ_7	$\mathrm{Spin}(9)\times\langle\tau\rangle$
$\mathfrak{e}_6$	Γ_8	$((\mathrm{Spin}(5)\times\mathrm{Spin}(5))/\langle(-1,-1)\rangle)\rtimes\langle\tau',z\rangle$, $\mathrm{Ad}(z)(x,y)=(y,x)$
$\mathfrak{e}_7$	Γ_1	$((SU(6)\times U(1)\times U(1))/\langle(e^{\frac{2\pi i}{3}}I,e^{\frac{-2\pi i}{3}},1),(-I,1,1)\rangle)\rtimes\langle z\rangle$, $\mathrm{Ad}(z)(X,\lambda,\mu)=(J_3\overline{X}J_3^{-1},\lambda^{-1},\mu^{-1})$
$\mathfrak{e}_7$	Γ_2	$(\mathrm{Spin}(8)\times\mathrm{Sp}(1)^3)/\langle(c,-1,1,1),(1,-1,-1,-1),(-1,-1,-1,1)\rangle$
$\mathfrak{e}_7$	Γ_3	$((\mathrm{Spin}(10)\times U(1)\times U(1))/\langle(c,i,1)\rangle)\rtimes\langle z\rangle$, $\mathrm{Ad}(z)(x,\lambda,\mu)=(e_1xe_1^{-1},\lambda^{-1},\mu^{-1})$
$\mathfrak{e}_7$	Γ_4	$((SU(6)\times\mathrm{Sp}(1)\times U(1))/\langle(e^{\frac{2\pi i}{3}}I,1,e^{\frac{-2\pi i}{3}}),(-I,-1,1)\rangle)\rtimes\langle z\rangle$, $\mathrm{Ad}(z)(X,y,\lambda)=(J_3\overline{X}J_3^{-1},y,\lambda^{-1})$
$\mathfrak{e}_7$	Γ_5	$((\mathrm{Spin}(6)\times\mathrm{Spin}(6)\times U(1))/\langle(c,c',1),(1,-1,-1)\rangle)\rtimes\langle z_1,z_2\rangle$, $\mathrm{Ad}(z_1)(x,y,\lambda)=(y,x,\lambda^{-1})$, $\mathrm{Ad}(z_2)(x,y,\lambda)=(e_1xe_1^{-1},e_1ye_1^{-1},\lambda^{-1})$
$\mathfrak{e}_7$	Γ_6	$F_4\times\langle\tau,\omega\rangle$
$\mathfrak{e}_7$	Γ_7	$(\mathrm{Sp}(4)/\langle-I\rangle)\times\langle\tau,\omega'\rangle$
$\mathfrak{e}_7$	Γ_8	$(SO(8)/\langle-I\rangle)\times\langle\tau',\omega'\rangle$
$\mathfrak{e}_8$	Γ_1	$((E_6\times U(1)\times U(1))/\langle(c,e^{\frac{2\pi i}{3}},1)\rangle)\rtimes\langle z\rangle$, $\mathfrak{l}_0^z=\mathfrak{f}_4\oplus 0\oplus 0$
$\mathfrak{e}_8$	Γ_2	$(\mathrm{Spin}(12)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\langle(c,-1,1),(-1,-1,-1)\rangle$
$\mathfrak{e}_8$	Γ_3	$((SU(8)\times U(1))/\langle(-I,1),(iI,-1)\rangle)\rtimes\langle z\rangle$, $\mathfrak{l}_0^z=\mathfrak{sp}(4)\oplus 0$
$\mathfrak{e}_8$	Γ_4	$((\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\langle(-1,-1),(c,c)\rangle)\rtimes\langle z\rangle$, $\mathrm{Ad}(z)(x,y)=(y,x)$
$\mathfrak{f}_4$	Γ_1	$((SU(3)\times U(1)\times U(1))/\langle(e^{\frac{2\pi i}{3}}I,e^{\frac{-2\pi i}{3}},1)\rangle)\rtimes\langle z\rangle$, $\mathfrak{l}_0^z=\mathfrak{so}(3)\oplus 0\oplus 0$
$\mathfrak{f}_4$	Γ_2	$((\mathrm{Sp}(2)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\langle(-I,-1,-1)\rangle$
$\mathfrak{f}_4$	Γ_3	$\mathrm{Spin}(8)$
$\mathfrak{g}_2$	Γ	$(U(1)\times U(1))\rtimes\langle z\rangle$, $\mathrm{Ad}(z)(\lambda,\mu)=(\lambda^{-1},\mu^{-1})$

Table 6. Fixed point subgroups of Klein four-subgroups: exceptional cases.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 262 No. 2 April 2013

0030-8730(201304)262:2;1-8