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We consider biharmonic hypersurfaces in complete Riemannian manifolds and prove that, under some additional assumptions, they are minimal.

1. Introduction

According to a definition first given by B. Y. Chen [1991], an isometrically immersed oriented hypersurface in Euclidean space, \( \varphi : M \to \mathbb{R}^{m+1} \) is biharmonic if its mean curvature vector field \( H \) satisfies

\[
\Delta H = 0,
\]

where \( \Delta \) denotes the Laplacian on the hypersurface. It is well known that for submanifolds of Euclidean space, \( \text{trace}(B) = mH = \Delta \varphi \), where \( B \) is the second fundamental form of the immersion. Hence, for any fixed unit vector \( a \) of \( \mathbb{R}^{m+1} \),

\[
m\Delta \langle H, a \rangle = \Delta^2 \langle \varphi, a \rangle
\]

and the hypersurface is biharmonic if and only if each component of the immersion \( \varphi \) is a biharmonic function. Chen [1991; 1996] conjectured that a biharmonic hypersurface (in fact any biharmonic submanifold) of \( \mathbb{R}^{m+1} \) is minimal, the converse being, of course trivially true. This statement is of a local nature and the conjecture holds for hypersurfaces in \( \mathbb{R}^3 \) [Chen 1991] and \( \mathbb{R}^4 \) [Hasanis and Vlachos 1995; Defever 1998]. However, in general, it has been shown to be true only under some additional assumptions, sometimes of a global nature: see for instance [Akutagawa and Maeta 2013] and [Nakauchi and Urakawa 2011].

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This problem can be considered in a more general perspective. Indeed, let \((M, g)\) and \((N, h)\) be Riemannian manifolds and \(\varphi : (M, g) \rightarrow (N, h)\) a smooth map. Let \(\tau(\varphi)\) denote its tension field, that is,

\[
\tau(\varphi) = \text{trace}(\nabla d\varphi) = \sum_{i=1}^{m} (\nabla d\varphi)(e_i, e_i), \quad m = \dim M,
\]

where \(\nabla d\varphi\) is the generalized second fundamental tensor and \(\{e_1, \ldots, e_m\}\) is a local orthonormal frame on \((M, g)\). Given a relatively compact domain \(\Omega \subset M\) one introduces the bienergy functional \(E^\varphi (\Omega)\) on \(\Omega\) by setting

\[
E^\varphi (\Omega) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2,
\]

where integration is understood with respect to the volume element of \((M, g)\). Then \(\varphi\) is a biharmonic map (meaning a critical point of this functional on \(M\) — i.e., on each relatively compact domain \(\Omega \subset M\), if and only if the bitension field

\[
\tau_2(\varphi) = \Delta \tau(\varphi) - \sum_{i} R^N (\tau(\varphi), \varphi_*(e_i)) \varphi_*(e_i)
\]

vanishes identically. Here \(R^N\) denotes the \((3,1)\) curvature tensor of \((N, h)\).

When \(\varphi : (M^m, g) \rightarrow (N^{m+1}, h)\) is an isometric immersion of an \(m\)-dimensional hypersurface and \(\nu\) is a local unit normal vector field along \(\varphi\), writing the mean curvature vector as

\[
H = H \nu
\]

and indicating with \(B\) the second fundamental form in the direction of \(\nu\), a heavy computation shows that (2) is equivalent to the system

\[
\begin{align*}
\Delta H - |B|^2 H + \text{Ric}^N(\nu, \nu) H &= 0, \\
2B(\nabla H, \cdot)^\# + \frac{1}{2} m \nabla H^2 - 2H(\text{Ric}^N(\nu, \cdot)^\#, \cdot)^T &= 0,
\end{align*}
\]

where \(^\#: TM^* \rightarrow TM\) denotes the musical isomorphism, \(^T\) the tangential component and \(\text{Ric}^N\) the Ricci tensor of \((N, h)\) [Ou 2010, Theorem 2.1].

At this point one easily verifies that a biharmonic hypersurface in \(\mathbb{R}^{m+1}\) in the sense of Chen is exactly a biharmonic hypersurface as defined in this more general setting. In this new perspective Chen’s conjecture has been generalized to the following [Caddeo et al. 2001; 2002]:

\[
\text{Let } \varphi : (M, g) \rightarrow (N, h) \text{ be an isometric immersion into a Riemannian manifold of nonpositive sectional curvature. If } \varphi \text{ is biharmonic then it is minimal.}
\]
This new conjecture has been shown to be true if $M$ is compact [Jiang 1986] or if $H$ is constant [Ou 2010], but false in general [Ou and Tang 2012]. Here we restrict ourselves to complete noncompact biharmonic hypersurfaces and in fact we concentrate our efforts on the consequences of (4a) alone.

To avoid confusion with a terminology used for biharmonic submanifolds, we underline that in what follows by a proper immersion we mean an immersion that is topologically proper: preimages of compact sets are compact sets.

2. Statement of main results

Our first main result is the following.

**Theorem 1.** Let $\varphi : M \to (N, \langle \cdot, \cdot \rangle)$ be an oriented, proper, isometrically immersed, biharmonic hypersurface in the complete manifold $N$. For some origin $o \in N$ assume that

$$\varphi(M) \cap \text{cut}(o) = \emptyset.$$  

Having set $\varphi = \text{dist}_N(\cdot, o)$, suppose that the radial sectional curvature $K^N_{\text{rad}}$ of $N$ satisfies

$$K^N_{\text{rad}} \geq -G(\varphi)$$  

for $\varphi \gg 1$ and some $G \in \ell^2(\mathbb{R}_0^+) \text{ such that } G(0) > 0$, $G'(t) \geq 0$ and $G(t) = o(t^2)$ as $t \to +\infty$. Let $\nu$ be a unit normal vector field along $\varphi$ and suppose

$$\text{Ric}^N(\nu, \nu) \leq 0$$  

along $\varphi$. Then $\varphi$ is minimal. In particular if the sectional curvature $K^N_{\text{sect}}$ is nonpositive, $\varphi(M)$ is unbounded in $N$.

As an immediate consequence of Theorem 1, using [Mari and Rigoli 2010] and [Alías et al. 2009], we obtain:

**Corollary 2.** Let $\varphi : M \to \mathbb{R}^{m+1}$ be an oriented, isometrically immersed, biharmonic hypersurface. If the image $\varphi(M)$ is contained in a nondegenerate open cone of $\mathbb{R}^{m+1}$ or the hypersurface is cylindrically bounded as $\varphi(M) \subset B_r(o) \times \mathbb{R}^{m-1} \subset \mathbb{R}^2 \times \mathbb{R}^{m-1}$, then the immersion cannot be proper.

We recall here that, fixed an origin $o \in \mathbb{R}^{m+1}$, the nondegenerate cone with vertex $o$, direction $a$ and width $\theta$ is the subset

$$\mathcal{C} = \mathcal{C}_{o,a,\theta} = \left\{ p \in \mathbb{R}^{m+1} \setminus \{o\} : \left| \frac{p - o}{|p - o|}, a \right| \geq \cos \theta \right\},$$

where $a \in \mathbb{S}^m$ is a unit vector and $\theta \in (0, \pi/2)$. By nondegenerate we mean that it is strictly smaller than a half-space. On the other hand, following the definition introduced in [Alfás et al. 2009], an immersed hypersurface $\varphi : M \to \mathbb{R}^{m+1}$ is said
to be cylindrically bounded if $\varphi(M) \subset B_r(o) \times \mathbb{R}^{m+1-p} \subset \mathbb{R}^p \times \mathbb{R}^{m+1-p}$, where $p \geq 2$ and $B_r(o) \subset \mathbb{R}^p$ denotes the ball of radius $r$. In particular, $p = 2$ gives the weakest requirement.

To introduce the next result we consider the operator

$$L = \Delta + \text{Ric}^N(v, v)$$

where $v$ is a unit normal vector field along the hypersurface $\varphi : M \to (N, \langle \cdot, \cdot \rangle)$ and we let $\lambda_1^L(M)$ denote its spectral radius. Clearly if $\text{Ric}^N(v, v) \leq 0$ then $\lambda_1^L(M) \geq 0$ but this latter fact can be true even if $\text{Ric}^N(v, v) > 0$ provided this positivity compensate with the geometry of $M$. (For a detailed discussion see [Bianchini et al. 2012]). Thus $\lambda_1^L(M) \geq 0$ is weaker than $\text{Ric}^N(v, v) \leq 0$.

**Theorem 3.** Let $\varphi : M \to (N, \langle \cdot, \cdot \rangle)$ be a biharmonic, complete, oriented hypersurface with mean curvature $H$. Suppose that the operator $L$ in (7) satisfies

$$\lambda_1^L(M) \geq 0.$$ 

If $H \in L^2(M)$ then $\varphi$ is minimal.

This result is extended to a different class of integrability for $H$ in Theorem 7 of Section 3 below.

Next, we consider the case when $(N, \langle \cdot, \cdot \rangle)$ is a Cartan–Hadamard manifold, that is, $N$ is complete, simply connected and with nonpositive sectional curvature. What follows is a gap theorem.

**Theorem 4.** Let $\varphi : M \to (N, \langle \cdot, \cdot \rangle)$ be an isometrically immersed, oriented, biharmonic hypersurface of dimension $m \geq 3$ into a Cartan–Hadamard manifold. Suppose that the mean curvature $H$ satisfies

$$(9) \quad \|H\|_{L^m(M)} < \frac{\omega_m^{1/m}}{\pi^{2(m-1)/2} m^{1+1/m}(m+1)^{1+1/m}},$$

where $\omega_m$ is the volume of the unit ball of $\mathbb{R}^m$. Then $\varphi$ is a minimal hypersurface.

### 3. Proof of the main theorems and some further results

With the notations of Theorem 1 we consider the function $v = \varphi^2 \circ \varphi$. The assumption $\varphi(M) \cap \text{cut}(o) = \emptyset$ implies that $v$ is smooth on $M$. Clearly,

$$|\nabla v| \leq 2\sqrt{v}.$$ 

Since $M$ is complete and noncompact and $\varphi$ is proper we have

$$v(x) \to +\infty \quad \text{as} \quad x \to \infty \quad \text{in} \quad M.$$
To compute $\Delta v$ we recall (see, for instance, [Jorge and Koutroufiotis 1981]) that
\begin{equation}
\Delta (\varphi^2 \circ \varphi) = (\text{Hess } \varphi^2)(\varphi_*(e_i), \varphi_*(e_i)) + \langle \nabla \varphi^2, mH \rangle
\end{equation}
with $\{e_i\}$ a local orthonormal frame on $M$. Let $G \in C^\infty(\mathbb{R}_0^+)$ satisfy
\begin{equation}
G(0) > 0 \quad \text{and} \quad G'(t) \geq 0 \quad \text{on } \mathbb{R}_0^+.
\end{equation}
(In particular, $G$ can be chosen to agree, for $t$ large, with the function $ct^d$, where $0 < d < 2$, or with $ct^2(\log t)^{-\varepsilon}$, where $\varepsilon > 0$.)

If $K_r^N \geq -G$, by the Hessian comparison theorem (see Theorem 2.3 and Remark 2.3 of [Pigola et al. 2008] for the appropriate statement that we are using here) we get
\begin{equation}
\text{Hess}(\varphi^2) \leq C \varphi \sqrt{G(\varphi)} (\cdot, \cdot)
\end{equation}
outside a compact set and for some appropriate constant $C > 0$. Up to modifying $C$ we can assume that (14) is true on $M$. Hence, from (12) and (14) we deduce that
\begin{equation}
\Delta v \leq C^2 \sqrt{v} \sqrt{G(\sqrt{v})} + 2m \sqrt{v} |H|
\end{equation}
on $M$. Next, from (4a), letting $u = H^2$ we get
\begin{equation}
\Delta u = 2H \Delta H + 2|\nabla H|^2 = 2|B|^2 u - 2 \text{Ric}^N(v, v)u + 2|\nabla H|^2.
\end{equation}
Using Newton’s inequality,
\begin{equation}
|B|^2 \geq m|H|^2,
\end{equation}
we obtain
\begin{equation}
\Delta u + 2 \text{Ric}^N(v, v)u - 2mu^2 \geq 2|\nabla H|^2 \geq 0,
\end{equation}
and we are left with a solution $u \geq 0$ of the differential inequality
\begin{equation}
\Delta u + a(x)u - 2mu^2 \geq 0
\end{equation}
with
\begin{equation}
a(x) = 2 \text{Ric}^N(v, v) \circ \varphi(x).
\end{equation}

**Proof of Theorem 1.** First observe that since $\varphi$ is proper and $N$ is complete, the induced metric on $M$ is complete. Next we follow an idea introduced in [Akutagawa and Maeta 2013]. Since $\varphi$ is proper, for every $T \in \mathbb{R}^+$, the set
\begin{equation}
D_T = v^{-1}([0, T])
\end{equation}
is compact. Suppose $u \neq 0$. Then there exists $x_0 \in M$ such that $u(x_0) > 0$ and we can suppose to have chosen $T$ sufficiently large that $x_0 \in D_{T/2} \setminus \partial D_{T/2}$. 
We define
\begin{equation}
F(x) = (T - v(x))^2 u(x)
\end{equation}
on $D_T$. Note that $F \geq 0$, $F \equiv 0$ on $\partial D_T$ and $F(x_0) > 0$. It follows that there exists a positive absolute maximum for $F(x)$ at some point $\tilde{x} \in D_T \setminus \partial D_T$. At this point we have
\begin{equation}
\nabla F \bigg( \frac{\tilde{x}}{D_T} \bigg) = 0 \quad \text{and} \quad \frac{\Delta F}{F} \bigg( \frac{\tilde{x}}{D_T} \bigg) \leq 0.
\end{equation}
From (22), a straightforward computation yields
\begin{equation}
\frac{\nabla u(\tilde{x})}{u(\tilde{x})} = \frac{2}{T - v(\tilde{x})} \nabla v(\tilde{x})
\end{equation}
and
\begin{equation}
\frac{\Delta u(\tilde{x})}{u(\tilde{x})} \leq \frac{2}{T - v(\tilde{x})} \Delta v(\tilde{x}) - \frac{2}{(T - v(\tilde{x}))^2} |\nabla v(\tilde{x})|^2 + \frac{4}{T - v(\tilde{x})} \frac{|\nabla u(\tilde{x})|}{u(\tilde{x})} |\nabla v(\tilde{x})|.
\end{equation}
We use (23), (15) at $\tilde{x}$ with $\sqrt{u} = |H|$, and (10) at $\tilde{x}$ into the above inequality to obtain (omitting $\tilde{x}$ for the ease of notation)
\begin{align*}
\frac{\Delta u}{u} & \leq \frac{2}{T - v} \left[ C^2 \sqrt{G(\sqrt{v})} + 2m \sqrt{u} \right] \sqrt{v} + \frac{6}{(T - v)^2} |\nabla v|^2 \\
& \leq \frac{2}{T - v} \left[ C^2 \sqrt{G(\sqrt{v})} + 2m \sqrt{u} \right] \sqrt{v} + \frac{24}{(T - v)^2} v.
\end{align*}
From (19) we then deduce
\begin{equation}
u \leq \frac{a}{2m} + \frac{C^2 \sqrt{v}}{m(T - v)} \sqrt{G(\sqrt{v})} + \frac{2 \sqrt{v}}{T - v} \sqrt{u} + \frac{12}{m(T - v)^2} v.
\end{equation}
Multiplying by $(T - v(x))^2$ both sides of (24) and using that $a(x) = a_+(x) - a_-(x)$, that $G$ is nondecreasing, and that $\tilde{x} \in D_T$ we have
\begin{align*}
F(\tilde{x}) & \leq \frac{a_+(\tilde{x})}{2m}(T - v(\tilde{x}))^2 + \frac{C^2}{m} \frac{\sqrt{v(\tilde{x})}}{m(T - v(\tilde{x}))} \sqrt{G(\sqrt{v(\tilde{x})})} \\
& \quad + 2 \sqrt{v(\tilde{x})} \sqrt{F(\tilde{x})} + \frac{12}{m} v(\tilde{x}) \\
& \leq \frac{T^2}{2m} a_+(\tilde{x}) + \frac{C^2 T^{3/2}}{m} \sqrt{G(\sqrt{T})} + 2 \sqrt{T} \sqrt{F(\tilde{x})} + \frac{12}{m} T.
\end{align*}
Therefore
\begin{align*}
F(\tilde{x}) - 2 \sqrt{T} \sqrt{F(\tilde{x})} - T Z(T) & \leq 0,
\end{align*}
where
\begin{align*}
Z(T) &= \frac{T}{2m} \sup_{D_T} a_+ + \frac{C^2}{m} \sqrt{T} \sqrt{G(\sqrt{T})} + \frac{12}{m}.
\end{align*}
Note that $Z(T) \geq 0$. Then
\[ F(x_0) \leq F(\tilde{x}) \leq T \left(1 + \sqrt{1 + Z(T)}\right)^2 \leq C^2 T (1 + Z(T)) \]
and therefore, since $x_0 \in D_{T/2}$,
\[ u(x_0) \leq \frac{C^2 T}{(T - v(x_0))^2} \left(T \sup_{D_T} a_+ + \sqrt{T} \sqrt{G(\sqrt{T})}\right) \]
\[ \leq \frac{C^2}{T} \left(T \sup_{D_T} a_+ + \sqrt{T} \sqrt{G(\sqrt{T})}\right) = C^2 \left(\sup_{D_T} a_+ + \frac{1}{\sqrt{T}} \sqrt{G(\sqrt{T})}\right). \]
However, by assumption $a_+ \equiv 0$ and using $G(t) = o(t^2)$ as $t \to +\infty$ we have
\[ T^{-1/2} \sqrt{G(v)} = o(1) \quad \text{as } T \to +\infty. \]
Thus, letting $T \to +\infty$ in (25), we deduce $u(x_0) \leq 0$ which contradicts the assumption $u(x_0) > 0$. The contradiction shows that $u = H^2 \equiv 0$ on $M$, that is, $\varphi$ is minimal.

Suppose now that $K_{\text{sect}}^N \leq 0$. Since $\varphi$ is minimal (15) becomes
\[ \Delta v \leq C^2 \sqrt{v} \sqrt{G(\sqrt{v})}. \]
This, together with (10) and (11), guarantees the validity of the Omori–Yau maximum principle on $M$ (see Theorem 1.9 of [Pigola et al. 2005]). Now the result follows from Theorem 3.9 of [Pigola et al. 2005].

For the proof of Theorem 3 we need the next proposition which is a version, adapted to the present purposes, of Lemma 3.1 in [Brandolini et al. 1998].

**Proposition 5.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold and let $a(x), b(x) \in \mathcal{C}^0(M)$ and suppose that
\[ b(x) \geq 0 \]
and
\[ \lambda_1^L(M) \geq 0 \quad \text{with } L = \Delta + a(x). \]
Let $u \in C^2(M)$ be a solution of
\[ \Delta u + a(x)u - b(x)u = 0 \quad \text{on } M. \]
If $u \in L^2(M)$ then $u \equiv 0$ on $\text{supp}(b(x))$. In particular, if $u$ does not change sign and $b(x) \not\equiv 0$, then $u \equiv 0$.

**Proof.** We suppose $b(x) \not\equiv 0$ otherwise there is nothing to prove. Next, we reason by contradiction and we assume the existence of $x_0 \in \text{supp}(b(x)) \subset M$ such that $u(x_0) \neq 0$ and $b(x_0) \neq 0$. (Note that if $u(x_0) \neq 0$ and $b(x_0) = 0$ by continuity...
we can always find \( x'_0 \) sufficiently close to \( x_0 \) so that \( u(x'_0) \neq 0 \) and \( b(x'_0) \neq 0 \). Choose \( R \gg 1 \) such that \( x_0 \in B_R \). Let \( \psi \) be a cut-off function \( 0 \leq \psi \leq 1 \) satisfying
\[
\psi \equiv 1 \quad \text{on} \; B_R, \quad \text{supp}(\psi) \subseteq B_{R+1}, \quad |\nabla \psi| \leq 2.
\]
Then \( u\psi \in C^2_0(M) \), \( u\psi \neq 0 \) and by the variational characterization of \( \lambda^L_1(B_{R+1}) \) we have
\[
\lambda^L_1(B_{R+1}) \leq \frac{\int_{B_{R+1}} (|\nabla (u\psi)|^2 - a(x)(u\psi)^2)}{\int_{B_{R+1}} (u\psi)^2}.
\]
Since \( \lambda^L_1(M) \geq 0 \) the monotonicity property of eigenvalues yields \( \lambda^L_1(B_{R+1}) > 0 \). Next, we consider the vector field \( W = u\psi^2 \nabla u \). A direct computation using (28) gives
\[
\text{div}(W) = b(x)u^2\psi^2 - a(x)u^2\psi^2 + |\nabla (u\psi)|^2 - u^2|\nabla \psi|^2.
\]
Hence by (29) and the divergence theorem
\[
0 \geq \lambda^L_1(B_{R+1}) \int_{B_{R+1}} u^2\psi^2 - \int_{B_{R+1}} u^2|\nabla \psi|^2 + \int_{B_{R+1}} b(x)u^2\psi^2.
\]
Rearranging, using the properties of \( \psi \) and (26) we obtain
\[
\lambda^L_1(B_{R+1}) \int_{B_R} u^2 - \int_{B_R} b(x)u^2 \leq 4 \int_{B_{R+1}\setminus B_R} u^2.
\]
Letting \( R \to +\infty \) and using the fact that \( u \in L^2(M) \) we deduce
\[
\lambda^L_1(M) \int_{\mathcal{M}} u^2 - \int_{\mathcal{M}} b(x)u^2 \leq 0.
\]
We reach a contradiction by observing that \( \lambda^L_1(M) \geq 0 \) and in a neighborhood of \( x_0 \), \( b(x) \) and \( u^2(x) \) are strictly positive.

The last statement follows immediately from the strong maximum principle and (28) (see the remark after the proof of Theorem 3.5 on page 35 of [Gilbarg and Trudinger 1983]).

**Proof of Theorem 3.** We apply Proposition 5 to the solution \( H \) of (4a) with \( a(x) = \text{Ric}^N(v, v) \) and \( b(x) = |B|^2 \). By Newton’s inequality (17), \( \text{supp}(H) \subseteq \text{supp}(b(x)) \), which gives a contradiction to the conclusion of Proposition 5 unless \( H \equiv 0 \); thus \( \varphi : M \to (N, \langle , \rangle) \) is minimal.

**Corollary 6.** Any biharmonic, isometrically immersed, complete oriented hypersurface \( M \) with mean curvature satisfying \( H \in L^2(M) \) in a space with nonpositive Ricci tensor is minimal.
For the proof of this corollary simply observe that since $\text{Ric}^N(v, v) \leq 0$ then $\lambda_1^L(M) \geq 0$ for $L = \Delta + \text{Ric}^N(v, v)$.

With the aid of Theorem 4.6 in [Pigola et al. 2008] we can extend the range of integrability of $H$ as follows.

**Theorem 7.** Let $\varphi : M \to (N, \langle , \rangle)$ be a biharmonic, isometrically immersed, oriented hypersurface. For some $\Lambda \geq \frac{1}{2}$ let $L_\Lambda = \Delta + 2\Lambda \text{Ric}^N(v, v)$ and suppose that

$$\lambda_1^{L_\Lambda}(M) \geq 0.$$  \hfill (30)

Let $-\frac{1}{2} \leq \beta \leq \Lambda - 1$ and assume that

$$H \in L^{4(\beta+1)}(M).$$  \hfill (31)

Then $\varphi$ is minimal.

**Remark 8.** If $\Lambda = \frac{1}{2}$, $L_\Lambda = L = \Delta + \text{Ric}^N(v, v)$ and $\beta = -\frac{1}{2}$ so that condition (31) becomes $H \in L^2(M)$. In this way, we recover Theorem 3.

**Proof of Theorem 7.** We let $u = H^2$. From the differential inequality (18) and

$$|\nabla H|^2 = \frac{1}{4} \frac{|\nabla u|^2}{u},$$

we deduce that $u$ is a nonnegative solution of

$$u \Delta u + 2 \text{Ric}^N(v, v)u^2 - 2mu^3 \geq \frac{1}{2} |\nabla u|^2.$$  \hfill (32)

By Theorem 1 of [Fischer-Colbrie and Schoen 1980], inequality (30) implies the existence of a positive solution $\psi$ on $M$ of

$$\Delta \psi + 2\Lambda \text{Ric}^N(v, v)\psi = 0.$$  

We can thus apply Theorem 4.6 of [Pigola et al. 2008] with $\varphi = \psi$, $A = -\frac{1}{2}$, $|H| = \Lambda$, $K = 0$, $a(x) = 2\text{Ric}^N(v, v)$, $b(x) = 2m$ and $\sigma = 2$. Note that assumption (4.43) of Theorem 4.6 of [Pigola et al. 2008] is true by (31). It follows that $u \equiv 0$, that is, $\varphi : M \to (N, \langle , \rangle)$ is minimal. \hfill $\square$

We remark that if we let $L_{m/4} = \Delta + (m/2) \text{Ric}^N(v, v)$ and we assume

$$\lambda_1^{L_{m/4}}(M) \geq 0,$$  \hfill (33)

as a consequence of Theorem 7, if $H \in L^m(M)$ then $\varphi$ is minimal.

As a matter of fact, we can avoid assumption (33) and obtain the same conclusion in case $(N, \langle , \rangle)$ is a Cartan–Hadamard manifold. This is the content of Theorem 4. Towards this end, we observe that if $\varphi : M \to (N, \langle , \rangle)$ is an isometric immersion
of dimension $m \geq 2$, Hoffman and Spruck [1974] have shown the validity of the following $L^1$-Sobolev inequality: for every $u \in W^{1,1}_0(M)$,

$$S_1(m)^{-1} \left( \int_M |u|^{m/(m-1)} \right)^{(m-1)/m} \leq \int_M (|\nabla u| + m|H||u|)$$

(34)

with

$$S_1(m) = \frac{\pi^{2m-1} (m + 1)^{1+\frac{1}{m}}}{\omega_m^{1/m} m - 1}$$

(35)

where $\omega_m$ is the volume of the unit ball of $\mathbb{R}^m$ (observe that in [Hoffman and Spruck 1974] the mean curvature vector field is not normalized). Having fixed $\varepsilon > 0$, from (34) we immediately deduce (see for instance [Pigola et al. 2008, pp. 175–176]) that for every $v \in W^{1,2}_0(M)$

$$S_2(m, \varepsilon)^{-1} \left( \int_M |v|^{2m/(m-2)} \right)^{(m-2)/m} \leq \int_M (|\nabla v|^2 + \frac{\varepsilon^2}{4} \left( \frac{m-2}{m-1} \right)^2 m^2 |H|^2 v^2)$$

(36)

with

$$S_2(m, \varepsilon) = \frac{4(m-1)^2}{(m-2)^2} \frac{1 + \varepsilon^2}{\varepsilon^2} S_1(m)^2.$$

(37)

Proof of Theorem 4. In the assumptions of the theorem and by the above discussion we have the validity of (36) on $M$. Next, for $u = H^2$ we rewrite (16) in the form

$$u \Delta u + 2 \operatorname{Ric}^N(v, v) u^2 - 2 |B|^2 u^2 = \frac{1}{2} |\nabla u|^2.$$  

(38)

Since $N$ is Cartan–Hadamard,

$$2(\operatorname{Ric}^N(v, v) - |B|^2) \leq 0.$$  

(39)

From (9) and the fact that $H \in L^m(M)$ we have

$$u \in L^{m/2}(M) \quad \text{with } m/2 > \frac{1}{2},$$

(40)

because $m \geq 3$. Applying Theorem 9.12 of [Pigola et al. 2008] with $\sigma = m/2$, $\alpha = 2/m$ and $A = -\frac{1}{2}$ to (38) we deduce that either $u$ is identically zero or, by formula (9.41) of [Pigola et al. 2008],

$$\left( \int_M |H|^m \right)^{2/m} \geq \frac{1}{(1 + \varepsilon^2)m^2 S_1(m)^2}.$$  

Note that to obtain this inequality we use (37). Thus, letting $\varepsilon \downarrow 0^+$ we obtain

$$\|H\|_{L^m(M)} \geq \frac{1}{mS_1(m)} = \frac{\omega_m^{1/m}}{\pi^{2m-1} m(m+1)^{1+\frac{1}{m}}}.$$  

(41)
Using (35) in this latter we contradict (9). Thus $u \equiv 0$ and $\varphi : M \to (N, \langle \cdot , \cdot \rangle)$ is minimal. 

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References


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