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We consider biharmonic hypersurfaces in complete Riemannian manifolds and prove that, under some additional assumptions, they are minimal.

1. Introduction

According to a definition first given by B. Y. Chen [1991], an isometrically immersed oriented hypersurface in Euclidean space, $\varphi : M \to \mathbb{R}^{m+1}$ is biharmonic if its mean curvature vector field $\boldsymbol{H}$ satisfies

$$\Delta \boldsymbol{H} = 0,$$

where Δ denotes the Laplacian on the hypersurface. It is well known that for submanifolds of Euclidean space, $\operatorname{trace}(B) = m\boldsymbol{H} = \Delta\varphi$, where B is the second fundamental form of the immersion. Hence, for any fixed unit vector $\boldsymbol{a}$ of $\mathbb{R}^{m+1}$,

$$m\Delta\langle \boldsymbol{H}, \boldsymbol{a}\rangle = \Delta^2\langle \varphi, \boldsymbol{a}\rangle \tag{1}$$

and the hypersurface is biharmonic if and only if each component of the immersion φ is a biharmonic function. Chen [1991; 1996] conjectured that a biharmonic hypersurface (in fact any biharmonic submanifold) of $\mathbb{R}^{m+1}$ is minimal, the converse being, of course trivially true. This statement is of a local nature and the conjecture holds for hypersurfaces in $\mathbb{R}^3$ [Chen 1991] and $\mathbb{R}^4$ [Hasanis and Vlachos 1995; Defever 1998]. However, in general, it has been shown to be true only under some additional assumptions, sometimes of a global nature: see for instance [Akutagawa and Maeta 2013] and [Nakauchi and Urakawa 2011].

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This problem can be considered in a more general perspective. Indeed, let (M, g) and (N, h) be Riemannian manifolds and $\varphi : (M, g) \to (N, h)$ a smooth map. Let $\tau(\varphi)$ denote its tension field, that is,

$$\tau(\varphi) = \operatorname{trace}(\nabla d\varphi) = \sum_{i=1}^{m} (\nabla d\varphi)(e_i, e_i), \quad m = \dim M,$$

where $\nabla d\varphi$ is the generalized second fundamental tensor and $\{e_1, \dots, e_m\}$ is a local orthonormal frame on (M, g). Given a relatively compact domain $\Omega \subset M$ one introduces the bienergy functional $E_\tau^\varphi(\Omega)$ on Ω by setting

$$E_\tau^\varphi(\Omega) = \frac{1}{2} \int_\Omega |\tau(\varphi)|^2,$$

where integration is understood with respect to the volume element of (M, g). Then φ is a biharmonic map (meaning a critical point of this functional on M — i.e., on each relatively compact domain $\Omega \subset M$), if and only if the bitension field

$$\tau_2(\varphi) = \Delta\tau(\varphi) - \sum_i R^N\big(\tau(\varphi), \varphi_*(e_i)\big)\varphi_*(e_i) \tag{2}$$

vanishes identically. Here R^N denotes the $(3,1)$ curvature tensor of (N, h).

When $\varphi : (M^m, g) \to (N^{m+1}, h)$ is an isometric immersion of an m-dimensional hypersurface and ν is a local unit normal vector field along φ, writing the mean curvature vector as

$$\boldsymbol{H} = H\nu \tag{3}$$

and indicating with B the second fundamental form in the direction of ν, a heavy computation shows that (2) is equivalent to the system

$$\Delta H - |B|^2 H + \operatorname{Ric}^N(\nu, \nu) H = 0, \tag{4a}$$

$$2B(\nabla H, \cdot)^\sharp + \tfrac{1}{2} m \nabla H^2 - 2H(\operatorname{Ric}^N(\nu, \cdot)^\sharp)^T = 0, \tag{4b}$$

where $^\sharp : TM^* \to TM$ denotes the musical isomorphism, ^{T} the tangential component and Ric^N the Ricci tensor of (N, h) [Ou 2010, Theorem 2.1].

At this point one easily verifies that a biharmonic hypersurface in $\mathbb{R}^{m+1}$ in the sense of Chen is exactly a biharmonic hypersurface as defined in this more general setting. In this new perspective Chen's conjecture has been generalized to the following [Caddeo et al. 2001; 2002]:

> *Let $\varphi : (M, g) \to (N, h)$ be an isometric immersion into a Riemannian manifold of nonpositive sectional curvature. If φ is biharmonic then it is minimal.*

This new conjecture has been shown to be true if M is compact [Jiang 1986] or if H is constant [Ou 2010], but false in general [Ou and Tang 2012]. Here we restrict ourselves to complete noncompact biharmonic hypersurfaces and in fact we concentrate our efforts on the consequences of (4a) alone.

To avoid confusion with a terminology used for biharmonic submanifolds, we underline that in what follows by a *proper immersion* we mean an immersion that is topologically proper: preimages of compact sets are compact sets.

2. Statement of main results

Our first main result is the following.

Theorem 1. *Let $\varphi : M \to (N, \langle\ ,\ \rangle)$ be an oriented, proper, isometrically immersed, biharmonic hypersurface in the complete manifold N. For some origin $o \in N$ assume that*

$$\varphi(M) \cap \mathrm{cut}(o) = \varnothing.$$

Having set $\varrho = \mathrm{dist}_N(\,\cdot\,, o)$, suppose that the radial sectional curvature K^N_{rad} of N satisfies

$$K^N_{\mathrm{rad}} \geq -G(\varrho) \tag{5}$$

for $\varrho \gg 1$ and some $G \in \mathscr{C}^2(\mathbb{R}^+_0)$ such that $G(0) > 0$, $G'(t) \geq 0$ and $G(t) = o(t^2)$ as $t \to +\infty$. Let ν be a unit normal vector field along φ and suppose

$$\mathrm{Ric}^N(\nu, \nu) \leq 0 \tag{6}$$

along φ. Then φ is minimal. In particular if the sectional curvature K^N_{sect} is nonpositive, $\varphi(M)$ is unbounded in N.

As an immediate consequence of Theorem 1, using [Mari and Rigoli 2010] and [Alías et al. 2009], we obtain:

Corollary 2. *Let $\varphi : M \to \mathbb{R}^{m+1}$ be an oriented, isometrically immersed, biharmonic hypersurface. If the image $\varphi(M)$ is contained in a nondegenerate open cone of $\mathbb{R}^{m+1}$ or the hypersurface is cylindrically bounded as $\varphi(M) \subset B_r(o) \times \mathbb{R}^{m-1} \subset \mathbb{R}^2 \times \mathbb{R}^{m-1}$, then the immersion cannot be proper.*

We recall here that, fixed an origin $o \in \mathbb{R}^{m+1}$, the nondegenerate cone with vertex o, direction a and width θ is the subset

$$\mathscr{C} = \mathscr{C}_{o,a,\theta} = \left\{ p \in \mathbb{R}^{m+1} \backslash \{o\} : \left\langle \frac{p-o}{|p-o|}, a \right\rangle \geq \cos\theta \right\},$$

where $a \in \mathbb{S}^m$ is a unit vector and $\theta \in (0, \pi/2)$. By nondegenerate we mean that it is strictly smaller than a half-space. On the other hand, following the definition introduced in [Alías et al. 2009], an immersed hypersurface $\varphi : M \to \mathbb{R}^{m+1}$ is said to be cylindrically bounded if $\varphi(M) \subset B_r(o) \times \mathbb{R}^{m+1-p} \subset \mathbb{R}^p \times \mathbb{R}^{m+1-p}$, where $p \geq 2$ and $B_r(o) \subset \mathbb{R}^p$ denotes the ball of radius r. In particular, $p = 2$ gives the weakest requirement.

To introduce the next result we consider the operator

$$L = \Delta + \mathrm{Ric}^N(\nu, \nu) \tag{7}$$

where ν is a unit normal vector field along the hypersurface $\varphi : M \to (N, \langle\,,\rangle)$ and we let $\lambda_1^L(M)$ denote its spectral radius. Clearly if $\mathrm{Ric}^N(\nu, \nu) \leq 0$ then $\lambda_1^L(M) \geq 0$ but this latter fact can be true even if $\mathrm{Ric}^N(\nu, \nu) > 0$ provided this positivity compensate with the geometry of M. (For a detailed discussion see [Bianchini et al. 2012]). Thus $\lambda_1^L(M) \geq 0$ is weaker than $\mathrm{Ric}^N(\nu, \nu) \leq 0$.

Theorem 3. *Let $\varphi : M \to (N, \langle\,,\rangle)$ be a biharmonic, complete, oriented hypersurface with mean curvature H. Suppose that the operator L in (7) satisfies*

$$\lambda_1^L(M) \geq 0. \tag{8}$$

If $H \in L^2(M)$ then φ is minimal.

This result is extended to a different class of integrability for H in Theorem 7 of Section 3 below.

Next, we consider the case when $(N, \langle\,,\rangle)$ is a Cartan–Hadamard manifold, that is, N is complete, simply connected and with nonpositive sectional curvature. What follows is a gap theorem.

Theorem 4. *Let $\varphi : M \to (N, \langle\,,\rangle)$ be an isometrically immersed, oriented, biharmonic hypersurface of dimension $m \geq 3$ into a Cartan–Hadamard manifold. Suppose that the mean curvature H satisfies*

$$\|H\|_{L^m(M)} < \frac{\omega_m^{1/m}}{\pi 2^{m-1}} \frac{m-1}{m(m+1)^{1+\frac{1}{m}}}, \tag{9}$$

where ω_m is the volume of the unit ball of $\mathbb{R}^m$. Then φ is a minimal hypersurface.

3. Proof of the main theorems and some further results

With the notations of Theorem 1 we consider the function $v = \varrho^2 \circ \varphi$. The assumption $\varphi(M) \cap \mathrm{cut}(o) = \varnothing$ implies that v is smooth on M. Clearly,

$$|\nabla v| \leq 2\sqrt{v}. \tag{10}$$

Since M is complete and noncompact and φ is proper we have

$$v(x) \to +\infty \quad \text{as } x \to \infty \quad \text{in } M. \tag{11}$$

To compute Δv we recall (see, for instance, [Jorge and Koutroufiotis 1981]) that

$$\Delta(\varrho^2 \circ \varphi) = (\mathrm{Hess}\,\varrho^2)\big(\varphi_*(e_i), \varphi_*(e_i)\big) + \langle \nabla \varrho^2, m\boldsymbol{H} \rangle \tag{12}$$

with $\{e_i\}$ a local orthonormal frame on M. Let $G \in C^\infty(\mathbb{R}_0^+)$ satisfy

$$G(0) > 0 \qquad \text{and} \qquad G'(t) \geq 0 \ \text{ on } \mathbb{R}_0^+. \tag{13}$$

(In particular, G can be chosen to agree, for t large, with the function ct^d, where $0 < d < 2$, or with $ct^2(\log t)^{-\varepsilon}$, where $\varepsilon > 0$.)

If $K_{\mathrm{rad}}^N \geq -G$, by the Hessian comparison theorem (see Theorem 2.3 and Remark 2.3 of [Pigola et al. 2008] for the appropriate statement that we are using here) we get

$$\mathrm{Hess}(\varrho^2) \leq C\varrho\sqrt{G(\varrho)}\langle\, ,\rangle \tag{14}$$

outside a compact set and for some appropriate constant $C > 0$. Up to modifying C we can assume that (14) is true on M. Hence, from (12) and (14) we deduce that

$$\Delta v \leq C^2\sqrt{v}\sqrt{G(\sqrt{v})} + 2m\sqrt{v}|H| \tag{15}$$

on M. Next, from (4a), letting $u = H^2$ we get

$$\Delta u = 2H\Delta H + 2|\nabla H|^2 = 2|B|^2 u - 2\,\mathrm{Ric}^N(\nu,\nu)u + 2|\nabla H|^2. \tag{16}$$

Using Newton's inequality,

$$|B|^2 \geq m|H|^2, \tag{17}$$

we obtain

$$\Delta u + 2\,\mathrm{Ric}^N(\nu,\nu)u - 2mu^2 \geq 2|\nabla H|^2 \geq 0, \tag{18}$$

and we are left with a solution $u \geq 0$ of the differential inequality

$$\Delta u + a(x)u - 2mu^2 \geq 0 \tag{19}$$

with

$$a(x) = 2\,\mathrm{Ric}^N(\nu,\nu) \circ \varphi(x). \tag{20}$$

Proof of Theorem 1. First observe that since φ is proper and N is complete, the induced metric on M is complete. Next we follow an idea introduced in [Akutagawa and Maeta 2013]. Since φ is proper, for every $T \in \mathbb{R}^+$, the set

$$D_T = v^{-1}([0, T])$$

is compact. Suppose $u \not\equiv 0$. Then there exists $x_0 \in M$ such that $u(x_0) > 0$ and we can suppose to have chosen T sufficiently large that $x_0 \in D_{T/2} \backslash \partial D_{T/2}$.

We define

$$F(x) = (T - v(x))^2 u(x) \tag{21}$$

on D_T. Note that $F \geq 0$, $F \equiv 0$ on ∂D_T and $F(x_0) > 0$. It follows that there exists a positive absolute maximum for $F(x)$ at some point $\bar{x} \in D_T \backslash \partial D_T$. At this point we have

$$\frac{\nabla F}{F}(\bar{x}) = 0 \quad \text{and} \quad \frac{\Delta F}{F}(\bar{x}) \leq 0. \tag{22}$$

From (22), a straightforward computation yields

$$\frac{\nabla u(\bar{x})}{u(\bar{x})} = \frac{2}{T - v(\bar{x})} \nabla v(\bar{x}) \tag{23}$$

and

$$\frac{\Delta u(\bar{x})}{u(\bar{x})} \leq \frac{2}{T - v(\bar{x})} \Delta v(\bar{x}) - \frac{2}{(T - v(\bar{x}))^2} |\nabla v(\bar{x})|^2 + \frac{4}{T - v(\bar{x})} \frac{|\nabla u(\bar{x})|}{u(\bar{x})} |\nabla v(\bar{x})|.$$

We use (23), (15) at $\bar{x}$ with $\sqrt{u} = |H|$, and (10) at $\bar{x}$ into the above inequality to obtain (omitting $\bar{x}$ for the ease of notation)

$$\begin{aligned} \frac{\Delta u}{u} &\leq \frac{2}{T - v} \big[C^2 \sqrt{G(\sqrt{v})} + 2m\sqrt{u}\big] \sqrt{v} + \frac{6}{(T - v)^2} |\nabla v|^2 \\ &\leq \frac{2}{T - v} \big[C^2 \sqrt{G(\sqrt{v})} + 2m\sqrt{u}\big] \sqrt{v} + \frac{24}{(T - v)^2} v. \end{aligned}$$

From (19) we then deduce

$$u \leq \frac{a}{2m} + \frac{C^2 \sqrt{v}}{m(T - v)} \sqrt{G(\sqrt{v})} + \frac{2\sqrt{v}}{T - v} \sqrt{u} + \frac{12}{m(T - v)^2} v. \tag{24}$$

Multiplying by $(T - v(x))^2$ both sides of (24) and using that $a(x) = a_+(x) - a_-(x)$, that G is nondecreasing, and that $\bar{x} \in D_T$ we have

$$\begin{aligned} F(\bar{x}) &\leq \frac{a_+(\bar{x})}{2m} (T - v(\bar{x}))^2 + \frac{C^2 \sqrt{v(\bar{x})}}{m} (T - v(\bar{x})) \sqrt{G(\sqrt{v(\bar{x})})} \\ &\qquad + 2\sqrt{v(\bar{x})} \sqrt{F(\bar{x})} + \frac{12}{m} v(\bar{x}) \\ &\leq \frac{T^2}{2m} a_+(\bar{x}) + \frac{C^2 T^{3/2}}{m} \sqrt{G(\sqrt{T})} + 2\sqrt{T} \sqrt{F(\bar{x})} + \frac{12}{m} T. \end{aligned}$$

Therefore

$$F(\bar{x}) - 2\sqrt{T} \sqrt{F(\bar{x})} - T Z(T) \leq 0,$$

where

$$Z(T) = \frac{T}{2m} \sup_{D_T} a_+ + \frac{C^2}{m} \sqrt{T} \sqrt{G(\sqrt{T})} + \frac{12}{m}.$$

Note that $Z(T) \geq 0$. Then

$$F(x_0) \leq F(\bar{x}) \leq T\big(1+\sqrt{1+Z(T)}\big)^2 \leq C^2 T(1+Z(T))$$

and therefore, since $x_0 \in D_{T/2}$,

$$\begin{aligned} u(x_0) &\leq \frac{C^2 T}{(T-v(x_0))^2}\big(T \sup\nolimits_{D_T} a_+ + \sqrt{T}\sqrt{G(\sqrt{T})}\big) \\ &\leq \frac{C^2}{T}\big(T \sup\nolimits_{D_T} a_+ + \sqrt{T}\sqrt{G(\sqrt{T})}\big) = C^2\big(\sup\nolimits_{D_T} a_+ + \frac{1}{\sqrt{T}}\sqrt{G(\sqrt{T})}\big). \end{aligned}$$

However, by assumption $a_+ \equiv 0$ and using $G(t) = o(t^2)$ as $t \to +\infty$ we have

$$T^{-1/2}\sqrt{G(\sqrt{T})} = o(1) \quad \text{as } T \to +\infty.$$

Thus, letting $T \to +\infty$ in (25), we deduce $u(x_0) \leq 0$ which contradicts the assumption $u(x_0) > 0$. The contradiction shows that $u = H^2 \equiv 0$ on M, that is, φ is minimal.

Suppose now that $K^N_{\mathrm{sect}} \leq 0$. Since φ is minimal (15) becomes

$$\Delta v \leq C^2\sqrt{v}\sqrt{G(\sqrt{v})}. \tag{25}$$

This, together with (10) and (11), guarantees the validity of the Omori–Yau maximum principle on M (see Theorem 1.9 of [Pigola et al. 2005]). Now the result follows from Theorem 3.9 of [Pigola et al. 2005]. □

For the proof of Theorem 3 we need the next proposition which is a version, adapted to the present purposes, of Lemma 3.1 in [Brandolini et al. 1998].

Proposition 5. *Let $(M, \langle\,,\rangle)$ be a complete manifold and let $a(x), b(x) \in \mathscr{C}^0(M)$ and suppose that*

$$b(x) \geq 0 \tag{26}$$

and

$$\lambda_1^L(M) \geq 0 \quad \textit{with } L = \Delta + a(x). \tag{27}$$

Let $u \in C^2(M)$ be a solution of

$$\Delta u + a(x)u - b(x)u = 0 \quad \textit{on } M. \tag{28}$$

If $u \in L^2(M)$ then $u \equiv 0$ on $\operatorname{supp}(b(x))$. *In particular, if u does not change sign and $b(x) \not\equiv 0$, then $u \equiv 0$.*

Proof. We suppose $b(x) \not\equiv 0$ otherwise there is nothing to prove. Next, we reason by contradiction and we assume the existence of $x_0 \in \operatorname{supp}(b(x)) \subset M$ such that $u(x_0) \neq 0$ and $b(x_0) \neq 0$. (Note that if $u(x_0) \neq 0$ and $b(x_0) = 0$ by continuity we can always find x_0' sufficiently close to x_0 so that $u(x_0') \neq 0$ and $b(x_0') \neq 0$). Choose $R \gg 1$ such that $x_0 \in B_R$. Let ψ be a cut-off function $0 \leq \psi \leq 1$ satisfying

$$\psi \equiv 1 \quad \text{on } B_R, \quad \operatorname{supp}(\psi) \subseteq B_{R+1}, \quad |\nabla \psi| \leq 2.$$

Then $u\psi \in \mathscr{C}_0^2(M)$, $u\psi \neq 0$ and by the variational characterization of $\lambda_1^L(B_{R+1})$ we have

$$\lambda_1^L(B_{R+1}) \leq \frac{\int_{B_{R+1}} \left(|\nabla(u\psi)|^2 - a(x)(u\psi)^2\right)}{\int_{B_{R+1}} (u\psi)^2}. \tag{29}$$

Since $\lambda_1^L(M) \geq 0$ the monotonicity property of eigenvalues yields $\lambda_1^L(B_{R+1}) > 0$. Next, we consider the vector field $W = u\psi^2 \nabla u$. A direct computation using (28) gives

$$\operatorname{div}(W) = b(x)u^2\psi^2 - a(x)u^2\psi^2 + |\nabla(u\psi)|^2 - u^2|\nabla\psi|^2.$$

Hence by (29) and the divergence theorem

$$0 \geq \lambda_1^L(B_{R+1}) \int_{B_{R+1}} u^2\psi^2 - \int_{B_{R+1}} u^2|\nabla\psi|^2 + \int_{B_{R+1}} b(x)u^2\psi^2.$$

Rearranging, using the properties of ψ and (26) we obtain

$$\lambda_1^L(B_{R+1}) \int_{B_R} u^2 - \int_{B_R} b(x)u^2 \leq 4 \int_{B_{R+1} \setminus B_R} u^2.$$

Letting $R \to +\infty$ and using the fact that $u \in L^2(M)$ we deduce

$$\lambda_1^L(M) \int_M u^2 - \int_M b(x)u^2 \leq 0.$$

We reach a contradiction by observing that $\lambda_1^L(M) \geq 0$ and in a neighborhood of x_0, $b(x)$ and $u^2(x)$ are strictly positive.

The last statement follows immediately from the strong maximum principle and (28) (see the remark after the proof of Theorem 3.5 on page 35 of [Gilbarg and Trudinger 1983]). $\square$

Proof of Theorem 3. We apply Proposition 5 to the solution H of (4a) with $a(x) = \operatorname{Ric}^N(\nu, \nu)$ and $b(x) = |B|^2$. By Newton's inequality (17), $\operatorname{supp}(H) \subseteq \operatorname{supp}(b(x))$, which gives a contradiction to the conclusion of Proposition 5 unless $H \equiv 0$; thus $\varphi : M \to (N, \langle\,,\rangle)$ is minimal. $\square$

Corollary 6. *Any biharmonic, isometrically immersed, complete oriented hypersurface M with mean curvature satisfying $H \in L^2(M)$ in a space with nonpositive Ricci tensor is minimal.*

For the proof of this corollary simply observe that since $\mathrm{Ric}^N(\nu,\nu)\le 0$ then $\lambda_1^L(M)\ge 0$ for $L=\Delta+\mathrm{Ric}^N(\nu,\nu)$.

With the aid of Theorem 4.6 in [Pigola et al. 2008] we can extend the range of integrability of H as follows.

Theorem 7. *Let $\varphi : M \to (N,\langle\,,\rangle)$ be a biharmonic, isometrically immersed, oriented hypersurface. For some $\Lambda\ge\frac{1}{2}$ let $L_\Lambda=\Delta+2\Lambda\,\mathrm{Ric}^N(\nu,\nu)$ and suppose that*

$$\lambda_1^{L_\Lambda}(M)\ge 0. \tag{30}$$

Let $-\frac{1}{2}\le\beta\le\Lambda-1$ and assume that

$$H\in L^{4(\beta+1)}(M). \tag{31}$$

Then φ is minimal.

Remark 8. If $\Lambda=\frac{1}{2}$, $L_\Lambda=L=\Delta+\mathrm{Ric}^N(\nu,\nu)$ and $\beta=-\frac{1}{2}$ so that condition (31) becomes $H\in L^2(M)$. In this way, we recover Theorem 3.

Proof of Theorem 7. We let $u=H^2$. From the differential inequality (18) and

$$|\nabla H|^2=\frac{1}{4}\frac{|\nabla u|^2}{u}$$

we deduce that u is a nonnegative solution of

$$u\Delta u+2\,\mathrm{Ric}^N(\nu,\nu)u^2-2mu^3\ge\tfrac{1}{2}|\nabla u|^2. \tag{32}$$

By Theorem 1 of [Fischer-Colbrie and Schoen 1980], inequality (30) implies the existence of a positive solution ψ on M of

$$\Delta\psi+2\Lambda\,\mathrm{Ric}^N(\nu,\nu)\psi=0.$$

We can thus apply Theorem 4.6 of [Pigola et al. 2008] with $\varphi=\psi$, $A=-\frac{1}{2}$, $|\boldsymbol{H}|=\Lambda$, $K=0$, $a(x)=2\,\mathrm{Ric}^N(\nu,\nu)$, $b(x)=2m$ and $\sigma=2$. Note that assumption (4.43) of Theorem 4.6 of [Pigola et al. 2008] is true by (31). It follows that $u\equiv 0$, that is, $\varphi : M\to(N,\langle\,,\rangle)$ is minimal. □

We remark that if we let $L_{m/4}=\Delta+(m/2)\,\mathrm{Ric}^N(\nu,\nu)$ and we assume

$$\lambda_1^{L_{m/4}}(M)\ge 0, \tag{33}$$

as a consequence of Theorem 7, if $H\in L^m(M)$ then φ is minimal.

As a matter of fact, we can avoid assumption (33) and obtain the same conclusion in case $(N,\langle\,,\rangle)$ is a Cartan–Hadamard manifold. This is the content of Theorem 4. Towards this end, we observe that if $\varphi : M\to(N,\langle\,,\rangle)$ is an isometric immersion of dimension $m \geq 2$, Hoffman and Spruck [1974] have shown the validity of the following L^1-Sobolev inequality: for every $u \in W_0^{1,1}(M)$,

$$S_1(m)^{-1}\left(\int_M |u|^{m/(m-1)}\right)^{(m-1)/m} \leq \int_M \left(|\nabla u| + m|H||u|\right) \tag{34}$$

with

$$S_1(m) = \frac{\pi 2^{m-1}}{\omega_m^{1/m}} \frac{(m+1)^{1+\frac{1}{m}}}{m-1} \tag{35}$$

where ω_m is the volume of the unit ball of $\mathbb{R}^m$ (observe that in [Hoffman and Spruck 1974] the mean curvature vector field is not normalized). Having fixed $\varepsilon > 0$, from (34) we immediately deduce (see for instance [Pigola et al. 2008, pp. 175–176]) that for every $v \in W_0^{1,2}(M)$

(36)

$$S_2(m,\varepsilon)^{-1}\left(\int_M |v|^{2m/(m-2)}\right)^{(m-2)/m} \leq \int_M \left(|\nabla v|^2 + \frac{\varepsilon^2}{4}\left(\frac{m-2}{m-1}\right)^2 m^2 |H|^2 v^2\right)$$

with

$$S_2(m,\varepsilon) = \frac{4(m-1)^2}{(m-2)^2} \frac{1+\varepsilon^2}{\varepsilon^2} S_1(m)^2. \tag{37}$$

Proof of Theorem 4. In the assumptions of the theorem and by the above discussion we have the validity of (36) on M. Next, for $u = H^2$ we rewrite (16) in the form

$$u\Delta u + 2\,\mathrm{Ric}^N(\nu,\nu)u^2 - 2|B|^2 u^2 = \tfrac{1}{2}|\nabla u|^2. \tag{38}$$

Since N is Cartan–Hadamard,

$$2(\mathrm{Ric}^N(\nu,\nu) - |B|^2) \leq 0. \tag{39}$$

From (9) and the fact that $H \in L^m(M)$ we have

$$u \in L^{m/2}(M) \quad \text{with } m/2 > \tfrac{1}{2}, \tag{40}$$

because $m \geq 3$. Applying Theorem 9.12 of [Pigola et al. 2008] with $\sigma = m/2$, $\alpha = 2/m$ and $A = -\frac{1}{2}$ to (38) we deduce that either u is identically zero or, by formula (9.41) of [Pigola et al. 2008],

$$\left(\int_M |H|^m\right)^{2/m} \geq \frac{1}{(1+\varepsilon^2)m^2 S_1(m)^2}.$$

Note that to obtain this inequality we use (37). Thus, letting $\varepsilon \downarrow 0^+$ we obtain

$$\|H\|_{L^m(M)} \geq \frac{1}{mS_1(m)} = \frac{\omega_m^{1/m}}{\pi 2^{m-1}} \frac{m-1}{m(m+1)^{1+\frac{1}{m}}}.$$

Using (35) in this latter we contradict (9). Thus $u \equiv 0$ and $\varphi : M \to (N, \langle\, , \rangle)$ is minimal. □

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