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**FIXED POINTS OF ENDOMORPHISMS
OF VIRTUALLY FREE GROUPS**

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A fixed point theorem is proved for inverse transducers, which leads to an automata-theoretic proof of the fixed point subgroup of an endomorphism of a finitely generated virtually free group being finitely generated. If the endomorphism is uniformly continuous for the hyperbolic metric, it is proved that the set of regular fixed points in the hyperbolic boundary has finitely many orbits under the action of the finite fixed points. In the automorphism case, it is shown that these regular fixed points are either exponentially stable attractors or exponentially stable repellers.

1. Introduction

Throughout the paper, the ambient groups are assumed to be finitely generated.

Gersten [1987] proved that the fixed point subgroup of a free group automorphism φ is finitely generated. Using a different approach, Cooper [1987] gave an alternative proof, proving also that the fixed points of the continuous extension of φ to the boundary of the free group is, in some sense, finitely generated. Bestvina and Handel [1992] achieved a major breakthrough with their innovative train track techniques, bounding the rank of the fixed point subgroup and the generating set for the infinite fixed points. Their approach was pursued by Maslakova [2003], who considered the problem of effectively computing a basis for the fixed point subgroup. The paper turned out to contain some errors, and subsequently a new paper by Bogopolski and Maslakova [2012] was posted on arXiv with the purpose of correcting these errors.

Gersten's result was generalized to further classes of groups and endomorphisms in subsequent years. Goldstein and Turner extended it to monomorphisms of free groups [1985] and to arbitrary endomorphisms [1986]. Collins and Turner extended it to automorphisms of free products of freely indecomposable groups [1994];

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see the survey by Ventura [2002]. With respect to automorphisms, the widest generalization is to hyperbolic groups and is due to Paulin [1989].

Sykiotis [2002] extended Collins and Turner's result to arbitrary endomorphisms of virtually free groups using symmetric endomorphisms; see also [Sykiotis 2007] for further results on symmetric endomorphisms. In [Silva 2012], we generalized Goldstein and Turner's automata-theoretic proof to arbitrary endomorphisms of free products of cyclic groups. In the present paper, this result is extended to arbitrary endomorphisms of virtually free groups, providing an automata-theoretic alternative to Sykiotis' result.

This is done by reducing the problem to the rationality of some languages associated to a finite inverse transducer, and subsequent application of Anisimov and Seifert's theorem.

Infinite fixed points of automorphisms of free groups were discussed by Gaboriau, Jaeger, Levitt, and Lustig [Gaboriau et al. 1998], where it is remarked in particular that some of the results would hold for virtually free groups with some adaptations.

In [Silva 2010], we discussed infinite fixed points for monomorphisms of free products of cyclic groups, the group case of a more general setting based on the concept of special confluent rewriting system. These results are now extended to endomorphisms with finite kernel of virtually free groups (which are precisely the uniformly continuous endomorphisms for the hyperbolic metric), and we discuss the dynamical nature of the regular fixed points in the automorphism case, generalizing the results of [Gaboriau et al. 1998] on free groups.

The paper is organized as follows. Section 2 is devoted to preliminaries on groups and automata. We discuss inverse transducers in Section 3, proving a useful fixed point theorem. In Section 4 we prove that the fixed point subgroup is finitely generated for arbitrary endomorphisms of a (finitely generated) virtually free group G .

In Section 5 we get a rewriting system with good properties to represent the elements of G , and in Section 6 we use it to construct a simple model for the hyperbolic boundary of G . We study uniformly continuous endomorphisms in Section 7 and in Section 8 we prove that the infinite fixed points of such endomorphisms are, in some sense, finitely generated.

The classification of the infinite fixed points of automorphisms is performed in Section 9, and Section 10 includes an example and some open problems.

2. Preliminaries

Throughout the paper, we assume alphabets to be *finite*. We start with some group-theoretic definitions. Given an alphabet A , we denote by A^{-1} a set of *formal inverses* of A , and write $\tilde{A} = A \cup A^{-1}$. We extend the mapping $a \mapsto a^{-1}$ to an

involution of the free monoid \tilde{A}^* in the obvious way. As usual, the *free group on A* is the quotient of \tilde{A}^* by the congruence generated by the relation $\{(aa^{-1}, 1) : a \in \tilde{A}\}$. We denote by $\theta : \tilde{A}^* \rightarrow F_A$ the canonical morphism.

Let

$$R_A = \tilde{A}^* \setminus \left(\bigcup_{a \in \tilde{A}} \tilde{A}^* a a^{-1} \tilde{A}^* \right)$$

be the subset of all *reduced words* in \tilde{A}^* . It is well known that, for every $g \in F_A$, $g\theta^{-1}$ contains a unique reduced word, denoted by \bar{g} . We also write $\bar{u} = \overline{u\theta}$ for every $u \in \tilde{A}^*$. Note that the equivalence $u\theta = v\theta \Leftrightarrow \bar{u} = \bar{v}$ holds for all $u, v \in \tilde{A}^*$.

A group G is *virtually free* if G has a free subgroup F of finite index. In view of Nielsen's theorem, it is well-known that F can be assumed to be normal, and is finitely generated if G is finitely generated itself. Therefore every finitely generated virtually free group G admits a decomposition as a disjoint union

$$G = F \cup Fb_1 \cup \dots \cup Fb_m,$$

where $F \trianglelefteq G$ is a free group of finite rank and $b_1, \dots, b_m \in G$.

We shall need also some basic concepts from automata theory.

Let A be a (finite) alphabet. A subset of A^* is called an *A-language*. We say that $\mathcal{A} = (Q, q_0, T, \delta)$ is a (finite) *deterministic A-automaton* if

- Q is a (finite) set,
- $q_0 \in Q$ and $T \subseteq Q$,
- $\delta : Q \times A \rightarrow Q$ is a partial mapping.

We extend δ to a partial mapping $Q \times A^* \rightarrow Q$ by induction through

$$(q, 1)\delta = q, \quad (q, ua)\delta = ((q, u)\delta, a)\delta \quad (u \in A^*, a \in A).$$

When the automaton is clear from the context, we write $qu = (q, u)\delta$. We can view \mathcal{A} as a directed graph with edges labeled by letters $a \in A$ by identifying $(p, a)\delta = q$ with the edge $p \xrightarrow{a} q$. We denote by $E(\mathcal{A}) \subseteq Q \times A \times Q$ the set of all such edges.

A *finite nontrivial path* in \mathcal{A} is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} p_n$$

with $(p_{i-1}, a_i, p_i) \in E(\mathcal{A})$ for $i = 1, \dots, n$. Its *label* is the word $a_1 \dots a_n \in A^*$. It is said to be a *successful path* if $p_0 = q_0$ and $p_n \in T$. We also consider the *trivial path* $p \xrightarrow{1} p$ for $p \in Q$. It is successful if $p = q_0 \in T$.

The *language L recognized by \mathcal{A}* is the set of all labels of successful paths in \mathcal{A} . Equivalently, $L(\mathcal{A}) = \{u \in A^* : q_0 u \in T\}$. If $(p_{i-1}, a_i, p_i) \in E(\mathcal{A})$ for every

$i \in \mathbb{N}$, we may also consider the *infinite path*

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \cdots .$$

Its label is the (right) infinite word $a_1 a_2 a_3 \cdots$. We denote by A^ω the set of all (right) infinite words on the alphabet A , and also write $A^\infty = A^* \cup A^\omega$. We denote by $L_\omega(\mathcal{A})$ the set of labels of all infinite paths $q_0 \longrightarrow \cdots$ in \mathcal{A} .

Given $u \in A^*$ and $\alpha \in A^\infty$, we say that u is a *prefix* of α and write $u \leq \alpha$ if $\alpha = u\beta$ for some $\beta \in A^\infty$. By convention, this includes the case $\alpha \leq \alpha$ for $\alpha \in A^\omega$. For every $n \in \mathbb{N}$, we denote by $\alpha^{[n]}$ the prefix of length n of α , applying the convention that $\alpha^{[n]} = \alpha$ if $n > |\alpha|$.

It is immediate that (A^∞, \leq) is a complete \wedge -semilattice: given $\alpha, \beta \in A^\infty$, $\alpha \wedge \beta$ is the longest common prefix of α and β (or α if $\alpha = \beta \in A^\omega$). The \wedge operator will play a crucial role in later sections of the paper.

The *star* operator on A -languages is defined by

$$L^* = \bigcup_{n \geq 0} L^n,$$

where $L^0 = \{1\}$. An A -language L is said to be *rational* if L can be obtained from finite A -languages using finitely many times the union, product, and star operators (this is called a *rational expression*). Alternatively, by Kleene's theorem [Berstel 1979, Section III], L is rational if and only if it is recognized by a finite deterministic A -automaton \mathcal{A} . The definition through rational expressions generalizes to subsets of an arbitrary group in the obvious way. Moreover, if we fix a homomorphism $\pi : A^* \rightarrow G$, the rational subsets of G are the images by π of the rational A -languages. For obvious reasons, we shall be dealing mostly with matched homomorphisms. A homomorphism $\pi : \tilde{A}^* \rightarrow G$ is said to be *matched* if $a^{-1}\pi = (a\pi)^{-1}$ for every $a \in A$. For details on rational languages and subsets, the reader is referred to [Berstel 1979; Sakarovitch 2003].

We shall need also the following classical result of Anisimov and Seifert.

Proposition 2.1 [Sakarovitch 2003, Proposition II.6.2]. *Let H be a subgroup of a group G . Then H is a rational subset of G if and only if H is finitely generated.*

We end this section with an elementary observation that helps us to establish that fixed point subgroups are finitely generated.

Proposition 2.2. *Let $\pi : \tilde{A}^* \rightarrow G$ be a matched epimorphism and let $X \subseteq G$. Let \mathcal{A} be a finite \tilde{A} -automaton such that*

- (i) $L(\mathcal{A}) \subseteq X\pi^{-1}$,
- (ii) $L(\mathcal{A}) \cap x\pi^{-1} \neq \emptyset$ for every $x \in X$.

Then X is a rational subset of G .

Proof. It follows immediately that $X = (L(\mathcal{A}))\pi$, so X is a rational subset of G . \square

3. Inverse transducers

Given a finite alphabet A , we say that $\mathcal{T} = (Q, q_0, \delta, \lambda)$ is a (finite) *deterministic A -transducer* if

- Q is a (finite) set,
- $q_0 \in Q$,
- $\delta : Q \times A \rightarrow Q$ and $\lambda : Q \times A \rightarrow A^*$ are mappings.

As in the automaton case, we may extend δ to a mapping $Q \times A^* \rightarrow Q$. Similarly, we extend λ to a mapping $Q \times A^* \rightarrow A^*$ through

$$(q, 1)\lambda = 1, \quad (q, ua)\lambda = (q, u)\lambda((q, u)\delta, a)\lambda \quad (u \in A^*, a \in A).$$

When the transducer is clear from the context, we write $qa = (q, a)\delta$. We can view \mathcal{T} as a directed graph with edges labeled by elements of $A \times A^*$ (represented in the form $a|w$) by identifying $(p, a)\delta = q$, $(p, a)\lambda = w$ with the edge $p \xrightarrow{a|w} q$. The set of all such edges is denoted by $E(\mathcal{T}) \subseteq Q \times A \times A^* \times Q$. If $pu = q$ and $(p, u)\lambda = v$, we also write $p \xrightarrow{u|v} q$ and call it a path in \mathcal{T} .

It is immediate that, given $u \in A^*$, there exists exactly one path in \mathcal{T} of the form $q_0 \xrightarrow{u|v} q$. We write $u\widehat{\mathcal{T}} = v$, thus defining a mapping $\widehat{\mathcal{T}} : A^* \rightarrow A^*$.

Assume now that $\mathcal{T} = (Q, q_0, \delta, \lambda)$ is a deterministic \widetilde{A} -transducer such that

$$p \xrightarrow{a|u} q \text{ is an edge of } \mathcal{T} \text{ if and only if } q \xrightarrow{a^{-1}|u^{-1}} p \text{ is an edge of } \mathcal{T}.$$

Then \mathcal{T} is said to be *inverse*.

Proposition 3.1. *Let $\mathcal{T} = (Q, q_0, \delta, \lambda)$ be an inverse \widetilde{A} -transducer. Then*

- (i) $\delta : Q \times \widetilde{A}^* \rightarrow Q$ induces a mapping $\widehat{\delta} : Q \times F_A \rightarrow Q$ by $(q, u\theta)\widehat{\delta} = (q, u)\delta$,
- (ii) $\widehat{\mathcal{T}} : \widetilde{A}^* \rightarrow \widetilde{A}^*$ induces a mapping $\widetilde{\mathcal{T}} : F_A \rightarrow F_A$ by $u\theta\widetilde{\mathcal{T}} = u\widehat{\mathcal{T}}\theta$.

Proof. (i) Since the free group congruence \sim is generated by the pairs $(aa^{-1}, 1)$, it suffices to show that $(q, vaa^{-1}w)\delta = (q, vw)\delta$ for all $q \in Q$; $v, w \in \widetilde{A}^*$ and $a \in \widetilde{A}$.

Since δ is a full mapping, we have a path

$$(1) \quad q \xrightarrow{v|v'} q_1 \xrightarrow{a|u} q_2 \xrightarrow{a^{-1}|u'} q_3 \xrightarrow{w|w'} q_4$$

in \mathcal{T} . Since \mathcal{T} is inverse (in particular deterministic), we must have $u' = u^{-1}$ and $q_3 = q_1$. Hence we also have a path

$$q \xrightarrow{v|v'} q_1 \xrightarrow{w|w'} q_4$$

and so $(q, vaa^{-1}w)\delta = q_4 = (q, vw)\delta$ as required.

(ii) Similarly to part (i), it suffices to show that $(vaa^{-1}w)\widehat{\mathcal{T}}\theta = (vw)\widehat{\mathcal{T}}\theta$ for all $v, w \in \widetilde{A}^*$ and $a \in \widetilde{A}$.

We consider the path (1) for $q = q_0$. Since $u' = u^{-1}$ and $q_3 = q_1$, we get

$$(vaa^{-1}w)\widehat{\mathcal{T}}\theta = (v'uu^{-1}w')\theta = (v'w')\theta = (vw)\widehat{\mathcal{T}}\theta,$$

as required. \square

We now prove one of our main results, generalizing Goldstein and Turner's proof [1986] to mappings induced by inverse transducers.

Theorem 3.2. *Let \mathcal{T} be a finite inverse \widetilde{A} -transducer and let $z \in F_A$. Then*

$$L = \{g \in F_A : g\widetilde{\mathcal{T}} = gz\}$$

is rational.

Proof. Write $\mathcal{T} = (Q, q_0, \delta, \lambda)$. For every $g \in F_A$, let $P_1(g) = g^{-1}(g\widetilde{\mathcal{T}}) \in F_A$ and write $q_0g = (q_0, g)\widehat{\delta}$, $P(g) = (P_1(g), q_0g)$. Note that $g \in L$ if and only if $P_1(g) = z$. We define a deterministic \widetilde{A} -automaton $\mathcal{A}_{\mathcal{T}} = (P, (1, q_0), S, E)$ by

$$P = \{P(g) : g \in F_A\};$$

$$S = P \cap (\{z\} \times Q);$$

$$E = \{(P(g), a, P(ga)) : g \in F_A, a \in \widetilde{A}\}.$$

Clearly, $\mathcal{A}_{\mathcal{T}}$ is a possibly infinite automaton. Note that, since \mathcal{T} is inverse, we have $qaa^{-1} = q$ for all $q \in Q$ and $a \in \widetilde{A}$. It follows that, whenever $(p, a, p') \in E$, $(p', a^{-1}, p) \in E$. We say that such edges are the *inverses* of each other.

Since every $w \in \widetilde{A}^*$ labels a unique path $P(1) \xrightarrow{w} P(w\theta)$, it follows that

$$L(\mathcal{A}_{\mathcal{T}}) = L\theta^{-1}.$$

In view of Proposition 2.2, to prove that L is rational it suffices to construct a finite subautomaton $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{A}_{\mathcal{T}}$ such that $\bar{L} \subseteq L(\mathcal{B}_{\mathcal{T}})$.

We now fix

$$M = \max\{|(q, a)\lambda| : q \in Q, a \in \widetilde{A}\}, \quad N = \max\{2M + 1, |z|\}$$

and

$$P' = \{P(g) \in P : |P_1(g)| \leq N\}.$$

Since A and \mathcal{T} are finite, so is P' . However, infinitely many $g \in F_A$ may yield the same state $P(g)$.

Given $g \in F_A$, write $g\iota = \bar{g}^{[1]}$. Given $p = (g, q) \in P$, we also write $p\iota = g\iota$. We say that an edge $(p_1, a, p_2) \in E$ is

- *central* if $p_1, p_2 \in P'$,
- *compatible* if it is not central and $p_1\iota = a$.

Lemma 3.3. (i) *There are only finitely many central edges in $\mathcal{A}_{\mathcal{T}}$.*

(ii) *If $(p_1, a, p_2) \in E$ is not central, either (p_1, a, p_2) or (p_2, a^{-1}, p_1) is compatible.*

(iii) *For every $p \in P$, there is at most one compatible edge leaving p .*

Proof. (i) A and P' are both finite.

(ii) Assume that (p_1, a, p_2) is neither central nor compatible. Write $p_1 = (g_1, q_1)$ and $p_2 = (g_2, q_2)$. Suppose that $g_1 = 1$. Then $g_2 = P_1(a) = a^{-1}(a\tilde{\mathcal{T}})$ and so $|g_2| \leq 1 + M \leq N$, in contradiction with (p_1, a, p_2) being noncentral.

Thus $\bar{g}_1 = bu$ for some $b \in \tilde{A} \setminus \{a\}$ and $u \in R_A$. On the other hand, we have $g_2 = a^{-1}g_1(q_1, a)\lambda$, and so

$$\bar{g}_2 = \overline{a^{-1}bu(q_1, a)\lambda}.$$

If $|u| < M$, then $|g_1|, |g_2| \leq 2M + 1 \leq N$ and (p_1, a, p_2) is central, a contradiction. Thus $|u| \geq M \geq |(q_1, a)\lambda|$ and so $g_2u = a^{-1}$. Thus (p_2, a^{-1}, p_1) is compatible.

(iii) Any compatible edge leaving p must be labeled by $p\iota$, and $\mathcal{A}_{\mathcal{T}}$ is deterministic. \square

A (possibly infinite) path $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots$ in $\mathcal{A}_{\mathcal{T}}$ is

- *central* if all the vertices in it are in P' ,
- *compatible* if all the edges in it are compatible and no intermediate vertex is in P' .

Lemma 3.4. *Let $u \in \bar{L}$. Then there exists a path*

$$(1, q_0) = p'_0 \xrightarrow{u_0} p''_0 \xrightarrow{v_1} p_1 \xrightarrow{w_1^{-1}} p'_1 \xrightarrow{u_1} \dots \xrightarrow{v_n} p_n \xrightarrow{w_n^{-1}} p'_n \xrightarrow{u_n} p''_n \in S$$

in $\mathcal{A}_{\mathcal{T}}$ such that

- (i) $u = u_0v_1w_1^{-1}u_1 \dots v_nw_n^{-1}u_n$,
- (ii) the paths $p'_j \xrightarrow{u_j} p''_j$ are central,
- (iii) the paths $p''_{j-1} \xrightarrow{v_j} p_j$ and $p'_j \xrightarrow{w_j} p_j$ are compatible,
- (iv) $p_j \notin P'$ if both v_j and w_j are nonempty.

Proof. Since $S \subseteq P'$ by definition of N , there exists a path

$$(2) \quad (1, q_0) = p'_0 \xrightarrow{u_0} p''_0 \xrightarrow{x_1} p'_1 \xrightarrow{u_1} \dots \xrightarrow{x_n} p'_n \xrightarrow{u_n} p''_n \in S$$

in $\mathcal{A}_{\mathcal{T}}$ such that $u = u_0x_1u_1 \dots x_nu_n$ and the paths $p'_j \xrightarrow{u_j} p''_j$ (which may be trivial) collect all the occurrences of vertices in P' (and are therefore central).

By Lemma 3.3(ii), if (p, a, r) occurs in a path $p''_{j-1} \xrightarrow{x_j} p'_j$, either (p, a, r) or (r, a^{-1}, p) is compatible. On the other hand, since x_j is reduced, it follows from Lemma 3.3(iii) that $p''_{j-1} \xrightarrow{x_j} p'_j$ can be factored as

$$p''_{j-1} \xrightarrow{v_j} p_j \xrightarrow{w_j^{-1}} p'_j$$

with $p''_{j-1} \xrightarrow{v_j} p_j$ and $p_j \xrightarrow{w_j^{-1}} p'_j$ compatible. Clearly (iv) holds since no intermediate vertex of $p''_{j-1} \xrightarrow{v_j} p'_j$ belongs to P' by construction. \square

We say that a compatible path is *maximal* if it is infinite or cannot be extended (to the right) to produce another compatible path.

Lemma 3.5. *For every $p \in P'$, there exists in $\mathcal{A}_{\mathcal{T}}$ a unique maximal compatible path M_p starting at p .*

Proof. Clearly, every compatible path can be extended to a maximal compatible path. Uniqueness follows from Lemma 3.3(iii). \square

We now define

$$P'_1 = \{p \in P' : M_p \text{ has finitely many distinct edges} \}$$

and $P'_2 = P' \setminus P'_1$. Hence M_p contains no cycles if $p \in P'_2$. By Lemma 3.5, if M_p and $M_{p'}$ intersect at vertex $r_{pp'}$, they coincide from $r_{pp'}$ onwards. In particular, if M_p and $M_{p'}$ intersect, then $p \in P'_1$ if and only if $p' \in P'_1$. Let

$$Y = \{(p, p') \in P'_2 \times P'_2 : M_p \text{ intersects } M_{p'}\}.$$

For every $(p, p') \in Y$, let $M_p \setminus M_{p'}$ denote the (finite) subpath $p \longrightarrow r_{pp'}$ of M_p . In particular, if $p' = p$, $M_p \setminus M_{p'}$ is the trivial path at p .

Let $\mathcal{B}_{\mathcal{T}}$ be the subautomaton of $\mathcal{A}_{\mathcal{T}}$ containing

- all vertices in P' and all central edges,
- all vertices and edges in the paths M_p ($p \in P'_1$) and their inverses,
- all vertices and edges in the paths $M_p \setminus M_{p'}$ ($(p, p') \in Y$) and their inverses.

It follows easily from Lemma 3.3(i) and the definitions of P'_1 and $M_p \setminus M_{p'}$ that $\mathcal{B}_{\mathcal{T}}$ is a finite subautomaton of $\mathcal{A}_{\mathcal{T}}$. As remarked before, it suffices to show that $\bar{L} \subseteq L(\mathcal{B}_{\mathcal{T}})$.

Let $u \in \bar{L}$. Since $\mathcal{B}_{\mathcal{T}}$ contains all the central edges of $\mathcal{A}_{\mathcal{T}}$, it suffices to show that all subpaths

$$p''_{j-1} \xrightarrow{v_j} p_j \xrightarrow{w_j^{-1}} p'_j$$

appearing in the factorization provided by Lemma 3.4 are paths in $\mathcal{B}_{\mathcal{T}}$.

Without loss of generality, we may assume that $v_j \neq 1$. If $w_j = 1$, $p''_{j-1} \in P'_1$ and we are done. Hence we may also assume that $w_j \neq 1$. Now, if one of the vertices p''_{j-1}, p'_j is in P'_1 , so is the other and we are done, since $\mathcal{B}_{\mathcal{T}}$ contains all

the edges in the paths M_p ($p \in P'_1$) and their inverses. Hence we may assume that $p''_{j-1}, p'_j \in P'_2$. It follows that $p_j = r_{p''_{j-1}, p'_j}$. (Since $v_j w_j^{-1} \in R_A$, the paths $M_{p''_{j-1}}$ and $M_{p'_j}$ cannot meet before p_j .) Thus $p''_{j-1} \xrightarrow{v_j} p_j$ is $M_{p''_{j-1}} \setminus M_{p'_j}$ and $p'_j \xrightarrow{w_j} p_j$ is $M_{p'_j} \setminus M_{p''_{j-1}}$, and so these are also paths in $\mathcal{B}_{\mathcal{T}}$ as required. \square

4. The fixed point subgroup

We can now produce an automata-theoretic proof to Sykiotis' theorem.

Theorem 4.1 [Sykiotis 2002, Proposition 3.4]. *Let φ be an endomorphism of a finitely generated virtually free group. Then $\text{Fix } \varphi$ is finitely generated.*

Proof. We consider a decomposition of G as a disjoint union

$$(3) \quad G = Fb_0 \cup Fb_1 \cup \dots \cup Fb_m,$$

where $F = F_A \trianglelefteq G$ is a free group with A finite and $b_0, \dots, b_m \in G$ with $b_0 = 1$.

Let $\varphi_0 : F_A \rightarrow F_A$ and $\eta : F_A \rightarrow \{0, \dots, m\}$ be defined by

$$g\varphi = (g\varphi_0)b_{g\eta} \quad (g \in F_A).$$

Since the decomposition (3) is disjoint, $g\varphi_0$ and $g\eta$ are both uniquely determined by $g\varphi$, and so both mappings are well defined.

Write $Q = \{0, \dots, m\}$. For all $i \in Q$ and $a \in \tilde{A}$, we have $b_i(a\varphi) = h_{i,a}b_{(i,a)\delta}$ for some (unique) $h_{i,a} \in F_A$ and $(i, a)\delta \in Q$. It follows that, for every $j \in Q$, $\mathcal{A}_j = (Q, 0, j, \delta)$ is a well-defined finite deterministic \tilde{A} -automaton. We define also a finite deterministic \tilde{A} -transducer $\mathcal{T} = (Q, 0, \delta, \lambda)$ by taking $(i, a)\lambda = \overline{h_{i,a}}$ for all $i \in Q$ and $a \in \tilde{A}$.

Assume that

$$i \xrightarrow{a|\overline{h_{i,a}}} (i, a)\delta = j$$

is an edge of \mathcal{T} . Then $b_i(a\varphi) = h_{i,a}b_j$ and so

$$b_i = b_i(a\varphi)(a^{-1}\varphi) = h_{i,a}b_j(a^{-1}\varphi) = h_{i,a}h_{j,a^{-1}}b_{(j,a^{-1})\delta}.$$

This yields $h_{i,a}h_{j,a^{-1}} = 1$ and $(j, a^{-1})\delta = i$. Thus there is an edge

$$j \xrightarrow{a^{-1}|\overline{h_{i,a}^{-1}}} (j, a^{-1})\delta = i$$

in \mathcal{T} and so \mathcal{T} is an inverse transducer. We claim that $\tilde{\mathcal{T}} = \varphi_0$. Indeed, let $g = a_1 \dots a_n$ ($a_i \in \tilde{A}_i$). Then there exists a (unique) path in \mathcal{T} of the form

$$0 = i_0 \xrightarrow{a_1|\overline{h_{i_0,a_1}}} i_1 \xrightarrow{a_2|\overline{h_{i_1,a_2}}} \dots \xrightarrow{a_n|\overline{h_{i_{n-1},a_n}}} i_n.$$

Moreover, $i_j = (i_{j-1}, a_j)\delta$ for $j = 1, \dots, n$. It follows that

$$\begin{aligned} g\varphi &= b_{i_0}(a_1\varphi) \cdots (a_n\varphi) = h_{i_0, a_1} b_{i_1}(a_2\varphi) \cdots (a_n\varphi) \\ &= h_{i_0, a_1} h_{i_1, a_2} b_{i_2}(a_3\varphi) \cdots (a_n\varphi) = \cdots = h_{i_0, a_1} \cdots h_{i_{n-1}, a_n} b_{i_n} \end{aligned}$$

and so

$$g\varphi_0 = h_{i_0, a_1} \cdots h_{i_{n-1}, a_n} = (\overline{h_{i_0, a_1}} \cdots \overline{h_{i_{n-1}, a_n}})\theta = g\tilde{\mathcal{F}}.$$

Thus $\tilde{\mathcal{F}} = \varphi_0$.

Note that we have also shown that $g\eta = i_n = (0, a_1 \cdots a_n)\delta$. Hence

$$(4) \quad L(\mathcal{A}_j) = \{u \in \tilde{A}^* : u\theta\eta = j\}.$$

Next let

$$Y = \{(i, j) \in Q \times Q : b_j(b_i\varphi) \in F_A b_i\}.$$

For every $(i, j) \in Y$, let $z_{i,j} \in F_A$ be such that $b_j(b_i\varphi) = z_{i,j}b_i$ and define

$$X_{i,j} = \{g \in F_A : gb_i \in \text{Fix } \varphi \text{ and } g\eta = j\}.$$

We claim that $X_{i,j}$ is a rational subset of F_A for every $(i, j) \in Y$. Indeed, $(gb_i)\varphi = (g\varphi_0)(b_i\varphi) = (g\varphi_0)b_{g\eta}(b_i\varphi)$. Hence

$$\begin{aligned} X_{i,j} &= \{g \in F_A : (g\varphi_0)b_j(b_i\varphi) = gb_i \text{ and } g\eta = j\} \\ &= \{g \in F_A : (g\varphi_0)z_{i,j}b_i = gb_i \text{ and } g\eta = j\} \\ &= \{g \in F_A : g\varphi_0 = gz_{i,j}^{-1}\} \cap \{g \in F_A : g\eta = j\}. \end{aligned}$$

Writing

$$L_{i,j} = \{g \in F_A : g\varphi_0 = gz_{i,j}^{-1}\},$$

it follows from (4) that $X_{i,j} = L_{i,j} \cap (L(\mathcal{A}_j))\theta$. Since $\varphi_0 = \tilde{\mathcal{F}}$, it follows from [Theorem 3.2](#) that $X_{i,j}$ is an intersection of two rational subsets of F_A , and is hence rational itself; see [\[Berstel 1979, Corollary III.2.10\]](#).

Now it is easy to check that

$$(5) \quad \text{Fix } \varphi = \bigcup_{i \in Q} \left(\bigcup \{X_{i,j} : (i, j) \in Y\} \right) b_i.$$

Indeed, for every $(i, j) \in Y$, we have $X_{i,j}b_i \subseteq \text{Fix } \varphi$ by definition of $X_{i,j}$. Conversely, let $gb_i \in \text{Fix } \varphi$ for some $g \in F_A$ and $i \in Q$. Then $gb_i = (gb_i)\varphi = (g\varphi_0)b_{g\eta}(b_i\varphi)$ and so $b_{g\eta}(b_i\varphi) \in F_A b_i$. Hence $(i, g\eta) \in Y$. Since $g \in X_{i, g\eta}$, (5) holds. Since the $X_{i,j}$ are rational subsets of F_A and therefore of G , it follows that $\text{Fix } \varphi$ is a rational subset of G and is thus finitely generated by [Proposition 2.1](#). \square

Unfortunately, our approach does not lead directly to an algorithm to compute a basis of $\text{Fix } \varphi$ (see [Bogopolski and Maslakova 2012]) because it is not clear how to decide in Section 3 whether $p \in P'$ belongs to P'_1 or P'_2 and how to compute the paths M_p and $M_p \setminus M_{p'}$.

5. A good rewriting system

We recall that a (finite) *rewriting system* on A is a (finite) subset \mathcal{R} of $A^* \times A^*$. Given $u, v \in A^*$, we write $u \longrightarrow_{\mathcal{R}} v$ if there exist $(r, s) \in \mathcal{R}$ and $x, y \in A^*$ such that $u = xry$ and $v = xsy$. The reflexive and transitive closure of $\longrightarrow_{\mathcal{R}}$ is denoted by $\longrightarrow_{\mathcal{R}}^*$.

We say that \mathcal{R} is

- *length-reducing* if $|r| > |s|$ for every $(r, s) \in \mathcal{R}$,
- *length-nonincreasing* if $|r| \geq |s|$ for every $(r, s) \in \mathcal{R}$,
- *noetherian* if, for every $u \in A^*$, there is a bound on the length of a chain

$$u \longrightarrow_{\mathcal{R}} v_1 \longrightarrow_{\mathcal{R}} \dots \longrightarrow_{\mathcal{R}} v_n,$$

- *confluent* if, whenever $u \longrightarrow_{\mathcal{R}}^* v$ and $u \longrightarrow_{\mathcal{R}}^* w$, there exists some $z \in A^*$ such that $v \longrightarrow_{\mathcal{R}}^* z$ and $w \longrightarrow_{\mathcal{R}}^* z$.

A word $u \in A^*$ is *irreducible* if no $v \in A^*$ satisfies $u \longrightarrow_{\mathcal{R}} v$. We denote by $\text{Irr } \mathcal{R}$ the set of all irreducible words in A^* with respect to \mathcal{R} .

We introduce now some basic concepts and results from the theory of hyperbolic groups. For details on this class of groups, the reader is referred to [Ghys and de la Harpe 1990].

Let $\pi : \tilde{A}^* \rightarrow G$ be a matched epimorphism with A finite. The *Cayley graph* $\Gamma_A(G)$ of G with respect to π has vertex set G and edges $(g, a, g(a\pi))$ for all $g \in G$ and $a \in \tilde{A}$. We say that a path $p \xrightarrow{u} q$ in $\Gamma_A(G)$ is a *geodesic* if it has shortest length among all the paths connecting p to q in $\Gamma_A(G)$. We denote by $\text{Geo}_A(G)$ the set of labels of all geodesics in $\Gamma_A(G)$. Note that, since $\Gamma_A(G)$ is vertex-transitive, it is irrelevant whether or not we fix a basepoint.

The *geodesic distance* d_1 on G is defined by taking $d_1(g, h)$ to be the length of a geodesic from g to h . Given $X \subseteq G$ nonempty and $g \in G$, we define

$$d_1(g, X) = \min\{d_1(g, x) : x \in X\}.$$

A *geodesic triangle* in $\Gamma_A(G)$ is a collection of three geodesics

$$P_1 : g_1 \longrightarrow g_2, \quad P_2 : g_2 \longrightarrow g_3, \quad P_3 : g_3 \longrightarrow g_1$$

connecting three vertices $g_1, g_2, g_3 \in G$. Let $V(P_i)$ denote the set of vertices occurring in the path P_i . We say that $\Gamma_A(G)$ is δ -hyperbolic for some $\delta \geq 0$ if

$$\forall g \in V(P_1) : d_1(g, V(P_2) \cup V(P_3)) < \delta$$

for every geodesic triangle $\{P_1, P_2, P_3\}$ in $\Gamma_A(G)$. If this happens for some δ , we say that G is hyperbolic. It is well known that the concept is independent from both alphabet and matched epimorphism, but the hyperbolicity constant δ may change. Virtually free groups are among the most important examples of hyperbolic groups.

We now use a theorem of Gilman, Hermiller, Holt, and Rees [Gilman et al. 2007] to prove the following result.

Lemma 5.1. *Let G be a finitely generated virtually free group. Then there exist a finite alphabet A , a matched epimorphism $\pi : \tilde{A}^* \rightarrow G$, and a positive integer N_0 such that, for all $u \in \text{Geo}_A(G)$ and $v \in \tilde{A}^*$,*

(i) *there exists some $w \in \text{Geo}_A(G)$ such that $w\pi = (uv)\pi$ and*

$$|u \wedge w| \geq |u| - N_0|v|;$$

(ii) *there exists some $z \in \text{Geo}_A(G)$ such that $z\pi = (vu)\pi$ and $|u^{-1} \wedge z^{-1}| \geq |u| - N_0|v|$.*

Proof. (i) By [Gilman et al. 2007, Theorem 1], there exists a finite alphabet A , a matched epimorphism $\pi : \tilde{A}^* \rightarrow G$, and a finite length-reducing rewriting system \mathcal{R} such that $\text{Geo}_A(G) = \text{Irr } \mathcal{R}$. The authors also prove that this property characterizes (finitely generated) virtually free groups.

Let $N_0 = 2 \max\{|r| : (r, s) \in \mathcal{R}\}$. Suppose that

$$uv = w_0 \longrightarrow_{\mathcal{R}} w_1 \longrightarrow_{\mathcal{R}} \cdots \longrightarrow_{\mathcal{R}} w_n = w$$

is a sequence of reductions leading to a geodesic w . Then $(wv^{-1})\pi = u\pi$ and since u is a geodesic we get $|u| \leq |v| + |w|$ and so $|u| - |w| \leq |v|$. On the other hand, since \mathcal{R} is length-reducing, we get $|u| + |v| = |uv| \geq |w| + n$ and so $n - |v| \leq |u| - |w| \leq |v|$. Thus $n \leq 2|v|$.

Trivially, $|u \wedge w_0| \geq |u|$. Since $u \wedge w_{i-1} \in \text{Geo}_A(G)$, it is immediate that $|u \wedge w_i| > |u \wedge w_{i-1}| - N_0/2$ and so

$$|u \wedge w| = |u \wedge w_n| \geq |u| - n \frac{N_0}{2} \geq |u| - N_0|v|.$$

(ii) The inverse of a geodesic is still a geodesic. By applying (i) to u^{-1} and v^{-1} , we get $(u^{-1}v^{-1})\pi = x\pi$ for some $x \in \text{Geo}_A(G)$ satisfying $|u^{-1} \wedge x| \geq |u^{-1}| - N_0|v^{-1}|$. Then we take $z = x^{-1}$. \square

We assume for the remainder of the paper that G is a finitely generated virtually free group, $\pi : \tilde{A}^* \rightarrow G$ a matched epimorphism, and N_0 a positive integer satisfying

the conditions of [Lemma 5.1](#). Since G is hyperbolic, it follows from [\[Epstein et al. 1992, Theorem 3.4.5\]](#) that $\text{Geo}_A(G)$ is an automatic structure for G with respect to π (see [\[Epstein et al. 1992\]](#) for definitions), and so the *fellow traveler property* holds for some constant $K_0 > 0$ (which can be taken as $2(\delta + 1)$, if δ is the hyperbolicity constant). This amounts to saying that

$$\forall u, v \in \text{Geo}_A(G) : d_1(u\pi, v\pi) \leq 1 \implies \forall n \in \mathbb{N} : d_1(u^{[n]}\pi, v^{[n]}\pi) \leq K_0.$$

We fix a total ordering of \tilde{A} . The *shortlex ordering* of \tilde{A}^* is defined by

$$u \leq_{sl} v \text{ if } \begin{cases} |u| < |v|, & \text{or} \\ |u| = |v| \text{ and } u = wau', v = wbv' \text{ with } a < b \text{ in } \tilde{A}. \end{cases}$$

This is a well-known well-ordering of \tilde{A}^* , compatible with multiplication on the left and on the right. Let

$$(6) \quad L = \{u \in \text{Geo}_A(G) : u \leq_{sl} v \text{ for every } v \in u\pi\pi^{-1}\}.$$

By [\[Epstein et al. 1992, Theorem 2.5.1\]](#), L is also an automatic structure for G with respect to π , and therefore rational. We note that L is *factorial* (a factor of a word in L is still in L).

Given $g \in G$, let \bar{g} denote the unique word of L representing g . This corresponds precisely to free group reduction if $G = F_A$ and $\pi = \theta$. Since we shall not need free group reduction from now on, we also write $\bar{u} = \overline{u\pi}$ for every $u \in \tilde{A}^*$ to simplify notation.

Theorem 5.2. *Consider the finite rewriting system \mathcal{R}' on A defined by*

$$\mathcal{R}' = \{(u, \bar{u}) : u \in \tilde{A}^*, |u| \leq K_0N_0 + 1, u \neq \bar{u}\}.$$

Then

- (i) \mathcal{R}' is length-nonincreasing, noetherian and confluent,
- (ii) $\text{Irr}\mathcal{R}' = L$.

Proof. (i) \mathcal{R}' is trivially length-nonincreasing, and that it is noetherian follows from

$$(7) \quad (u, \bar{u}) \in \mathcal{R}' \implies u >_{sl} \bar{u}$$

and \tilde{A}^* being well-ordered by \leq_{sl} , plus compatibility of \leq_{sl} with multiplication.

Next we show that

$$(8) \quad u \xrightarrow{\mathcal{R}'}^* \bar{u} \text{ holds for every } u \in \tilde{A}^*.$$

We use induction on $|u|$. The case $|u| \leq K_0N_0 + 1$ follows from the definition of \mathcal{R}' . Hence assume that $|u| > K_0N_0 + 1$ and (8) holds for shorter words. Write

$u = avb$ with $a, b \in \tilde{A}$. If $av \notin L$, we have $u \xrightarrow{*}_R \overline{avb}$ and $\bar{u} = \overline{avb}$. Hence $u \xrightarrow{*}_R \bar{u}$ follows from $\overline{avb} \xrightarrow{*}_R \overline{avb}$. Hence we may assume that $av \in L$.

Suppose that $u \notin \text{Geo}_A(G)$. By Lemma 5.1(i), there exists some $w \in \text{Geo}_A(G)$ such that $w\pi = (avb)\pi$ and $|av \wedge w| \geq |av| - N_0 \geq K_0 N_0 + 1 - N_0 > 0$. Hence we may write $w = aw'$ and we get $(vb)\pi = (a^{-1}w)\pi = w'\pi$. Since $|w'| < |vb|$ due to $u \notin \text{Geo}_A(G)$, we get $|v\bar{b}| < |vb|$, and so we may apply the induction hypothesis twice to get

$$u = avb \xrightarrow{*}_R \overline{avb} \xrightarrow{*}_R \overline{avb} = \bar{u}.$$

Thus we may assume that $u \in \text{Geo}_A(G)$. We claim that $\bar{u}^{[1]} = a$. Let $p = K_0 N_0 + 1 < |u|$. Since $u, \bar{u} \in \text{Geo}_A(G)$ and $u\pi = \bar{u}\pi$, the fellow traveler property yields $d_1(u^{[p]}\pi, \bar{u}^{[p]}\pi) \leq K_0$, and so $u^{[p]}\pi = (\bar{u}^{[p]}x)\pi$ for some x of length $\leq K_0$. Thus, by Lemma 5.1(i), there exists some $w \in \text{Geo}_A(G)$ such that $w\pi = (\bar{u}^{[p]}x)\pi = u^{[p]}\pi$ and

$$|\bar{u}^{[p]} \wedge w| \geq |\bar{u}^{[p]}| - N_0|x| \geq p - K_0 N_0 = 1.$$

Hence $\bar{u}^{[1]} = w^{[1]}$. Now $av \in L$ by assumption; hence $u^{[p]} \in L$, and so $u^{[p]} = \overline{u^{[p]}}$. Since $w\pi = u^{[p]}\pi$ and $w \in \text{Geo}_A(G)$, we get $a = u^{[1]} \leq w^{[1]} = \bar{u}^{[1]}$ in (\tilde{A}, \leq) . On the other hand, $\bar{u} \leq_{sl} u$ yields $\bar{u}^{[1]} \leq a$ in (\tilde{A}, \leq) , and so $\bar{u}^{[1]} = a$ as claimed.

Now it follows easily that $\bar{u} = aa^{-1}u = \overline{avb}$ and the induction hypothesis yields $vb \xrightarrow{*}_R \overline{vb}$ and therefore $u = avb \xrightarrow{*}_R \overline{avb} = \bar{u}$. Therefore (8) holds.

Assume now that $u \xrightarrow{*}_R v$ and $u \xrightarrow{*}_R w$. By (8), we get $v \xrightarrow{*}_R \bar{v} = \bar{u}$ and $w \xrightarrow{*}_R \bar{w} = \bar{u}$. Hence \mathcal{R}' is confluent.

(ii) It follows from (8) that $\text{Irr } \mathcal{R}' \subseteq L$. The converse inclusion follows from the implication

$$u \xrightarrow{*}_R v \Rightarrow u >_{sl} v,$$

which follows in turn from (7). \square

We now establish some technical results which are useful in later sections.

Lemma 5.3. *Let $u, v \in L$ and let $w \in \tilde{A}^*$ be such that $vw \in \text{Geo}_A(G)$ and $(vw)\pi = u\pi$. Then $|u \wedge v| \geq |v| - K_0 N_0$.*

Proof. Let $k = |v|$ and write $u = u^{[k]}u'$. Since $v = (vw)^{[k]}$, it follows from the fellow traveler property that $d_1(v\pi, u^{[k]}\pi) \leq K_0$. Hence we may write $v\pi = (u^{[k]}z)\pi$ with $|z| \leq K_0$. Since $u^{[k]}$ is itself a geodesic, it follows from Lemma 5.1(i) that there exists a geodesic $u^{[p]}z'$ satisfying $(u^{[p]}z')\pi = (u^{[k]}z)\pi = v\pi$ and

$$p = |u^{[k]} \wedge u^{[p]}z'| \geq |u^{[k]}| - N_0|z| \geq |v| - K_0 N_0.$$

Now $v \in L$ yields $v \leq_{sl} u^{[p]}z'$, and so $v^{[p]} \leq_{sl} u^{[p]}$. On the other hand, $u \in L$ yields $u \leq_{sl} vw$, and so $u^{[p]} \leq_{sl} v^{[p]}$. Thus $u^{[p]} = v^{[p]}$, and so $|u \wedge v| \geq p \geq |v| - K_0 N_0$. \square

Proposition 5.4. (i) *Let $uv \in L$ and let $w \in \tilde{A}^*$ be such that $|v| \geq K_0 N_0 + N_0|w|$. Then $\overline{uvw} = u\bar{v}\bar{w}$.*

(ii) Let $u \in \tilde{A}^*$ and let $vw, vw' \in L$. Then $|\overline{uvw} \wedge \overline{uvw'}| \geq |v| - K_0 N_0 - N_0 |u|$.

Proof. (i) Write $v = v_1 v_2$ with $|v_2| = N_0 |w|$. By Lemma 5.1(i), there exists some $uv_1 z \in \text{Geo}_A(G)$ such that $(uv_1 z)\pi = (uvw)\pi$. Let $x = \overline{uvw}$. By Lemma 5.3, we get $|x \wedge uv_1| \geq |uv_1| - K_0 N_0$. Since $|v_1| = |v| - |v_2| \geq K_0 N_0$, $u \leq x$ and we may write $x = uy$ for some y . Since L is factorial, we have $y \in L$. In view of $y\pi = (u^{-1}x)\pi = (vw)\pi$, we get $y = \overline{vw}$ and so $\overline{uvw} = uv\overline{w}$.

(ii) We may assume that $|v| > K_0 N_0 + N_0 |u|$. Write $v = v_1 v_2$ with $|v_1| = N_0 |u|$. Let $x = \overline{uv_1}$ and write $p = |x| + |v_2|$. By the proof of Lemma 5.1, we have $xv_2 w, xv_2 w' \in \text{Geo}_A(G)$.

Let $y = \overline{uvw}$. Since $(xv_2 w)\pi = y\pi$, it follows from the fellow traveler property that $d_1((xv_2)\pi, y^{[p]}\pi) \leq K_0$. Hence we may write $(xv_2)\pi = (y^{[p]s})\pi$ with $|s| \leq K_0$. Since $y^{[p]}$ is itself a geodesic, it follows from Lemma 5.1(i) that there exists a geodesic $y^{[p-K_0 N_0]s'}$ satisfying $(y^{[p-K_0 N_0]s'})\pi = (y^{[p]s})\pi = (xv_2)\pi$. To complete the proof, it suffices to show that

$$(9) \quad |y \wedge \overline{xv_2}| \geq p - K_0 N_0.$$

Indeed, together with the corresponding inequality for $y' = \overline{uvw'}$, this implies

$$|\overline{uvw} \wedge \overline{uvw'}| \geq p - K_0 N_0 \geq |v_2| - K_0 N_0 = |v| - K_0 N_0 - N_0 |u|$$

and we obtain the desired inequality.

To prove (9), we consider the geodesic $y^{[p-K_0 N_0]s'}$. Since $(y^{[p-K_0 N_0]s'})\pi = (xv_2)\pi$, we get $\overline{xv_2} \leq_{sl} y^{[p-K_0 N_0]s'}$, and so $\overline{xv_2}^{[p-K_0 N_0]} \leq_{sl} y^{[p-K_0 N_0]}$. On the other hand, $xv_2 w$ is also a geodesic. Hence $y = \overline{uvw} = \overline{xv_2 w} \leq_{sl} \overline{xv_2} w$ yields $y^{[p-K_0 N_0]} \leq_{sl} \overline{xv_2}^{[p-K_0 N_0]}$. Therefore $y^{[p-K_0 N_0]} = \overline{xv_2}^{[p-K_0 N_0]}$, so (9) holds. \square

6. A new model for the boundary

We can now present a new model for the boundary of a finitely generated virtually free group which proves useful in studying infinite fixed points. The notion of boundary is indeed one of the important features associated to hyperbolic groups. To present it, we define a second distance in G by means of the *Gromov product* (taking 1 as basepoint). We keep all the notation introduced in Section 5. In particular, G is a finitely generated virtually free group and $L = \text{Irr } \mathcal{R}'$.

Given $g, h \in G$, we define

$$(g|h) = \frac{1}{2}(d_1(1, g) + d_1(1, h) - d_1(g, h)).$$

Fix $\varepsilon > 0$ such that $\varepsilon\delta \leq 1/5$, where δ is the hyperbolicity constant from Section 5. Write $z = e^\varepsilon$ and define

$$\rho(g, h) = \begin{cases} z^{-(g|h)} & \text{if } g \neq h, \\ 0 & \text{otherwise} \end{cases}$$

for all $g, h \in G$. In general, ρ is not a distance because it fails the triangular inequality. This problem is overcome by defining

$$d_2(g, h) = \inf\{\rho(g_0, g_1) + \dots + \rho(g_{n-1}, g_n) : g_0 = g, g_n = h; g_1, \dots, g_{n-1} \in G\}.$$

By [Väisälä 2005, Proposition 5.16] (see also [Ghys and de la Harpe 1990, Proposition 7.10]), d_2 is a distance on G and the inequalities

$$(10) \quad \frac{1}{2}\rho(g, h) \leq d_2(g, h) \leq \rho(g, h)$$

hold for all $g, h \in G$.

In general, the metric space (G, d_2) is not complete. Its completion $(\widehat{G}, \widehat{d}_2)$ is essentially unique, and $\partial G = \widehat{G} \setminus G$ is the *boundary* of G . The elements of the boundary admit several standard descriptions, such as equivalence classes of rays (infinite words whose finite factors are geodesics) where two rays are equivalent if the Hausdorff distance between them is finite [Ghys and de la Harpe 1990, Section 7.1]. We won't need precise definitions for these concepts or \widehat{d}_2 since, as we shall see next, we can get a simpler description of \widehat{G} for virtually free groups.

Lemma 6.1. *There exists some $M_0 > 0$ such that, for all $g, h \in G$,*

- (i) $|\bar{g}| \leq |\bar{g} \wedge \bar{gh}| + K_0 N_0 + N_0 |\bar{h}|$,
- (ii) $d_1(g, h) \geq \frac{|\bar{g}| - |\bar{g} \wedge \bar{h}|}{N_0} - K_0$,
- (iii) $|\bar{g} \wedge \bar{h}| \leq (g|h) \leq |\bar{g} \wedge \bar{h}| + M_0$.

Proof. (i) By applying Lemma 5.1 to the product $\bar{g}\bar{h}$, there exists some factorization $\bar{g} = v\bar{z}$ and some geodesic $vw \in (gh)\pi^{-1}$ such that $|v| \geq |\bar{g}| - N_0|\bar{h}|$. Now we apply Lemma 5.3 to $u = \bar{gh}$ and vw to get $|u \wedge v| \geq |v| - K_0 N_0$. Hence

$$|\bar{g} \wedge \bar{gh}| = |u \wedge v| \geq |v| - K_0 N_0 \geq |\bar{g}| - N_0|\bar{h}| - K_0 N_0.$$

(ii) Let $u = \bar{g} \wedge \bar{h}$. Applying (i) to g and $g^{-1}h$, and in view of $d_1(g, h) = |\overline{g^{-1}h}|$, we get

$$|\bar{g}| \leq |\bar{g} \wedge \bar{h}| + K_0 N_0 + N_0 d_1(g, h).$$

(iii) We define $M_0 = \delta + (2\delta + 1 + K_0)N_0 - 1/2$, assuming that $\Gamma_A(G)$ is δ -hyperbolic. Let $u = \bar{g} \wedge \bar{h}$, and write $\bar{g} = uv, \bar{h} = uw$. It is easy to check that

$$(g|h) = \frac{1}{2}(d_1(1, g) + d_1(1, h) - d_1(g, h)) = \frac{1}{2}(|u| + d_1(u\pi, g) + |u| + d_1(u\pi, h) - d_1(g, h)).$$

Since $d_1(g, h) \leq d_1(g, u\pi) + d_1(u\pi, h)$, we get $|\bar{g} \wedge \bar{h}| = |u| \leq (g|h)$.

Consider now the geodesic triangle determined by the paths

$$P_1 : u\pi \xrightarrow{v} g, \quad P_2 : u\pi \xrightarrow{w} h, \quad P_3 : g \xrightarrow{\overline{g^{-1}h}} h.$$

Since $\Gamma_A(G)$ is δ -hyperbolic,

$$(11) \quad d_1(q, V(P_1) \cup V(P_2)) < \delta \text{ for every } q \in V(P_3).$$

Assume that $P_3 : g = q_0 \xrightarrow{a_1} \dots \xrightarrow{a_n} q_n = h$ with $a_i \in \tilde{A}$. Since

$$d_1(q_0, V(P_1)) = 0 < \delta \quad \text{and} \quad d_1(q_n, V(P_2)) = 0 < \delta,$$

it follows from (11) that there exist some $j \in \{0, \dots, n-1\}$ and $p_1 \in V(P_1)$, $p_2 \in V(P_2)$ such that $d_1(q_j, p_1), d_1(q_{j+1}, p_2) \leq \delta$. Since P_1 and P_2 are geodesics, we get

$$\begin{aligned} (g|h) &= \frac{1}{2}(d_1(1, g) + d_1(1, h) - d_1(g, h)) \\ &= \frac{1}{2}(|u| + d_1(u\pi, p_1) + d_1(p_1, g) \\ &\quad + |u| + d_1(u\pi, p_2) + d_1(p_2, h) - d_1(g, q_j) - 1 - d_1(q_{j+1}, h)) \\ &= |\bar{g} \wedge \bar{h}| + \frac{1}{2}(d_1(u\pi, p_1) + d_1(u\pi, p_2)) \\ &\quad + \frac{1}{2}(d_1(p_1, g) - d_1(g, q_j)) + \frac{1}{2}(d_1(p_2, h) - d_1(q_{j+1}, h)) - \frac{1}{2}. \end{aligned}$$

Since $d_1(p_1, g) \leq d_1(p_1, q_j) + d_1(q_j, g) \leq \delta + d_1(q_j, g)$, we have

$$\frac{1}{2}(d_1(p_1, g) - d_1(g, q_j)) \leq \frac{\delta}{2}.$$

Similarly,

$$\frac{1}{2}(d_1(p_2, h) - d_1(q_{j+1}, h)) \leq \frac{\delta}{2}.$$

Out of symmetry, it suffices to show that $d_1(u\pi, p_1) \leq (2\delta + 1 + K_0)N_0$.

Applying (ii) to p_1 and p_2 , we get

$$d_1(p_1, p_2) \geq \frac{|\overline{p_1}| - |\overline{p_1} \wedge \overline{p_2}|}{N_0} - K_0.$$

Since $\overline{p_1}$ (respectively $\overline{p_2}$) is a prefix of \bar{g} (respectively \bar{h}), it follows easily that $\overline{p_1} \wedge \overline{p_2} = u$ and $|\overline{p_1}| - |\overline{p_1} \wedge \overline{p_2}| = d_1(u\pi, p_1)$. Hence

$$\begin{aligned} d_1(u\pi, p_1) &\leq (d_1(p_1, p_2) + K_0)N_0 \leq (d_1(p_1, q_j) + d_1(q_j, q_{j+1}) + d_1(q_{j+1}, p_2) + K_0)N_0 \\ &\leq (2\delta + 1 + K_0)N_0. \quad \square \end{aligned}$$

The language L introduced in (6) was noted to be rational. We recall that an automaton is said to be trim if every vertex occurs in some successful path. Let $\mathcal{A} = (Q, q_0, T, E)$ be a finite trim deterministic \tilde{A} -automaton recognizing L (for example, the minimal automaton of L ; see [Berstel 1979]). Since L is factorial, we must have $T = Q$. Let

$$\partial L = \{\alpha \in \tilde{A}^\omega : \alpha^{[n]} \in L \text{ for every } n \in \mathbb{N}\}.$$

Equivalently, since \mathcal{A} is trim and deterministic and $T = Q$, we have $\partial L = L_\omega(\mathcal{A})$. Write $\hat{L} = L \cup \partial L$. We define a mapping $d_3 : \hat{L} \times \hat{L} \rightarrow \mathbb{R}_0^+$ by

$$d_3(\alpha, \beta) = \begin{cases} 2^{-|\alpha \wedge \beta|} & \text{if } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that d_3 is a distance in \hat{L} . Indeed, an ultrametric distance since

$$|\alpha \wedge \gamma| \geq \min\{|\alpha \wedge \beta|, |\beta \wedge \gamma|\}$$

holds for all $\alpha, \beta, \gamma \in \hat{L}$. We commit a slight abuse of notation by also denoting by d_3 the restriction of d_3 to $L \times L$.

Proposition 6.2. (i) *The mutually inverse mappings $(G, d_2) \rightarrow (L, d_3) : g \mapsto \bar{g}$ and $(L, d_3) \rightarrow (G, d_2) : u \mapsto u\pi$ are uniformly continuous;*

(ii) *(\hat{L}, d_3) is the completion of (L, d_3) ;*

(iii) *$(\partial L, d_3)$ is homeomorphic to the boundary of G .*

Proof. (i) In view of (10), it suffices to show that

$$\forall M > 0 \exists N > 0 : ((g|h) > N \Rightarrow |\bar{g} \wedge \bar{h}| > M),$$

$$\forall M > 0 \exists N > 0 : (|\bar{g} \wedge \bar{h}| > N \Rightarrow (g|h) > M).$$

Now we apply Lemma 6.1(iii).

(ii) Let $(\alpha_n)_n$ be a Cauchy sequence in (\hat{L}, d_3) . For every $k \in \mathbb{N}$, the sequence $(\alpha_n^{[k]})_n$ stabilizes when $n \rightarrow +\infty$. Moreover, $\lim_{n \rightarrow +\infty} \alpha_n^{[k]}$ is a prefix of

$$\lim_{n \rightarrow +\infty} \alpha_n^{[k+1]}.$$

Let $\beta \in A^\infty$ be the unique word satisfying $\beta^{[k]} = \lim_{n \rightarrow +\infty} \alpha_n^{[k]}$ for every $k \in \mathbb{N}$. It is immediate that $\beta \in \hat{L}$ and $\beta = \lim_{n \rightarrow +\infty} \alpha_n$. Hence (\hat{L}, d_3) is complete. Since $\alpha = \lim_{n \rightarrow +\infty} \alpha^{[n]}$ for every $\alpha \in \partial L$, (\hat{L}, d_3) is the completion of (L, d_3) .

(iii) By (i) and (ii), the uniformly continuous mappings $(G, d_2) \rightarrow (L, d_3) : g \mapsto \bar{g}$ and $(L, d_3) \rightarrow (G, d_2) : u \mapsto u\pi$ admit (unique) continuous extensions to their completions (see [Dugundji 1966, Section XIV.6]), say

$$\Phi : \widehat{G} \rightarrow \hat{L}, \quad \Psi : \hat{L} \rightarrow \widehat{G}.$$

Hence $\Phi\Psi$ is a continuous extension of the identity on G to its completion \widehat{G} . Since such an extension is unique, $\Phi\Psi$ must be the identity mapping on \widehat{G} . Similarly, $\Psi\Phi$ must be the identity mapping on \hat{L} , and so Φ and Ψ are mutually inverse homeomorphisms. Therefore the restriction $\Phi|_{\partial G} : \partial G \rightarrow \partial L$ must also be a homeomorphism. \square

We have just proved that our construction of \hat{L} constitutes a model for the hyperbolic completion of G . But we must also import to \hat{L} the algebraic operations of \widehat{G} since we shall be considering homomorphisms soon. Clearly, the binary operation on L is defined as

$$L \times L \rightarrow L : (u, v) \mapsto \overline{uv},$$

so that $(G, d_2) \rightarrow (L, d_3) : g \mapsto \bar{g}$ is also a group isomorphism. But there is another important algebraic operation involved. Indeed, for every $g \in G$, the left translation $\tau_g : G \rightarrow G : x \mapsto gx$ is uniformly continuous for d_2 and so admits a continuous extension $\hat{\tau}_g : \widehat{G} \rightarrow \widehat{G}$. It follows that the left action of G in its boundary, $G \times \partial G \rightarrow \partial G : (g, \alpha) \mapsto \alpha \hat{\tau}_g$, is continuous. We can also replicate this operation in \hat{L} as follows.

Proposition 6.3. *Let $u \in L$. Then $\tau_u : L \rightarrow L : v \mapsto \overline{uv}$ is uniformly continuous.*

Proof. It suffices to show that

$$\forall M > 0 \exists N > 0 : (|v \wedge w| > N \Rightarrow |\overline{uv} \wedge \overline{vw}| > M).$$

By Proposition 5.4(ii), we can take $N = M + K_0 N_0 + N_0 |u|$. □

Therefore τ_u admits a continuous extension $\hat{\tau}_u : \hat{L} \rightarrow \hat{L}$ and the left action $L \times \partial L \rightarrow \partial L : (u, \alpha) \mapsto \alpha \hat{\tau}_u$ is continuous. Write $\overline{u\alpha} = \alpha \hat{\tau}_u$. For every $\alpha \in \partial L$, we have

$$\overline{u\alpha} = \overline{u \lim_{n \rightarrow +\infty} \alpha^{[n]}} = \lim_{n \rightarrow +\infty} \overline{u\alpha^{[n]}}.$$

Hence (\hat{L}, d_3) serves as a model for (\widehat{G}, \hat{d}_2) both topologically and algebraically. From now on, we pursue our work within (\hat{L}, d_3) .

7. Uniformly continuous endomorphisms

We keep all the notation introduced in Section 5. In particular, G is a finitely generated virtually free group and $L = \text{Irr } \mathcal{R}'$. Following the program announced above, we work within (\hat{L}, d_3) .

Given an endomorphism φ of G , we denote by $\bar{\varphi}$ the corresponding endomorphism of L for the binary operation induced by the product in G , that is, $u\bar{\varphi} = \overline{(u\pi)\varphi}$. To simplify notation, we often write $u\varphi$ instead of $u\pi\varphi$ for $u \in \tilde{A}^*$.

We say that φ satisfies the *bounded cancellation property* if

$$\{|u\bar{\varphi}| - |u\bar{\varphi} \wedge (uv)\bar{\varphi}| : uv \in L\}$$

is bounded. In that case, we denote its maximum by B_φ . This property was considered originally for free group automorphisms by Cooper [1987].

We also fix the notation $D_\varphi = \max\{|\overline{a\varphi}| : a \in \tilde{A}\}$ and recall that a homomorphism with finite kernel is called *virtually injective*.

Theorem 7.1. *Let φ be a virtually injective endomorphism φ of G . Then φ satisfies the bounded cancellation property.*

Proof. Suppose that φ does not satisfy the bounded cancellation property. Then

$$\forall m \in \mathbb{N} \exists u_m v_m \in L : |u_m \bar{\varphi}| - |u_m \bar{\varphi} \wedge (u_m v_m) \bar{\varphi}| > m.$$

Let $X_0 = (K_0 + D_\varphi)N_0$. We claim that

$$(12) \quad \forall m \in \mathbb{N} \exists u'_m v'_m \in L : (|u'_m \bar{\varphi}| - |(u'_m v'_m) \bar{\varphi}| > m \\ \text{and } |(u'_m v'_m) \bar{\varphi}| - |u'_m \bar{\varphi} \wedge (u'_m v'_m) \bar{\varphi}| \leq X_0).$$

Indeed, let $m \in \mathbb{N}$. Take $n = m + X_0$ and write $v_n = a_1 \cdots a_k$ ($a_i \in \tilde{A}$). For $i = 0, \dots, k$, let $w_i = (u_n a_1 \cdots a_i) \bar{\varphi}$. Let j denote the smallest i such that

$$|u_n \bar{\varphi} \wedge w_i| \leq |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}|.$$

Take $u'_m = u_n$ and $v'_m = a_1 \cdots a_{j-1}$ (since $j > 0$). Since L is factorial, we have $u'_m v'_m \in L$.

Now, by the minimality of j , we get

$$|u_n \bar{\varphi} \wedge w_{j-1}| > |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}|.$$

Since $|u_n \bar{\varphi} \wedge w_j| \leq |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}|$, it follows that

$$|w_{j-1} \wedge w_j| \leq |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}|.$$

Applying [Lemma 6.1](#)(i) to $w_{j-1}\pi$ and $a_j\varphi$, we get

$$|w_{j-1}| \leq |w_{j-1} \wedge w_j| + K_0 N_0 + N_0 |\overline{a_j \varphi}| \leq |w_{j-1} \wedge w_j| + X_0 \\ \leq |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}| + X_0 < |u_n \bar{\varphi}| - n + X_0 = |u_n \bar{\varphi}| - m,$$

and so $|u'_m \bar{\varphi}| - |(u'_m v'_m) \bar{\varphi}| = |u_n \bar{\varphi}| - |w_{j-1}| > m$.

Suppose that $|w_{j-1}| - |u_n \bar{\varphi} \wedge w_{j-1}| > X_0$. Since we have seen above that $|w_{j-1}| \leq |w_{j-1} \wedge w_j| + X_0$, we get $|u_n \bar{\varphi} \wedge w_{j-1}| < |w_{j-1} \wedge w_j|$, in contradiction with $|w_{j-1} \wedge w_j| \leq |u_n \bar{\varphi} \wedge (u_n v_n) \bar{\varphi}| < |u_n \bar{\varphi} \wedge w_{j-1}|$. Thus

$$|(u'_m v'_m) \bar{\varphi}| - |u'_m \bar{\varphi} \wedge (u'_m v'_m) \bar{\varphi}| = |w_{j-1}| - |u_n \bar{\varphi} \wedge w_{j-1}| \leq X_0,$$

and so (12) holds.

We prove that

$$(13) \quad \forall m \in \mathbb{N} \exists u''_m v''_m \in L : |u''_m \bar{\varphi}| > m \text{ and } |(u''_m v''_m) \bar{\varphi}| \leq X_0 + N_0 D_\varphi.$$

Indeed, let $m \in \mathbb{N}$. We have in $\Gamma_A(G)$ geodesics

$$\begin{array}{ccc} 1 & \xrightarrow{p} & g \\ & & \searrow^q \\ & & u'_m \varphi \\ & & \searrow^r \\ & & (u'_m v'_m) \varphi, \end{array}$$

where $pq = u'_m \bar{\varphi}$, $pr = (u'_m v'_m) \bar{\varphi}$, and $p = u'_m \bar{\varphi} \wedge (u'_m v'_m) \bar{\varphi}$. Assume that $u'_m = a_1 \cdots a_k$ ($a_i \in A$). Let

$I = \{i \in \{0, \dots, k\} : \text{there exists a geodesic } (a_1 \cdots a_i) \varphi \xrightarrow{q} u'_m \varphi \text{ in } \Gamma_A(G)\}$.

Clearly, $0 \in I$. We claim that

$$(14) \quad (i - 1 \in I \text{ and } d_1((a_1 \cdots a_{i-1}) \varphi, g) > N_0 D_\varphi) \Rightarrow i \in I$$

holds for $i = 1, \dots, k$. Indeed, assume $i - 1 \in I$ and $(a_1 \cdots a_{i-1}) \varphi \xrightarrow{y} g \xrightarrow{q} u'_m \varphi$ is a geodesic with $y \in L$. Applying [Lemma 5.1\(ii\)](#) to the word $a_i^{-1} \bar{\varphi}$ and the geodesic $u = yq$, it follows that there exists some geodesic $(a_1 \cdots a_i) \varphi \xrightarrow{z} u'_m \varphi$ such that z and u share a suffix of length $\geq |yq| - N_0 |a_i^{-1} \bar{\varphi}| \geq |yq| - N_0 D_\varphi > |q|$. Since $\Gamma_A(G)$ is deterministic, our geodesic $(a_1 \cdots a_i) \varphi \xrightarrow{z} u'_m \varphi$ factors through g , and so (14) holds.

Since $k \notin I$ due to $|q| > 0$, it follows from (14) that $d_1((a_1 \cdots a_i) \varphi, g) \leq N_0 D_\varphi$ for some $i \in \{1, \dots, k\}$. Let j denote the smallest such i . We define $u''_m = a_{j+1} \cdots a_k$ and $v''_m = v'_m$. Since L is factorial and $u'_m v'_m \in L$, we also have $u''_m v''_m \in L$.

By minimality of j , we have $d_1((a_1 \cdots a_i) \varphi, g) > N_0 D_\varphi$ for $i = 0, \dots, j - 1$. By (14), we get $1, \dots, j \in I$ and so there exists a geodesic $(a_1 \cdots a_j) \varphi \xrightarrow{q} u'_m \varphi$ in $\Gamma_A(G)$. Hence

$$|u''_m \bar{\varphi}| = d_1(1, u''_m \varphi) = d_1((a_1 \cdots a_j) \varphi, u'_m \varphi) \geq |q| \geq |u'_m \bar{\varphi}| - |(u'_m v'_m) \bar{\varphi}| > m.$$

Finally,

$$\begin{aligned} |(u''_m v''_m) \bar{\varphi}| &= d_1(1, (u''_m v''_m) \varphi) = d_1((a_1 \cdots a_j) \varphi, (u'_m v'_m) \varphi) \\ &\leq d_1((a_1 \cdots a_j) \varphi, g) + d_1(g, (u'_m v'_m) \varphi) \leq N_0 D_\varphi + |r| \\ &= N_0 D_\varphi + |(u'_m v'_m) \bar{\varphi}| - |u'_m \bar{\varphi} \wedge (u'_m v'_m) \bar{\varphi}| \leq N_0 D_\varphi + X_0 \end{aligned}$$

and so (13) holds.

Now, since $|(u''_m v''_m) \bar{\varphi}|$ is bounded, $u''_m v''_m \in L$, and $\text{Ker } \varphi$ is finite, $|u''_m v''_m|$ must be bounded and so must be $|u''_m|$. This implies that $|u''_m \bar{\varphi}|$ must be bounded, contradicting $|u''_m \bar{\varphi}| > m$. Thus φ satisfies the bounded cancellation property. \square

Proposition 7.2. *The following conditions are equivalent for a nontrivial endomorphism φ of G :*

- (i) φ is uniformly continuous for d_2 ;
- (ii) φ is virtually injective.

Proof. (i) \Rightarrow (ii). Suppose that $\text{Ker } \varphi$ is infinite. In view of (10), it suffices to show that there exists some $\eta > 0$ such that

$$\forall \xi > 0 \exists g, h \in G : (\rho(g, h) < \xi \text{ and } \rho(g\varphi, h\varphi) \geq \eta).$$

By (10), we only need to show that there exists some $M \in \mathbb{N}$ such that

$$\forall N \in \mathbb{N} \exists g, h \in G : ((g|h) > N \text{ and } g\varphi \neq h\varphi \text{ and } ((g\varphi)|(h\varphi)) \leq M).$$

Take $M = (g_0\varphi|1) = 0$ and fix $g_0 \in G \setminus \text{Ker } \varphi$. We prove the claim by showing that

$$(15) \quad \forall N \in \mathbb{N} \exists h \in \text{Ker } \varphi : ((hg_0)|h) > N.$$

Let $N \in \mathbb{N}$. By Lemma 6.1(iii), we have $|\overline{hg_0} \wedge \overline{h}| \leq ((hg_0)|h)$ for every $h \in G$; hence we only need to find out $h \in \text{Ker } \varphi$ satisfying $|\overline{hg_0} \wedge \overline{h}| > N$. By Lemma 6.1(i), we have $|\overline{hg_0} \wedge \overline{h}| \geq |\overline{h}| - K_0 N_0 - N_0 |\overline{g_0}|$. Hence it suffices that $|\overline{h}| > N + K_0 N_0 + N_0 |\overline{g_0}|$ for some $h \in \text{Ker } \varphi$, and that is ensured by $\text{Ker } \varphi$ being infinite. Thus (15) holds as required.

(ii) \Rightarrow (i). Suppose that φ is not uniformly continuous for d_2 . In view of (10), there exists some $\eta > 0$ such that

$$\forall \xi > 0 \exists g, h \in G : (\rho(g, h) < \xi \text{ and } \rho(g\varphi, h\varphi) \geq \eta).$$

Hence, by (10), there exists some $M \in \mathbb{N}$ such that

$$\forall N \in \mathbb{N} \exists g, h \in G : ((g|h) > N \text{ and } g\varphi \neq h\varphi \text{ and } ((g\varphi)|(h\varphi)) \leq M).$$

In view of Lemma 6.1(iii), we have that

$$\forall n \in \mathbb{N} \exists u_n, v_n \in L : (|u_n \wedge v_n| > n \text{ and } u_n \bar{\varphi} \neq v_n \bar{\varphi} \text{ and } |u_n \bar{\varphi} \wedge v_n \bar{\varphi}| \leq M).$$

Let $w_n = u_n \wedge v_n \in L$. Then either $w_n \bar{\varphi} \neq u_n \bar{\varphi}$ or $w_n \bar{\varphi} \neq v_n \bar{\varphi}$. Without loss of generality, we may assume that $w_n \bar{\varphi} \neq u_n \bar{\varphi}$. Suppose that $|w_n \bar{\varphi}| > M + B_\varphi$. By definition of B_φ , we get $|w_n \bar{\varphi}| - |w_n \bar{\varphi} \wedge u_n \bar{\varphi}| \leq B_\varphi$, and so $|w_n \bar{\varphi} \wedge u_n \bar{\varphi}| > M$. Similarly, $|w_n \bar{\varphi} \wedge v_n \bar{\varphi}| > M$, and so $|u_n \bar{\varphi} \wedge v_n \bar{\varphi}| > M$, a contradiction. Therefore $|w_n \bar{\varphi}| \leq M + B_\varphi$ for every n . Since $|w_n| > n$ and L is a cross-section for π , it follows that $\text{Ker } \varphi$ is infinite. \square

Given a uniformly continuous endomorphism φ of (G, d_2) , $\bar{\varphi} : L \rightarrow L$ is uniformly continuous for d_3 . Since \hat{L} is the completion of (L, d_3) , $\bar{\varphi}$ admits a unique continuous extension $\Phi : \hat{L} \rightarrow \hat{L}$. By continuity, we have

$$(16) \quad \alpha\Phi = \left(\lim_{n \rightarrow +\infty} \alpha^{[n]} \right) \Phi = \lim_{n \rightarrow +\infty} \alpha^{[n]} \bar{\varphi}.$$

Corollary 7.3. *Let φ be a uniformly continuous endomorphism of G and $u\alpha \in \partial L$. Then $|u\bar{\varphi}| - |u\bar{\varphi} \wedge (u\alpha)\Phi| \leq B_\varphi$.*

Proof. We have $(u\alpha)\Phi = \lim_{n \rightarrow +\infty} (u\alpha^{[n]})\bar{\varphi}$ by (16). In view of Proposition 7.2, we have $\lim_{n \rightarrow +\infty} |(u\alpha^{[n]})\bar{\varphi}| = +\infty$. Hence $|u\bar{\varphi} \wedge (u\alpha)\Phi| = |u\bar{\varphi} \wedge (u\alpha^{[m]})\bar{\varphi}|$ for sufficiently large m . Since $u\alpha^{[m]} \in L$, the claim follows by definition of B_φ . \square

8. Infinite fixed points

Keeping all the notation and assumptions introduced in the preceding sections, we fix now a uniformly continuous endomorphism φ of the finitely generated virtually free group G . We adapt notation introduced in [Ladra and Silva 2011] for free groups, and the proofs are adaptations of proofs in [Silva 2010].

Given $u \in L$, let $u\sigma = u \wedge u\bar{\varphi}$ and write

$$u = (u\sigma)(u\tau), \quad u\bar{\varphi} = (u\sigma)(u\rho).$$

Also define

$$u\sigma' = \bigwedge \{(uv)\sigma : uv \in L\}$$

and write $u\sigma = (u\sigma')(u\sigma'')$.

Lemma 8.1. *Let $uv \in L$. Then*

- (i) $|u\sigma''| \leq B_\varphi$,
- (ii) $|u\sigma| - |u\sigma \wedge (uv)\bar{\varphi}| \leq |u\sigma''|$,
- (iii) $(uv)\bar{\varphi} = (u\sigma')\overline{(u\sigma'')(u\rho)(v\bar{\varphi})}$,
- (iv) $(uv)\sigma' = (u\sigma')\left(\bigwedge_{uvz \in L} \overline{((u\sigma'')(u\rho)((vz)\bar{\varphi})} \wedge (u\sigma'')(u\tau)vz)\right)$.

Proof. (i) We may assume that $|u\sigma| > B_\varphi$. Let v denote the suffix of length B_φ of $u\sigma$ and write $u\sigma = u'v$. Suppose that $uw \in L$. It suffices to show that u' is a prefix of $(uw)\bar{\varphi}$, and this follows from

$$|u'v(u\rho)| - |u'v(u\tau) \wedge (uw)\bar{\varphi}| = |u\bar{\varphi}| - |u\bar{\varphi} \wedge (uw)\bar{\varphi}| \leq B_\varphi$$

and $|v| = B_\varphi$.

- (ii) $u\sigma'$ is a prefix of $u\sigma \wedge (uv)\bar{\varphi}$.
- (iii) $u\sigma'$ is a prefix of $(uv)\bar{\varphi}$ and both sides of the equality are equivalent in G .
- (iv) $u\sigma'$ is a prefix of $(uv)\sigma'$ by (iii). \square

For every $u \in L$, we define

$$u\xi = (u\sigma'', u\tau, u\rho, q_0u).$$

Note that there exists precisely one path of the form $q_0 \xrightarrow{u} q_0u$ in \mathcal{A} .

Lemma 8.2. *Let $u, v \in L$ be such that $u\xi = v\xi$ and let $a \in \tilde{A}, \alpha \in \tilde{A}^\infty$. Then*

- (i) $ua \in L$ if and only if $va \in L$;
- (ii) if $ua \in L$, $(ua)\xi = (va)\xi$;
- (iii) $\overline{uv^{-1}} \in \text{Fix } \bar{\varphi}$;
- (iv) $u\alpha \in \hat{L}$ if and only if $v\alpha \in \hat{L}$;
- (v) $u\alpha \in \text{Fix } \Phi$ if and only if $v\alpha \in \text{Fix } \Phi$;
- (vi) if $\alpha \in \hat{L}$, $\alpha = \lim_{n \rightarrow +\infty} \overline{\alpha^{[n]}u}$.

Proof. (i) $u\xi = v\xi$ implies $q_0u = q_0v$.

(ii) Clearly, $q_0u = q_0v$ yields $q_0ua = q_0va$. Considering $v = a$ in [Lemma 8.1\(iii\)](#), we may write $(ua)\sigma = (u\sigma')u'$ and deduce that u' , $(ua)\tau$, and $(ua)\rho$ are all determined by $u\xi$. Hence $(ua)\tau = (va)\tau$, $(ua)\rho = (va)\rho$, and $u' = v'$.

Finally, since $q_0u = q_0v$, we have $uaz \in L$ if and only if $vaz \in L$. It follows from [Lemma 8.1\(iv\)](#) that there exists a word $x \in L$ which satisfies both $(ua)\sigma' = (u\sigma')x$ and $(va)\sigma' = (v\sigma')x$. Now $(u\sigma')u' = (ua)\sigma = ((ua)\sigma')((ua)\sigma'') = (u\sigma')x((ua)\sigma'')$. Hence $u' = x((ua)\sigma'')$. Similarly, $v' = x((va)\sigma'')$. Since $u' = v'$, we get $(ua)\sigma'' = (va)\sigma''$, and so $(ua)\xi = (va)\xi$.

$$\begin{aligned} \text{(iii)} \quad \overline{(uv^{-1})\varphi} &= \overline{(u\varphi)(v\varphi)^{-1}} = \overline{(u\sigma)(u\rho)(v\rho)^{-1}(v\sigma)^{-1}} = \overline{(u\sigma)(v\sigma)^{-1}} \\ &= \overline{(u\sigma)(u\tau)(v\tau)^{-1}(v\sigma)^{-1}} = \overline{uv^{-1}}. \end{aligned}$$

(iv) We have $u\alpha \in \hat{L}$ if and only if $u\alpha^{[n]} \in L$ for every $n \in \mathbb{N}$. Now we use (i) and induction on n .

(v) We have $u\alpha = (u\sigma')(u\sigma'')(u\tau)\alpha$ and, in view of [Corollary 7.3](#) and (16), also

$$(u\alpha)\Phi = (u\sigma') \lim_{n \rightarrow +\infty} \overline{(u\sigma'')(u\rho)(\alpha^{[n]}\bar{\varphi})}.$$

Hence $u\alpha \in \text{Fix } \Phi$ depends just on $u\xi$ and α .

(vi) Let $m = K_0N_0 + N_0|u|$. By [Lemma 6.1\(i\)](#), we have $|\alpha^{[n]} \wedge \overline{\alpha^{[n]}u}| \geq n - m$ for every n . Hence $\alpha = \lim_{n \rightarrow +\infty} \alpha^{[n-m]} = \lim_{n \rightarrow +\infty} \overline{\alpha^{[n]}u}$. \square

Given $X \subseteq A^\infty$, write

$$\text{Pref } X = \{u \in A^* : u\alpha \in X \text{ for some } \alpha \in A^\infty\}.$$

Recall the finite trim deterministic \tilde{A} -automaton $\mathcal{A} = (Q, q_0, Q, E)$ recognizing L . We build a (possibly infinite) \tilde{A} -automaton $\mathcal{A}'_\varphi = (Q', q'_0, T', E')$ by taking

$$\begin{aligned} Q' &= \{u\xi : u \in \text{Pref Fix } \Phi\}, \\ q'_0 &= 1\xi, \\ T' &= \{u\xi \in Q' : u\tau = u\rho = 1\}, \\ E' &= \{(u\xi, a, v\xi) \in Q' \times \tilde{A} \times Q' : v = ua \in \text{Pref Fix } \Phi\}. \end{aligned}$$

We note that \mathcal{A}'_φ is deterministic by [Lemma 8.2\(ii\)](#) and is also *accessible*: if $u \in \text{Pref Fix } \Phi$, there exists a path $q'_0 \xrightarrow{u} u\xi$, and so every vertex can be reached from the initial vertex.

Let S denote the set of all vertices $q \in Q'$ such that there exist at least two edges in \mathcal{B}'_φ leaving q . Let Q'' denote the set of all vertices $q \in Q'$ such that there exists some path $q \rightarrow p \in S \cup T'$. We define $\mathcal{A}''_\varphi = (Q'', q''_0, T'', E'')$ by taking $q''_0 = q'_0$, $T'' = T'$, and $E'' = E' \cap (Q'' \times \tilde{A} \times Q'')$.

Lemma 8.3. *S is finite.*

Proof. In view of [Lemma 8.1](#), the unique components of $u\xi$ that may assume infinitely many values are $u\tau$ and $u\rho$. Moreover, we claim that

$$(17) \quad u\tau \neq 1 \Rightarrow |u\rho| \leq B_\varphi$$

holds for every $u \in \text{Pref Fix } \Phi$. Indeed, suppose that $u\tau \neq 1$ and $|u\rho| > B_\varphi$. Write $\alpha = u\beta$ for some $\alpha \in \text{Fix } \Phi$. In view of [Corollary 7.3](#), $|u\rho| > B_\varphi$ yields $|(u\beta)\Phi \wedge u\bar{\varphi}| > |u\sigma|$ and now $u\tau \neq 1$ yields $((u\beta)\Phi \wedge u\beta) = (u\bar{\varphi} \wedge u) = u\sigma$. Since $\beta \neq 1$, this contradicts $\alpha \in \text{Fix } \Phi$. Therefore (17) holds.

It is also easy to see that

$$(18) \quad |u\rho| > B_\varphi \Rightarrow u\xi \notin S$$

for every $u \in \text{Pref Fix } \Phi$. Indeed, if $|u\rho| > B_\varphi$ and a is the first letter of $u\rho$, then, by definition of B_φ , $(u\sigma)a$ is a prefix of $(u\alpha)\Phi$ whenever $u\alpha \in \text{Fix } \Phi$. Therefore any edge leaving $u\xi$ in \mathcal{A}'_φ must have label a , and so (18) holds.

In view of [Proposition 7.2](#), we can define

$$W_0 = \max\{|u| : u \in L, |u\bar{\varphi}| \leq 2(B_\varphi + D_\varphi - 1)\}.$$

Let $Z_0 = B_\varphi + N_0(K_0 + W_0)D_\varphi$. To complete the proof, it suffices to prove that

$$(19) \quad |u\tau| > Z_0 \Rightarrow u\xi \notin S$$

for every $u \in \text{Pref Fix } \Phi$.

Suppose that $|u\tau| > Z_0$ and

$$(u\xi, a, (ua)\xi), (u\xi, b, (ub)\xi) \in E'$$

for some $u \in \text{Pref Fix } \Phi$, where $a, b \in \tilde{A}$ are distinct. We have $(ua)\xi = v\xi$ for some $v \in \text{Pref Fix } \Phi$. By [Lemma 8.2\(v\)](#), we get $ua\alpha \in \text{Fix } \Phi$ for some $\alpha \in \hat{L}$. By [\(16\)](#), we get $ua\alpha = \lim_{n \rightarrow +\infty} (ua\alpha^{[n]})\bar{\varphi}$, and so $|(ua\alpha^{[n]})\bar{\varphi}| \geq |u|$ for sufficiently large n . Let

$$p = \min\{n \in \mathbb{N} : |(ua\alpha^{[n]})\bar{\varphi}| \geq |u|\}.$$

Note that $p > 0$ since $|u\tau| > Z_0$ and by [\(17\)](#). Since $|(ua\alpha^{[p-1]})\bar{\varphi}| < |u|$ by the minimality of p , we get

$$(20) \quad |(ua\alpha^{[p]})\bar{\varphi}| \leq |(ua\alpha^{[p-1]})\bar{\varphi}| + D_\varphi < |u| + D_\varphi.$$

On the other hand,

$$(21) \quad |u| - |(ua\alpha^{[p]})\bar{\varphi} \wedge u| \leq B_\varphi.$$

Otherwise, by definition of B_φ , $ua\alpha$ and $(ua\alpha)\Phi$ would differ at position

$$|(ua\alpha^{[p]})\bar{\varphi} \wedge u| + 1.$$

Similarly, $ub\beta \in \text{Fix } \Phi$ for some $\beta \in \hat{L}$. Defining

$$q = \min\{n \in \mathbb{N} : |(ub\beta^{[n]})\bar{\varphi}| \geq |u|\},$$

we get

$$(22) \quad |(ub\beta^{[q]})\bar{\varphi}| < |u| + D_\varphi$$

and

$$(23) \quad |u| - |(ub\beta^{[q]})\bar{\varphi} \wedge u| \leq B_\varphi.$$

Write $u = u_1u_2$ with $|u_2| = B_\varphi$. Then by [\(20\)](#) and [\(21\)](#) we may write $(ua\alpha^{[p]})\bar{\varphi} = u_1x$ for some x such that $|x| < B_\varphi + D_\varphi$. Similarly, [\(22\)](#) and [\(23\)](#) yield $(ub\beta^{[q]})\bar{\varphi} = u_1y$ for some x such that $|x| < B_\varphi + D_\varphi$. Writing $w = (\beta^{[q]})^{-1}b^{-1}a\alpha^{[p]}$, it follows that $w\varphi = (y^{-1}x)\pi$, and so $|w\bar{\varphi}| \leq 2(B_\varphi + D_\varphi - 1)$. Hence $|w| \leq W_0$. Applying [Lemma 6.1\(i\)](#) to $g = (ub\beta^{[q]})\pi$ and $h = w\pi$, we get

$$|ub\beta^{[q]}| \leq |ub\beta^{[q]} \wedge ua\alpha^{[p]}| + N_0(K_0 + |w|) \leq |u| + N_0(K_0 + W_0),$$

and so $q < N_0(K_0 + W_0)$. Hence, in view of [\(17\)](#), we get

$$\begin{aligned} |u\tau| &= |u| - |u\sigma| \leq |(ub\beta^{[q]})\bar{\varphi}| - |u\sigma| \leq |u\bar{\varphi}| + |(b\beta^{[q]})\bar{\varphi}| - |u\sigma| \\ &\leq |u\rho| + N_0(K_0 + W_0)D_\varphi \leq B_\varphi + N_0(K_0 + W_0)D_\varphi, \end{aligned}$$

contradicting $|u\tau| > Z_0$. Thus [\(19\)](#) holds and the lemma is proved. \square

We say that an infinite fixed point $\alpha \in \text{Fix } \Phi \cap \partial L$ is *singular* if α belongs to the topological closure $(\text{Fix } \varphi)^c$ of $\text{Fix } \varphi$. Otherwise, α is said to be *regular*. We denote by $\text{Sing } \Phi$ (respectively $\text{Reg } \Phi$) the set of all singular (respectively regular) infinite fixed points of Φ .

Theorem 8.4. *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G . Then*

- (i) *the automaton \mathcal{A}''_φ is finite;*
- (ii) *$L(\mathcal{A}''_\varphi) = \text{Fix } \bar{\varphi}$;*
- (iii) *$L_\omega(\mathcal{A}''_\varphi) = \text{Sing } \Phi$.*

Proof. (i) The set T' is finite and S is finite by [Lemma 8.3](#). On the other hand, by definition of S , there are only finitely many paths in \mathcal{A}'_φ of the form $v_j : p' \rightarrow q'$ with $p', q' \in S \cup T' \cup \{q'_0\}$ and no intermediate vertex in $S \cup T' \cup \{q'_0\}$. Now recall that \mathcal{A}'_φ is accessible. Hence every path of the form $q \xrightarrow{u} p \in S \cup T'$ can be extended to some path $q'_0 \xrightarrow{v} q \xrightarrow{u} p \in S \cup T'$ which is itself a concatenation of the finitely many paths v_j . Therefore Q'' is finite and so is \mathcal{A}''_φ .

(ii) Every $u \in L$ labels at most a unique path $q'_0 = 1\xi \xrightarrow{u} u\xi$ out of the initial vertex in \mathcal{A}'_φ . On the other hand, if $q'_0 = 1\xi \xrightarrow{u} q'$ is a path in \mathcal{A}'_φ , the fourth component of ξ yields a path $q_0 \xrightarrow{u} q$ in \mathcal{A} , and so $u \in L$. Hence

$$L(\mathcal{A}'_\varphi) = \{u \in L : u\xi \in T'\} = \{u \in L : u\tau = u\rho = 1\} = \text{Fix } \bar{\varphi}.$$

Since $L(\mathcal{A}''_\varphi) = L(\mathcal{A}'_\varphi)$, (ii) holds.

(iii) Let $\alpha \in L_\omega(\mathcal{A}''_\varphi)$. Then there exists some $q'' \in Q''$ and some infinite sequence $(i_n)_n$ such that $q''_0 \xrightarrow{\alpha^{[i_n]}} q''$ is a path in \mathcal{A}''_φ for every n . Write $u = \alpha^{[i_1]}$ and let $v_n = \overline{\alpha^{[i_n]}u^{-1}}$. By [Lemma 8.2](#)(iii), we have $v_n \in \text{Fix } \bar{\varphi}$ for every n . It follows from [Lemma 8.2](#)(vi) that $\alpha = \lim_{n \rightarrow +\infty} v_n$. Thus $\alpha \in \text{Sing } \Phi$.

Conversely, let $\alpha \in \text{Sing } \Phi$. Then we may write $\alpha = \lim_{n \rightarrow +\infty} v_n$ for some sequence $(v_n)_n$ in $\text{Fix } \bar{\varphi}$. Let $k \in \mathbb{N}$. For large enough n , we have $\alpha^{[k]} = v_n^{[k]}$, and so there is some path

$$q''_0 \xrightarrow{\alpha^{[k]}} q''_k \xrightarrow{w} t''_k \in T'',$$

where $\alpha^{[k]}w = v_n$. Thus $\alpha \in L_\omega(\mathcal{A}''_\varphi)$ as required. \square

Recall now the continuous extensions $\hat{\tau}_u : \hat{L} \rightarrow \hat{L}$ of the uniformly continuous mappings $\tau_u : L \rightarrow L : v \mapsto \overline{uv}$ defined for each $u \in L$ (see [Proposition 6.3](#)). As remarked before, this is equivalent to saying that the left action

$$L \times \partial L \rightarrow \partial L : (u, \alpha) \mapsto \overline{u\alpha}$$

is continuous. Identifying L with G and ∂L with ∂G , we have a continuous action (on the left) of G on ∂G . Clearly, this action restricts to a left action of $\text{Fix } \varphi$ on

$\text{Fix } \Phi \cap \partial G$: if $g \in \text{Fix } \varphi$ and $\alpha \in \text{Fix } \Phi \cap \partial G$ with $\alpha = \lim_{n \rightarrow +\infty} g_n$ ($g_n \in G$),

$$\begin{aligned} (g\alpha)\Phi &= (g \lim_{n \rightarrow +\infty} g_n)\Phi = (\lim_{n \rightarrow +\infty} gg_n)\Phi = \lim_{n \rightarrow +\infty} (gg_n)\varphi \\ &= \lim_{n \rightarrow +\infty} (g\varphi)(g_n\varphi) = (g\varphi) \lim_{n \rightarrow +\infty} g_n\varphi = g(\lim_{n \rightarrow +\infty} g_n)\Phi \\ &= g(\alpha\Phi) = g\alpha. \end{aligned}$$

Moreover, the $(\text{Fix } \varphi)$ -orbits of $\text{Sing } \Phi$ and $\text{Reg } \Phi$ are disjoint: if $\alpha \in \text{Sing } \Phi$, we can write $\alpha = \lim_{n \rightarrow +\infty} g_n$ with the $g_n \in \text{Fix } \varphi$ and get $g\alpha = \lim_{n \rightarrow +\infty} gg_n$ with $gg_n \in \text{Fix } \varphi$ for every n ; hence $\alpha \in \text{Sing } \Phi \Rightarrow g\alpha \in \text{Sing } \Phi$ and the action of g^{-1} yields the converse implication.

We can now prove the main result of this section.

Theorem 8.5. *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G . Then $\text{Reg } \Phi$ has finitely many $(\text{Fix } \varphi)$ -orbits.*

Proof. Let P be the set of all infinite paths $s'_0 \xrightarrow{a_1} s'_1 \xrightarrow{a_2} \dots$ in \mathcal{A}'_φ such that

$$s'_0 \in S \cup \{q_0\}, \quad s'_n \notin S \cup \{q_0\} \text{ for every } n > 0, \quad s'_n \neq s'_m \text{ whenever } n \neq m.$$

By [Lemma 8.3](#), there are only finitely many choices for s'_0 . Since A is finite and \mathcal{A}'_φ is deterministic, there are only finitely many choices for s'_1 , and from that vertex onwards, the path is uniquely determined due to $s'_n \notin S$ ($n \geq 1$). Hence P is finite, and we may assume that it consists of paths $p'_i \xrightarrow{\alpha_i} \dots$ for $i = 1, \dots, m$. Fix a path $q'_0 \xrightarrow{u_i} p'_i$ for each i and let $X = \{u_1\alpha_1, \dots, u_m\alpha_m\} \subseteq \partial L$. We claim that $X \subseteq \text{Reg } \Phi$.

Let $i \in \{1, \dots, m\}$ and write $\beta = u_i\alpha_i$. To show that $\beta \in \text{Fix } \Phi$, it suffices to show that $\lim_{n \rightarrow +\infty} \beta^{[n]}\bar{\varphi} = \beta$. Let $k \in \mathbb{N}$. We must show that there exists some $r \in \mathbb{N}$ such that

$$(24) \quad n \geq r \Rightarrow |\beta^{[n]}\bar{\varphi} \wedge \beta| > k.$$

In view of [Proposition 7.2](#), there exists some $r > k$ such that

$$n \geq r \Rightarrow |\beta^{[n]}\bar{\varphi}| > k + B_\varphi.$$

Suppose that $|\beta^{[n]}\bar{\varphi} \wedge \beta| \leq k$ for some $n \geq r$. Then $|\beta^{[n]}\sigma| \leq k$. Since $k < r \leq n$, it follows that $\beta^{[n]}\tau \neq 1$. On the other hand, since $|\beta^{[n]}\bar{\varphi}| > k + B_\varphi$, we get $|\beta^{[n]}\rho| > B_\varphi$. In view of [\(17\)](#), this contradicts $\beta^{[n]}\xi \in Q'$. Therefore [\(24\)](#) holds for our choice of r and so $X \subseteq \text{Fix } \Phi$. Since the path

$$q'_0 \xrightarrow{\beta} \dots$$

can visit only finitely often a given vertex, $\beta \notin L_\omega(\mathcal{A}''_\varphi)$, and so $X \subseteq \text{Reg } \Phi$ by [Theorem 8.4\(iii\)](#).

By the previous comments on $(\text{Fix } \varphi)$ -orbits, the $(\text{Fix } \varphi)$ -orbits of the elements of X must be contained in $\text{Reg } \Phi$. We complete the proof of the theorem by proving the opposite inclusion.

Let $\beta \in \text{Reg } \Phi$. By [Theorem 8.4\(iii\)](#), we have $\beta \notin L_\omega(\mathcal{A}''_\varphi)$, and so there exists a factorization $\beta = u\alpha$ and a path

$$q'_0 \xrightarrow{u} p' \xrightarrow{\alpha} \dots$$

in \mathcal{A}'_φ such that p' signals the last occurrence of a vertex from $S \cup \{q'_0\}$. We claim that no vertex is repeated after p' . Otherwise, since no vertex of S appears after p' , we would get a factorization of $p' \xrightarrow{\alpha} \dots$ as

$$p' \xrightarrow{v} q' \xrightarrow{w} q' \xrightarrow{w} \dots$$

and by [Lemma 8.2\(iii\)](#) and (iv) we would get $(uvw^n v^{-1} u^{-1})\pi \in \text{Fix } \varphi$ and

$$\beta = \lim_{n \rightarrow +\infty} \overline{uvw^n v^{-1} u^{-1}},$$

contradicting $\beta \in \text{Reg } \Phi$. Thus no vertex is repeated after p' , and so we must have $p' = p'_i$ and $\alpha = \alpha_i$ for some $i \in \{1, \dots, m\}$. It follows that $\beta = u\alpha_i$. By [Lemma 8.2\(iii\)](#), we get

$$\overline{uu_i^{-1}} \in \text{Fix } \bar{\varphi},$$

and we are done. □

[Theorem 8.5](#) is somehow a version for infinite fixed points of [Theorem 4.1](#), which we proved before for finite fixed points. Note however that $\text{Sing } \Phi$ does *not* in general have finitely many $(\text{Fix } \varphi)$ -orbits since $\text{Sing } \Phi$ may be uncountable (take for instance the identity automorphism on a free group of rank 2).

Since every finite set is closed in a metric space, we obtain the following corollary from [Theorem 8.5](#).

Corollary 8.6. *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G with $\text{Fix } \varphi$ finite. Then $\text{Fix } \Phi$ is finite.*

9. Classification of the infinite fixed points

We can now investigate the nature of the infinite fixed points of Φ when φ is an automorphism. Since, by [Proposition 7.2](#), both φ and φ^{-1} are then uniformly continuous, they extend to continuous mappings Φ and Ψ which turn out to be mutually inverse in view of the uniqueness of continuous extensions to the completion. Therefore Φ is a bijection. We say that $\alpha \in \text{Reg } \Phi$ is

- an *attractor* if $\exists \varepsilon > 0 \forall \beta \in \hat{L} : (d_3(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \rightarrow +\infty} \beta \Phi^n = \alpha)$;
- a *repeller* if $\exists \varepsilon > 0 \forall \beta \in \hat{L} : (d_3(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \rightarrow +\infty} \beta \Phi^{-n} = \alpha)$.

The latter amounts to saying that α is an attractor for Φ^{-1} . There exist other types, but they do not occur in our context as we shall see.

We say that an attractor $\alpha \in \text{Reg } \Phi$ is *exponentially stable* if

$$\exists \varepsilon, k, \ell > 0 \forall \beta \in \hat{L} \forall n \in \mathbb{N} : (d_3(\alpha, \beta) < \varepsilon \Rightarrow d_3(\alpha, \beta \Phi^n) \leq k 2^{-\ell n} d_3(\alpha, \beta)).$$

This is equivalent to saying that

$$(25) \quad \exists M, N, \ell > 0 \forall \beta \in \hat{L} \forall n \in \mathbb{N} : \\ (|\alpha \wedge \beta| > M \Rightarrow |\alpha \wedge \beta \Phi^n| + N > \ell n + |\alpha \wedge \beta|).$$

A repeller $\alpha \in \text{Reg } \Phi$ is *exponentially stable* if it is an exponentially stable attractor for Φ^{-1} .

Theorem 9.1. *Let φ be an automorphism of a finitely generated virtually free group G . Then $\text{Reg } \Phi$ contains only exponentially stable attractors and exponentially stable repellers.*

Proof. Let $\alpha \in \text{Reg } \Phi$ and write $\alpha = a_1 a_2 \cdots$ with $a_i \in \tilde{A}$. Then there exists a path

$$1\xi \xrightarrow{a_1} \alpha^{[1]}\xi \xrightarrow{a_2} \alpha^{[2]}\xi \xrightarrow{a_3} \cdots$$

in \mathcal{A}'_φ . Let $Y_0 = B_\varphi(D_{\varphi^{-1}} + 1) + B_{\varphi^{-1}}(D_\varphi + 1)$ and let

$$V = \{u\xi \in Q' : |u\tau| > Y_0 \text{ or } |u\rho| > Y_0\}.$$

It is easy to see that $Q' \setminus V$ is finite. We saw in the proof of [Theorem 8.5](#) that there are only finitely many repetitions of vertices in a path in \mathcal{A}'_φ labeled by a regular fixed point. Hence there exists some $n_0 \in \mathbb{N}$ such that

$$(26) \quad \alpha^{[n]}\xi \in V \text{ for every } n \geq n_0.$$

Now we consider two cases.

Case I: $\alpha^{[n_0]}\tau = 1$. We claim that

$$(27) \quad \alpha^{[n]}\tau = 1 \text{ for every } n \geq n_0.$$

The case $n = n_0$ holds in Case I, so assume that $\alpha^{[n]}\tau = 1$ for some $n \geq n_0$. Then $\alpha^{[n]}\xi \in V$, and so $|\alpha^{[n]}\rho| > Y_0 > 2B_\varphi$. Since $|\alpha^{[n+1]}\bar{\varphi}| \geq |\alpha^{[n]}\bar{\varphi}| - B_\varphi$ by definition of B_φ ,

$$|\alpha^{[n+1]}\rho| \geq |\alpha^{[n+1]}\bar{\varphi}| - |\alpha^{[n+1]}| \geq |\alpha^{[n]}\bar{\varphi}| - B_\varphi - |\alpha^{[n]}| - 1 = |\alpha^{[n]}\rho| - B_\varphi - 1 \\ > Y_0 - B_\varphi - 1 > B_\varphi.$$

By (17), we get $\alpha^{[n+1]}\tau = 1$, and so (27) holds.

Next we show that

$$(28) \quad ((\alpha^{[n]}\gamma)\Phi)^{[n+1]} = \alpha^{[n+1]}$$

if $n \geq n_0$ and $\alpha^{[n]}\gamma \in \hat{L}$. Indeed, by (27) we have $\alpha^{[n]}\bar{\varphi} = \alpha^{[n]}(\alpha^{[n]}\rho)$ and $|\alpha^{[n]}\rho| > Y_0 > B_\varphi$. By the definition of B_φ and Corollary 7.3, we get $((\alpha^{[n]}\gamma)\Phi)^{[n+1]} = \alpha^{[n]}(\alpha^{[n]}\rho)^{[1]}$. Considering the particular case $\gamma = a_{n+1}$, we also get

$$(\alpha^{[n+1]}\bar{\varphi})^{[n+1]} = \alpha^{[n]}(\alpha^{[n]}\rho)^{[1]} = ((\alpha^{[n]}\gamma)\Phi)^{[n+1]}.$$

Since $\alpha^{[n+1]}\tau = 1$ by (27), we have $(\alpha^{[n+1]}\bar{\varphi})^{[n+1]} = \alpha^{[n+1]}$, and so (28) holds.

Hence we may write $(\alpha^{[n]}\gamma)\Phi = \alpha^{[n+1]}\gamma'$ whenever $\alpha^{[n]}\gamma \in \hat{L}$. Iterating, it follows that, for all $k \geq n_0$ and $n \in \mathbb{N}$, $\alpha^{[k]}\gamma \in \hat{L}$ implies $(\alpha^{[k]}\gamma)\Phi^n = \alpha^{[k+n]}\gamma'$ for some γ' . By considering $\beta = \alpha^{[k]}\gamma$ and $\alpha^{[k]} = \alpha \wedge \beta$, we deduce that

$$|\alpha \wedge \beta| \geq n_0 \Rightarrow |\alpha \wedge \beta \Phi^n| \geq n + |\alpha \wedge \beta|$$

holds for all $\beta \in \hat{L}$ and $n \in \mathbb{N}$. Therefore (25) holds, and so α is an exponentially stable attractor in this case.

Now, if $|\alpha^{[t]}\tau| = 1$ for some $t > n_0$, we can always replace n_0 by t and deduce by Case I that α is an exponentially stable attractor. Thus we may assume the following.

Case II: $\alpha^{[n]}\tau \neq 1$ for every $n \geq n_0$. By replacing n_0 by a larger integer if necessary, we may assume that (26) is also satisfied when we consider the equivalents of ξ and V for φ^{-1} .

Since φ is injective, there exists some $n_1 \geq n_0$ such that

$$|\alpha^{[n_1]}\bar{\varphi}| \geq n_0 + B_\varphi.$$

Since $\alpha^{[n_1]}\tau \neq 1$, it follows from (17) that $|\alpha^{[n_1]}\rho| \leq B_\varphi$; hence $\alpha^{[n_1]}\sigma = \alpha^{[n_2]}$ for some $n_2 \geq n_0$. Write $x = \alpha^{[n_1]}\rho$. Then $\alpha^{[n_1]}\bar{\varphi} = \alpha^{[n_2]}x$ yields

$$\alpha^{[n_1]} = \overline{(\alpha^{[n_2]}\bar{\varphi}^{-1})(x\bar{\varphi}^{-1})},$$

and so

$$n_1 = |\alpha^{[n_1]}| \leq |\alpha^{[n_2]}\bar{\varphi}^{-1}| + |x\bar{\varphi}^{-1}| \leq |\alpha^{[n_2]}\bar{\varphi}^{-1}| + B_\varphi D_{\varphi^{-1}}.$$

On the other hand, $|\alpha^{[n_1]}\rho| \leq B_\varphi < Y_0$ and $\alpha^{[n_1]} \in V$ together yield $Y_0 < |\alpha^{[n_1]}\tau| = n_1 - n_2$, and so

$$n_2 + B_{\varphi^{-1}} < n_1 - Y_0 + B_{\varphi^{-1}} < n_1 - B_\varphi D_{\varphi^{-1}} \leq |\alpha^{[n_2]}\bar{\varphi}^{-1}|.$$

In view of (17), we can apply Case I to φ^{-1} . Hence α is an exponentially stable attractor for φ^{-1} and, therefore, an exponentially stable repeller for φ . \square

10. Example and open problems

We include a simple example which illustrates some of the constructions introduced earlier.

Example. Let $G = \mathbb{Z} \times \mathbb{Z}_2$ and let $A = \{a, b, c\}$. We note that this is not the canonical set of generators, which would not work. Then the matched homomorphism $\pi : \tilde{A}^* \rightarrow G$ defined by

$$a\pi = (1, 0), \quad b\pi = (0, 1), \quad c\pi = (1, 1)$$

yields

$$\text{Geo}_A(G) = (a \cup c)^* \cup (a^{-1} \cup c^{-1})^* \cup \{b, b^{-1}\},$$

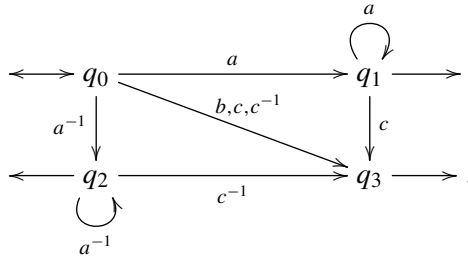
and we can take

$$\begin{aligned} \mathcal{R} = \{ & (xx^{-1}, 1) : x \in \tilde{A}\} \cup \{(a^\varepsilon b^\delta, c^\varepsilon), (b^\delta a^\varepsilon, c^\varepsilon), (c^\varepsilon b^\delta, a^\varepsilon), (b^\delta c^\varepsilon, a^\varepsilon) : \delta, \varepsilon = \pm 1\} \\ & \cup \{(ac^{-1}, b), (c^{-1}a, b), (a^{-1}c, b), (ca^{-1}, b), (b^2, 1), (b^{-2}, 1)\} \end{aligned}$$

to get $\text{Geo}_A(G) = \text{Irr } \mathcal{R}$. Ordering \tilde{A} by $a < c < a^{-1} < c^{-1} < b < b^{-1}$, we get

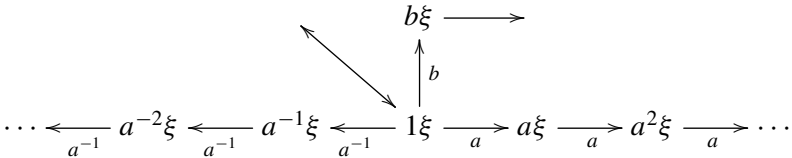
$$L = a^*(1 \cup c) \cup (a^{-1})^*(1 \cup c^{-1}) \cup b,$$

recognized by the automaton \mathcal{A} depicted by



Hence $\partial L = L_\omega(\mathcal{A}) = \{a^\omega, (a^{-1})^\omega\}$.

Let φ be the endomorphism of G defined by $(m, n)\varphi = (2m, n)$. Then φ is injective and therefore uniformly continuous, admitting a continuous extension Φ to \hat{L} . Since $B_\varphi = 0$, it is easy to check that \mathcal{A}'_φ is the automaton



and

$$1\xi = (1, 1, 1, q_0), \quad b\xi = (1, 1, 1, q_3), \quad a^n\xi = (1, 1, a^n, q_1), \quad a^{-n}\xi = (1, 1, a^{-n}, q_2)$$

for $n \geq 1$. Note that, in general, we ignore how to compute \mathcal{A}'_φ , our proofs being far from constructive!

It is immediate that $\text{Fix } \Phi = \{1, b, a^\omega, (a^{-1})^\omega\}$. Moreover, the regular infinite fixed points a^ω and $(a^{-1})^\omega$ are both exponentially stable attractors.

Finally, we end the paper with some easily predictable open problems.

Problem 10.1. Is it possible to generalize Theorems 4.1, 8.5, and 9.1 to arbitrary finitely generated hyperbolic groups?

Paulin proved [1989] that Theorem 4.1 holds for automorphisms of hyperbolic groups.

Problem 10.2. Is $\text{Fix } \varphi$ effectively computable when φ is an endomorphism of a finitely generated virtually free group?

For the moment, only the case of free group automorphisms is known; see [Bogopolski and Maslakova 2012].

Another natural question to ask in this context is whether similar results hold for *equalizers*. Given homomorphisms $\varphi, \psi : G \rightarrow G'$, let

$$\text{Eq}(\varphi, \psi) = \{x \in G : x\varphi = x\psi\}.$$

Problem 10.3. Given homomorphisms $\varphi, \psi : G \rightarrow G'$ of finitely generated virtually free groups with φ injective, is $\text{Eq}(\varphi, \psi)$ finitely generated?

This question has been solved by Goldstein and Turner for free groups [1986]. The restriction to the case where at least one of the homomorphisms is injective is required even in the free group case (see [Gersten 1987] and [Ventura 2002, Section 3] for counterexamples).

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Biharmonic hypersurfaces in complete Riemannian manifolds	1
LUIS J. ALÍAS, S. CAROLINA GARCÍA-MARTÍNEZ and MARCO RIGOLI	
Half-commutative orthogonal Hopf algebras	13
JULIEN BICHON and MICHEL DUBOIS-VIOLETTE	
Superdistributions, analytic and algebraic super Harish-Chandra pairs	29
CLAUDIO CARMELI and RITA FIORESI	
Orbifolds with signature $(0; k, k^{n-1}, k^n, k^n)$	53
ANGEL CAROCCA, RUBÉN A. HIDALGO and RUBÍ E. RODRÍGUEZ	
Explicit isogeny theorems for Drinfeld modules	87
IMIN CHEN and YOONJIN LEE	
Topological pressures for ϵ -stable and stable sets	117
XIANFENG MA and ERCAI CHEN	
Lipschitz and bilipschitz maps on Carnot groups	143
WILLIAM MEYERSON	
Geometric inequalities in Carnot groups	171
FRANCESCO PAOLO MONTEFALCONE	
Fixed points of endomorphisms of virtually free groups	207
PEDRO V. SILVA	
The sharp lower bound for the first positive eigenvalue of the Folland–Stein operator on a closed pseudohermitian $(2n + 1)$ -manifold	241
CHIN-TUNG WU	
Remark on “Maximal functions on the unit n -sphere” by Peter M. Knopf (1987)	253
HONG-QUAN LI	