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THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF THE FOLLAND–STEIN OPERATOR ON A CLOSED PSEUDOHERMITIAN $(2n+1)$-MANIFOLD

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In this paper, we obtain a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator $\mathscr{L}_c$, $|c| \le n$, on a closed pseudohermitian $(2n+1)$-manifold M. This generalizes the first nonzero eigenvalue estimates of the sublaplacian and Kohn Laplacian.

1. Introduction

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold (see the next section for basic notions in pseudohermitian geometry). A. Greenleaf [1985], S.-Y. Li and H.-S. Luk [2004], and H.-L. Chiu [2006] proved the sharp lower bound of the first positive eigenvalue λ_1^0 of the sublaplacian Δ_b on a pseudohermitian $(2n+1)$-manifold M. More precisely, it was proved that

$$\lambda_1^0 \ge \frac{nk}{n+1}$$

if $[\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}](Z, Z) \ge k\langle Z, Z\rangle$ for all $Z \in T_{1,0}$, some positive constant k, on a closed pseudohermitian $(2n+1)$-manifold with the nonnegative CR Paneitz operator P_0 if $n = 1$ (also see [Chang and Wu 2010]).

Very recently, S. Chanillo, H.-L. Chiu and P. Yang [Chanillo et al. 2012] obtained the sharp lower bound of the first positive eigenvalue λ_1^n of the Kohn Laplacian $\Box_b$ on a pseudohermitian $(2n+1)$-manifold M with $n = 1, 2$. Later, S.-C. Chang and the author [Chang and Wu $\ge$ 2013] proved the same result for $n \ge 3$. They showed that

$$\lambda_1^n \ge \frac{2nk}{n+1}$$

if $\mathrm{Ric}(Z, Z) \ge k\langle Z, Z\rangle$ for all $Z \in T_{1,0}$, some positive constant k, on a closed pseudohermitian $(2n+1)$-manifold M with nonnegative CR Paneitz operator P_0 if $n = 1$. Note that there is no assumption involving the pseudohermitian torsion.

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In this paper, we generalize the first nonzero eigenvalue estimates of the sublaplacian Δ_b and Kohn Laplacian $\Box_b$ to the Folland–Stein operator $\mathcal{L}_c$. First we need some definitions.

Definition 1.1. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Define

$$P\varphi = \sum_{\alpha=1}^{n} \big(\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha}\big)\theta^{\beta} = (P_\beta\varphi)\theta^{\beta}, \quad \beta = 1, 2, \ldots, n,$$

which is an operator that characterizes CR-pluriharmonic functions ([Lee 1988] for $n=1$ and [Graham and Lee 1988] for $n \geq 2$). Here $P_\beta\varphi = \sum_{\alpha=1}^{n}(\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})$ and $\bar{P}\varphi = (\bar{P}_\beta\varphi)\theta^{\bar{\beta}}$, the conjugate of P. Moreover, we define

$$P_0\varphi = \delta_b(P\varphi),$$

which is the so-called CR Paneitz operator P_0. Here δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_\alpha\theta^\alpha) = \sigma_\alpha{,}^\alpha$ and $\bar{\delta}_b(\sigma_{\bar{\alpha}}\theta^{\bar{\alpha}}) = \sigma_{\bar{\alpha}}{,}^{\bar{\alpha}}$. If we define $\partial_b\varphi = \varphi_\alpha\theta^\alpha$ and $\bar{\partial}_b\varphi = \varphi_{\bar{\alpha}}\theta^{\bar{\alpha}}$, then the formal adjoint of ∂_b on functions (with respect to the Levi form and the volume form $\theta \wedge (d\theta)^n$) is $\partial_b^* = -\delta_b$.

We observe that P_0 is a real and symmetric operator and

$$\int \langle P\varphi, \partial_b\varphi\rangle = -\int (P_0\varphi)\bar{\varphi}.$$

Definition 1.2. We say that the Paneitz operator P_0 with respect to (J, θ) is nonnegative if, for all C^∞ smooth functions φ,

$$\int (P_0\varphi)\bar{\varphi} \geq 0.$$

Remark 1.3. When (M, J, θ) is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator is nonnegative [Chang et al. 2007]. Unlike $n = 1$, let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. The corresponding CR Paneitz operator is always nonnegative as in (3-4).

Definition 1.4 [Graham and Lee 1988]. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by

$$Q\varphi = 2i(A^{\alpha\beta}\varphi_\alpha)_{,\beta}\,.$$

Note that $[T, \Delta_b] = 2\operatorname{Im} Q$ and

$$4P_0 = \Delta_b^2 + n^2T^2 - 2n\operatorname{Re} Q = (\Delta_b + inT)(\Delta_b - inT) - 2nQ \tag{1-1}$$
$$= (\Delta_b - inT)(\Delta_b + inT) - 2n\bar{Q}.$$

Now we consider, for $c \in \mathbb{R}$, the self-adjoint operators

$$\mathcal{L}_c = \Delta_b + icT,$$

with $|c| \leq n$. By a result in [Folland and Stein 1974], each $\mathcal{L}_c$, $|c| < n$, is a subelliptic operator of order $\frac{1}{2}$; hence $\mathcal{L}_c$ has a discrete spectrum tending to $+\infty$.

In the following we can obtain a sharp lower bound for the first nonzero eigenvalue λ_1^c of the Folland–Stein operator $\mathcal{L}_c$, $c \in \mathbb{R}$ with $|c| \leq n$, on a closed pseudohermitian $(2n+1)$-manifold.

Theorem 1.5. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\begin{cases} \left[\mathrm{Ric} - \dfrac{(n-c)(n+1)}{2(n+c)}\,\mathrm{Tor}\right](Z, Z) \geq k\langle Z, Z\rangle & \text{if } c \geq 0, \\ \left[\mathrm{Ric} - \dfrac{(n+c)(n+1)}{2(n-c)}\,\mathrm{Tor}\right](\overline{Z}, \overline{Z}) \geq k\langle Z, Z\rangle & \text{if } c < 0, \end{cases} \tag{1-2}$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n+|c|}{n+1}k.$$

Note that the constant in the torsion tensor term in assumption (1-2) depends on the variable c. In the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ with the induced CR structure $\hat{J}$ from $\mathbb{C}^{n+1}$ and the standard contact form $\hat{\theta}$, we can show that the lower bound in Theorem 1.5 is sharp (see Section 4).

In particular, when (M, J, θ) is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator P_0 is nonnegative.

Corollary 1.6. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with vanishing pseudohermitian torsion. Suppose that*

$$\begin{cases} \mathrm{Ric}(Z, Z) \geq k\langle Z, Z\rangle & \text{if } c \geq 0, \\ \mathrm{Ric}(\overline{Z}, \overline{Z}) \geq k\langle Z, Z\rangle & \text{if } c < 0, \end{cases}$$

for a positive constant k and for all $Z \in T_{1,0}$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n+|c|}{n+1}k.$$

Moreover, when $c = n$, the operator $\mathscr{L}_n$ is just the Kohn Laplacian: $\mathscr{L}_n = \Box_b$.

Corollary 1.7. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\mathrm{Ric}(Z, Z) \geq k\langle Z, Z\rangle$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of the Kohn Laplacian $\Box_b$ must satisfy

$$\lambda_1^n \geq \frac{2nk}{n+1}.$$

When $c = 0$, the operator $\mathscr{L}_0$ is just the sublaplacian Δ_b; i.e., $\mathscr{L}_0 = \Delta_b$.

Corollary 1.8. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\left[\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}\right](Z, Z) \geq k\langle Z, Z\rangle$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of the sublaplacian Δ_b must satisfy

$$\lambda_1^0 \geq \frac{nk}{n+1}.$$

Further, we study the case when a sharp lower bound estimate of $\mathscr{L}_c$, $|c| \leq n$, is achieved in Section 4.

Proposition 1.9. *Under the same conditions as in Theorem 1.5, if we assume the first nonzero eigenvalue of $\mathscr{L}_c$, $0 < |c| \leq n$, satisfies*

$$\lambda_1^c = \frac{n+|c|}{n+1}k,$$

$$\int A^{\alpha\beta}\varphi_{c\alpha}\bar{\varphi}_{c\beta} = 0 \tag{1-3}$$

for a corresponding eigenfunction φ_c of $\mathscr{L}_c$ with respect to λ_1^c and with $\int\langle\varphi_c, \varphi_c\rangle = 1$, then the eigenfunction φ_c will satisfy

$$\int |\bar{\partial}_b\varphi_c|^2 = \frac{n(n+c)}{2(n^2+c^2)}\lambda_1^c \quad \textit{and} \quad \int |\partial_b\varphi_c|^2 = \frac{n(n-c)}{2(n^2+c^2)}\lambda_1^c; \tag{1-4}$$

thus we also have

$$\int \langle\Delta_b\varphi_c, \varphi_c\rangle = \frac{n^2}{n^2+c^2}\lambda_1^c \quad \textit{and} \quad \int i\langle T\varphi_c, \varphi_c\rangle = \frac{c}{n^2+c^2}\lambda_1^c.$$

Letting $c \to 0^+$, we see that $\int |\bar\partial_b \varphi_c|^2 = \int |\partial_b \varphi_c|^2 = \frac{1}{2}\lambda_1^0$ and $\int i\langle T\varphi_c, \varphi_c\rangle = 0$ for $c = 0$. When $c = n$, from (1-4), we get that $\partial_b \varphi_n = 0$ and thus $\overline{\Box}_b \varphi_n = 0$. This implies that the corresponding eigenfunction φ_n of $\mathscr{L}_n = \Box_b$ with respect to λ_1^n will also satisfy

$$\Delta_b \varphi_n = \frac{nk}{n+1}\varphi_n.$$

This yields that φ_n achieves a sharp lower bound for the first nonzero eigenvalue of the sublaplacian Δ_b. Furthermore, it can be showed the pseudohermitian torsion $A_{\alpha\beta}$ of M is zero; thus (M, J, θ) is the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ (see [Chang and Wu $\geq$ 2013] for details).

2. Basic materials

Let us give a brief introduction to pseudohermitian geometry (see [Lee 1988] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: if X and Y are in ξ, then so is $[JX, Y] + [X, JY]$, and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$. A CR structure J can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of J with respect to i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathscr{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar\alpha}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha} = \overline{Z}_\alpha \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar\alpha}\}$, satisfies

$$d\theta = i h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$. Actually we can always choose Z_α such that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$; hence, throughout this paper, we assume $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$.

The Levi form $\langle\ ,\ \rangle$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle = -i\big\langle d\theta, Z \wedge \overline{W}\big\rangle.$$

We can extend $\langle\ ,\ \rangle$ to $T_{0,1}$ by defining $\langle \overline{Z}, \overline{W}\rangle = \overline{\langle Z, W\rangle}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, also denoted by $\langle\ ,\ \rangle$, and hence on all the induced tensor bundles.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha\in T_{1,0}$ by

$$\nabla Z_\alpha=\omega_\alpha{}^\beta\otimes Z_\beta,\quad \nabla Z_{\bar\alpha}=\omega_{\bar\alpha}{}^{\bar\beta}\otimes Z_{\bar\beta},\quad \nabla T=0,$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$d\theta^\beta=\theta^\alpha\wedge\omega_\alpha{}^\beta+\theta\wedge\tau^\beta,\quad \tau_\alpha\wedge\theta^\alpha=0,\quad \omega_\alpha{}^\beta+\omega_{\bar\beta}{}^{\bar\alpha}=0.$$

We can write $\tau_\alpha=A_{\alpha\beta}\theta^\beta$ with $A_{\alpha\beta}=A_{\beta\alpha}$. The curvature of the Webster–Stanton connection, expressed in terms of the coframe $\{\theta=\theta^0,\theta^\alpha,\theta^{\bar\alpha}\}$, is

$$\Pi_\beta{}^\alpha=\overline{\Pi_{\bar\beta}{}^{\bar\alpha}}=d\omega_\beta{}^\alpha-\omega_\beta{}^\gamma\wedge\omega_\gamma{}^\alpha,$$
$$\Pi_0{}^\alpha=\Pi_\alpha{}^0=\Pi_0{}^{\bar\beta}=\Pi_{\bar\beta}{}^0=\Pi_0{}^0=0.$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written as

$$\Pi_\beta{}^\alpha=R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma}+W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta-W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho}\wedge\theta+i\theta_\beta\wedge\tau^\alpha-i\tau_\beta\wedge\theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma}=\overline{R_{\alpha\bar\beta\sigma\bar\rho}}=R_{\bar\alpha\beta\bar\sigma\rho}=R_{\rho\bar\alpha\beta\bar\sigma},\quad W_{\beta\bar\alpha\gamma}=W_{\gamma\bar\alpha\beta}.$$

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar\alpha\}$ indicate derivatives with respect to $\{T,Z_\alpha,Z_{\bar\alpha}\}$. For derivatives of a function, we will often omit the comma, for instance, $\varphi_\alpha=Z_\alpha\varphi$, $\varphi_{\alpha\bar\beta}=Z_{\bar\beta}Z_\alpha\varphi-\omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma\varphi$, $\varphi_0=T\varphi$ for a (smooth) function φ. Let the Cauchy–Riemann operator ∂_b be defined locally by $\partial_b\varphi=\varphi_\alpha\theta^\alpha$, and let $\bar\partial_b$ be the conjugate of ∂_b. For a function φ, the subgradient ∇_b is defined locally by $\nabla_b\varphi=\varphi^\alpha Z_\alpha+\varphi^{\bar\alpha}Z_{\bar\alpha}$. The sublaplacian Δ_b, the Kohn Laplacian $\Box_b$, and the Folland–Stein operator $\mathscr{L}_c$ on functions are defined by

$$\Delta_b\varphi=-(\varphi_\alpha{}^\alpha+\varphi_{\bar\alpha}{}^{\bar\alpha}),\quad \Box_b\varphi=(\Delta_b+inT)\varphi,\quad \mathscr{L}_c\varphi=(\Delta_b+icT)\varphi.$$

The Webster–Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$\mathrm{Ric}(X,Y)=R_{\alpha\bar\beta}X^\alpha Y^{\bar\beta},$$
$$\mathrm{Tor}(X,Y)=i\sum_{\alpha,\beta}\big(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta}-A_{\alpha\beta}X^\alpha Y^\beta\big),$$

where $X=X^\alpha Z_\alpha$, $Y=Y^\beta Z_\beta$, $R_{\alpha\bar\beta}=R_\gamma{}^\gamma{}_{\alpha\bar\beta}$. The Webster scalar curvature is $R=R_\alpha{}^\alpha=h^{\alpha\bar\beta}R_{\alpha\bar\beta}$.

3. Proof of Theorem 1.5

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. In this section, we can obtain lower bound estimates for the first nonzero eigenvalue of the Folland–Stein operator $\mathcal{L}_c$, $|c| \le n$, on a closed pseudohermitian $(2n+1)$-manifold.

First we need the following Bochner formula for the Kohn Laplacian [Chanillo et al. 2012, Equation (2.8)]).

Lemma 3.1. *For any complex-valued function φ, we have*

$$-\frac{1}{2}\Box_b\big|\bar\partial_b\varphi\big|^2 = \sum_{\alpha,\beta}\big(\varphi_{\bar\alpha\bar\beta}\overline{\varphi}_{\alpha\beta} + \varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta}\big) + \mathrm{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) - \frac{1}{2n}\big\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\big\rangle - \frac{n+1}{2n}\big\langle\bar\partial_b\Box_b\varphi, \bar\partial_b\varphi\big\rangle - \frac{1}{n}\big\langle\overline{P}\varphi, \bar\partial_b\varphi\big\rangle + \frac{n-1}{n}\big\langle P\overline{\varphi}, \partial_b\overline{\varphi}\big\rangle, \tag{3-1}$$

where $(\nabla_b\varphi)_{\mathbb{C}} = \varphi^\alpha Z_\alpha$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$.

First we derive some useful identities which we need in the proof of Theorem 1.5. Let φ be a smooth complex-valued function on M. By integrating the Bochner formula (3-1), we have

$$0 = \int\sum_{\alpha,\beta}\big(\varphi_{\bar\alpha\bar\beta}\overline{\varphi}_{\alpha\beta} + \varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta}\big) - \frac{n+2}{2n}\int\big\langle\Box_b\varphi, \Box_b\varphi\big\rangle + \frac{2-n}{n}\int(P_0\varphi)\overline{\varphi} + \int\mathrm{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big). \tag{3-2}$$

We also have

$$\int\sum_{\alpha,\beta}\varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta} = \int\sum_{\alpha,\beta}\Big|\overline{\varphi}_{\alpha\bar\beta} - \frac{1}{n}\overline{\varphi}_\gamma{}^\gamma h_{\alpha\bar\beta}\Big|^2 + \frac{1}{4n}\int\big\langle\Box_b\varphi, \Box_b\varphi\big\rangle = \frac{n-1}{n}\int(P_0\varphi)\overline{\varphi} + \frac{1}{4n}\int\big\langle\Box_b\varphi, \Box_b\varphi\big\rangle. \tag{3-3}$$

Here we used the following divergence formula [Graham and Lee 1988] for the trace-free part of $\overline{\varphi}_{\alpha\bar\beta}$:

$$B_{\alpha\bar\beta}\overline{\varphi} = \overline{\varphi}_{\alpha\bar\beta} - \frac{1}{n}\overline{\varphi}_\gamma{}^\gamma h_{\alpha\bar\beta}.$$

That is,

$$\begin{aligned}\big(B^{\alpha\bar\beta}\varphi\big)\big(B_{\alpha\bar\beta}\overline{\varphi}\big) &= \varphi^{\alpha\bar\beta}\big(B_{\alpha\bar\beta}\overline{\varphi}\big) = \big(\varphi^\alpha B_{\alpha\bar\beta}\overline{\varphi}\big),^{\bar\beta} - \frac{n-1}{n}\varphi^\alpha P_\alpha\overline{\varphi}\\ &= \big(\varphi^\alpha B_{\alpha\bar\beta}\overline{\varphi}\big),^{\bar\beta} - \frac{n-1}{n}(\varphi P_\alpha\overline{\varphi}),^\alpha + \frac{n-1}{n}(P_0\overline{\varphi})\varphi.\end{aligned}$$

Then we integrate both sides to get

$$\int \sum_{\alpha,\beta} \left| B_{\alpha\bar\beta}\overline{\varphi} \right|^2 = \frac{n-1}{n} \int (P_0\varphi)\overline{\varphi}. \tag{3-4}$$

Taking together the two formulas (3-2) and (3-3), we get

$$\frac{n+1}{4n}\int\langle\Box_b\varphi, \Box_b\varphi\rangle=\int\sum_{\alpha,\beta}\varphi_{\bar\alpha\bar\beta}\overline{\varphi}_{\alpha\beta}+\frac{1}{n}\int(P_0\varphi)\overline{\varphi}+\int\mathrm{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big). \tag{3-5}$$

By taking complex conjugate to (3-5) and replacing $\overline{\varphi}$ by φ, one obtains

$$\frac{n+1}{4n}\int\langle\overline{\Box}_b\varphi, \overline{\Box}_b\varphi\rangle=\int\sum_{\alpha,\beta}\varphi_{\alpha\beta}\overline{\varphi}_{\bar\alpha\bar\beta}+\frac{1}{n}\int(P_0\varphi)\overline{\varphi}+\int\mathrm{Ric}\big((\nabla_b\overline{\varphi})_{\mathbb{C}}, (\nabla_b\overline{\varphi})_{\mathbb{C}}\big). \tag{3-6}$$

From the formula (1-1), we have

$$\begin{aligned} 4\int (P_0\varphi)\overline{\varphi} &= \int \langle (\Delta_b + inT)(\Delta_b - inT)\varphi - 2nQ\varphi, \varphi\rangle \\ &= \int \langle \overline{\Box}_b\varphi, \Box_b\varphi\rangle - 2n\int \langle Q\varphi, \varphi\rangle. \end{aligned} \tag{3-7}$$

By (1-1), we can also obtain

$$4\int (P_0\varphi)\overline{\varphi} = \int \langle \Box_b\varphi, \overline{\Box}_b\varphi\rangle - 2n\int \langle \overline{Q}\varphi, \varphi\rangle. \tag{3-8}$$

Proof of Theorem 1.5. Let φ_c be an eigenfunction of the Folland–Stein operator $\mathscr{L}_c$, $c \in \mathbb{R}$ with $|c| \le n$, with respect to the first nonzero eigenvalue λ_1^c; i.e., $\mathscr{L}_c\varphi_c = \lambda_1^c\varphi_c$.

When $0 \le c \le n$, from (3-6) and (3-7) for

$$\mathscr{L}_c = \frac{n+c}{2n}\Box_b + \frac{n-c}{2n}\overline{\Box}_b,$$

we have

$$\begin{aligned} \frac{1}{2}\int \langle \Box_b\varphi_c, \mathscr{L}_c\varphi_c\rangle &= \frac{n+c}{4n}\int\langle \Box_b\varphi_c, \Box_b\varphi_c\rangle + \frac{n-c}{4n}\int\langle \Box_b\varphi_c, \overline{\Box}_b\varphi_c\rangle \\ &= \frac{n+c}{n+1}\int\sum_{\alpha,\beta}\varphi_{c\bar\alpha\bar\beta}\overline{\varphi}_{c\alpha\beta} + \frac{n+2-c}{n+1}\int (P_0\varphi_c)\overline{\varphi}_c \\ &\quad + \frac{n+c}{n+1}\int \mathrm{Ric}\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big) + \frac{n-c}{2}\int\langle \overline{Q}\varphi_c, \varphi_c\rangle \\ &= \frac{n+c}{n+1}\int\sum_{\alpha,\beta}\varphi_{c\bar\alpha\bar\beta}\overline{\varphi}_{c\alpha\beta} + \frac{n+2-c}{n+1}\int (P_0\varphi_c)\overline{\varphi}_c \\ &\quad + \frac{n+c}{n+1}\int \left[\mathrm{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\,\mathrm{Tor}\right]\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big), \end{aligned}$$

where we used the equation

$$\int \langle \overline{Q}\varphi_c, \varphi_c \rangle = -\int \operatorname{Tor}\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big),$$

since $\int \langle \overline{Q}\varphi_c, \varphi_c\rangle$ is real, and thus $\int \langle \overline{Q}\varphi_c, \varphi_c\rangle = 2\int i A^{\bar\alpha\bar\beta}\varphi_{c\bar\alpha}\bar\varphi_{c\bar\beta} = -2\int i A^{\alpha\beta}\varphi_{c\alpha}\bar\varphi_{c\beta}$.

Hence, if P_0 is nonnegative and

$$[\operatorname{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\operatorname{Tor}]((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}) \geq k|\bar\partial_b\varphi_c|^2,$$

we have

$$\begin{aligned}\lambda_1^c \int |\bar\partial_b\varphi_c|^2 &= \frac{n+c}{n+1}\int \sum_{\alpha,\beta}\varphi_{c\bar\alpha\bar\beta}\bar\varphi_{c\alpha\beta} + \frac{n+2-c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n+c}{n+1}\int \Big[\operatorname{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\operatorname{Tor}\Big]\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big) \\ &\geq \frac{n+c}{n+1}k\int |\bar\partial_b\varphi_c|^2,\end{aligned} \tag{3-9}$$

which shows that $\lambda_1^c \geq \frac{n+c}{n+1}k$.

When $-n \leq c < 0$, from (3-5) and (3-8), the same computation shows that

$$\begin{aligned}\frac{1}{2}\int \langle \overline{\Box}_b\varphi_c, \mathscr{L}_c\varphi_c\rangle &= \frac{n+c}{4n}\int \langle \overline{\Box}_b\varphi_c, \Box_b\varphi_c\rangle + \frac{n-c}{4n}\int \langle \overline{\Box}_b\varphi_c, \overline{\Box}_b\varphi_c\rangle \\ &= \frac{n-c}{n+1}\int \sum_{\alpha,\beta}\varphi_{c\alpha\beta}\bar\varphi_{c\bar\alpha\bar\beta} + \frac{n+2+c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n-c}{n+1}\int \Big[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}\Big]\big((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}\big).\end{aligned}$$

Thus, if P_0 is nonnegative and

$$[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}]((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}) \geq k|\partial_b\varphi_c|^2,$$

we get

$$\begin{aligned}\lambda_1^c \int |\partial_b\varphi_c|^2 &= \frac{n-c}{n+1}\int \sum_{\alpha,\beta}\varphi_{c\alpha\beta}\bar\varphi_{c\bar\alpha\bar\beta} + \frac{n+2+c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n-c}{n+1}\int \Big[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}\Big]\big((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}\big) \\ &\geq \frac{n-c}{n+1}k\int |\partial_b\varphi_c|^2,\end{aligned}$$

which implies that $\lambda_1^c \geq \dfrac{n-c}{n+1}k$. This completes the proof of Theorem 1.5. □

4. Example and proof of Proposition 1.9

In this section, we calculate the eigenvalues of sublaplacian Δ_b, Kohn Laplacian $\Box_b$, and the Folland–Stein operator $\mathscr{L}_c$, $|c| \le n$, of the standard pseudohermitian $(2n+1)$-sphere S^{2n+1}. We show that the lower bound in Theorem 1.5 is sharp. We also study the case when a sharp lower bound estimate of $\mathscr{L}_c$, $|c| \le n$, is achieved.

Let $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \mid \sum_{j=0}^{n} z_j \bar{z}_j = 1\} \subset \mathbb{C}^{n+1}$ with the induced CR structure from $\mathbb{C}^{n+1}$ and the contact form $\theta = \frac{i}{2}(\bar{\partial} u - \partial u)|_{S^{2n+1}}$ where $u = \big(\sum_{j=0}^{n} z_j \bar{z}_j\big) - 1$ is a defining function. It can be shown that the pseudohermitian torsion is free and the Webster–Ricci tensor is given by $R_{\alpha\bar{\beta}} = (n+1)h_{\alpha\bar{\beta}}$.

We write

$$\partial_j = \frac{\partial}{\partial z_j},\ \bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j} \quad (0 \le j \le n), \qquad \partial_{j\bar{k}} = \partial_j \partial_{\bar{k}} \quad (0 \le j, k \le n),$$

and $z = (z_0, z_1, \dots, z_n)$, $\delta = (\partial_0, \partial_1, \dots, \partial_n)$. We let $\cdot$ denote the dot product. Then, by the computation in Section 1 of [Geller 1980], we have

$$\mathscr{L}_c = 2\Big(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Big) + (n+c)\bar{z} \cdot \bar{\delta} + (n-c) z \cdot \delta,$$

where $\Delta = \sum_{j=0}^{n} \partial_j \partial_{\bar{j}}$ is the standard Laplacian on $\mathbb{C}^{n+1}$. In particular, we have

$$\Delta_b = 2\Big(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Big) + n\big(\bar{z} \cdot \bar{\delta} + z \cdot \delta\big),$$
$$\Box_b = 2\Big(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Big) + 2n\bar{z} \cdot \bar{\delta}.$$

If Y is a bigraded spherical harmonic of type (p, q) on $\mathbb{C}^{n+1}$ (a harmonic polynomial which is a linear combination in terms of the form $z^{\rho}\bar{z}^{\gamma}$, where ρ, γ are multiindices with $|\rho| = p$, $|\gamma| = q$), then $\mathscr{L}_c Y = \big(2pq + (n+c)q + (n-c)p\big)Y$. Similarly,

$$\Delta_b Y = (2pq + n(p+q))Y, \quad \Box_b Y = 2q(p+n)Y.$$

This example shows that the lower bound in Theorem 1.5 is sharp.

Now we study the case when a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator $\mathscr{L}_c$, $|c| \le n$, on a pseudohermitian $(2n+1)$-manifold M is achieved. We only consider the case when the constant c is nonnegative. The same computation follows when c is negative.

First, from (3-9), we have the following observation.

Lemma 4.1. *Under the same conditions as in Theorem 1.5, when the first nonzero eigenvalue of $\mathscr{L}_c$, $0 \le c \le n$, satisfies*

$$\lambda_1^c = \frac{n+c}{n+1}k,$$

then the corresponding eigenfunction φ_c will satisfy

$$\varphi_{c\bar{\alpha}\bar{\beta}} = 0 \quad \textit{for all } \alpha, \beta, \tag{4-1}$$

$$\left[\mathrm{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\,\mathrm{Tor}\right]\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big) = k\big|\bar{\partial}_b\varphi_c\big|^2, \tag{4-2}$$

$$P_0\varphi_c = 0. \tag{4-3}$$

Proof of Proposition 1.9. The integral condition (1-3) says that

$$\int \langle Q\varphi_c, \varphi_c\rangle = -2i\int A^{\alpha\beta}\varphi_{c\alpha}\bar{\varphi}_{c\beta} = 0,$$

and then by integration by parts, we obtain

$$\int \big\langle \bar{Q}\varphi_c, \varphi_c\big\rangle = \int \langle \varphi_c, Q\varphi_c\rangle = \int \langle Q\varphi_c, \varphi_c\rangle = 0. \tag{4-4}$$

From (1-1), one can see that

$$4P_0 = [\Delta_b - i(n^2/c)T][\Delta_b + icT] - \frac{1}{2c}\big[(2nc+n+c)\bar{Q} + (2nc-n-c)Q\big].$$

Then, from (4-3) and (4-4), one obtains

$$\begin{aligned} 0 = 4\int (P_0\varphi_c)\bar{\varphi}_c &= \lambda_1^c \int \big\langle [\Delta_b - i(n^2/c)T]\varphi_c, \varphi_c\big\rangle \\ &= \frac{1}{2}\lambda_1^c \int \big\langle [(1-n/c)\Box_b + (1+n/c)\bar{\Box}_b]\varphi_c, \varphi_c\big\rangle \\ &= \lambda_1^c \int \big[(1-n/c)|\bar{\partial}_b\varphi_c|^2 + (1+n/c)|\partial_b\varphi_c|^2\big], \end{aligned}$$

which is

$$(n-c)\int \big|\bar{\partial}_b\varphi_c\big|^2 = (n+c)\int |\partial_b\varphi_c|^2. \tag{4-5}$$

On the other hand, the equation $\mathscr{L}_c\varphi_c = (\Delta_b + icT)\varphi_c = \lambda_1^c\varphi_c$ yields

$$\begin{aligned} \lambda_1^c = \lambda_1^c \int \langle \varphi_c, \varphi_c\rangle &= \int \langle \mathscr{L}_c\varphi_c, \varphi_c\rangle \\ &= \frac{1}{2n}\int \big\langle \big[(n+c)\Box_b + (n-c)\bar{\Box}_b\big]\varphi_c, \varphi_c\big\rangle \\ &= \int (1+n/c)\big|\bar{\partial}_b\varphi_c\big|^2 + (1-n/c)|\partial_b\varphi_c|^2. \end{aligned} \tag{4-6}$$

The equations (1-4) follow from (4-5) and (4-6) easily. □

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