THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF THE FOLLAND–STEIN OPERATOR ON A CLOSED PSEUDOHERMITIAN $(2n + 1)$-MANIFOLD

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In this paper, we obtain a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator \(L_c, |c| \leq n\), on a closed pseudohermitian \((2n+1)\)-manifold \(M\). This generalizes the first nonzero eigenvalue estimates of the sublaplacian and Kohn Laplacian.

1. Introduction

Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold (see the next section for basic notions in pseudohermitian geometry). A. Greenleaf [1985], S.-Y. Li and H.-S. Luk [2004], and H.-L. Chiu [2006] proved the sharp lower bound of the first positive eigenvalue \(\lambda_1^0\) of the sublaplacian \(\Delta_b\) on a pseudohermitian \((2n+1)\)-manifold \(M\). More precisely, it was proved that

\[
\lambda_1^0 \geq \frac{nk}{n+1}
\]

if \(\text{Ric} - \frac{n+1}{2}\text{Tor}(Z, Z) \geq k\langle Z, Z \rangle\) for all \(Z \in T_{1,0}\), some positive constant \(k\), on a closed pseudohermitian \((2n+1)\)-manifold with the nonnegative CR Paneitz operator \(P_0\) if \(n = 1\) (also see [Chang and Wu 2010]).

Very recently, S. Chanillo, H.-L. Chiu and P. Yang [Chanillo et al. 2012] obtained the sharp lower bound of the first positive eigenvalue \(\lambda_1^n\) of the Kohn Laplacian \(\square_b\) on a pseudohermitian \((2n+1)\)-manifold \(M\) with \(n = 1, 2\). Later, S.-C. Chang and the author [Chang and Wu \geq 2013] proved the same result for \(n \geq 3\). They showed that

\[
\lambda_1^n \geq \frac{2nk}{n+1}
\]

if \(\text{Ric}(Z, Z) \geq k\langle Z, Z \rangle\) for all \(Z \in T_{1,0}\), some positive constant \(k\), on a closed pseudohermitian \((2n+1)\)-manifold \(M\) with nonnegative CR Paneitz operator \(P_0\) if \(n = 1\). Note that there is no assumption involving the pseudohermitian torsion.

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In this paper, we generalize the first nonzero eigenvalue estimates of the sublaplacian $\Delta_b$ and Kohn Laplacian $\Box_b$ to the Folland–Stein operator $\mathcal{L}_c$. First we need some definitions.

**Definition 1.1.** Let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold. Define

$$P\varphi = \sum_{\alpha=1}^{n} (\varphi_{\overline{\alpha} \beta} + i n A_{\beta \alpha} \varphi^\alpha) \theta^\beta = (P_\beta \varphi) \theta^\beta,$$

which is an operator that characterizes CR-pluriharmonic functions ([Lee 1988] for $n = 1$ and [Graham and Lee 1988] for $n \geq 2$). Here $P_\beta \varphi = \sum_{\alpha=1}^{n} (\varphi_{\overline{\alpha} \beta} + i n A_{\beta \alpha} \varphi^\alpha)$ and $\overline{P} \varphi = (\overline{P}_\beta \varphi) \theta^\overline{\beta}$, the conjugate of $P$. Moreover, we define

$$P_0 \varphi = \delta_b(P \varphi),$$

which is the so-called CR Paneitz operator $P_0$. Here $\delta_b$ is the divergence operator that takes $(1, 0)$-forms to functions by $\delta_b(\sigma_{\alpha} \theta^\alpha) = \sigma_{\alpha}$ and $\overline{\delta}_b(\sigma_{\overline{\alpha}} \theta^\overline{\alpha}) = \sigma_{\overline{\alpha}}$. If we define $\partial_b \varphi = \varphi_{\alpha} \theta^\alpha$ and $\overline{\partial}_b \varphi = \varphi_{\overline{\alpha}} \theta^\overline{\alpha}$, then the formal adjoint of $\partial_b$ on functions (with respect to the Levi form and the volume form $\theta \wedge (d\theta)^n$) is $\partial_b^* = -\delta_b$.

We observe that $P_0$ is a real and symmetric operator and

$$\int \langle P \varphi, \partial_b \varphi \rangle = -\int (P_0 \varphi) \overline{\varphi}.$$

**Definition 1.2.** We say that the Paneitz operator $P_0$ with respect to $(J, \theta)$ is nonnegative if, for all $C^\infty$ smooth functions $\varphi$,

$$\int (P_0 \varphi) \overline{\varphi} \geq 0.$$

**Remark 1.3.** When $(M, J, \theta)$ is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator is nonnegative [Chang et al. 2007]. Unlike $n = 1$, let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold with $n \geq 2$. The corresponding CR Paneitz operator is always nonnegative as in (3-4).

**Definition 1.4** [Graham and Lee 1988]. Let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold. We define the purely holomorphic second-order operator $Q$ by

$$Q\varphi = 2i (A_{\alpha \beta} \varphi_{\alpha}^\beta) .$$

Note that $[T, \Delta_b] = 2 \text{Im } Q$ and

$$4P_0 = \Delta_b^2 + n^2 T^2 - 2n \text{Re } Q = (\Delta_b + inT)(\Delta_b - inT) - 2n Q = (\Delta_b - inT)(\Delta_b + inT) - 2n \overline{Q} .$$

(1-1)
Now we consider, for $c \in \mathbb{R}$, the self-adjoint operators

$$\mathcal{L}_c = \Delta_b + ic T,$$

with $|c| \leq n$. By a result in [Folland and Stein 1974], each $\mathcal{L}_c$ with $|c| < n$, is a subelliptic operator of order $\frac{1}{2}$; hence $\mathcal{L}_c$ has a discrete spectrum tending to $+\infty$.

In the following we can obtain a sharp lower bound for the first nonzero eigenvalue $\lambda_1^c$ of the Folland–Stein operator $\mathcal{L}_c$, $c \in \mathbb{R}$ with $|c| \leq n$, on a closed pseudohermitian $(2n + 1)$-manifold.

**Theorem 1.5.** Let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold. Suppose that

\[
\begin{cases}
\text{Ric} - \frac{(n - c)(n + 1)}{2(n + c)} \text{Tor} (Z, Z) \geq k \langle Z, Z \rangle & \text{if } c \geq 0, \\
\text{Ric} - \frac{(n + c)(n + 1)}{2(n - c)} \text{Tor} (\bar{Z}, \bar{Z}) \geq k \langle Z, Z \rangle & \text{if } c < 0,
\end{cases}
\]

(1-2)

for a positive constant $k$ and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator $P_0$ is nonnegative if $n = 1$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n + |c|}{n + 1} k.$$

Note that the constant in the torsion tensor term in assumption (1-2) depends on the variable $c$. In the standard pseudohermitian $(2n + 1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ with the induced CR structure $\hat{J}$ from $\mathbb{C}^{n+1}$ and the standard contact form $\hat{\theta}$, we can show that the lower bound in Theorem 1.5 is sharp (see Section 4).

In particular, when $(M, J, \theta)$ is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator $P_0$ is nonnegative.

**Corollary 1.6.** Let $(M, J, \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold with vanishing pseudohermitian torsion. Suppose that

\[
\begin{cases}
\text{Ric}(Z, Z) \geq k \langle Z, Z \rangle & \text{if } c \geq 0, \\
\text{Ric}(\bar{Z}, \bar{Z}) \geq k \langle Z, Z \rangle & \text{if } c < 0,
\end{cases}
\]

for a positive constant $k$ and for all $Z \in T_{1,0}$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n + |c|}{n + 1} k.$$
Moreover, when \( c = n \), the operator \( \mathcal{L}_n \) is just the Kohn Laplacian: \( \mathcal{L}_n = \Box_b \).

**Corollary 1.7.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold. Suppose that

\[
\text{Ric}(Z, Z) \geq k\langle Z, Z \rangle
\]

for a positive constant \( k \) and for all \( Z \in T_{1,0} \). In addition we assume the Paneitz operator \( P_0 \) is nonnegative if \( n = 1 \). Then the first nonzero eigenvalue of the Kohn Laplacian \( \Box_b \) must satisfy

\[
\lambda_1^n \geq \frac{2nk}{n+1}.
\]

When \( c = 0 \), the operator \( \mathcal{L}_0 \) is just the sublaplacian \( \Delta_b \); i.e., \( \mathcal{L}_0 = \Delta_b \).

**Corollary 1.8.** Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold. Suppose that

\[
\left[ \text{Ric} - \frac{n+1}{2} \text{Tor} \right](Z, Z) \geq k\langle Z, Z \rangle
\]

for a positive constant \( k \) and for all \( Z \in T_{1,0} \). In addition we assume the Paneitz operator \( P_0 \) is nonnegative if \( n = 1 \). Then the first nonzero eigenvalue of the sublaplacian \( \Delta_b \) must satisfy

\[
\lambda_1^0 \geq \frac{nk}{n+1}.
\]

Further, we study the case when a sharp lower bound estimate of \( \mathcal{L}_c, |c| \leq n \), is achieved in Section 4.

**Proposition 1.9.** Under the same conditions as in Theorem 1.5, if we assume the first nonzero eigenvalue of \( \mathcal{L}_c, 0 < |c| \leq n \), satisfies

\[
\lambda_1^c = \frac{n + |c|}{n+1} k,
\]

(1-3)

\[
\int A^{\alpha\beta} \varphi_c \bar{\varphi}_c \beta = 0
\]

for a corresponding eigenfunction \( \varphi_c \) of \( \mathcal{L}_c \) with respect to \( \lambda_1^c \) and with \( \int \langle \varphi_c, \varphi_c \rangle = 1 \), then the eigenfunction \( \varphi_c \) will satisfy

\[
|\partial_b \varphi_c|^2 = \frac{n(n+c)}{2(n^2 + c^2)} \lambda_1^c
\]

(1-4)

\[
|\partial_b \varphi_c|^2 = \frac{n(n-c)}{2(n^2 + c^2)} \lambda_1^c;
\]

thus we also have

\[
\int \langle \Delta_b \varphi_c, \varphi_c \rangle = \frac{n^2}{n^2 + c^2} \lambda_1^c \quad \text{and} \quad \int i \langle T \varphi_c, \varphi_c \rangle = \frac{c}{n^2 + c^2} \lambda_1^c.
\]
Letting $c \to 0^+$, we see that $\int |\bar{\partial}_b \varphi_c|^2 = \int |\partial_b \varphi_c|^2 = \frac{1}{2} \lambda_1^0$ and $\int i \langle T \varphi_c, \varphi_c \rangle = 0$ for $c = 0$. When $c = n$, from (1-4), we get that $\partial_b \varphi_n = 0$ and thus $\Box_b \varphi_n = 0$. This implies that the corresponding eigenfunction $\varphi_n$ of $\mathcal{L}_n = \Box_b$ with respect to $\lambda_n^1$ will also satisfy
\[
\Delta_b \varphi_n = \frac{n k}{n + 1} \varphi_n.
\]
This yields that $\varphi_n$ achieves a sharp lower bound for the first nonzero eigenvalue of the sublaplacian $\Delta_b$. Furthermore, it can be showed the pseudohermitian torsion $A_{\alpha \beta}$ of $M$ is zero; thus $(M, J, \theta)$ is the standard pseudohermitian $(2n + 1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ (see [Chang and Wu 2013] for details).

2. Basic materials

Let us give a brief introduction to pseudohermitian geometry (see [Lee 1988] for more details). Let $(M, \xi)$ be a $(2n + 1)$-dimensional, orientable, contact manifold with contact structure $\xi$, $\dim_R \xi = 2n$. A CR structure compatible with $\xi$ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that $J$ satisfies the following integrability condition: if $X$ and $Y$ are in $\xi$, then so is $[JX, Y] + [X, JY]$, and $J((JX, Y) + [X, JY]) = [JX, JY] - [X, Y]$. A CR structure $J$ can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of $J$ with respect to $i$ and $-i$, respectively. A pseudohermitian structure compatible with $\xi$ is a CR structure $J$ compatible with $\xi$ together with a choice of contact form $\theta$. Such a choice determines a unique real vector field $T$ transverse to $\xi$, called the characteristic vector field of $\theta$, such that $\theta(T) = 1$ and $\mathcal{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where $Z_\alpha$ is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \bar{Z}_\alpha \in T_{0,1}$ and $T$ is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies
\[
d\theta = ih_{\alpha \bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}
\]
for some positive definite hermitian matrix of functions $(h_{\alpha \bar{\beta}})$. Actually we can always choose $Z_\alpha$ such that $h_{\alpha \bar{\beta}} = \delta_{\alpha \bar{\beta}}$; hence, throughout this paper, we assume $h_{\alpha \bar{\beta}} = \delta_{\alpha \bar{\beta}}$.

The Levi form $\langle , \rangle$ is the Hermitian form on $T_{1,0}$ defined by
\[
\langle Z, W \rangle = -i \langle d\theta, Z \wedge \bar{W} \rangle.
\]
We can extend $\langle , \rangle$ to $T_{0,1}$ by defining $\langle \bar{Z}, \bar{W} \rangle = \overline{\langle Z, W \rangle}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, also denoted by $\langle , \rangle$, and hence on all the induced tensor bundles.
The pseudohermitian connection of \((J, \theta)\) is the connection \(\nabla\) on \(TM \otimes \mathbb{C}\) (and extended to tensors) given in terms of a local frame \(Z_{\alpha} \in T_{1,0}\) by
\[
\nabla Z_{\alpha} = \omega_{\alpha}^{\beta} \otimes Z_{\beta}, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,
\]
where \(\omega_{\alpha}^{\beta}\) are the 1-forms uniquely determined by the following equations:
\[
d\theta^{\beta} = \theta^{\alpha} \wedge \omega_{\alpha}^{\beta} + \theta \wedge \tau^{\beta}, \quad \tau_{\alpha} \wedge \theta^{\alpha} = 0, \quad \omega_{\alpha}^{\beta} + \omega_{\bar{\alpha}}^{\bar{\beta}} = 0.
\]
We can write \(\tau_{\alpha} = A_{\alpha\beta} \theta^{\beta}\) with \(A_{\alpha\beta} = A_{\bar{\alpha}\bar{\beta}}\). The curvature of the Webster–Stanton connection, expressed in terms of the coframe \(\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}\), is
\[
\Pi^\alpha_{\beta} = \nabla^\alpha_{\beta} = d\omega^\alpha_{\rho} - \omega^\beta_{\rho} \wedge \omega^\alpha_{\gamma} + \omega^\gamma_{\beta} \wedge \omega^\alpha_{\rho} = 0,
\]
\[
\Pi^0_\alpha = \Pi^0_{\bar{\alpha}} = \Pi_{\bar{\beta}}^\beta = \Pi^\beta_{\alpha} = \Pi^0_0 = 0.
\]
Webster showed that \(\Pi^\alpha_{\beta}\) can be written as
\[
\Pi^\alpha_{\beta} = R^\alpha_{\beta \rho \sigma} \theta^\rho \wedge \theta^\sigma + W^\alpha_{\beta \rho} \theta^\rho \wedge \theta - W^\alpha_{\beta \rho} \theta^\rho \wedge \theta + i\theta_{\beta} \wedge \tau^\alpha - i\tau_{\beta} \wedge \theta^\alpha,
\]
where the coefficients satisfy
\[
R^\alpha_{\beta \rho \sigma} = \overline{R^\alpha_{\beta \rho \sigma}} = R_{\rho \sigma \beta \alpha} = R^\rho_{\sigma \beta \alpha}, \quad W^\alpha_{\beta \gamma} = W^\gamma_{\alpha \bar{\beta}}.
\]
We will denote components of covariant derivatives with indices preceded by comma; thus write \(A_{\alpha \beta \gamma}\). The indices \(\{0, \alpha, \bar{\alpha}\}\) indicate derivatives with respect to \(\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}\). For derivatives of a function, we will often omit the comma, for instance, \(\varphi_{\alpha} = Z_{\alpha} \varphi, \varphi_{\bar{\alpha}} = Z_{\bar{\alpha}} \varphi - \omega^\gamma_{\bar{\alpha}} (Z^\gamma_{\bar{\beta}})Z_{\gamma} \varphi, \varphi_0 = T \varphi\) for a (smooth) function \(\varphi\). Let the Cauchy–Riemann operator \(\partial_b\) be defined locally by \(\partial_b \varphi = \varphi_{\alpha} \theta^\alpha\), and let \(\partial_{\bar{b}}\) be the conjugate of \(\partial_b\). For a function \(\varphi\), the subgradient \(\nabla_b\) is defined locally by \(\nabla_b \varphi = \varphi^\alpha Z_{\alpha} + \varphi^{\bar{\alpha}} Z_{\bar{\alpha}}\). The sublaplacian \(\Delta_b\), the Kohn Laplacian \(\Box_b\), and the Folland–Stein operator \(\mathcal{L}_c\) on functions are defined by
\[
\Delta_b \varphi = - (\varphi_{\alpha}^\alpha + \varphi_{\bar{\alpha}}^{\bar{\alpha}}), \quad \Box_b \varphi = (\Delta_b + inT) \varphi, \quad \mathcal{L}_c \varphi = (\Delta_b + icT) \varphi.
\]
The Webster–Ricci tensor and the torsion tensor on \(T_{1,0}\) are defined by
\[
\text{Ric}(X, Y) = R_{\alpha \beta} X^\alpha Y^\beta, \quad \text{Tor}(X, Y) = i \sum_{\alpha, \beta} (A_{\alpha \beta} X^\alpha Y^\beta - A_{\alpha \beta} X^\beta Y^\alpha),
\]
where \(X = X^\alpha Z_{\alpha}, Y = Y^\beta Z_{\beta}, R_{\alpha \beta} = R_{\gamma \alpha \beta}\). The Webster scalar curvature is \(R = R_{\alpha}^{\alpha} = h_{\alpha \beta} R_{\alpha \beta}\).
3. Proof of Theorem 1.5

Let \((M, J, \theta)\) be a closed pseudohermitian \((2n+1)\)-manifold. In this section, we can obtain lower bound estimates for the first nonzero eigenvalue of the Folland–Stein operator \(\mathcal{L}_c\), \(|c| \leq n\), on a closed pseudohermitian \((2n + 1)\)-manifold.

First we need the following Bochner formula for the Kohn Laplacian [Chanillo et al. 2012, Equation (2.8)]).

**Lemma 3.1.** For any complex-valued function \(\varphi\), we have

\[
\begin{align*}
-\frac{1}{2} \Box_b |\bar{\partial}_b \varphi|^2 &= \sum_{\alpha, \beta} (\varphi_{\alpha \bar{\beta}} \bar{\varphi}_{\alpha \beta} + \varphi_{\alpha \bar{\beta}} \bar{\varphi}_{\alpha \beta}) + \text{Ric}((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) \\
&\quad - \frac{1}{2n} \langle \bar{\partial}_b \varphi, \bar{\partial}_b \Box_b \varphi \rangle - \frac{n+1}{2n} \langle \bar{\partial}_b \Box_b \varphi, \bar{\partial}_b \varphi \rangle \\
&\quad - \frac{1}{n} \langle \bar{P} \varphi, \bar{\partial}_b \varphi \rangle + \frac{n-1}{n} \langle P \varphi, \partial_b \varphi \rangle,
\end{align*}
\]

where \((\nabla_b \varphi)_C = \varphi^\alpha Z_\alpha\) is the corresponding complex \((1, 0)\)-vector field of \(\nabla_b \varphi\).

First we derive some useful identities which we need in the proof of Theorem 1.5. Let \(\varphi\) be a smooth complex-valued function on \(M\). By integrating the Bochner formula (3-1), we have

\[
\begin{align*}
0 &= \int \sum_{\alpha, \beta} (\varphi_{\alpha \bar{\beta}} \bar{\varphi}_{\alpha \beta} + \varphi_{\alpha \bar{\beta}} \bar{\varphi}_{\alpha \beta}) - \frac{n+2}{2n} \int \langle \Box_b \varphi, \Box_b \varphi \rangle \\
&\quad + \frac{2-n}{n} \int (P_0 \varphi) \bar{\varphi} + \int \text{Ric}((\nabla_b \varphi)_C, (\nabla_b \varphi)_C).
\end{align*}
\]

We also have

\[
\begin{align*}
\int \sum_{\alpha, \beta} \varphi_{\alpha \bar{\beta}} \bar{\varphi}_{\alpha \beta} &= \int \sum_{\alpha, \beta} \left| \bar{\varphi}_{\alpha \bar{\beta}} - \frac{1}{n} \bar{\varphi} \gamma h_{\alpha \bar{\beta}} \right|^2 + \frac{1}{4n} \int \langle \Box_b \varphi, \Box_b \varphi \rangle \\
&= \frac{n-1}{n} \int (P_0 \varphi) \bar{\varphi} + \frac{1}{4n} \int \langle \Box_b \varphi, \Box_b \varphi \rangle.
\end{align*}
\]

Here we used the following divergence formula [Graham and Lee 1988] for the trace-free part of \(\bar{\varphi}_{\alpha \bar{\beta}}\):

\[
B_{\alpha \bar{\beta}} \bar{\varphi} = \bar{\varphi}_{\alpha \bar{\beta}} - \frac{1}{n} \bar{\varphi} \gamma h_{\alpha \bar{\beta}}.
\]

That is,

\[
\begin{align*}
(B^\alpha_{\bar{\beta}} \varphi)(B_{\alpha \bar{\beta}} \bar{\varphi}) &= (\varphi^\alpha_{\bar{\beta}})(B_{\alpha \bar{\beta}} \bar{\varphi}) = (\varphi^\alpha_{\bar{\beta}} B_{\alpha \bar{\beta}} \bar{\varphi}) + \frac{n-1}{n} \varphi^\alpha P_0 \bar{\varphi} \\
&= (\varphi^\alpha B_{\alpha \bar{\beta}} \bar{\varphi})_{\bar{\beta}} - \frac{n-1}{n} \varphi^\alpha P_0 \bar{\varphi},
\end{align*}
\]
Then we integrate both sides to get

\[(3-4) \quad \int \sum_{\alpha, \beta} |B_{\alpha\beta} \bar{\varphi}|^2 = \frac{n - 1}{n} \int (P_0 \varphi) \bar{\varphi}.
\]

Taking together the two formulas (3-2) and (3-3), we get

\[(3-5) \quad \frac{n + 1}{4n} \int [\square_{b \varphi} \cdot \square_{b \varphi}] = \int \sum_{\alpha, \beta} \varphi_{\alpha\beta} \bar{\varphi}_{\alpha\beta} + \frac{1}{n} \int (P_0 \varphi) \bar{\varphi} + \int \text{Ric}(\nabla_b \varphi, \nabla_b \varphi)_C, (\nabla_b \varphi)_C).
\]

By taking complex conjugate to (3-5) and replacing \(\varphi\) by \(\varphi\), one obtains

\[(3-6) \quad \frac{n + 1}{4n} \int [\square_{b \varphi} \cdot \square_{b \varphi}] = \int \sum_{\alpha, \beta} \varphi_{\alpha\beta} \bar{\varphi}_{\alpha\beta} + \frac{1}{n} \int (P_0 \varphi) \bar{\varphi} + \int \text{Ric}(\nabla_b \varphi, \nabla_b \varphi)_C, (\nabla_b \varphi)_C).
\]

From the formula (1-1), we have

\[(3-7) \quad \frac{4}{n + 1} \int (P_0 \varphi) \bar{\varphi} = \int \langle (\Delta_b + i \nabla T)(\Delta_b - i \nabla T) \varphi - 2n Q \varphi, \varphi \rangle
\]
\[= \int \langle \square_{b \varphi}, \square_{b \varphi} \rangle - 2n \int \langle Q \varphi, \varphi \rangle.
\]

By (1-1), we can also obtain

\[(3-8) \quad \frac{4}{n + 1} \int (P_0 \varphi) \bar{\varphi} = \int \langle \square_{b \varphi}, \square_{b \varphi} \rangle - 2n \int \langle Q \varphi, \varphi \rangle.
\]

**Proof of Theorem 1.5.** Let \(\varphi_c\) be an eigenfunction of the Folland–Stein operator \(\mathcal{L}_c\), \(c \in \mathbb{R}\) with \(|c| \leq n\), with respect to the first nonzero eigenvalue \(\lambda_c^1\); i.e., \(\mathcal{L}_c \varphi_c = \lambda_c^1 \varphi_c\).

When \(0 \leq c \leq n\), from (3-6) and (3-7) for

\[\mathcal{L}_c = \frac{n + c}{2n} \square_b + \frac{n - c}{2n} \square_b,
\]

we have

\[
\frac{1}{2} \int \langle \square_b \varphi_c, \mathcal{L}_c \varphi_c \rangle = \frac{n + c}{4n} \int \langle \square_b \varphi_c, \square_b \varphi_c \rangle + \frac{n - c}{4n} \int \langle \square_b \varphi_c, \square_b \varphi_c \rangle
\]
\[= \frac{n + c}{n + 1} \int \sum_{\alpha, \beta} \varphi_{c\alpha\beta} \bar{\varphi}_{c\alpha\beta} + \frac{n + 2 - c}{n + 1} \int (P_0 \varphi_c) \bar{\varphi}_c
\]
\[+ \frac{n + c}{n + 1} \int \text{Ric}(\nabla_b \varphi_c)_C, (\nabla_b \varphi_c)_C) + \frac{n - c}{2} \int \langle Q \varphi_c, \varphi_c \rangle
\]
\[= \frac{n + c}{n + 1} \int \sum_{\alpha, \beta} \varphi_{c\alpha\beta} \bar{\varphi}_{c\alpha\beta} + \frac{n + 2 - c}{n + 1} \int (P_0 \varphi_c) \bar{\varphi}_c
\]
\[+ \frac{n + c}{n + 1} \left[ \text{Ric} - \frac{(n - c)(n + 1)}{2(n + c)} \text{Tor} \right](\nabla_b \varphi_c)_C, (\nabla_b \varphi_c)_C),
\]
where we used the equation
\[
\int \langle \bar{Q} \phi_c, \phi_c \rangle = - \int \text{Tor}((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C),
\]

since \(\int \langle \bar{Q} \phi_c, \phi_c \rangle\) is real, and thus \(\int \langle \bar{Q} \phi_c, \phi_c \rangle = 2 \int i A^{\alpha \beta} \phi_c \bar{\varphi} \bar{\varphi}_C = -2 \int i A^{\alpha \beta} \phi_c \bar{\varphi} \bar{\varphi}_C\).

Hence, if \(P_0\) is nonnegative and
\[
[Ric - \frac{(n-c)(n+1)}{2(n+c)} \text{Tor}((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C) \geq k|\bar{\partial}_b \phi_c|^2,
\]

we have
\[
(3-9) \quad \lambda^c_1 \int |\bar{\partial}_b \phi_c|^2 = \frac{n+c}{n+1} \int \sum_{\alpha, \beta} \phi_c \bar{\varphi} \bar{\varphi}_c + \frac{n+2-c}{n+1} \int (P_0 \phi_c) \bar{\varphi}_c
\]
\[
+ \frac{n+c}{n+1} \int \left[ Ric - \frac{(n-c)(n+1)}{2(n+c)} \text{Tor} \right]((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C)
\]
\[
\geq \frac{n+c}{n+1} k \int |\bar{\partial}_b \phi_c|^2,
\]

which shows that \(\lambda^c_1 \geq \frac{n+c}{n+1} k\).

When \(-n \leq c < 0\), from (3-5) and (3-8), the same computation shows that
\[
\frac{1}{2} \int \langle \square_b \phi_c, \mathcal{L}_c \phi_c \rangle = \frac{n+c}{4n} \int \langle \square_b \phi_c, \square_b \phi_c \rangle + \frac{n-c}{4n} \int \langle \square_b \phi_c, \square_b \phi_c \rangle
\]
\[
= \frac{n-c}{n+1} \int \sum_{\alpha, \beta} \phi_c \bar{\varphi} \bar{\varphi}_c + \frac{n+2+c}{n+1} \int (P_0 \phi_c) \bar{\varphi}_c
\]
\[
+ \frac{n-c}{n+1} \int \left[ Ric - \frac{(n+c)(n+1)}{2(n-c)} \text{Tor} \right]((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C).
\]

Thus, if \(P_0\) is nonnegative and
\[
[Ric - \frac{(n+c)(n+1)}{2(n-c)} \text{Tor}((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C) \geq k|\bar{\partial}_b \phi_c|^2,
\]

we get
\[
\lambda^c_1 \int |\partial_b \phi_c|^2 = \frac{n-c}{n+1} \int \sum_{\alpha, \beta} \phi_c \bar{\varphi} \bar{\varphi}_c + \frac{n+2+c}{n+1} \int (P_0 \phi_c) \bar{\varphi}_c
\]
\[
+ \frac{n-c}{n+1} \int \left[ Ric - \frac{(n+c)(n+1)}{2(n-c)} \text{Tor} \right]((\nabla_b \phi_c)_C, (\nabla_b \phi_c)_C)
\]
\[
\geq \frac{n-c}{n+1} k \int |\bar{\partial}_b \phi_c|^2,
\]

which implies that \(\lambda^c_1 \geq \frac{n-c}{n+1} k\). This completes the proof of Theorem 1.5. \(\square\)
4. Example and proof of Proposition 1.9

In this section, we calculate the eigenvalues of sublaplacian $\Delta_b$, Kohn Laplacian $\Box_b$, and the Folland–Stein operator $\mathcal{L}_c$, $|c| \leq n$, of the standard pseudohermitian $(2n+1)$-sphere $S^{2n+1}$. We show that the lower bound in Theorem 1.5 is sharp. We also study the case when a sharp lower bound estimate of $\mathcal{L}_c$, $|c| \leq n$, is achieved.

Let $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) | \sum_{j=0}^{n} z_j \bar{z}_j = 1\} \subset \mathbb{C}^{n+1}$ with the induced CR structure from $\mathbb{C}^{n+1}$ and the contact form $\theta = \frac{i}{2} (\partial u - \bar{\partial} u)|_{S^{2n+1}}$ where $u = (\sum_{j=0}^{n} z_j \bar{z}_j) - 1$ is a defining function. It can be shown that the pseudohermitian torsion is free and the Webster–Ricci tensor is given by $R_{\alpha \bar{\beta}} = (n+1) h_{\alpha \bar{\beta}}$.

We write \( \partial_j = \frac{\partial}{\partial z_j}, \quad \bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j} \quad (0 \leq j \leq n), \quad \partial_{j \bar{k}} = \partial_j \partial_{\bar{k}} \quad (0 \leq j, k \leq n), \) and $z = (z_0, z_1, \ldots, z_n)$, $\delta = (\partial_0, \partial_1, \ldots, \partial_n)$. We let $\cdot$ denote the dot product. Then, by the computation in Section 1 of [Geller 1980], we have

$$\mathcal{L}_c = 2 \left( -\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}} \right) + (n + c) \bar{z} \cdot \delta + (n - c) z \cdot \delta,$$

where $\Delta = \sum_{j=0}^{n} \partial_j \partial_j$ is the standard Laplacian on $\mathbb{C}^{n+1}$. In particular, we have

$$\Delta_b = 2 \left( -\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}} \right) + n (\bar{z} \cdot \delta + z \cdot \delta),$$

$$\Box_b = 2 \left( -\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}} \right) + 2n \bar{z} \cdot \delta.$$

If $Y$ is a bigraded spherical harmonic of type $(p, q)$ on $\mathbb{C}^{n+1}$ (a harmonic polynomial which is a linear combination in terms of the form $z^\rho \bar{z}^\gamma$, where $\rho$, $\gamma$ are multiindices with $|\rho| = p$, $|\gamma| = q$), then $\mathcal{L}_c Y = (2pq + (n + c)q + (n - c)p) Y$. Similarly,

$$\Delta_b Y = (2pq + n(p + q)) Y, \quad \Box_b Y = 2q(p + n) Y.$$

This example shows that the lower bound in Theorem 1.5 is sharp.

Now we study the case when a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator $\mathcal{L}_c$, $|c| \leq n$, on a pseudohermitian $(2n+1)$-manifold $M$ is achieved. We only consider the case when the constant $c$ is nonnegative. The same computation follows when $c$ is negative.

First, from (3-9), we have the following observation.
Lemma 4.1. Under the same conditions as in Theorem 1.5, when the first nonzero eigenvalue of $L_c$, $0 \leq c \leq n$, satisfies

$$\lambda_1^c = \frac{n + c}{n + 1} k,$$

then the corresponding eigenfunction $\varphi_c$ will satisfy

\begin{equation}
\varphi_{c\bar{\alpha}\beta} = 0 \quad \text{for all } \alpha, \beta,
\end{equation}

\begin{equation}
\left[ \text{Ric} - \frac{(n - c)(n + 1)}{2(n + c)} \text{Tor} \right] (\nabla_b \varphi_c, \nabla_b \varphi_c) = k |\bar{\partial}_b \varphi_c|^2,
\end{equation}

\begin{equation}
P_0 \varphi_c = 0.
\end{equation}

Proof of Proposition 1.9. The integral condition (1-3) says that

$$\int \langle Q \varphi_c, \varphi_c \rangle = -2i \int A^{a\beta} \varphi_c \bar{\varphi}_{\bar{c}\beta} = 0,$$

and then by integration by parts, we obtain

\begin{equation}
\int \langle \overline{Q} \varphi_c, \varphi_c \rangle = \int \langle \varphi_c, Q \varphi_c \rangle = \int \langle Q \varphi_c, \varphi_c \rangle = 0.
\end{equation}

From (1-1), one can see that

$$4P_0 = [\Delta_b - i(n^2/c)T][\Delta_b + icT] - \frac{1}{2c} [(2nc + n + c) \overline{Q} + (2nc - n - c) Q].$$

Then, from (4-3) and (4-4), one obtains

$$0 = 4 \int (P_0 \varphi_c) \overline{\varphi}_c = \lambda_1^c \int \langle [\Delta_b - i(n^2/c)T] \varphi_c, \varphi_c \rangle$$

$$= \frac{1}{2} \lambda_1^c \int \langle [(1 - n/c) \square_b + (1 + n/c) \overline{\square}_b] \varphi_c, \varphi_c \rangle$$

$$= \lambda_1^c \int [(1 - n/c) |\bar{\partial}_b \varphi_c|^2 + (1 + n/c) |\partial_b \varphi_c|^2],$$

which is

\begin{equation}
(n - c) \int |\bar{\partial}_b \varphi_c|^2 = (n + c) \int |\partial_b \varphi_c|^2.
\end{equation}

On the other hand, the equation $L_c \varphi_c = (\Delta_b + icT) \varphi_c = \lambda_1^c \varphi_c$ yields

\begin{equation}
\lambda_1^c = \lambda_1^c \int \langle \varphi_c, \varphi_c \rangle = \int \langle L_c \varphi_c, \varphi_c \rangle$$

$$= \frac{1}{2n} \int \langle [(n + c) \square_b + (n - c) \overline{\square}_b] \varphi_c, \varphi_c \rangle$$

$$= \int [(1 + n/c) |\bar{\partial}_b \varphi_c|^2 + (1 - n/c) |\partial_b \varphi_c|^2].$$
The equations (1-4) follow from (4-5) and (4-6) easily.

References


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