Pacific Journal of Mathematics

THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF THE FOLLAND–STEIN OPERATOR ON A CLOSED PSEUDOHERMITIAN $(2n+1)$-MANIFOLD

CHIN-TUNG WU

Volume 263 No. 1 May 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 263, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.263.241

THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF THE FOLLAND–STEIN OPERATOR ON A CLOSED PSEUDOHERMITIAN $(2n+1)$-MANIFOLD

CHIN-TUNG WU

In this paper, we obtain a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator $\mathscr{L}_c$, $|c| \leq n$, on a closed pseudohermitian $(2n+1)$-manifold M. This generalizes the first nonzero eigenvalue estimates of the sublaplacian and Kohn Laplacian.

1. Introduction

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold (see the next section for basic notions in pseudohermitian geometry). A. Greenleaf [1985], S.-Y. Li and H.-S. Luk [2004], and H.-L. Chiu [2006] proved the sharp lower bound of the first positive eigenvalue λ_1^0 of the sublaplacian Δ_b on a pseudohermitian $(2n+1)$-manifold M. More precisely, it was proved that

$$\lambda_1^0 \geq \frac{nk}{n+1}$$

if $[\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}](Z, Z) \geq k\langle Z, Z\rangle$ for all $Z \in T_{1,0}$, some positive constant k, on a closed pseudohermitian $(2n+1)$-manifold with the nonnegative CR Paneitz operator P_0 if $n = 1$ (also see [Chang and Wu 2010]).

Very recently, S. Chanillo, H.-L. Chiu and P. Yang [Chanillo et al. 2012] obtained the sharp lower bound of the first positive eigenvalue λ_1^n of the Kohn Laplacian $\Box_b$ on a pseudohermitian $(2n+1)$-manifold M with $n = 1, 2$. Later, S.-C. Chang and the author [Chang and Wu ≥ 2013] proved the same result for $n \geq 3$. They showed that

$$\lambda_1^n \geq \frac{2nk}{n+1}$$

if $\mathrm{Ric}(Z, Z) \geq k\langle Z, Z\rangle$ for all $Z \in T_{1,0}$, some positive constant k, on a closed pseudohermitian $(2n+1)$-manifold M with nonnegative CR Paneitz operator P_0 if $n = 1$. Note that there is no assumption involving the pseudohermitian torsion.

Research supported in part by NSC.
MSC2010: primary 32V05, 32V20; secondary 53C56.
Keywords: Folland–Stein operator, sublaplacian, Kohn Laplacian, CR Paneitz operator, pseudohermitian manifold, pseudohermitian Ricci curvature, pseudohermitian torsion.

In this paper, we generalize the first nonzero eigenvalue estimates of the sublaplacian Δ_b and Kohn Laplacian $\Box_b$ to the Folland–Stein operator $\mathscr{L}_c$. First we need some definitions.

Definition 1.1. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Define

$$P\varphi = \sum_{\alpha=1}^{n} \big(\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha}\big)\theta^{\beta} = (P_{\beta}\varphi)\theta^{\beta}, \quad \beta = 1, 2, \ldots, n,$$

which is an operator that characterizes CR-pluriharmonic functions ([Lee 1988] for $n=1$ and [Graham and Lee 1988] for $n \geq 2$). Here $P_{\beta}\varphi = \sum_{\alpha=1}^{n}(\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})$ and $\bar{P}\varphi = (\bar{P}_{\beta}\varphi)\theta^{\bar{\beta}}$, the conjugate of P. Moreover, we define

$$P_0\varphi = \delta_b(P\varphi),$$

which is the so-called CR Paneitz operator P_0. Here δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_{\alpha}\theta^{\alpha}) = \sigma_{\alpha},{}^{\alpha}$ and $\bar{\delta}_b(\sigma_{\bar{\alpha}}\theta^{\bar{\alpha}}) = \sigma_{\bar{\alpha}},{}^{\bar{\alpha}}$. If we define $\partial_b\varphi = \varphi_{\alpha}\theta^{\alpha}$ and $\bar{\partial}_b\varphi = \varphi_{\bar{\alpha}}\theta^{\bar{\alpha}}$, then the formal adjoint of ∂_b on functions (with respect to the Levi form and the volume form $\theta \wedge (d\theta)^n$) is $\partial_b^* = -\delta_b$.

We observe that P_0 is a real and symmetric operator and

$$\int \langle P\varphi, \partial_b\varphi\rangle = -\int (P_0\varphi)\overline{\varphi}.$$

Definition 1.2. We say that the Paneitz operator P_0 with respect to (J, θ) is nonnegative if, for all C^{∞} smooth functions φ,

$$\int (P_0\varphi)\overline{\varphi} \geq 0.$$

Remark 1.3. When (M, J, θ) is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator is nonnegative [Chang et al. 2007]. Unlike $n=1$, let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. The corresponding CR Paneitz operator is always nonnegative as in (3-4).

Definition 1.4 [Graham and Lee 1988]. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by

$$Q\varphi = 2i(A^{\alpha\beta}\varphi_{\alpha}),_{\beta}\,.$$

Note that $[T, \Delta_b] = 2\,\mathrm{Im}\,Q$ and

$$\text{(1-1)} \qquad 4P_0 = \Delta_b^2 + n^2T^2 - 2n\,\mathrm{Re}\,Q = (\Delta_b + inT)(\Delta_b - inT) - 2nQ = (\Delta_b - inT)(\Delta_b + inT) - 2n\overline{Q}.$$

Now we consider, for $c \in \mathbb{R}$, the self-adjoint operators

$$\mathcal{L}_c = \Delta_b + icT,$$

with $|c| \leq n$. By a result in [Folland and Stein 1974], each $\mathcal{L}_c$, $|c| < n$, is a subelliptic operator of order $\frac{1}{2}$; hence $\mathcal{L}_c$ has a discrete spectrum tending to $+\infty$.

In the following we can obtain a sharp lower bound for the first nonzero eigenvalue λ_1^c of the Folland–Stein operator $\mathcal{L}_c$, $c \in \mathbb{R}$ with $|c| \leq n$, on a closed pseudohermitian $(2n+1)$-manifold.

Theorem 1.5. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\begin{cases} \left[\operatorname{Ric} - \dfrac{(n-c)(n+1)}{2(n+c)} \operatorname{Tor}\right](Z, Z) \geq k\langle Z, Z\rangle & \text{if } c \geq 0, \\ \left[\operatorname{Ric} - \dfrac{(n+c)(n+1)}{2(n-c)} \operatorname{Tor}\right](\overline{Z}, \overline{Z}) \geq k\langle Z, Z\rangle & \text{if } c < 0, \end{cases} \tag{1-2}$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n+|c|}{n+1} k.$$

Note that the constant in the torsion tensor term in assumption (1-2) depends on the variable c. In the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ with the induced CR structure $\hat{J}$ from $\mathbb{C}^{n+1}$ and the standard contact form $\hat{\theta}$, we can show that the lower bound in Theorem 1.5 is sharp (see Section 4).

In particular, when (M, J, θ) is a closed pseudohermitian 3-manifold with vanishing pseudohermitian torsion, the corresponding CR Paneitz operator P_0 is nonnegative.

Corollary 1.6. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with vanishing pseudohermitian torsion. Suppose that*

$$\begin{cases} \operatorname{Ric}(Z, Z) \geq k\langle Z, Z\rangle & \text{if } c \geq 0, \\ \operatorname{Ric}(\overline{Z}, \overline{Z}) \geq k\langle Z, Z\rangle & \text{if } c < 0, \end{cases}$$

for a positive constant k and for all $Z \in T_{1,0}$. Then the first nonzero eigenvalue of $\mathcal{L}_c$, $|c| \leq n$, must satisfy

$$\lambda_1^c \geq \frac{n+|c|}{n+1} k.$$

Moreover, when $c = n$, the operator $\mathscr{L}_n$ is just the Kohn Laplacian: $\mathscr{L}_n = \Box_b$.

Corollary 1.7. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\mathrm{Ric}(Z, Z) \geq k\langle Z, Z\rangle$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of the Kohn Laplacian $\Box_b$ must satisfy

$$\lambda_1^n \geq \frac{2nk}{n+1}.$$

When $c = 0$, the operator $\mathscr{L}_0$ is just the sublaplacian Δ_b; i.e., $\mathscr{L}_0 = \Delta_b$.

Corollary 1.8. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\left[\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}\right](Z, Z) \geq k\langle Z, Z\rangle$$

for a positive constant k and for all $Z \in T_{1,0}$. In addition we assume the Paneitz operator P_0 is nonnegative if $n = 1$. Then the first nonzero eigenvalue of the sublaplacian Δ_b must satisfy

$$\lambda_1^0 \geq \frac{nk}{n+1}.$$

Further, we study the case when a sharp lower bound estimate of $\mathscr{L}_c$, $|c| \leq n$, is achieved in Section 4.

Proposition 1.9. *Under the same conditions as in Theorem 1.5, if we assume the first nonzero eigenvalue of $\mathscr{L}_c$, $0 < |c| \leq n$, satisfies*

$$\lambda_1^c = \frac{n+|c|}{n+1}k,$$

$$\int A^{\alpha\beta}\varphi_{c\alpha}\bar{\varphi}_{c\beta} = 0 \tag{1-3}$$

for a corresponding eigenfunction φ_c of $\mathscr{L}_c$ with respect to λ_1^c and with $\int\langle\varphi_c, \varphi_c\rangle = 1$, then the eigenfunction φ_c will satisfy

$$\int |\bar{\partial}_b\varphi_c|^2 = \frac{n(n+c)}{2(n^2+c^2)}\lambda_1^c \quad \textit{and} \quad \int |\partial_b\varphi_c|^2 = \frac{n(n-c)}{2(n^2+c^2)}\lambda_1^c; \tag{1-4}$$

thus we also have

$$\int \langle\Delta_b\varphi_c, \varphi_c\rangle = \frac{n^2}{n^2+c^2}\lambda_1^c \quad \textit{and} \quad \int i\langle T\varphi_c, \varphi_c\rangle = \frac{c}{n^2+c^2}\lambda_1^c.$$

Letting $c \to 0^+$, we see that $\int |\bar\partial_b \varphi_c|^2 = \int |\partial_b \varphi_c|^2 = \frac{1}{2}\lambda_1^0$ and $\int i\langle T\varphi_c, \varphi_c\rangle = 0$ for $c = 0$. When $c = n$, from (1-4), we get that $\partial_b \varphi_n = 0$ and thus $\overline{\Box}_b \varphi_n = 0$. This implies that the corresponding eigenfunction φ_n of $\mathscr{L}_n = \Box_b$ with respect to λ_1^n will also satisfy

$$\Delta_b \varphi_n = \frac{nk}{n+1}\varphi_n.$$

This yields that φ_n achieves a sharp lower bound for the first nonzero eigenvalue of the sublaplacian Δ_b. Furthermore, it can be showed the pseudohermitian torsion $A_{\alpha\beta}$ of M is zero; thus (M, J, θ) is the standard pseudohermitian $(2n+1)$-sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$ (see [Chang and Wu ≥ 2013] for details).

2. Basic materials

Let us give a brief introduction to pseudohermitian geometry (see [Lee 1988] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: if X and Y are in ξ, then so is $[JX, Y] + [X, JY]$, and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$. A CR structure J can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of J with respect to i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathscr{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar\alpha}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha} = \overline{Z}_\alpha \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar\alpha}\}$, satisfies

$$d\theta = i h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$. Actually we can always choose Z_α such that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$; hence, throughout this paper, we assume $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$.

The Levi form $\langle\ ,\ \rangle$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle = -i\big\langle d\theta, Z \wedge \overline{W}\big\rangle.$$

We can extend $\langle\ ,\ \rangle$ to $T_{0,1}$ by defining $\langle \overline{Z}, \overline{W}\rangle = \overline{\langle Z, W\rangle}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, also denoted by $\langle\ ,\ \rangle$, and hence on all the induced tensor bundles.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = \omega_{\bar\alpha}{}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta + \theta \wedge \tau^\beta, \quad \tau_\alpha \wedge \theta^\alpha = 0, \quad \omega_\alpha{}^\beta + \omega_{\bar\beta}{}^{\bar\alpha} = 0.$$

We can write $\tau_\alpha = A_{\alpha\beta}\theta^\beta$ with $A_{\alpha\beta} = A_{\beta\alpha}$. The curvature of the Webster–Stanton connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$, is

$$\Pi_\beta{}^\alpha = \overline{\Pi_{\bar\beta}{}^{\bar\alpha}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha,$$
$$\Pi_0{}^\alpha = \Pi_\alpha{}^0 = \Pi_0{}^{\bar\beta} = \Pi_{\bar\beta}{}^0 = \Pi_0{}^0 = 0.$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written as

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a function, we will often omit the comma, for instance, $\varphi_\alpha = Z_\alpha\varphi$, $\varphi_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha\varphi - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma\varphi$, $\varphi_0 = T\varphi$ for a (smooth) function φ. Let the Cauchy–Riemann operator ∂_b be defined locally by $\partial_b\varphi = \varphi_\alpha\theta^\alpha$, and let $\bar\partial_b$ be the conjugate of ∂_b. For a function φ, the subgradient ∇_b is defined locally by $\nabla_b\varphi = \varphi^\alpha Z_\alpha + \varphi^{\bar\alpha} Z_{\bar\alpha}$. The sublaplacian Δ_b, the Kohn Laplacian $\Box_b$, and the Folland–Stein operator $\mathscr{L}_c$ on functions are defined by

$$\Delta_b\varphi = -(\varphi_\alpha{}^\alpha + \varphi_{\bar\alpha}{}^{\bar\alpha}), \quad \Box_b\varphi = (\Delta_b + inT)\varphi, \quad \mathscr{L}_c\varphi = (\Delta_b + icT)\varphi.$$

The Webster–Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$\mathrm{Ric}(X, Y) = R_{\alpha\bar\beta}X^\alpha Y^{\bar\beta},$$
$$\mathrm{Tor}(X, Y) = i\sum_{\alpha,\beta}\big(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta} - A_{\alpha\beta}X^\alpha Y^\beta\big),$$

where $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$, $R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}$. The Webster scalar curvature is $R = R_\alpha{}^\alpha = h^{\alpha\bar\beta}R_{\alpha\bar\beta}$.

3. Proof of Theorem 1.5

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. In this section, we can obtain lower bound estimates for the first nonzero eigenvalue of the Folland–Stein operator $\mathscr{L}_c$, $|c| \le n$, on a closed pseudohermitian $(2n+1)$-manifold.

First we need the following Bochner formula for the Kohn Laplacian [Chanillo et al. 2012, Equation (2.8)]).

Lemma 3.1. *For any complex-valued function φ, we have*

$$-\frac{1}{2}\Box_b\left|\bar\partial_b\varphi\right|^2 = \sum_{\alpha,\beta}\left(\varphi_{\bar\alpha\bar\beta}\overline{\varphi}_{\alpha\beta} + \varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta}\right) + \operatorname{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) - \frac{1}{2n}\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\rangle - \frac{n+1}{2n}\langle\bar\partial_b\Box_b\varphi, \bar\partial_b\varphi\rangle - \frac{1}{n}\langle\overline{P}\varphi, \bar\partial_b\varphi\rangle + \frac{n-1}{n}\langle P\overline{\varphi}, \partial_b\overline{\varphi}\rangle, \tag{3-1}$$

where $(\nabla_b\varphi)_{\mathbb{C}} = \varphi^\alpha Z_\alpha$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$.

First we derive some useful identities which we need in the proof of Theorem 1.5. Let φ be a smooth complex-valued function on M. By integrating the Bochner formula (3-1), we have

$$0 = \int\sum_{\alpha,\beta}\left(\varphi_{\bar\alpha\bar\beta}\overline{\varphi}_{\alpha\beta} + \varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta}\right) - \frac{n+2}{2n}\int\langle\Box_b\varphi, \Box_b\varphi\rangle + \frac{2-n}{n}\int(P_0\varphi)\overline{\varphi} + \int\operatorname{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big). \tag{3-2}$$

We also have

$$\int\sum_{\alpha,\beta}\varphi_{\bar\alpha\beta}\overline{\varphi}_{\alpha\bar\beta} = \int\sum_{\alpha,\beta}\left|\overline{\varphi}_{\alpha\bar\beta} - \frac{1}{n}\overline{\varphi}_\gamma{}^\gamma h_{\alpha\bar\beta}\right|^2 + \frac{1}{4n}\int\langle\Box_b\varphi, \Box_b\varphi\rangle = \frac{n-1}{n}\int(P_0\varphi)\overline{\varphi} + \frac{1}{4n}\int\langle\Box_b\varphi, \Box_b\varphi\rangle. \tag{3-3}$$

Here we used the following divergence formula [Graham and Lee 1988] for the trace-free part of $\overline{\varphi}_{\alpha\bar\beta}$:

$$B_{\alpha\bar\beta}\overline{\varphi} = \overline{\varphi}_{\alpha\bar\beta} - \frac{1}{n}\overline{\varphi}_\gamma{}^\gamma h_{\alpha\bar\beta}.$$

That is,

$$\begin{aligned}\big(B^{\alpha\bar\beta}\varphi\big)\big(B_{\alpha\bar\beta}\overline{\varphi}\big) = \varphi^{\alpha\bar\beta}\big(B_{\alpha\bar\beta}\overline{\varphi}\big) &= \big(\varphi^\alpha B_{\alpha\bar\beta}\overline{\varphi}\big),{}^{\bar\beta} - \frac{n-1}{n}\varphi^\alpha P_\alpha\overline{\varphi}\\ &= \big(\varphi^\alpha B_{\alpha\bar\beta}\overline{\varphi}\big),{}^{\bar\beta} - \frac{n-1}{n}(\varphi P_\alpha\overline{\varphi}),{}^\alpha + \frac{n-1}{n}(P_0\overline{\varphi})\varphi.\end{aligned}$$

Then we integrate both sides to get

$$\int \sum_{\alpha,\beta} \left| B_{\alpha\bar{\beta}} \overline{\varphi} \right|^2 = \frac{n-1}{n} \int (P_0 \varphi) \overline{\varphi}. \tag{3-4}$$

Taking together the two formulas (3-2) and (3-3), we get

$$\frac{n+1}{4n} \int \langle \Box_b \varphi, \Box_b \varphi \rangle = \int \sum_{\alpha,\beta} \varphi_{\bar{\alpha}\bar{\beta}} \overline{\varphi}_{\alpha\beta} + \frac{1}{n} \int (P_0 \varphi) \overline{\varphi} + \int \mathrm{Ric}\big((\nabla_b \varphi)_{\mathbb{C}}, (\nabla_b \varphi)_{\mathbb{C}}\big). \tag{3-5}$$

By taking complex conjugate to (3-5) and replacing $\overline{\varphi}$ by φ, one obtains

$$\frac{n+1}{4n} \int \langle \overline{\Box}_b \varphi, \overline{\Box}_b \varphi \rangle = \int \sum_{\alpha,\beta} \varphi_{\alpha\beta} \overline{\varphi}_{\bar{\alpha}\bar{\beta}} + \frac{1}{n} \int (P_0 \varphi) \overline{\varphi} + \int \mathrm{Ric}\big((\nabla_b \overline{\varphi})_{\mathbb{C}}, (\nabla_b \overline{\varphi})_{\mathbb{C}}\big). \tag{3-6}$$

From the formula (1-1), we have

$$\begin{aligned} 4 \int (P_0 \varphi) \overline{\varphi} &= \int \big\langle (\Delta_b + inT)(\Delta_b - inT)\varphi - 2nQ\varphi, \varphi \big\rangle \\ &= \int \langle \overline{\Box}_b \varphi, \Box_b \varphi \rangle - 2n \int \langle Q\varphi, \varphi \rangle. \end{aligned} \tag{3-7}$$

By (1-1), we can also obtain

$$4 \int (P_0 \varphi) \overline{\varphi} = \int \langle \Box_b \varphi, \overline{\Box}_b \varphi \rangle - 2n \int \langle \overline{Q} \varphi, \varphi \rangle. \tag{3-8}$$

Proof of Theorem 1.5. Let φ_c be an eigenfunction of the Folland–Stein operator $\mathscr{L}_c$, $c \in \mathbb{R}$ with $|c| \le n$, with respect to the first nonzero eigenvalue λ_1^c; i.e., $\mathscr{L}_c \varphi_c = \lambda_1^c \varphi_c$.

When $0 \le c \le n$, from (3-6) and (3-7) for

$$\mathscr{L}_c = \frac{n+c}{2n} \Box_b + \frac{n-c}{2n} \overline{\Box}_b,$$

we have

$$\begin{aligned} \frac{1}{2} \int \langle \Box_b \varphi_c, \mathscr{L}_c \varphi_c \rangle &= \frac{n+c}{4n} \int \langle \Box_b \varphi_c, \Box_b \varphi_c \rangle + \frac{n-c}{4n} \int \langle \Box_b \varphi_c, \overline{\Box}_b \varphi_c \rangle \\ &= \frac{n+c}{n+1} \int \sum_{\alpha,\beta} \varphi_{c\bar{\alpha}\bar{\beta}} \overline{\varphi}_{c\alpha\beta} + \frac{n+2-c}{n+1} \int (P_0 \varphi_c) \overline{\varphi}_c \\ &\quad + \frac{n+c}{n+1} \int \mathrm{Ric}\big((\nabla_b \varphi_c)_{\mathbb{C}}, (\nabla_b \varphi_c)_{\mathbb{C}}\big) + \frac{n-c}{2} \int \langle \overline{Q} \varphi_c, \varphi_c \rangle \\ &= \frac{n+c}{n+1} \int \sum_{\alpha,\beta} \varphi_{c\bar{\alpha}\bar{\beta}} \overline{\varphi}_{c\alpha\beta} + \frac{n+2-c}{n+1} \int (P_0 \varphi_c) \overline{\varphi}_c \\ &\quad + \frac{n+c}{n+1} \int \left[\mathrm{Ric} - \frac{(n-c)(n+1)}{2(n+c)} \mathrm{Tor} \right] \big((\nabla_b \varphi_c)_{\mathbb{C}}, (\nabla_b \varphi_c)_{\mathbb{C}}\big), \end{aligned}$$

where we used the equation

$$\int \langle \overline{Q}\varphi_c, \varphi_c\rangle = -\int \operatorname{Tor}\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big),$$

since $\int\langle \overline{Q}\varphi_c, \varphi_c\rangle$ is real, and thus $\int\langle \overline{Q}\varphi_c, \varphi_c\rangle = 2\int i A^{\bar\alpha\bar\beta}\varphi_{c\bar\alpha}\bar\varphi_{c\bar\beta} = -2\int i A^{\alpha\beta}\varphi_{c\alpha}\bar\varphi_{c\beta}$.

Hence, if P_0 is nonnegative and

$$[\operatorname{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\operatorname{Tor}]((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}) \geq k|\bar\partial_b\varphi_c|^2,$$

we have

$$\begin{aligned}\lambda_1^c \int |\bar\partial_b\varphi_c|^2 &= \frac{n+c}{n+1}\int \sum_{\alpha,\beta}\varphi_{c\bar\alpha\bar\beta}\bar\varphi_{c\alpha\beta} + \frac{n+2-c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n+c}{n+1}\int\Big[\operatorname{Ric} - \frac{(n-c)(n+1)}{2(n+c)}\operatorname{Tor}\Big]\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big) \\ &\geq \frac{n+c}{n+1}k\int |\bar\partial_b\varphi_c|^2,\end{aligned} \tag{3-9}$$

which shows that $\lambda_1^c \geq \frac{n+c}{n+1}k$.

When $-n \leq c < 0$, from (3-5) and (3-8), the same computation shows that

$$\begin{aligned}\frac{1}{2}\int \langle \overline{\Box}_b\varphi_c, \mathcal{L}_c\varphi_c\rangle &= \frac{n+c}{4n}\int \langle \overline{\Box}_b\varphi_c, \Box_b\varphi_c\rangle + \frac{n-c}{4n}\int\langle \overline{\Box}_b\varphi_c, \overline{\Box}_b\varphi_c\rangle \\ &= \frac{n-c}{n+1}\int\sum_{\alpha,\beta}\varphi_{c\alpha\beta}\bar\varphi_{c\bar\alpha\bar\beta} + \frac{n+2+c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n-c}{n+1}\int\Big[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}\Big]\big((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}\big).\end{aligned}$$

Thus, if P_0 is nonnegative and

$$[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}]((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}) \geq k|\partial_b\varphi_c|^2,$$

we get

$$\begin{aligned}\lambda_1^c\int |\partial_b\varphi_c|^2 &= \frac{n-c}{n+1}\int\sum_{\alpha,\beta}\varphi_{c\alpha\beta}\bar\varphi_{c\bar\alpha\bar\beta} + \frac{n+2+c}{n+1}\int (P_0\varphi_c)\bar\varphi_c \\ &\quad + \frac{n-c}{n+1}\int\Big[\operatorname{Ric} - \frac{(n+c)(n+1)}{2(n-c)}\operatorname{Tor}\Big]\big((\nabla_b\bar\varphi_c)_{\mathbb{C}}, (\nabla_b\bar\varphi_c)_{\mathbb{C}}\big) \\ &\geq \frac{n-c}{n+1}k\int |\partial_b\varphi_c|^2,\end{aligned}$$

which implies that $\lambda_1^c \geq \dfrac{n-c}{n+1}k$. This completes the proof of Theorem 1.5. □

4. Example and proof of Proposition 1.9

In this section, we calculate the eigenvalues of sublaplacian Δ_b, Kohn Laplacian $\Box_b$, and the Folland–Stein operator $\mathscr{L}_c$, $|c| \leq n$, of the standard pseudohermitian $(2n+1)$-sphere S^{2n+1}. We show that the lower bound in Theorem 1.5 is sharp. We also study the case when a sharp lower bound estimate of $\mathscr{L}_c$, $|c| \leq n$, is achieved.

Let $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \mid \sum_{j=0}^n z_j \bar{z}_j = 1\} \subset \mathbb{C}^{n+1}$ with the induced CR structure from $\mathbb{C}^{n+1}$ and the contact form $\theta = \frac{i}{2}(\bar{\partial} u - \partial u)|_{S^{2n+1}}$ where $u = \left(\sum_{j=0}^n z_j \bar{z}_j\right) - 1$ is a defining function. It can be shown that the pseudohermitian torsion is free and the Webster–Ricci tensor is given by $R_{\alpha\bar{\beta}} = (n+1) h_{\alpha\bar{\beta}}$.

We write

$$\partial_j = \frac{\partial}{\partial z_j}, \ \bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j} \quad (0 \leq j \leq n), \qquad \partial_{j\bar{k}} = \partial_j \partial_{\bar{k}} \quad (0 \leq j, k \leq n),$$

and $z = (z_0, z_1, \dots, z_n)$, $\delta = (\partial_0, \partial_1, \dots, \partial_n)$. We let $\cdot$ denote the dot product. Then, by the computation in Section 1 of [Geller 1980], we have

$$\mathscr{L}_c = 2\Bigl(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Bigr) + (n+c)\bar{z} \cdot \bar{\delta} + (n-c) z \cdot \delta,$$

where $\Delta = \sum_{j=0}^n \partial_j \partial_{\bar{j}}$ is the standard Laplacian on $\mathbb{C}^{n+1}$. In particular, we have

$$\Delta_b = 2\Bigl(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Bigr) + n\bigl(\bar{z} \cdot \bar{\delta} + z \cdot \delta\bigr),$$
$$\Box_b = 2\Bigl(-\Delta + \sum_{j,k=0}^{n} z_j \bar{z}_k \partial_j \partial_{\bar{k}}\Bigr) + 2n \bar{z} \cdot \bar{\delta}.$$

If Y is a bigraded spherical harmonic of type (p, q) on $\mathbb{C}^{n+1}$ (a harmonic polynomial which is a linear combination in terms of the form $z^\rho \bar{z}^\gamma$, where ρ, γ are multiindices with $|\rho| = p$, $|\gamma| = q$), then $\mathscr{L}_c Y = \bigl(2pq + (n+c)q + (n-c)p\bigr) Y$. Similarly,

$$\Delta_b Y = (2pq + n(p+q))Y, \quad \Box_b Y = 2q(p+n)Y.$$

This example shows that the lower bound in Theorem 1.5 is sharp.

Now we study the case when a sharp lower bound estimate for the first nonzero eigenvalue of the Folland–Stein operator $\mathscr{L}_c$, $|c| \leq n$, on a pseudohermitian $(2n+1)$-manifold M is achieved. We only consider the case when the constant c is nonnegative. The same computation follows when c is negative.

First, from (3-9), we have the following observation.

Lemma 4.1. *Under the same conditions as in Theorem 1.5, when the first nonzero eigenvalue of $\mathscr{L}_c$, $0 \le c \le n$, satisfies*

$$\lambda_1^c = \frac{n+c}{n+1} k,$$

then the corresponding eigenfunction φ_c will satisfy

$$\varphi_{c\bar{\alpha}\bar{\beta}} = 0 \quad \textit{for all } \alpha, \beta, \tag{4-1}$$

$$\left[\mathrm{Ric} - \frac{(n-c)(n+1)}{2(n+c)} \mathrm{Tor}\right]\big((\nabla_b\varphi_c)_{\mathbb{C}}, (\nabla_b\varphi_c)_{\mathbb{C}}\big) = k\big|\bar\partial_b\varphi_c\big|^2, \tag{4-2}$$

$$P_0\varphi_c = 0. \tag{4-3}$$

Proof of Proposition 1.9. The integral condition (1-3) says that

$$\int \langle Q\varphi_c, \varphi_c\rangle = -2i \int A^{\alpha\beta}\varphi_{c\alpha}\bar\varphi_{c\beta} = 0,$$

and then by integration by parts, we obtain

$$\int \big\langle \bar{Q}\varphi_c, \varphi_c\big\rangle = \int \langle \varphi_c, Q\varphi_c\rangle = \int \langle Q\varphi_c, \varphi_c\rangle = 0. \tag{4-4}$$

From (1-1), one can see that

$$4P_0 = [\Delta_b - i(n^2/c)T][\Delta_b + icT] - \frac{1}{2c}\big[(2nc+n+c)\bar{Q} + (2nc-n-c)Q\big].$$

Then, from (4-3) and (4-4), one obtains

$$\begin{aligned} 0 = 4\int (P_0\varphi_c)\bar\varphi_c &= \lambda_1^c \int \big\langle [\Delta_b - i(n^2/c)T]\varphi_c, \varphi_c\big\rangle \\ &= \frac{1}{2}\lambda_1^c \int \big\langle [(1-n/c)\Box_b + (1+n/c)\bar\Box_b]\varphi_c, \varphi_c\big\rangle \\ &= \lambda_1^c \int \big[(1-n/c)|\bar\partial_b\varphi_c|^2 + (1+n/c)|\partial_b\varphi_c|^2\big], \end{aligned}$$

which is

$$(n-c)\int \big|\bar\partial_b\varphi_c\big|^2 = (n+c)\int |\partial_b\varphi_c|^2. \tag{4-5}$$

On the other hand, the equation $\mathscr{L}_c\varphi_c = (\Delta_b + icT)\varphi_c = \lambda_1^c\varphi_c$ yields

$$\begin{aligned} \lambda_1^c = \lambda_1^c \int \langle \varphi_c, \varphi_c\rangle &= \int \langle \mathscr{L}_c\varphi_c, \varphi_c\rangle \\ &= \frac{1}{2n}\int \big\langle \big[(n+c)\Box_b + (n-c)\bar\Box_b\big]\varphi_c, \varphi_c\big\rangle \\ &= \int (1+n/c)\big|\bar\partial_b\varphi_c\big|^2 + (1-n/c)|\partial_b\varphi_c|^2. \end{aligned} \tag{4-6}$$

The equations (1-4) follow from (4-5) and (4-6) easily. □

References

[Chang and Wu 2010] S.-C. Chang and C.-T. Wu, "The entropy formulas for the CR heat equation and their applications on pseudohermitian $(2n+1)$-manifolds", *Pacific J. Math.* **246**:1 (2010), 1–29. MR 2011i:58038 Zbl 1206.32016

[Chang and Wu $\geq$ 2013] S.-C. Chang and C.-T. Wu, "On the CR Obata theorem for Kohn Laplacian in a closed pseudo-Hermitian hypersurface in $\mathbb{C}^{n+1}$", in preparation.

[Chang et al. 2007] S.-C. Chang, J.-H. Cheng, and H.-L. Chiu, "A fourth order curvature flow on a CR 3-manifold", *Indiana Univ. Math. J.* **56**:4 (2007), 1793–1826. MR 2009a:53111 Zbl 1129.53046

[Chanillo et al. 2012] S. Chanillo, H.-L. Chiu, and P. Yang, "Embeddability for 3-dimensional Cauchy–Riemann manifolds and CR Yamabe invariants", *Duke Math. J.* **161**:15 (2012), 2909–2921. MR 2999315 Zbl 06121401

[Chiu 2006] H.-L. Chiu, "The sharp lower bound for the first positive eigenvalue of the sublaplacian on a pseudohermitian 3-manifold", *Ann. Global Anal. Geom.* **30**:1 (2006), 81–96. MR 2007j:58034 Zbl 1098.32017

[Folland and Stein 1974] G. B. Folland and E. M. Stein, "Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group", *Comm. Pure Appl. Math.* **27** (1974), 429–522. MR 51 #3719 Zbl 0293.35012

[Geller 1980] D. Geller, "The Laplacian and the Kohn Laplacian for the sphere", *J. Differential Geom.* **15**:3 (1980), 417–435. MR 82i:35132 Zbl 0507.58049

[Graham and Lee 1988] C. R. Graham and J. M. Lee, "Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains", *Duke Math. J.* **57**:3 (1988), 697–720. MR 90c:32031 Zbl 0699.35112

[Greenleaf 1985] A. Greenleaf, "The first eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold", *Comm. Partial Differential Equations* **10**:2 (1985), 191–217. MR 86f:58157 Zbl 0563.58034

[Lee 1988] J. M. Lee, "Pseudo-Einstein structures on CR manifolds", *Amer. J. Math.* **110**:1 (1988), 157–178. MR 89f:32034 Zbl 0638.32019

[Li and Luk 2004] S.-Y. Li and H.-S. Luk, "The sharp lower bound for the first positive eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold", *Proc. Amer. Math. Soc.* **132**:3 (2004), 789–798. MR 2005c:58056 Zbl 1041.32024

Received May 8, 2011. Revised November 29, 2012.

CHIN-TUNG WU
DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL PINGTUNG UNIVERSITY OF EDUCATION
NO. 4-18 MINSHENG RD
PINGTUNG CITY 90003
TAIWAN

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 263 No. 1 May 2013

Biharmonic hypersurfaces in complete Riemannian manifolds 1
LUIS J. ALÍAS, S. CAROLINA GARCÍA-MARTÍNEZ and MARCO RIGOLI

Half-commutative orthogonal Hopf algebras 13
JULIEN BICHON and MICHEL DUBOIS-VIOLETTE

Superdistributions, analytic and algebraic super Harish-Chandra pairs 29
CLAUDIO CARMELI and RITA FIORESI

Orbifolds with signature $(0; k, k^{n-1}, k^n, k^n)$ 53
ANGEL CAROCCA, RUBÉN A. HIDALGO and RUBÍ E. RODRÍGUEZ

Explicit isogeny theorems for Drinfeld modules 87
IMIN CHEN and YOONJIN LEE

Topological pressures for ϵ-stable and stable sets 117
XIANFENG MA and ERCAI CHEN

Lipschitz and bilipschitz maps on Carnot groups 143
WILLIAM MEYERSON

Geometric inequalities in Carnot groups 171
FRANCESCOPAOLO MONTEFALCONE

Fixed points of endomorphisms of virtually free groups 207
PEDRO V. SILVA

The sharp lower bound for the first positive eigenvalue of the Folland–Stein operator on a closed pseudohermitian $(2n+1)$-manifold 241
CHIN-TUNG WU

Remark on “Maximal functions on the unit n-sphere” by Peter M. Knopf (1987) 253
HONG-QUAN LI