REMARK ON
“MAXIMAL FUNCTIONS ON THE UNIT n-SPHERE”
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The article in question contains an important result on the behavior of the
Hardy–Littlewood maximal function \( M_{S^n} \) on the unit \( n \)-sphere, providing
a weak-type linear bound that has not been improved on in the intervening
decades. Unfortunately, the proof has a gap, since it relies on an incorrect
intermediate result (Lemma 3). We correct the proof by providing a sharper
lower bound for a trigonometry integral than the one used by Knopf.

1. Introduction

Let \( S^{n-1} \) \((n \geq 2)\) denote the unit sphere of dimension \( n - 1 \), i.e., the \( n - 1 \) dimen-
sional, simply connected Riemannian manifold of constant sectional curvature 1.
Let \( d_{S^{n-1}} \) be the induced distance and \( \mu_{S^{n-1}} \) be the induced measure.

Consider the centered Hardy–Littlewood maximal function, \( M_{S^{n-1}} \), on \( S^{n-1} \),
i.e.,

\[
M_{S^{n-1}} f(x) = \sup_{0 < r \leq \pi} \frac{1}{\mu_{S^{n-1}}(B_{S^{n-1}}(x, r))} \int_{B_{S^{n-1}}(x, r)} |f(y)| \, d\mu_{S^{n-1}}(y),
\]

where \( B_{S^{n-1}}(x, r) \) is the open ball with center \( x \) and radius \( r > 0 \).

In [Knopf 1987], the following theorem is presented:

**Theorem 1.1.** There exists a constant \( A > 0 \) such that

\[
\|M_{S^{n-1}}\|_{L^1 \rightarrow L^{1,\infty}} \leq An \quad \text{for all } n \geq 2.
\]

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For other results concerning the estimates of type (1-1), see for example [Stein and Strömberg 1983] in the setting of $\mathbb{R}^n$, [Li 2009; Li and Qian 2011] in the setting of H-type groups, [Li 2010] for Grushin operators, [Li and Lohoué 2012] for the case of real hyperbolic spaces and [Naor and Tao 2010]. There is also a bound of type

$$\lim_{n \to +\infty} \|M_{\text{Cube}}\|_{L^1 \to L^{1,\infty}} = +\infty$$

about the centered maximal function associated to cubes in $\mathbb{R}^n$; see [Aldaz 2011] or [Aubrun 2009] for details.

Let $\omega_{n-1}$ denote the area of the unit sphere of $\mathbb{R}^n$; i.e., $\omega_{n-1} = 2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Recall that, for $x \in S^{n-1}$, $0 < t \leq 2$,

$$S(x, t) = \left\{ y \in S^{n-1} \subset \mathbb{R}^n; |x - y| \leq t \right\},$$

$$|S(x, t)| = \mu_{S^{n-1}}(S(x, r)).$$

There exist some mistakes in [Knopf 1987]. For example, near the end of the proof of Lemma 3, take

$$t = \sqrt{2(1 - n^{-\frac{1}{2}})},$$

and we find that Lemma 3 is wrong. Knopf uses the estimate that

$$|S(x, t)| = \omega_{n-2} \int_{0}^{2\arcsin(t/2)} \sin^{n-2} u du \geq \omega_{n-2} \int_{0}^{2\arcsin(t/2)} \sin^{n-2} u \cos u du,$$

which gives the lower bound

$$|S(x, t)| \geq \frac{c\omega_{n-1}}{\sqrt{n}} \left[ t^2 \left( 1 - \frac{t^2}{4} \right) \right]^{\frac{n-1}{2}}$$

for all $0 < t \leq \sqrt{2}$, $n \geq 2$.

This estimate is not sharp enough to obtain the desired result. In order to make the proof in [Knopf 1987] effective, we need the sharper and sufficient lower bound:

**Lemma 1.2.** There exists a constant $c > 0$ such that, for all $n \geq 2$ and $0 < t \leq \sqrt{2}$, we have

$$|S(x, t)| \geq c\omega_{n-1} \left[ n \left( 1 - t \sqrt{1 - \frac{t^2}{4}} \right) + t \sqrt{1 - \frac{t^2}{4}} \right]^{-\frac{1}{2}} \left[ t^2 \left( 1 - \frac{t^2}{4} \right) \right]^{\frac{n-1}{2}}.$$

More specifically, using the bound (1-3) instead of (1-2) in the proof of Knopf’s Lemma 1 yields an improved result to replace Lemma 1:
\( M_{S^{n-1}} f(x) \) \leq c \max \left\{ \sup_{n^{-\frac{1}{2}} \leq t \leq \sqrt{2(n-1)}} \left[ \frac{u \left( 1 - t \sqrt{\frac{1-t^2}{4}} \right)}{t} + \sqrt{\frac{n-1}{n}} \sin \left( 1 - \frac{t}{\sqrt{n}} \right) \right], \sup_{0 < t \leq n^{-\frac{1}{2}}} \left[ n \sup_{0 < t \leq n^{-\frac{1}{2}}} \left( 1 - t \sqrt{\frac{1-t^2}{4}} \right) u \left( 1 - t \sqrt{\frac{1-t^2}{4}} \right) \right] \right\}.

Using (1-4) instead of the original Lemma 1 estimate at the end of the proof of Lemma 3 in [Knopf 1987] gives

\( M_{S^{n-1}} f(x) \) \leq c \max \left\{ \sup_{n^{-\frac{1}{2}} \leq t \leq \sqrt{2(n-1)}} \left[ \frac{u \left( 1 - t \sqrt{\frac{1-t^2}{4}} \right)}{t} + \sqrt{\frac{n-1}{n}} \sin \left( 1 - \frac{t}{\sqrt{n}} \right) \right], \sup_{0 < t \leq n^{-\frac{1}{2}}} \left( 1 + \sqrt{n \ln \left( 1 - \frac{1-t^2}{2} \right)} \right) \right\} \left( 1 + \sqrt{n \ln n} \right) M_T f(x).

It is trivial to check that the right side of (1-5) is at most \( cn M_T f(x) \), and using this inequality the rest of the original proof works and gives the correct result.

2. Proof of Equation (1-3)

For \( 0 < t \leq \sqrt{2} \), set \( r = 2 \arcsin(t/2) \); then

\[
|S(x,t)| = \int_0^r \omega_{n-2} (\sin s)^{n-2} ds = \omega_{n-2} \int_0^{\sin r} y^{n-2} \frac{dy}{\sqrt{1-y^2}} = \omega_{n-2} (\sin r)^{n-1} \int_0^1 \frac{u^{n-2}}{\sqrt{1-u \sin r}} du.
\]

Observe that

\[
\int_0^1 \frac{u^{n-2}}{\sqrt{1-u \sin r}} du \geq \int_{1-\frac{1}{n}}^1 \frac{du}{\sqrt{1-u \sin r}} = 2e^{(n-2) \ln(1-\frac{1}{n})} \frac{1}{n} \sqrt{1-\sin r} + \sqrt{1 - (1 - \frac{1}{n}) \sin r} > c \frac{1}{\sqrt{n} \sqrt{n(1 - \sin r) + \sin r}}.
\]

Then Stirling’s formula implies (1-3).
Remark. By (1-3), a simple computation then leads to

\[(2-1) \quad |S(x, t)| \geq c \omega_{n-1} \] whenever \( \sqrt{2(1-n^{-1})} \leq t \leq 2 \) and \( n \geq 2 \).

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