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REMARK ON
“MAXIMAL FUNCTIONS ON THE UNIT n -SPHERE”
BY PETER M. KNOPF (1987)

HONG-QUAN LI

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The article in question contains an important result on the behavior of the Hardy–Littlewood maximal function M_{S^n} on the unit n -sphere, providing a weak-type linear bound that has not been improved on in the intervening decades. Unfortunately, the proof has a gap, since it relies on an incorrect intermediate result (Lemma 3). We correct the proof by providing a sharper lower bound for a trigonometry integral than the one used by Knopf.

1. Introduction

Let S^{n-1} ($n \geq 2$) denote the unit sphere of dimension $n - 1$, i.e., the $n - 1$ dimensional, simply connected Riemannian manifold of constant sectional curvature 1. Let $d_{S^{n-1}}$ be the induced distance and $\mu_{S^{n-1}}$ be the induced measure.

Consider the centered Hardy–Littlewood maximal function, $M_{S^{n-1}}$, on S^{n-1} , i.e.,

$$M_{S^{n-1}} f(x) = \sup_{0 < r \leq \pi} \frac{1}{\mu_{S^{n-1}}(B_{S^{n-1}}(x, r))} \int_{B_{S^{n-1}}(x, r)} |f(y)| d\mu_{S^{n-1}}(y),$$

$$x \in S^{n-1}, f \in L^1(S^{n-1}),$$

where $B_{S^{n-1}}(x, r)$ is the open ball with center x and radius $r > 0$.

In [Knopf 1987], the following theorem is presented:

Theorem 1.1. *There exists a constant $A > 0$ such that*

$$(1-1) \quad \|M_{S^{n-1}}\|_{L^1 \rightarrow L^{1,\infty}} \leq An \quad \text{for all } n \geq 2.$$

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For other results concerning the estimates of type (1-1), see for example [Stein and Strömberg 1983] in the setting of \mathbb{R}^n , [Li 2009; Li and Qian 2011] in the setting of H-type groups, [Li 2010] for Grushin operators, [Li and Lohoué 2012] for the case of real hyperbolic spaces and [Naor and Tao 2010]. There is also a bound of type

$$\lim_{n \rightarrow +\infty} \|M_{\text{Cube}}\|_{L^1 \rightarrow L^{1,\infty}} = +\infty$$

about the centered maximal function associated to cubes in \mathbb{R}^n ; see [Aldaz 2011] or [Aubrun 2009] for details.

Let ω_{n-1} denote the area of the unit sphere of \mathbb{R}^n ; i.e., $\omega_{n-1} = 2 \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Recall that, for $x \in S^{n-1}$, $0 < t \leq 2$,

$$\begin{aligned} S(x, t) &= \{y \in S^{n-1} \subset \mathbb{R}^n; |x - y| \leq t\}, \\ |S(x, t)| &= \mu_{S^{n-1}}(S(x, r)). \end{aligned}$$

There exist some mistakes in [Knopf 1987]. For example, near the end of the proof of Lemma 3, take

$$t = \sqrt{2(1 - n^{-\frac{1}{2}})},$$

and we find that Lemma 3 is wrong. Knopf uses the estimate that

$$|S(x, t)| = \omega_{n-2} \int_0^{2 \arcsin(t/2)} \sin^{n-2} u \, du \geq \omega_{n-2} \int_0^{2 \arcsin(t/2)} \sin^{n-2} u \cos u \, du,$$

which gives the lower bound

$$(1-2) \quad |S(x, t)| \geq \frac{c\omega_{n-1}}{\sqrt{n}} \left[t^2 \left(1 - \frac{t^2}{4} \right) \right]^{\frac{n-1}{2}} \quad \text{for all } 0 < t \leq \sqrt{2}, n \geq 2.$$

This estimate is not sharp enough to obtain the desired result. In order to make the proof in [Knopf 1987] effective, we need the sharper and sufficient lower bound:

Lemma 1.2. *There exists a constant $c > 0$ such that, for all $n \geq 2$ and $0 < t \leq \sqrt{2}$, we have*

$$(1-3) \quad |S(x, t)| \geq c\omega_{n-1} \left[n \left(1 - t \sqrt{1 - \frac{t^2}{4}} \right) + t \sqrt{1 - \frac{t^2}{4}} \right]^{-\frac{1}{2}} \left[t^2 \left(1 - \frac{t^2}{4} \right) \right]^{\frac{n-1}{2}}.$$

More specifically, using the bound (1-3) instead of (1-2) in the proof of Knopf’s Lemma 1 yields an improved result to replace Lemma 1:

$$(1-4) \quad M_{S^{n-1}} f(x) \leq c \max \left\{ \sup_{\substack{n^{-\frac{1}{2}} \leq t \\ \leq \sqrt{2(1-n^{-1})}}} \frac{\sqrt{n(1-t\sqrt{1-\frac{t^2}{4}})} + t\sqrt{1-\frac{t^2}{4}}}{t} u\left(\left(1-\frac{t^2}{2}\right)x\right), \right. \\ \left. n \sup_{0 < t \leq n^{-\frac{1}{2}}} u\left(\left(1-\frac{t}{\sqrt{n}}\right)x\right), \quad u(n^{-1}x) \right\}.$$

Using (1-4) instead of the original Lemma 1 estimate at the end of the proof of Lemma 3 in [Knopf 1987] gives

$$(1-5) \quad M_{S^{n-1}} f(x) \leq c \max \left\{ \sup_{\substack{n^{-\frac{1}{2}} \leq t \\ \leq \sqrt{2(1-n^{-1})}}} \frac{\sqrt{n(1-t\sqrt{1-\frac{t^2}{4}})} + t\sqrt{1-\frac{t^2}{4}}}{t} \left(1 + \sqrt{n \ln\left(1-\frac{t^2}{2}\right)^{-1}}\right), \right. \\ \left. n \sup_{0 < t \leq n^{-\frac{1}{2}}} \left(1 + \sqrt{n \ln\left(1-\frac{t}{\sqrt{n}}\right)^{-1}}\right), \quad 1 + \sqrt{n \ln n} \right\} M_T f(x).$$

It is trivial to check that the right side of (1-5) is at most $cnM_T f(x)$, and using this inequality the rest of the original proof works and gives the correct result.

2. Proof of Equation (1-3)

For $0 < t \leq \sqrt{2}$, set $r = 2 \arcsin(t/2)$; then

$$|S(x, t)| = \int_0^r \omega_{n-2}(\sin s)^{n-2} ds = \omega_{n-2} \int_0^{\sin r} y^{n-2} \frac{dy}{\sqrt{1-y^2}} \\ \geq \frac{\omega_{n-2}}{\sqrt{2}} \int_0^{\sin r} y^{n-2} \frac{dy}{\sqrt{1-y}} = \frac{\omega_{n-2}}{\sqrt{2}} (\sin r)^{n-1} \int_0^1 \frac{u^{n-2}}{\sqrt{1-u \sin r}} du.$$

Observe that

$$\int_0^1 \frac{u^{n-2}}{\sqrt{1-u \sin r}} du \geq \left(1 - \frac{1}{n}\right)^{n-2} \int_{1-\frac{1}{n}}^1 \frac{du}{\sqrt{1-u \sin r}} \\ = 2e^{(n-2)\ln(1-\frac{1}{n})} \frac{1}{n} \frac{1}{\sqrt{1-\sin r} + \sqrt{1-(1-\frac{1}{n})\sin r}} \\ > c \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n(1-\sin r) + \sin r}}.$$

Then Stirling’s formula implies (1-3). □

Remark. By (1-3), a simple computation then leads to

$$(2-1) \quad |S(x, t)| \geq c\omega_{n-1} \quad \text{whenever } \sqrt{2(1-n^{-1})} \leq t \leq 2 \text{ and } n \geq 2.$$

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