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REMARK ON “MAXIMAL FUNCTIONS ON THE UNIT *n*-SPHERE” BY PETER M. KNOPF (1987)

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The article in question contains an important result on the behavior of the Hardy–Littlewood maximal function M_{S^n} on the unit n-sphere, providing a weak-type linear bound that has not been improved on in the intervening decades. Unfortunately, the proof has a gap, since it relies on an incorrect intermediate result (Lemma 3). We correct the proof by providing a sharper lower bound for a trigonometry integral than the one used by Knopf.

1. Introduction

Let S^{n-1} ($n \geq 2$) denote the unit sphere of dimension $n-1$, i.e., the $n-1$ dimensional, simply connected Riemannian manifold of constant sectional curvature 1. Let $d_{S^{n-1}}$ be the induced distance and $\mu_{S^{n-1}}$ be the induced measure.

Consider the centered Hardy–Littlewood maximal function, $M_{S^{n-1}}$, on S^{n-1}, i.e.,

$$M_{S^{n-1}} f(x) = \sup_{0<r\leq\pi} \frac{1}{\mu_{S^{n-1}}\big(B_{S^{n-1}}(x,r)\big)} \int_{B_{S^{n-1}}(x,r)} |f(y)|\, d\mu_{S^{n-1}}(y), \qquad x \in S^{n-1},\ f \in L^1(S^{n-1}),$$

where $B_{S^{n-1}}(x,r)$ is the open ball with center x and radius $r > 0$.

In [Knopf 1987], the following theorem is presented:

Theorem 1.1. *There exists a constant $A > 0$ such that*

$$\|M_{S^{n-1}}\|_{L^1 \longrightarrow L^{1,\infty}} \leq An \quad \text{for all } n \geq 2. \tag{1-1}$$

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For other results concerning the estimates of type (1-1), see for example [Stein and Strömberg 1983] in the setting of $\mathbb{R}^n$, [Li 2009; Li and Qian 2011] in the setting of H-type groups, [Li 2010] for Grushin operators, [Li and Lohoué 2012] for the case of real hyperbolic spaces and [Naor and Tao 2010]. There is also a bound of type

$$\lim_{n\longrightarrow+\infty} \|M_{\mathrm{Cube}}\|_{L^1\longrightarrow L^{1,\infty}} = +\infty$$

about the centered maximal function associated to cubes in $\mathbb{R}^n$; see [Aldaz 2011] or [Aubrun 2009] for details.

Let ω_{n-1} denote the area of the unit sphere of $\mathbb{R}^n$; i.e., $\omega_{n-1} = 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$. Recall that, for $x \in S^{n-1}$, $0 < t \le 2$,

$$\begin{aligned} S(x,t) &= \{y \in S^{n-1} \subset \mathbb{R}^n;\ |x-y| \le t\}, \\ |S(x,t)| &= \mu_{S^{n-1}}(S(x,r)). \end{aligned}$$

There exist some mistakes in [Knopf 1987]. For example, near the end of the proof of Lemma 3, take

$$t = \sqrt{2\left(1-n^{-\frac{1}{2}}\right)},$$

and we find that Lemma 3 is wrong. Knopf uses the estimate that

$$|S(x,t)| = \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u\,du \ge \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u \cos u\,du,$$

which gives the lower bound

$$|S(x,t)| \ge \frac{c\omega_{n-1}}{\sqrt{n}}\left[t^2\left(1-\frac{t^2}{4}\right)\right]^{\frac{n-1}{2}} \quad \text{for all } 0 < t \le \sqrt{2},\ n \ge 2. \tag{1-2}$$

This estimate is not sharp enough to obtain the desired result. In order to make the proof in [Knopf 1987] effective, we need the sharper and sufficient lower bound:

Lemma 1.2. *There exists a constant $c > 0$ such that, for all $n \ge 2$ and $0 < t \le \sqrt{2}$, we have*

$$|S(x,t)| \ge c\omega_{n-1}\left[n\left(1-t\sqrt{1-\frac{t^2}{4}}\right)+t\sqrt{1-\frac{t^2}{4}}\right]^{-\frac{1}{2}}\left[t^2\left(1-\frac{t^2}{4}\right)\right]^{\frac{n-1}{2}}. \tag{1-3}$$

More specifically, using the bound (1-3) instead of (1-2) in the proof of Knopf's Lemma 1 yields an improved result to replace Lemma 1:

(1-4)
$$M_{S^{n-1}} f(x) \le c \max\Biggl\{ \sup_{\substack{n^{-\frac12}\le t \\ \le \sqrt{2(1-n^{-1})}}} \frac{\sqrt{n\Bigl(1-t\sqrt{1-\frac{t^2}{4}}\Bigr)+t\sqrt{1-\frac{t^2}{4}}}}{t} u\Bigl(\Bigl(1-\frac{t^2}{2}\Bigr)x\Bigr), \quad n \sup_{0<t\le n^{-\frac12}} u\Bigl(\Bigl(1-\frac{t}{\sqrt{n}}\Bigr)x\Bigr), \quad u(n^{-1}x) \Biggr\}.$$

Using (1-4) instead of the original Lemma 1 estimate at the end of the proof of Lemma 3 in [Knopf 1987] gives

(1-5)
$$M_{S^{n-1}} f(x) \le c \max\Biggl\{ \sup_{\substack{n^{-\frac12}\le t \\ \le \sqrt{2(1-n^{-1})}}} \frac{\sqrt{n\Bigl(1-t\sqrt{1-\frac{t^2}{4}}\Bigr)+t\sqrt{1-\frac{t^2}{4}}}}{t} \Biggl(1+\sqrt{n\ln\Bigl(1-\frac{t^2}{2}\Bigr)^{-1}}\Biggr), \quad n \sup_{0<t\le n^{-\frac12}} \Biggl(1+\sqrt{n\ln\Bigl(1-\frac{t}{\sqrt{n}}\Bigr)^{-1}}\Biggr), \quad 1+\sqrt{n\ln n}\Biggr\} M_T f(x).$$

It is trivial to check that the right side of (1-5) is at most $cnM_T f(x)$, and using this inequality the rest of the original proof works and gives the correct result.

2. Proof of Equation (1-3)

For $0 < t \le \sqrt{2}$, set $r = 2\arcsin(t/2)$; then

$$|S(x,t)| = \int_0^r \omega_{n-2}(\sin s)^{n-2}\,ds = \omega_{n-2}\int_0^{\sin r} y^{n-2}\frac{dy}{\sqrt{1-y^2}}$$
$$\ge \frac{\omega_{n-2}}{\sqrt2}\int_0^{\sin r} y^{n-2}\frac{dy}{\sqrt{1-y}} = \frac{\omega_{n-2}}{\sqrt2}(\sin r)^{n-1}\int_0^1 \frac{u^{n-2}}{\sqrt{1-u\sin r}}\,du.$$

Observe that

$$\int_0^1 \frac{u^{n-2}}{\sqrt{1-u\sin r}}\,du \ge \Bigl(1-\frac1n\Bigr)^{n-2}\int_{1-\frac1n}^1 \frac{du}{\sqrt{1-u\sin r}}$$
$$= 2e^{(n-2)\ln(1-\frac1n)}\frac1n \frac{1}{\sqrt{1-\sin r}+\sqrt{1-\bigl(1-\frac1n\bigr)\sin r}}$$
$$> c\frac{1}{\sqrt n}\frac{1}{\sqrt{n(1-\sin r)+\sin r}}.$$

Then Stirling’s formula implies (1-3). □

Remark. By (1-3), a simple computation then leads to

$$|S(x,t)| \geq c\omega_{n-1} \quad \text{whenever } \sqrt{2(1-n^{-1})} \leq t \leq 2 \text{ and } n \geq 2. \tag{2-1}$$

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