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SANDEEP BHARGAVA AND YUN GAO

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REALIZATIONS OF BC_r-GRADED INTERSECTION MATRIX ALGEBRAS WITH GRADING SUBALGEBRAS OF TYPE B_r, $r \geq 3$

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We study intersection matrix algebras $\mathfrak{im}(A^{[d]})$ that arise from affinizing a Cartan matrix A of type B_r with d arbitrary long roots in the root system Δ_{B_r}, where $r \geq 3$. We show that $\mathfrak{im}(A^{[d]})$ is isomorphic to the universal covering algebra of $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$, where $\mathfrak{a}$ is an associative algebra with involution η, and C is an $\mathfrak{a}$-module with hermitian form χ. We provide a description of all four of the components $\mathfrak{a}$, η, C, and χ.

1. Introduction

Peter Slodowy [1984; 1986] discovered that matrices like

$$M = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 1 \\ 0 & -2 & 2 & -2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

encode the intersection form on the second homology group of Milnor fibers for germs of holomorphic maps with an isolated singularity at the origin. These matrices were like the generalized Cartan matrices of Kac–Moody theory in that they had integer entries, 2's along the diagonal, and M_{ij} was negative if and only if M_{ji} was negative. What was new, however, was the presence of positive entries off the diagonal. Slodowy called such matrices generalized intersection matrices:

Definition 1 [Slodowy 1986]. An $n \times n$ integer-valued matrix M is called a *generalized intersection matrix* ($\mathfrak{gim}$) if the following conditions are satisfied. whenever $1 \leq i, j \leq n$ with $i \neq j$:

$$\begin{gathered} M_{ii} = 2; \\ M_{ij} < 0 \iff M_{ji} < 0; \\ M_{ij} > 0 \iff M_{ji} > 0. \end{gathered}$$

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Slodowy used these matrices to define a class of Lie algebras that encompassed all the Kac–Moody Lie algebras:

Definition 2 [Slodowy 1986; Berman and Moody 1992]. Given an $n \times n$ generalized intersection matrix $M = (M_{ij})$, define a Lie algebra over $\mathbb{C}$, called a *generalized intersection matrix (*$\mathfrak{gim}$*) algebra* and denoted by $\mathfrak{gim}(M)$, with

generators: $e_1, \dots, e_n,\ f_1, \dots, f_n,\ h_1, \dots h_n$,

relations:

(R1) For $1 \le i,\ j \le n$,
$$[h_i, e_j] = M_{ij} e_j,$$
$$[h_i, f_j] = -M_{ij} f_j,$$
$$[e_i, f_i] = h_i.$$

(R2) For $M_{ij} \le 0$,
$$[e_i, f_j] = 0 = [f_i, e_j],$$
$$(\operatorname{ad} e_i)^{-M_{ij}+1}\, e_j = 0 = (\operatorname{ad} f_i)^{-M_{ij}+1} f_j.$$

(R3) For $M_{ij} > 0$, $i \ne j$,
$$[e_i, e_j] = 0 = [f_i, f_j],$$
$$(\operatorname{ad} e_i)^{M_{ij}+1}\, f_j = 0 = (\operatorname{ad} f_i)^{M_{ij}+1} e_j.$$

If the M that we begin with is a generalized Cartan matrix, then the $3n$ generators and the first two groups of axioms, (R1) and (R2), provide a presentation of the Kac–Moody Lie algebras [Gabber and Kac 1981; Kac 1990; Carter 2005].

Slodowy [1986] and, later, Berman [1989] showed that the $\mathfrak{gim}$ algebras are also isomorphic to fixed point subalgebras of involutions on larger Kac–Moody algebras. So, in their words, the $\mathfrak{gim}$ algebras lie both "beyond and inside" Kac–Moody algebras.

Further progress came in the 1990s as a byproduct of work on the classification of root-graded Lie algebras [Berman and Moody 1992; Benkart and Zelmanov 1996; Neher 1996], which revealed that some families of intersection matrix ($\mathfrak{im}$) algebras, which are quotient algebras of $\mathfrak{gim}$ algebras, were universal covering algebras of well-understood Lie algebras. For instance the $\mathfrak{im}$ algebras that arise from multiply affinizing a Cartan matrix of type A_r, with $r \ge 3$, are the universal covering algebras of $\mathrm{sl}(\mathfrak{a})$, where $\mathfrak{a}$ is the associative algebra of noncommuting Laurent polynomials in several variables (the number of indeterminates depends on how many times the original Cartan matrix is affinized). A handful of other researchers also began engaging these new algebras. For example, Eswara Rao, Moody, and Yokonuma [Rao et al. 1992] used vertex operator representations to show that $\mathfrak{im}$ algebras were nontrivial. Gao [1996] examined compact forms of $\mathfrak{im}$ algebras arising from conjugations over the complex field. Peng [2002] found relations between $\mathfrak{im}$ algebras and the representations of tilted algebras via

Ringel–Hall algebras. Berman, Jurisich, and Tan [Berman et al. 2001] showed that the presentation of $\mathfrak{gim}$ algebras could be put into a broader framework that incorporated Borcherds algebras.

The chief objective of this paper is to continue advancing our understanding of $\mathfrak{gim}$ and $\mathfrak{im}$ algebras. We construct a generalized intersection matrix $A^{[d]}$ by adjoining d long roots to a base of a root system of type B_r, where $r \geq 3$. This is exactly the analogue of the affinization process in which a single root is adjoined to a Cartan matrix of a finite-dimensional Lie algebra to arrive at a generalized Cartan matrix and, eventually, an affine Kac–Moody algebra. The matrix $A^{[d]}$ is used to define a $\mathfrak{gim}$ algebra $\mathfrak{gim}(A^{[d]})$. Since $\mathfrak{gim}(A^{[d]})$ may possess roots with mixed signs, we quotient out by an ideal $\mathfrak{r}$ that is tailor-made to capture all such roots. The quotient algebra is called the intersection matrix algebra and is denoted by $\mathfrak{im}(A^{[d]})$.

We show that $\mathfrak{im}(A^{[d]})$ is a BC_r-graded Lie algebra, which, in turn, allows us to invoke Allison, Benkart, and Gao's recognition theorem [Allison et al. 2002] and relate $\mathfrak{im}(A^{[d]})$ to an algebraic structure that is better understood. Combining their theorem with the knowledge that $\mathfrak{im}(A^{[d]})$ is centrally closed, we conclude that, up to isomorphism, $\mathfrak{im}(A^{[d]})$ is the universal covering algebra of $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$. The algebra $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$ is like the usual matrix model $\mathrm{so}_{2r+1}(\mathbb{C})$ of a finite-dimensional Lie algebra of type B_r, except that we now replace the field $\mathbb{C}$ with an associative algebra $\mathfrak{a}$, which possesses an involution (that is, period two antiautomorphism) η, and we involve a right $\mathfrak{a}$-module C that has a hermitian form $\chi : C \times C \to \mathfrak{a}$. The defining relations of the generalized intersection matrix algebra and, hence, the intersection matrix algebra, in concert with the existence of a central, graded, surjective Lie algebra homomorphism ψ from $\mathfrak{im}(A^{[d]})$ to $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$ allow us to understand each of $\mathfrak{a}$, η, C, and χ. For example, we get (i) two generators of $\mathfrak{a}$, namely x and x^{-1}, for every long root of the form $\pm(\epsilon_i + \epsilon_{i+1})$, and (ii) four generators of $\mathfrak{a}$, namely y, y^{-1}, z, and z^{-1}, for every other type of long root that we adjoin. We are also able to study the relations among the generators, determine the action of the involution η, and discover that $C = 0$ and $\chi = 0$. Through constructing a surjective Lie algebra homomorphism $\varphi : \mathfrak{gim}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$ we verify that we indeed have a complete description of the "coordinate algebra" $\mathfrak{a}$.

Our work continues the line of research initiated by Berman, Moody, Benkart and Zelmanov. Berman and Moody [1992] were the first to find realizations of intersection matrix algebras over Lie algebras graded by root systems of types A_r ($r \geq 2$), D_r, E_6, E_7, and E_8. Benkart and Zelmanov [1996] found realizations of intersection matrix algebras over Lie algebras graded by root systems of types A_1, B_r, C_r, F_4, and G_2. In this paper, we find realizations of intersection matrix algebras over Lie algebras graded by root systems of type BC_r with grading subalgebras of type B_r ($r \geq 3$).

2. Multiply affinizing Cartan matrices

In this paper, we focus on generalized intersection matrix algebras that arise from multiply affinizing a Cartan matrix of type B_r, where $r \geq 3$, with long roots in the root system Δ_{B_r}.

Consider a root system of type B_r. Up to isomorphism, Δ_{B_r} may be described as

$$\Delta_{B_r} = \left\{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq r\right\} \cup \left\{\pm\epsilon_i : i = 1, \ldots, r\right\}.$$

Once we fix an ordering of the simple roots $\alpha_1, \ldots, \alpha_r$ in a base Π, the Cartan matrix A is described by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)_{\text{Killing}}}{(\alpha_i, \alpha_i)_{\text{Killing}}} \quad \text{for } 1 \leq i, j \leq r.$$

Choose any d long roots in Δ_{B_r}, say $\alpha_{r+1}, \ldots, \alpha_{r+d}$, and consider the $r+d$ by $r+d$ matrix $A^{[d]}$ given by

$$A^{[d]}_{ij} = \frac{2(\alpha_i, \alpha_j)_{\text{Killing}}}{(\alpha_i, \alpha_i)_{\text{Killing}}} \quad \text{for } 1 \leq i, j \leq r+d,$$

with respect to the ordering $(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r+d})$ of the r roots in the base Π plus the d "adjoined" roots. The axioms of a root system tell us that all the entries of $A^{[d]}$ are integers. Moreover, since the Killing form is symmetric, we have $A^{[d]}_{ji} = 0$ if $A^{[d]}_{ij} = 0$, or if $A^{[d]}_{ij}$ and $A^{[d]}_{ji}$ are nonzero, then they share the same sign. In other words, $A^{[d]}$ is a generalized intersection matrix.

Since the "d-affinized" Cartan matrix $A^{[d]}$ is a generalized intersection matrix, $\mathfrak{gim}(A^{[d]})$ is a generalized intersection matrix algebra.

Note that if we affinize the Cartan matrix A of type B_r with the negative of the highest long root of Δ_{B_r} then the resulting generalized intersection matrix algebra $\mathfrak{gim}(A^{[1]})$ is the affine Kac–Moody Lie algebra of type $B_r^{(1)}$.

3. Intersection matrix algebras

Fix a Cartan matrix A of type B_r ($r \geq 3$) with, say, $\alpha_1, \alpha_2, \ldots, \alpha_r$ being the simple roots in a base of Δ_{B_r} that were used to form A. Let

- $\Omega =$ set of all long roots of the form $\pm(\epsilon_i + \epsilon_{i+1})$ that we adjoin,
- $\Theta =$ set of all remaining long roots that are adjoined,
- $N_\mu =$ the number of copies of the long root μ we have adjoined, and
- $d = \sum_{\mu \in \Omega \cup \Theta} N_\mu$.

Let $A^{[d]}$ be the resulting generalized intersection matrix and $\mathfrak{gim}(A^{[d]})$ the corresponding generalized intersection matrix algebra.

We begin a move towards a quotient algebra of $\mathfrak{gim}(A^{[d]})$ using a slight generalization of the work done by Benkart and Zelmanov [1996]. Let Γ be the integer lattice generated by the Δ, where

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq r\} \cup \{\pm\epsilon_i, \pm 2\epsilon_i : i = 1, \dots, r\}$$

is a root system of type BC_r.

We define a Γ-grading on $\mathfrak{gim}(A^{[d]})$ as follows:

$$\deg e_i = \alpha_i = -\deg f_i, \quad \deg h_i = 0$$

for $i = 1, \dots, r$, and

$$\deg e_{\mu,i} = \mu = -\deg f_{\mu,i}, \quad \deg h_{\mu,i} = 0$$

for $\mu \in \Omega \cup \Theta$ and $i = 1, \dots, N_\mu$.

Next, we define the *radical* $\mathfrak{r}$ of $\mathfrak{gim}(A^{[d]})$ to be the ideal generated by the root spaces $\mathfrak{gim}(A^{[d]})_\gamma$ where $\gamma \notin \Delta \cup \{0\}$. Since the ideal $\mathfrak{r}$ is homogeneous, the resulting quotient algebra

$$\mathfrak{im}(A^{[d]}) = \mathfrak{gim}(A^{[d]})/\mathfrak{r}$$

is also Γ-graded. Moreover,

$$\mathfrak{im}(A^{[d]})_\gamma = 0 \quad \text{if } \gamma \notin \Delta \cup \{0\}.$$

We call $\mathfrak{im}(A^{[d]})$ the *intersection matrix* ($\mathfrak{im}$) *algebra* corresponding to the generalized intersection matrix algebra $\mathfrak{gim}(A^{[d]})$.

3.1. $\mathfrak{im}(A^{[d]})$ *is BC_r-graded.* Allison, Benkart, and Gao gave the following definition of a Lie algebra graded by a root system of type BC.

Definition 3 [Allison et al. 2002]. Let r be a positive integer greater than or equal to 3. A Lie algebra L over $\mathbb{C}$ is *graded by the root system BC_r* or is *BC_r-graded* with a *grading subalgebra of type B_r* if

(i) L contains, as a subalgebra, a finite-dimensional simple Lie algebra $\mathfrak{g}$ whose root system relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Δ_{B_r},

(ii) $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$ for $\mu \in \Delta \cup \{0\}$, and Δ is the root system of type BC_r, and

(iii) $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.

Proposition 4. *The algebra* $\mathfrak{im}(A^{[d]})$ *is BC_r-graded with a grading subalgebra of type B_r.*

Proof. The subalgebra in $\mathfrak{im}(A^{[d]})$ generated by $e_1 + \mathfrak{r}, \dots, h_r + \mathfrak{r}$, due to the relations on these elements induced by the relations on their preimages in $\mathfrak{gim}(A^{[d]})$, is isomorphic to a finite-dimensional simple Lie algebra $\mathfrak{g}$ of type B_r. We have already shown in Section 3.1 that $\mathfrak{im}(A^{[d]})$ is Γ-graded with $\mathfrak{im}(A^{[d]})_\gamma = 0$ if $\gamma \notin \Delta \cup \{0\}$. That is,

$$\mathfrak{im}(A^{[d]}) = \bigoplus_{\mu \in \Delta \cup \{0\}} \mathfrak{im}(A^{[d]})_\mu .$$

Finally, our initial degree assignments for the generators of $\mathfrak{gim}(A^{[d]})$, the $\mathfrak{gim}$ algebra relations like $h_i = [e_i, f_i]$ and $h_\mu = [e_\mu, f_\mu]$, and the fact that movement into the 0 root space can only occur by bracketing an element from an $\mathfrak{im}(A^{[d]})_\mu$ space with one from the $\mathfrak{im}(A^{[d]})_{-\mu}$ space all combine to lead us to the conclusion that

$$\mathfrak{im}(A^{[d]})_0 = \sum_{\mu \in \Delta} \big[\mathfrak{im}(A^{[d]})_\mu, \mathfrak{im}(A^{[d]})_{-\mu}\big]. \qquad \square$$

3.2. $\mathfrak{im}(A^{[d]})$ *is centrally closed.* Recall that a Lie algebra L is said to be perfect if it equals its derived algebra, that is, $L = [L, L]$. Furthermore, if L is perfect and is its own universal covering then we say that L is centrally closed [Moody and Pianzola 1995].

Proposition 5. *The algebra* $\mathfrak{gim}(A^{[d]})$ *is a perfect Lie algebra.*

Proof. Being a Lie algebra, $\mathfrak{gim}(A^{[d]})$ is closed under the operation of taking brackets; hence $\big[\mathfrak{gim}(A^{[d]}), \mathfrak{gim}(A^{[d]})\big] \subset \mathfrak{gim}(A^{[d]})$. To show the reverse inclusion, it suffices to show that all of the generators of $\mathfrak{gim}(A^{[d]})$ lie in $\big[\mathfrak{gim}(A^{[d]}), \mathfrak{gim}(A^{[d]})\big]$. But this is indeed the case because the generators e_i, f_i, h_i (for $1 \le i \le r$) and the $e_{\mu,i}, f_{\mu,i}, h_{\mu,i}$, which arise from adjoining the i-th copy of a long root μ, satisfy the relations (R1) of Definition 2. $\square$

Our next theorem is Proposition 1.6 in [Benkart and Zelmanov 1996] adapted to our context.

Theorem 6. *The algebra* $\mathfrak{im}(A^{[d]})$ *is centrally closed.*

Proof. Let $\big(\widetilde{U}, \phi\big)$ be the universal covering algebra of $\mathfrak{im}(A^{[d]})$. Let $\mathfrak{g}$ be the simple finite dimensional subalgebra of type B contained in $\mathfrak{im}(A^{[d]})$ with Cartan subalgebra $\mathfrak{h}$ whose root space decomposition induces a BC-gradation on $\mathfrak{im}(A^{[d]})$. The preimage $\phi^{-1}(\mathfrak{h})$ of $\mathfrak{h}$ contains $\ker\phi$. Since ϕ is a central map, $\ker\phi$ lies in the center of $\widetilde{U}$. So

$$\mathfrak{h}' = \phi^{-1}(\mathfrak{h}) / \ker\phi$$

acts on $\widetilde{U}$ via the adjoint action. If $h' \in \mathfrak{h}'$, $\phi(h') = h \in \mathfrak{h}$, and $\mu(t) \in \mathbb{C}[t]$ is the minimal polynomial of $\mathrm{ad}_{U_{\underline{L}}}(h)$, then

$$\mu\big(\mathrm{ad}_{\widetilde{U}}(h')\big)\big(\widetilde{U}\big) \subset \ker\phi .$$

So $\mathrm{ad}_{\widetilde{U}}(h')$ satisfies the polynomial $t\mu(t)$. Therefore $\widetilde{U}$ is a sum of root spaces with respect to $\mathrm{ad}_{\widetilde{U}}\,\mathfrak{h}'$, and $\widetilde{U}_\gamma \neq (0)$ if and only if $\gamma \in \Delta \cup \{0\}$. So ϕ induces an isomorphism between the nonzero root spaces of $\widetilde{U}$ and those of $\mathfrak{im}(A^{[d]})$. Moreover,

$$\widetilde{U}_0 = \sum_{\gamma\in\Delta} \big[\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma\big] + \ker\phi \quad \text{implies that} \quad \big[\widetilde{U}_0, \widetilde{U}_0\big] \subset \sum_{\gamma\in\Delta} \big[\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma\big].$$

Since $\widetilde{U} = \big[\widetilde{U}, \widetilde{U}\big]$, it follows that

$$\widetilde{U}_0 = \big[\widetilde{U}_0, \widetilde{U}_0\big] + \sum_{\gamma\in\Delta} \big[\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma\big] = \sum_{\gamma\in\Delta} \big[\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma\big].$$

Consequently, ϕ is an isomorphism. □

4. Recognition theorem

The following construction, given in Example 1.23 of [Allison et al. 2002], is a more general version of the classical construction of $\mathrm{so}_{2r+1}(\mathbb{C})$, the simple Lie algebra of type B_r.

Let r be a positive integer, $\mathfrak{a}$ be a unital associative algebra over $\mathbb{C}$ with an involution (that is, period two antiautomorphism) η, C be a right $\mathfrak{a}$-module with a hermitian form $\chi : C \times C \to \mathfrak{a}$, that is a biadditive map $\chi : C \times C \to \mathfrak{a}$ satisfying

$$\chi(c, c'\cdot a) = \chi(c, c')\cdot a, \quad \chi(c\cdot a, c') = \eta(a)\cdot\chi(c, c'), \quad \chi(c, c') = \eta\big(\chi(c', c)\big),$$

for $c, c' \in C$, $a \in \mathfrak{a}$, and G be the $(2r+1)\times(2r+1)$ matrix

$$G = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Also, given any $c \in C$, define $\chi_c \in C^*$ by $\chi_c(c') := \chi(c, c')$, for any $c' \in C$, and given any

$$\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in C^{2r+1}, \quad \text{define } \chi_{\underline{c}} := \begin{bmatrix} \chi_{c_1} \\ \vdots \\ \chi_{c_n} \end{bmatrix} \in \big(C^*\big)^{2r+1}.$$

Now set

$$\mathfrak{A}(\chi) := \big\{N \in \mathrm{End}_{\mathfrak{a}}(C) : \chi(Nc, c') + \chi(c, Nc') = 0 \text{ for all } c, c' \in C\big\},$$

$$\mathfrak{A} := \left\{ \begin{bmatrix} M & \chi_{\underline{c}} \\ \underline{c}^t G & N \end{bmatrix} : M \in \mathrm{M}_{2r+1}(\mathfrak{a}),\ (M^\eta)^t G + GM = 0,\ \underline{c} \in C^{2r+1},\ N \in \mathfrak{A}(\chi) \right\}.$$

It can be checked that $\mathfrak{A}$ is a Lie algebra that contains a simple Lie algebra

$$\mathfrak{g}=\left\{\begin{bmatrix} M & 0\\ 0 & 0\end{bmatrix} : M\in \mathrm{M}_{2r+1}(\mathbb{C}),\ M^tG+GM=0\right\},$$

of type B_r. If $\mathfrak{h}$ denotes the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$, then the adjoint action of $\mathfrak{h}$ on $\mathfrak{A}$ induces a root space decomposition

$$\mathfrak{A}=\bigoplus_{\mu\in\Delta\cup\{0\}}\mathfrak{A}_\mu,\qquad \text{where } \mathfrak{A}_\mu=\big\{T\in\mathfrak{A}:[h,T]=\mu(h)\,T \text{ for all } h\in\mathfrak{h}\big\}.$$

The following abbreviated notation helps describe these root spaces:

$$\text{for } v=\begin{bmatrix} v_1\\ \vdots\\ v_{2r+1}\end{bmatrix}\in\mathbb{C}^{2r+1}\quad\text{and}\quad c\in C,\ \text{let } vc:=\begin{bmatrix} v_1c\\ \vdots\\ v_{2r+1}c\end{bmatrix}\in C^{2r+1}.$$

Then $C^{2r+1}=\bigoplus_{i=1}^{2r+1}e_iC$, where $e_1,\dots,e_{2r+1}$ is the standard basis for $\mathbb{C}^{2r+1}$. Letting B denote the set of skew-symmetric elements of $\mathfrak{a}$ relative to the involution η, we have

$$\mathfrak{A}_{\epsilon_i-\epsilon_j}=\big\{E_{i,j}(a)+E_{2r+2-j,2r+2-i}(-\eta(a)):a\in\mathfrak{a}\big\},\qquad 1\le i\ne j\le r,$$

$$\mathfrak{A}_{\epsilon_i+\epsilon_j}=\big\{E_{i,2r+2-j}(a)+E_{j,2r+2-i}(-\eta(a)):a\in\mathfrak{a}\big\},\qquad 1\le i,\ j\le r,$$

$$\mathfrak{A}_{-\epsilon_i-\epsilon_j}=\big\{E_{2r+2-i,j}(a)+E_{2r+2-j,i}(-\eta(a)):a\in\mathfrak{a}\big\},\qquad 1\le i,\ j\le r,$$

$$\mathfrak{A}_{\epsilon_i}=\left\{\begin{bmatrix} 0\chi_{e_ic} & \\ (e_{2r+2-i}c)^t & 0\end{bmatrix}:c\in C\right\}$$
$$+\big\{E_{i,r+1}(a)+E_{r+1,2r+2-i}(-\eta(a)):a\in\mathfrak{a}\big\},\quad 1\le i\le r,$$

$$\mathfrak{A}_{-\epsilon_i}=\left\{\begin{bmatrix} 0 & \chi_{e_{2r+2-i}c}\\ (e_ic)^t & 0\end{bmatrix}:c\in C\right\}$$
$$+\big\{E_{r+1,i}(a)+E_{2r+2-i,r+1}(-\eta(a)):a\in\mathfrak{a}\big\},\quad 1\le i\le r,$$

$$\mathfrak{A}_0=\left\{\sum_{i=1}^{r}E_{ii}(a)+E_{2r+2-i,2r+2-i}(-\eta(a)):a\in\mathfrak{a}\right\}+\left\{\begin{bmatrix} 0 & 0\\ 0 & N\end{bmatrix}:N\in\mathfrak{A}(\chi)\right\}$$
$$+\big\{E_{r+1,r+1}(b):b\in B\big\}+\left\{\begin{bmatrix} 0 & \chi_{e_{r+1}c}\\ (e_{r+1}c)^t & 0\end{bmatrix}:c\in C\right\}.$$

The subalgebra

$$\mathrm{so}_{2r+1}(\mathfrak{a},\eta,C,\chi):=\sum_{\mu\in\Delta}\mathfrak{A}_\mu+\sum_{\mu\in\Delta}[\mathfrak{A}_\mu,\mathfrak{A}_{-\mu}]$$

of $\mathfrak{A}$ has the root spaces

$$\begin{aligned} \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)_0 &= \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi) \cap \mathfrak{A}_0, \\ \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)_\mu &= \mathfrak{A}_\mu \quad \text{for } \mu \in \Delta. \end{aligned}$$

In particular,

$$\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)_0 = \sum_{\mu \in \Delta} \big[\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)_\mu, \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)_{-\mu}\big].$$

Remark. In [Allison et al. 2002] the notation L is used to refer to the Lie algebra that we are calling $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$.

To shorten the description of elements in $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$, we use the following notation: Given any $1 \le k \le r$ and $a \in \mathfrak{a}$, let

$$\begin{aligned} E^{\boxed{\top}}_{k,r+1}(a) &:= E_{k,r+1}(a) + E_{r+1,2r+2-k}(-\eta(a)), \\ E^{\boxed{\vdash}}_{r+1,k}(a) &:= E_{r+1,k}(a) + E_{2r+2-k,r+1}(-\eta(a)), \end{aligned}$$

and for any $1 \le p, q \le r$ and $a \in \mathfrak{a}$, let

$$\begin{aligned} E^{\text{▚}}_{p,q}(a) &:= E_{p,q}(a) + E_{2r+2-q,2r+2-p}(-\eta(a)), \\ E^{\text{▝}}_{p,2r+2-q}(a) &:= E_{p,2r+2-q}(a) + E_{q,2r+2-p}(-\eta(a)), \\ E^{\text{▖}}_{2r+2-p,q}(a) &:= E_{2r+2-p,q}(a) + E_{2r+2-q,p}(-\eta(a)). \end{aligned}$$

We often also denote the involution η on $\mathfrak{a}$ by $\bar{\cdot}$. So, for example, we would write

$$E^{\text{▖}}_{2r+2-p,q}(a) \text{ (above) as } E_{2r+2-p,q}(a) + E_{2r+2-q,p}(-\bar{a}).$$

Allison, Benkart, and Gao's classification results on BC_r-graded Lie algebras [Allison et al. 2002] say the following in our setting:

Theorem 7 [Allison et al. 2002, Theorem 3.10]. *Let $r \ge 3$. Then L is BC_r-graded with grading subalgebra $\mathfrak{g}$ of type B_r if and only if there exists an associative algebra $\mathfrak{a}$ with involution η, and an $\mathfrak{a}$-module C with a hermitian form χ such that L is centrally isogenous to the BC_r-graded Lie algebra* $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$.

Since $\mathfrak{im}(A^{[d]})$ is BC_r-graded with a grading subalgebra of type B_r and is centrally closed, we have the following result.

Corollary 8. *The intersection matrix algebra* $\mathfrak{im}(A^{[d]})$ *is isomorphic to the universal covering algebra of the Lie algebra* $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$. *In particular, there exists a graded, central, surjective Lie algebra homomorphism*

$$\psi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi).$$

5. Arriving at a "minimal" understanding of $\mathfrak{a}$, η, C, and χ

The graded nature of the map $\psi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$ along with the relations among the generating elements of $\mathfrak{im}(A^{[d]})$ allow us to study each of components $\mathfrak{a}$, η, C, and χ involved in $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$.

Since the elements $e_1 + \mathfrak{r}, \dots, e_r + \mathfrak{r}$, $f_1 + \mathfrak{r}, \dots, f_r + \mathfrak{r}$, $h_1 + \mathfrak{r}, \dots, h_r + \mathfrak{r}$ in $\mathfrak{im}(A^{[d]})$ generate a subalgebra isomorphic to a simple Lie algebra of type B_r, and since ψ is a graded homomorphism, we may assume without loss of generality that (after relabeling the $e_i + \mathfrak{r}$, $f_i + \mathfrak{r}$, and $h_i + \mathfrak{r}$ as e_i, f_i, and h_i, respectively)

$$\psi(e_i) = E_{i,i+1}(1) \quad \text{for } 1 \le i \le r-1, \qquad \psi(e_r) = E_{r,r+1}(\sqrt{2}),$$
$$\psi(f_i) = E_{i+1,i}(1) \quad \text{for } 1 \le i \le r-1, \qquad \psi(f_r) = E_{r+1,r}(\sqrt{2}),$$
$$\psi(h_i) = E_{i,i}(1) + E_{i+1,i+1}(-1) \text{ for } 1 \le i \le r-1, \quad \psi(h_r) = E_{r,r}(2).$$

Remark. Here we are using the notation established in Section 4. The generators of $\mathfrak{im}(A^{[d]})$ coming from a simple root $\alpha_j \in \Pi$ are denoted by e_j, f_j, and h_j, while the generators coming from an i-th copy of an adjoined root $\alpha \in \Delta_{B_r}$ are denoted by $e_{\alpha,i}$, $f_{\alpha,i}$, and $h_{\alpha,si}$.

5.1. *Understanding the invertibility of some coordinates of* $\mathfrak{a}$.

Proposition 9. (i) *Let* $e_{\epsilon_p-\epsilon_q,i}$, $f_{\epsilon_p-\epsilon_q,i}$, $h_{\epsilon_p-\epsilon_q,i}$ *be the generators of* $\mathfrak{im}(A^{[d]})$ *that result from adjoining the* i*-th copy of a long root* $\epsilon_p - \epsilon_q$ $(1 \le p, q \le r$, $p \ne q)$. *If*

$$\psi(e_{\epsilon_p-\epsilon_q,i}) = E_{p,q}(a)$$

for some $a \in \mathfrak{a}$, *then* a *is an invertible element and*

$$\psi(f_{\epsilon_p-\epsilon_q,i}) = E_{q,p}(a^{-1}).$$

(ii) *Let* $e_{\epsilon_p+\epsilon_q,i}$, $f_{\epsilon_p+\epsilon_q,i}$, $h_{\epsilon_p+\epsilon_q,i}$ *be the generators of* $\mathfrak{im}(A^{[d]})$ *that result from adjoining the* i*-th copy of a long root* $\epsilon_p + \epsilon_q$ $(1 \le p, q \le r$, $p \ne q)$. *If*

$$\psi(e_{\epsilon_p+\epsilon_q,i}) = E_{p,2r+2-q}(b)$$

for some $b \in \mathfrak{a}$, *then* b *is an invertible element and*

$$\psi(f_{\epsilon_p+\epsilon_q,i}) = E_{2r+2-q,p}(b^{-1}).$$

(iii) *Let* $e_{-\epsilon_p-\epsilon_q,i}$, $f_{-\epsilon_p-\epsilon_q,i}$, $h_{-\epsilon_p-\epsilon_q,i}$ *be the generators of* $\mathfrak{im}(A^{[d]})$ *that result from adjoining the* i*-th copy of a long root* $-\epsilon_p - \epsilon_q$ $(1 \le p, q \le r$, $p \ne q)$. *If*

$$\psi(e_{-\epsilon_p-\epsilon_q,i}) = E_{2r+2-p,q}(c)$$

for some $c \in \mathfrak{a}$, then c is an invertible element and

$$\psi(f_{-\epsilon_p-\epsilon_q,i}) = E^{\text{◳}}_{q,2r+2-p}(c^{-1}).$$

Proof. (i) Since ψ is a graded homomorphism,

$$\psi(e_{\epsilon_p-\epsilon_q,i}) = E^{\text{▚}}_{p,q}(a) \quad \text{and} \quad \psi(f_{\epsilon_p-\epsilon_q,i}) = E^{\text{▚}}_{q,p}(a')$$

for some $a, a' \in \mathfrak{a}$. Without loss of generality, assume that $p < q$. Then

$$\begin{aligned}
\big[\big[\psi(e_{\epsilon_p-\epsilon_q,i}), \psi(f_{\epsilon_p-\epsilon_q,i})\big], \psi(e_q)\big] \\
&= \begin{cases} \big[\big[E^{\text{▚}}_{p,q}(a), E^{\text{▚}}_{q,p}(a')\big], E^{\text{▚}}_{q,q+1}(1)\big] & \text{if } q < r, \\ \big[\big[E^{\text{▚}}_{p,q}(a), E^{\text{▚}}_{q,p}(a')\big], E^{\square}_{r,r+1}(\sqrt{2})\big] & \text{if } q = r, \end{cases} \\
&= \begin{cases} E^{\text{▚}}_{q,q+1}(-a'a) & \text{if } q < r, \\ E^{\square}_{r,r+1}(-\sqrt{2}\, a'a) & \text{if } q = r. \end{cases}
\end{aligned}$$

But since

$$\big[e_{\epsilon_p-\epsilon_q,i}, f_{\epsilon_p-\epsilon_q,i}\big] = h_{\epsilon_p-\epsilon_q,i} = \big[h_{\epsilon_p-\epsilon_q,i}, e_q\big] = \begin{cases} A_{\epsilon_p-\epsilon_q,\epsilon_q-\epsilon_{q+1}}\, e_q & \text{if } q < r, \\ A_{\epsilon_p-\epsilon_q,\epsilon_r}\, e_q & \text{if } q = r \end{cases} = -e_q$$

and ψ is a homomorphism,

$$\big[\big[\psi(e_{\epsilon_p-\epsilon_q,i}), \psi(f_{\epsilon_p-\epsilon_q,i})\big], \psi(e_q)\big] = -\psi(e_q) = \begin{cases} E^{\text{▚}}_{q,q+1}(-1) & \text{if } q < r, \\ E^{\square}_{r,r+1}(-\sqrt{2}) & \text{if } q = r, \end{cases}$$

So whether $q < r$ or $q = r$, we have

$$a'a = 1. \tag{1}$$

We show that aa' also equals 1. Indeed,

$$\begin{aligned}
\big[\big[\psi(e_{\epsilon_p-\epsilon_q,i}), \psi(f_{\epsilon_p-\epsilon_q,i})\big], \psi(e_p)\big] &= \big[\big[E^{\text{▚}}_{p,q}(a), E^{\text{▚}}_{q,p}(a')\big], E^{\text{▚}}_{p,p+1}(1)\big] \\
&= \begin{cases} E^{\text{▚}}_{p,p+1}(aa') & \text{if } q - p \geq 2, \\ E^{\text{▚}}_{p,p+1}(aa' + a'a) & \text{if } q = p + 1. \end{cases}
\end{aligned}$$

But because

$$\begin{aligned}
\big[\big[e_{\epsilon_p-\epsilon_q,i}, f_{\epsilon_p-\epsilon_q,i}\big], e_p\big] &= A_{\epsilon_p-\epsilon_q,\epsilon_p-\epsilon_{p+1}} e_p \\
&= (1 + \delta_{q,p+1}) e_p = \begin{cases} e_p & \text{if } q \geq p + 2, \\ 2e_p & \text{if } q = p + 1, \end{cases}
\end{aligned}$$

we have

$$\big[\big[\psi(e_{\epsilon_p-\epsilon_q,i}),\,\psi(f_{\epsilon_p-\epsilon_q,i})\big],\,\psi(e_p)\big] = \begin{cases} E^{▚}_{p,p+1}(1) & \text{if } q \geq p+2, \\ E^{▚}_{p,p+1}(2) & \text{if } q = p+1. \end{cases}$$

So if $q \geq p+2$, then $aa' = 1$. If $q = p+1$, then $aa' + a'a = 2$. But, by (1), $a'a = 1$. Hence, in either case, $aa' = 1$.

(ii) Since ψ is a graded homomorphism,

$$\psi(e_{\epsilon_p+\epsilon_q,i}) = E^{▝}_{p,2r+2-q}(b) \quad \text{and} \quad \psi(f_{\epsilon_p+\epsilon_q,i}) = E^{▖}_{2r+2-q,p}(b')$$

for some $b, b' \in \mathfrak{a}$. Again without loss of generality, we may assume that $p < q$. Then $\big[\big[\psi(e_{\epsilon_p+\epsilon_q,i}),\,\psi(f_{\epsilon_p+\epsilon_q,i})\big],\,\psi(e_q)\big]$ equals

$$\begin{aligned} &\begin{cases} \big[\big[E^{▝}_{p,2r+2-q}(b),\,E^{▖}_{2r+2-q,p}(b')\big],\,E^{▚}_{q,q+1}(1)\big] & \text{if } q < r, \\ \big[\big[E^{▝}_{p,2r+2-q}(b),\,E^{▖}_{2r+2-q,p}(b')\big],\,E^{◫}_{r,r+1}(\sqrt{2})\big] & \text{if } q = r, \end{cases} \\ = &\begin{cases} E^{▚}_{q,q+1}\big(\eta(b)\,\eta(b')\big) & \text{if } q < r, \\ E^{◫}_{r,r+1}\big(\sqrt{2}\eta(b)\,\eta(b')\big) & \text{if } q = r. \end{cases} \end{aligned}$$

But it also equals

$$\psi\big(\big[\big[e_{\epsilon_p+\epsilon_q,i},\,f_{\epsilon_p+\epsilon_q,i}\big],\,e_q\big]\big) = \psi(e_q) = \begin{cases} E^{▚}_{q,q+1}(1) & \text{if } q < r, \\ E^{◫}_{r,r+1}(\sqrt{2}) & \text{if } q = r, \end{cases}$$

whence $\eta(b)\,\eta(b') = 1$. Applying (the antiautomorphism) η to both sides, we get that

$$b'b = 1. \tag{2}$$

To show that $bb' = 1$, we first compute that

$$\big[\big[\psi(e_{\epsilon_p+\epsilon_q,i}),\,\psi(f_{\epsilon_p+\epsilon_q,i})\big],\,\psi(e_p)\big] = \begin{cases} E^{▚}_{p,p+1}(bb') & \text{if } q \geq p+2, \\ E^{▚}_{p,p+1}(bb' - \eta(b)\,\eta(b')) & \text{if } q = p+1. \end{cases}$$

Since

$$\big[\big[e_{\epsilon_p+\epsilon_q,i},\,f_{\epsilon_p+\epsilon_q,i}\big],\,e_p\big] = A_{\epsilon_p+\epsilon_q,\epsilon_p-\epsilon_{p+1}}\,e_p = (1-\delta_{q,p+1})e_p = \begin{cases} e_p & \text{if } q \geq p+2, \\ 0 & \text{if } q = p+1, \end{cases}$$

we also have

$$\big[\big[\psi(e_{\epsilon_p+\epsilon_q,i}),\,\psi(f_{\epsilon_p+\epsilon_q,i})\big],\,\psi(e_p)\big] = \begin{cases} E^{▚}_{p,p+1}(1) & \text{if } q - p \geq 2, \\ 0 & \text{if } q = p+1. \end{cases}$$

So if $q \geq p+2$, then $bb' = 1$. If $q = p+1$, then $bb' - \eta(b)\,\eta(b') = 0$, which implies, using (2), that

$$bb' = \eta(b)\,\eta(b') = \eta(b'b) = \eta(1) = 1.$$

In either case, $bb' = 1$.

(iii) The proof follows using similar calculations as above. $\square$

5.2. *Understanding the involution η on $\mathfrak{a}$.*

Proposition 10. (i) *If*

$$\psi(e_{\epsilon_p+\epsilon_{p+1},i}) = E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-(p+1)}(a)$$

for some $1 \leq p \leq r-1$ and $a \in \mathfrak{a}$, then $\eta(a) = a$.

(ii) *If*

$$\psi(e_{-\epsilon_p-\epsilon_{p+1},i}) = E^{\begin{smallmatrix}\square&\square\\ \blacksquare&\square\end{smallmatrix}}_{2r+2-p,\,p+1}(b)$$

for some $1 \leq p \leq r-1$ and $b \in \mathfrak{a}$, then $\eta(b) = b$.

Proof. We prove (i). The proof of (ii) is similar. Observe that

$$\big[\psi(e_{\epsilon_p+\epsilon_{p+1},i}),\, \psi(e_p)\big] = \big[E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-(p+1)}(a),\, E^{\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}}_{p,p+1}(1)\big] = E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-p}(a).$$

But $A_{\epsilon_p+\epsilon_{p+1},\epsilon_p-\epsilon_{p+1}} = 0$ implies that $(\operatorname{ad} e_{\epsilon_p+\epsilon_{p+1},i})^{-0+1} e_p = \big[e_{\epsilon_p+\epsilon_{p+1},i},\, e_p\big] = 0$, which, in turn, implies that $\big[\psi(e_{\epsilon_p+\epsilon_{p+1},i}),\, \psi(e_p)\big] = 0$. So

$$E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-p}(a) = E_{p,2r+2-p}(a - \eta(a)) = 0$$

and thus

$$\eta(a) = a. \tag{3}$$

$\square$

5.3. *Understanding the relations on generators of $\mathfrak{a}$.*

Proposition 11. *If, as a consequence of adjoining an i-th copy of the long root $\epsilon_p - \epsilon_q$ and a j-th copy of the long root $\epsilon_p + \epsilon_q$, where $1 \leq p, q \leq r$ with $p \neq q$,*

$$\psi(e_{\epsilon_p-\epsilon_q,i}) = E^{\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}}_{p,q}(s) \text{ and } \psi\big(e_{\epsilon_p+\epsilon_q,j}\big) = E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-q}(t),$$

for some $s, t \in \mathfrak{a}$, then:

(a) *If $|p-q| = 1$, the elements s, t, and $\eta(s)$ in $\mathfrak{a}$ satisfy the relation*

$$s \cdot t = t \cdot \eta(s).$$

(b) *If $|p-q| \geq 2$, the elements s, t, $\eta(s)$, and $\eta(t)$ in $\mathfrak{a}$ satisfy the relation*

$$s \cdot \eta(t) = t \cdot \eta(s).$$

Proof. Observe that

$$\begin{aligned}
\big[\psi(e_{\epsilon_p-\epsilon_q,i}),\,\psi(e_{\epsilon_p+\epsilon_q,j})\big] &= \big[E^{\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}}_{p,q}(s),\,E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-q}(t)\big]\\
&= E^{\begin{smallmatrix}\square&\blacksquare\\ \square&\square\end{smallmatrix}}_{p,2r+2-p}(-s\cdot\eta(t))\\
&= E_{p,2r+2-p}(-s\cdot\eta(t)+t\cdot\eta(s))\\
&= \begin{cases} E_{p,2r+2-p}(-s\cdot t+t\cdot\eta(s)) & \text{if } |p-q|=1,\\ E_{p,2r+2-p}(-s\cdot\eta(t)+t\cdot\eta(s)) & \text{if } |p-q|\geq 2.\end{cases}
\end{aligned}$$

(The division into two cases in the last step follows from the use of (3).) But since $A_{\epsilon_p-\epsilon_q,\epsilon_p+\epsilon_q}=0$, the generalized intersection matrix algebra relations tell us that

$$\big(\operatorname{ad} e_{\epsilon_p-\epsilon_q,i}\big)^{-0+1} e_{\epsilon_p+\epsilon_q,j}=0.$$

That is, $\big[e_{\epsilon_p-\epsilon_q,i},\,e_{\epsilon_p+\epsilon_q,j}\big]=0$. So we must have $\big[\psi(e_{\epsilon_p-\epsilon_q,i}),\,\psi(e_{\epsilon_p+\epsilon_q,j})\big]=0$. This implies that

$$\begin{aligned}
-s\cdot t+t\cdot\eta(s)&=0 && \text{if } |p-q|=1,\\
-s\cdot\eta(t)+t\cdot\eta(s)&=0, && \text{if } |p-q|\geq 2.
\end{aligned}$$

□

Similarly:

Proposition 12. *If, as a consequence of adjoining an i-th copy of the long root $\epsilon_p-\epsilon_q$ and a j-th copy of the long root $-\epsilon_p-\epsilon_q$, where $1\leq p,q\leq r$ with $p\neq q$,*

$$\psi(e_{\epsilon_p-\epsilon_q,i})=E^{\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}}_{p,q}(s)\quad\textit{and}\quad\psi(e_{-\epsilon_p-\epsilon_q,j})=E^{\begin{smallmatrix}\square&\square\\ \blacksquare&\square\end{smallmatrix}}_{2r+2-p,q}(t),$$

for some $s,t\in\mathfrak{a}$, then:

(a) *If $|p-q|=1$, the elements s, t, and $\eta(s)$ in $\mathfrak{a}$ satisfy the relation*

$$\eta(s)\cdot t=t\cdot s.$$

(b) *If $|p-q|\geq 2$, the elements s, t, $\eta(s)$, and $\eta(t)$ in $\mathfrak{a}$ satisfy the relation*

$$\eta(s)\cdot t=\eta(t)\cdot s.$$

5.4. ***A description of the module C.*** Since ψ is a graded, surjective homomorphism from $\mathfrak{im}(A^{[d]})$ to $\mathrm{so}_{2r+1}(\mathfrak{a},\eta,C,\chi)$ and we are only adjoining long roots, we can examine the image of $\mathfrak{im}(A^{[d]})$ under ψ to help us understand C.

Proposition 13. *The module C is zero.*

Proof. The generators of $\psi\big(\mathfrak{im}(A^{[d]})\big)$ all have the form $\left[\begin{smallmatrix}M&0\\0&0\end{smallmatrix}\right]$, $M\in\mathrm{M}_{2r+1}(\mathfrak{a})$. Since the matrices of this form in $\mathrm{so}_{2r+1}(\mathfrak{a},\eta,C,\chi)$ form a subalgebra and since ψ is surjective, we have $C=0$. □

6. Achieving a "sufficient" understanding of $\mathfrak{a}$, η, C, and χ

In the previous section we used the homomorphism

$$\psi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi),$$

given by the recognition theorem of [Allison et al. 2002] to get a sense (i) of what the generators of $\mathfrak{a}$ ought to be, (ii) of what the involution η on $\mathfrak{a}$ ought to be, (iii) of what the relations on the generators of $\mathfrak{a}$ ought to be, and (iv) that $C = 0$ and $\chi = 0$.

In this section, we show that the understanding we have arrived at is complete. We do so as follows:

1. Take the 4-tuple of associative algebra, involution, module, and hermitian form as we presently understand it. That is:

(i) Let $\Omega =$ the set of all long roots of the form $\pm(\epsilon_i + \epsilon_{i+1})$ that we have adjoined, $\Theta =$ the set of all long roots in Δ_B which we have adjoined but that are not in Ω,

and let

$$X_e = \bigcup_{\omega\in\Omega} \{x_{\omega,1}, \dots, x_{\omega,N_\omega}\}, \quad X_f = \bigcup_{\omega\in\Omega} \{x^{-1}_{\omega,1}, \dots, x^{-1}_{\omega,N_\omega}\},$$
$$Y_e = \bigcup_{\theta\in\Theta} \{y_{\theta,1}, \dots, y_{\theta,N_\theta}\}, \quad Y_f = \bigcup_{\theta\in\Theta} \{y^{-1}_{\theta,1}, \dots, y^{-1}_{\theta,N_\theta}\},$$
$$Z_e = \bigcup_{\theta\in\Theta} \{z_{\theta,1}, \dots, z_{\theta,N_\theta}\}, \quad Z_f = \bigcup_{\theta\in\Theta} \{z^{-1}_{\theta,1}, \dots, z^{-1}_{\theta,N_\theta}\},$$

denote collections of indeterminates indexed by the sets Ω and Θ. Let $\mathfrak{b}$ be the unital associative $\mathbb{C}$-algebra generated by the indeterminates in

$$X_e \cup X_f \cup Y_e \cup Y_f \cup Z_e \cup Z_f,$$

subject to the relations

$$y_{\epsilon_p-\epsilon_q,i}\, x_{\epsilon_p+\epsilon_q,j} = x_{\epsilon_p+\epsilon_q,j}\, z_{\epsilon_p-\epsilon_q,i},$$
$$y_{\epsilon_p-\epsilon_q,i}\, z_{\epsilon_p+\epsilon_q,j} = y_{\epsilon_p+\epsilon_q,j}\, z_{\epsilon_p-\epsilon_q,i},$$
$$z_{\epsilon_p-\epsilon_q,i}\, x_{-\epsilon_p-\epsilon_q,k} = x_{-\epsilon_p-\epsilon_q,k}\, y_{\epsilon_p-\epsilon_q,i},$$
$$z_{\epsilon_p-\epsilon_q,i}\, y_{-\epsilon_p-\epsilon_q,k} = z_{-\epsilon_p-\epsilon_q,k}\, y_{\epsilon_p-\epsilon_q,i},$$

where $i = 1, \dots, N_{\epsilon_p-\epsilon_q}$ for $\epsilon_p - \epsilon_q \in \Theta$, $j = 1, \dots, N_{\epsilon_p+\epsilon_q}$ for $\epsilon_p + \epsilon_q \in \Omega \cup \Theta$, and $k = 1, \dots, N_{-\epsilon_p-\epsilon_q}$ for $-\epsilon_p - \epsilon_q \in \Omega \cup \Theta$.

(ii) Define an involution, which we also call η and sometimes denote by $\bar{\cdot}$, on $\mathfrak{b}$ by

$$\begin{aligned} \eta(x_{\omega,i}) &= x_{\omega,i} && \text{if } \omega \in \Omega \text{ and } 1 \le i \le N_\omega, \\ \eta(y_{\theta,i}) &= z_{\theta,i} && \text{if } \theta \in \Theta \text{ and } 1 \le i \le N_\theta, \\ \eta(z_{\theta,i}) &= y_{\theta,i} && \text{if } \theta \in \Theta \text{ and } 1 \le i \le N_\theta. \end{aligned}$$

(iii) Let $C = 0$ be the trivial $\mathfrak{b}$-module.

(iv) Let $\chi = 0$ be a hermitian form on C.

Remarks. (a) The indeterminates in $X_e \cup X_f \cup \cdots \cup Z_f$ are intended to capture the elements of the form a, a', b, b', c, and c' of $\mathfrak{a}$ that we studied in Section 5, which arose from the images of the map ψ.

(b) In the relations listed above, our use of the indeterminates $x_{\epsilon_p+\epsilon_q,j}$ and $x_{-\epsilon_p-\epsilon_q,j}$ signals that we are working with roots in Ω and, hence, $|p-q| = 1$ in this setting. Likewise, our use of the indeterminates $y_{\epsilon_p+\epsilon_q,j}$, $z_{\epsilon_p+\epsilon_q,j}$, $y_{-\epsilon_p-\epsilon_q,j}$, and $z_{-\epsilon_p-\epsilon_q,j}$ signals that we are working with roots in Θ and p, q such that $|p-q| \geq 2$.

2. Construct a map

$$\varphi : \mathfrak{gim}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$$

sending the generators

$$\begin{array}{lll} e_1, \ldots, e_r, & \bigcup_{\omega\in\Omega} \{e_{\omega,1}, \ldots, e_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{e_{\theta,1}, \ldots, e_{\theta,N_\theta}\}, \\ f_1, \ldots, f_r, & \bigcup_{\omega\in\Omega} \{f_{\omega,1}, \ldots, f_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{f_{\theta,1}, \ldots, f_{\theta,N_\theta}\}, \\ h_1, \ldots, h_r, & \bigcup_{\omega\in\Omega} \{h_{\omega,1}, \ldots, h_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{h_{\theta,1}, \ldots, h_{\theta,N_\theta}\}, \end{array}$$

of $\mathfrak{gim}(A^{[d]})$ to

$$\begin{array}{lll} \tilde{e}_1, \ldots, \tilde{e}_r, & \bigcup_{\omega\in\Omega} \{\tilde{e}_{\omega,1}, \ldots, \tilde{e}_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{\tilde{e}_{\theta,1}, \ldots, \tilde{e}_{\theta,N_\theta}\} \\ \tilde{f}_1, \ldots, \tilde{f}_r, & \bigcup_{\omega\in\Omega} \{\tilde{f}_{\omega,1}, \ldots, \tilde{f}_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{\tilde{f}_{\theta,1}, \ldots, \tilde{f}_{\theta,N_\theta}\}, \\ \tilde{h}_1, \ldots, \tilde{h}_r, & \bigcup_{\omega\in\Omega} \{\tilde{h}_{\omega,1}, \ldots, \tilde{h}_{\omega,N_\omega}\}, & \bigcup_{\theta\in\Theta} \{\tilde{h}_{\theta,1}, \ldots, \tilde{h}_{\theta,N_\theta}\}, \end{array}$$

respectively, where

$$\tilde{e}_i := E^{\text{▚}}_{i,i+1}(1), \quad 1 \leq i \leq r-1,$$

$$\tilde{e}_r := E^{\text{◫}}_{r,r+1}(\sqrt{2}),$$

$$\tilde{e}_{\omega,i} := \begin{cases} E^{\text{▝}}_{p,2r+2-(p+1)}(x_{\omega,i}) & \text{if } \omega = \epsilon_p + \epsilon_{p+1}, \\ E^{\text{▖}}_{2r+2-p,p+1}(x_{\omega,i}) & \text{if } \omega = -\epsilon_p - \epsilon_{p+1}, \end{cases} \quad \text{for } \omega \in \Omega \text{ and } 1 \leq i \leq N_\omega,$$

$$\tilde{e}_{\theta,i} := \begin{cases} E^{\text{▚}}_{p,q}(y_{\theta,i}) & \text{if } \theta = \epsilon_p - \epsilon_q, \\ E^{\text{▝}}_{p,2r+2-q}(y_{\theta,i}) & \text{if } \theta = \epsilon_p + \epsilon_q, \\ E^{\text{▖}}_{2r+2-p,q}(y_{\theta,i}) & \text{if } \theta = -\epsilon_p - \epsilon_q, \end{cases} \quad \text{for } \theta \in \Theta \text{ and } 1 \leq i \leq N_\theta,$$

$$\tilde{f}_i := E^{\square}_{i+1,i}(1), \quad 1 \le i \le r-1,$$

$$\tilde{f}_r := E^{\square}_{r+1,r}(\sqrt{2}),$$

$$\tilde{f}_{\omega,i} := \begin{cases} E^{\square}_{2r+2-(p+1),p}(x^{-1}_{\omega,i}) & \text{if } \omega = \epsilon_p + \epsilon_{p+1}, \\ E^{\square}_{p+1,2r+2-p}(x^{-1}_{\omega,i}) & \text{if } \omega = -\epsilon_p - \epsilon_{p+1}, \end{cases} \quad \text{for } \omega \in \Omega \text{ and } 1 \le i \le N_\omega,$$

$$\tilde{f}_{\theta,i} := \begin{cases} E^{\square}_{q,p}(y^{-1}_{\theta,i}) & \text{if } \theta = \epsilon_p - \epsilon_q, \\ E^{\square}_{2r+2-q,p}(y^{-1}_{\theta,i}) & \text{if } \theta = \epsilon_p + \epsilon_q, \\ E^{\square}_{q,2r+2-p}(y^{-1}_{\theta,i}) & \text{if } \theta = -\epsilon_p - \epsilon_q, \end{cases} \quad \text{for } \theta \in \Theta \text{ and } 1 \le i \le N_\theta,$$

$$\tilde{h}_i := E^{\square}_{i,i}(1) + E^{\square}_{i+1,i+1}, \quad 1 \le i \le r-1,$$

$$\tilde{h}_r := E^{\square}_{r,r}(2),$$

$$\tilde{h}_{\omega,i} := \begin{cases} E^{\square}_{p,p}(1) + E^{\square}_{p+1,p+1}(1) & \text{if } \omega = \epsilon_p + \epsilon_{p+1} \\ E^{\square}_{p,p}(-1) + E^{\square}_{p+1,p+1}(-1) & \text{if } \omega = -\epsilon_p - \epsilon_{p+1} \end{cases} \quad \text{for } \omega \in \Omega,\ 1 \le i \le N_\omega,$$

$$\tilde{h}_{\theta,i} := \begin{cases} E^{\square}_{p,p}(1) + E^{\square}_{q,q}(-1) & \text{if } \theta = \epsilon_p - \epsilon_q, \\ E^{\square}_{p,p}(1) + E^{\square}_{q,q}(1) & \text{if } \theta = \epsilon_p + \epsilon_q, \\ E^{\square}_{p,p}(-1) + E^{\square}_{q,q}(-1) & \text{if } \theta = -\epsilon_p - \epsilon_q, \end{cases} \quad \text{for } \theta \in \Theta \text{ and } 1 \le i \le N_\theta.$$

3. We show that φ is
   (a) a Lie algebra homomorphism (Theorem 14),
   (b) that is surjective (Proposition 15), and
   (c) graded (Proposition 16).

4. We show that the radical $\mathfrak{r}$ of $\mathfrak{gim}(A^{[d]})$ lies in the kernel of this map φ (see just before Proposition 17), hence inducing a surjective, graded, Lie algebra homomorphism

$$\phi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi).$$

5. Finally, we show that ϕ is a central map and that $\mathfrak{b} \cong \mathfrak{a}$ (Proposition 18).

Theorem 14. *The map $\varphi : \mathfrak{gim}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$ is a Lie algebra homomorphism.*

Proof. We show that the images in $\mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$ of the generators of $\mathfrak{gim}(A^{[d]})$, under the map φ, satisfy the relations (R1)–(R3) of Definition 2 with respect to the same $(r+d) \times (r+d)$ generalized intersection matrix $A^{[d]}$ as used in the construction of the algebra $\mathfrak{gim}(A^{[d]})$.

While working with the various long roots in our proof, we use labels like u or v to denote the indeterminates $x_{\omega,i}$ or $y_{\theta,i}$.

The reason that we can substitute u or v for the actual indeterminates is that the result of taking a bracket like

$$\left[\tilde{e}_{-\epsilon_p-\epsilon_q,i}, \tilde{e}_{-\epsilon_k-\epsilon_l,j}\right] = \left[E^{\square}_{2r+2-p,q}(y_{-\epsilon_p-\epsilon_q,i}), E^{\square}_{2r+2-k,l}(y_{-\epsilon_k-\epsilon_l,j})\right]$$

depends primarily on the indices p, q, k, and l rather than on the particular elements of the algebra $\mathfrak{b}$ being housed at these sites.

If we agree on this convention of using substitute variables like u, then we must recognize that

$$\bar{u} = \begin{cases} x_{\omega,i} & \text{if } u = x_{\omega,i}, \\ z_{\theta,i} & \text{if } u = y_{\theta,j}. \end{cases}$$

That is, the involution $\bar{\cdot}$ applied to u depends on whether u is substituting for a variable associated to a root in Ω or a root in Θ.

We show the computations for the interactions between the generators corresponding to the long roots $\epsilon_p - \epsilon_q$ and $\epsilon_k - \epsilon_l$. The remaining computations are similar.

Let $1 \le p, q, k, l \le r$ with $p \ne q$ and $k \ne l$, $u, v \in \{x_{\omega,i}, x_{\omega,j}, y_{\theta,i} y_{\theta,j}\}$ and $u^{-1}, v^{-1} \in \left\{x^{-1}_{\omega,i}, x^{-1}_{\omega,j}, y^{-1}_{\theta,i} y^{-1}_{\theta,j}\right\}$, where $\omega \in \Omega$, $\theta \in \Theta$, and $1 \le i, j \le N_\omega$ or $1 \le i, j \le N_\theta$.

Using the definition of $A_{\epsilon_p-\epsilon_q,\epsilon_k-\epsilon_l}$, we see that

$$A_{\epsilon_p-\epsilon_q,\epsilon_k-\epsilon_l} = \delta_{p,k} - \delta_{p,l} - \delta_{q,k} + \delta_{q,l} = \begin{cases} 0 & \text{if } p, q \notin \{k, l\}, \\ 1 & \text{if } p = k \text{ but } q \ne l, \\ -1 & \text{if } p = l \text{ but } q \ne k, \\ -1 & \text{if } p \ne l \text{ but } q = k, \\ 1 & \text{if } p \ne k \text{ but } q = l, \\ 2 & \text{if } p = k \text{ and } q = l, \\ -2 & \text{if } p = l \text{ and } q = k. \end{cases}$$

A. $\left[\tilde{e}_{\epsilon_p-\epsilon_q,i}, \tilde{e}_{\epsilon_k-\epsilon_l,j}\right] = \left[E^{\boxtimes}_{p,q}(u), E^{\boxtimes}_{k,l}(v)\right] = \delta_{q,k} E^{\boxtimes}_{p,l}(uv) + \delta_{l,p} E^{\boxtimes}_{k,q}(-vu)$

$$= \begin{cases} E^{\boxtimes}_{k,q}(-vu) & \text{if } p = l \text{ but } q \ne k, \\ E^{\boxtimes}_{p,l}(uv) & \text{if } p \ne l \text{ but } q = k, \\ E^{\boxtimes}_{p,p}(uv) + E^{\boxtimes}_{q,q}(-vu) & \text{if } p = l \text{ and } q = k, \\ 0 & \text{otherwise.} \end{cases}$$

- If $p = l$ but $q \ne k$, then $\left[E^{\boxtimes}_{p,q}(u), E^{\boxtimes}_{k,q}(-vu)\right] = 0$ because $q \ne k$ and $q \ne p$.

- If $p \neq l$ but $q = k$, then $\left[E^{\text{▚}}_{p,q}(u),\, E^{\text{▚}}_{p,l}(uv)\right] = 0$ because $q \neq p$ and $l \neq p$.
- If $p = l$ and $q = k$, then

$$\begin{aligned}\left[E^{\text{▚}}_{p,q}(u),\ E^{\text{▚}}_{p,p}(uv) + E^{\text{▚}}_{q,q}(-vu)\right] &= E^{\text{▚}}_{p,q}(-uvu) + E^{\text{▚}}_{p,q}(-uvu)\\ &= E^{\text{▚}}_{p,q}(-2uvu).\end{aligned}$$

So

$$\left(\operatorname{ad} \tilde{e}_{\epsilon_p-\epsilon_q,i}\right)^{1+1} \tilde{e}_{\epsilon_k-\epsilon_l,j} = \begin{cases} E^{\text{▚}}_{p,q}(-2uvu) & \text{if } p = l \text{ and } q = k,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\left[E^{\text{▚}}_{p,q}(u),\, E^{\text{▚}}_{p,q}(-2uvu)\right] = 0$, we get $\left(\operatorname{ad} \tilde{e}_{\epsilon_p-\epsilon_q,i}\right)^{2+1} \tilde{e}_{\epsilon_k-\epsilon_l,j} = 0$.

B. $$\begin{aligned}\left[\tilde{f}_{\epsilon_p-\epsilon_q,i},\, \tilde{f}_{\epsilon_k-\epsilon_l,j}\right] &= \left[E^{\text{▚}}_{q,p}(u^{-1}),\, E^{\text{▚}}_{l,k}(v^{-1})\right]\\ &= \delta_{p,l} E^{\text{▚}}_{q,k}(u^{-1}v^{-1}) + \delta_{k,q} E^{\text{▚}}_{l,p}(-v^{-1}u^{-1})\\ &= \begin{cases} E^{\text{▚}}_{q,k}(u^{-1}v^{-1}) & \text{if } p = l \text{ but } q \neq k,\\ E^{\text{▚}}_{l,p}(-v^{-1}u^{-1}) & \text{if } p \neq l \text{ but } q = k,\\ E^{\text{▚}}_{p,p}(-v^{-1}u^{-1}) + E^{\text{▚}}_{q,q}(u^{-1}v^{-1}) & \text{if } p = l \text{ and } q = k,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

- If $p = l$ but $q \neq k$, then $\left[E^{\text{▚}}_{q,p}(u^{-1}),\, E^{\text{▚}}_{q,k}(u^{-1}v^{-1})\right] = 0$ because $p \neq q$ and $k \neq q$.
- If $p \neq l$ but $q = k$, then $\left[E^{\text{▚}}_{q,p}(u^{-1}),\, E^{\text{▚}}_{l,p}(-v^{-1}u^{-1})\right] = 0$ because $p \neq l$ and $p \neq q$.
- If $p = l$ and $q = k$, then

$$\begin{aligned}&\left[E^{\text{▚}}_{q,p}(u^{-1}),\, E^{\text{▚}}_{p,p}(-v^{-1}u^{-1}) + E^{\text{▚}}_{q,q}(u^{-1}v^{-1})\right]\\ &\qquad = E^{\text{▚}}_{q,p}(-u^{-1}v^{-1}u^{-1}) + E^{\text{▚}}_{q,p}(-u^{-1}v^{-1}u^{-1})\\ &\qquad = E^{\text{▚}}_{q,p}\left(-2u^{-1}v^{-1}u^{-1}\right).\end{aligned}$$

So

$$\left(\operatorname{ad} \tilde{f}_{\epsilon_p-\epsilon_q,i}\right)^{1+1} \tilde{f}_{\epsilon_k-\epsilon_l,j} = \begin{cases} E^{\text{▚}}_{q,p}(-2u^{-1}v^{-1}u^{-1}) & \text{if } p = l \text{ and } q = k,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\left[E^{\text{▚}}_{q,p}(u^{-1}),\, E^{\text{▚}}_{q,p}(-2u^{-1}v^{-1}u^{-1})\right] = 0$, we get that

$$(\operatorname{ad} \tilde{f}_{\epsilon_p-\epsilon_q,i})^{2+1} \tilde{f}_{\epsilon_k-\epsilon_l,j} = 0.$$

C. $\left[\tilde{h}_{\epsilon_p-\epsilon_q,i}, \tilde{h}_{\epsilon_k-\epsilon_l,j}\right]$

$$\begin{aligned}
&= \left[E^{\boxbslash}_{p,p}(1) + E^{\boxbslash}_{q,q}(-1),\ \ E^{\boxbslash}_{k,k}(1) + E^{\boxbslash}_{l,l}(-1)\right] \\
&= \delta_{p,k} E^{\boxbslash}_{p,p}([1,1]) + \delta_{p,l} E^{\boxbslash}_{p,p}([1,-1]) + \delta_{q,k} E^{\boxbslash}_{q,q}([-1,1]) + \delta_{q,l} E^{\boxbslash}_{q,q}([-1,-1]) \\
&= \delta_{p,k} E^{\boxbslash}_{p,p}(0) + \delta_{p,l} E^{\boxbslash}_{p,p}(0) + \delta_{q,k} E^{\boxbslash}_{q,q}(0) + \delta_{q,l} E^{\boxbslash}_{q,q}(0) \\
&= 0
\end{aligned}$$

D. $$\begin{aligned}
\left[\tilde{e}_{\epsilon_p-\epsilon_q,i}, \tilde{f}_{\epsilon_k-\epsilon_l,j}\right] &= \left[E^{\boxbslash}_{p,q}(u),\ E^{\boxbslash}_{l,k}(v^{-1})\right] \\
&= \delta_{q,l} E^{\boxbslash}_{p,k}\left(u\, v^{-1}\right) + \delta_{k,p} E^{\boxbslash}_{l,q}\left(-v^{-1}\, u\right) \\
&= \begin{cases} E^{\boxbslash}_{l,q}(-v^{-1}u) & \text{if } p=k \text{ but } q \neq l, \\ E^{\boxbslash}_{p,k}(uv^{-1}) & \text{if } p \neq k \text{ but } q = l, \\ E^{\boxbslash}_{p,p}(uv^{-1}) + E^{\boxbslash}_{q,q}(-v^{-1}u) & \text{if } p=k \text{ and } q=l, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

- If $p=k$ but $q \neq l$, then $\left[E^{\boxbslash}_{p,q}(u),\ E^{\boxbslash}_{l,q}(-v^{-1}u)\right] = 0$ because $q \neq l$ and $q \neq p$.
- If $p \neq k$ but $q = l$, then $\left[E^{\boxbslash}_{p,q}(u),\ E^{\boxbslash}_{p,k}(uv^{-1})\right] = 0$ because $q \neq p$ and $k \neq p$.
- If $p=k$ and $q=l$, then

$$\begin{aligned}
\left[E^{\boxbslash}_{p,q}(u),\ E^{\boxbslash}_{p,p}(uv^{-1}) + E^{\boxbslash}_{q,q}(-v^{-1}u)\right] &= E^{\boxbslash}_{p,q}(-uv^{-1}u) + E^{\boxbslash}_{p,q}(-uv^{-1}u) \\
&= E^{\boxbslash}_{p,q}(-2uv^{-1}u).
\end{aligned}$$

So

$$\left(\operatorname{ad} \tilde{e}_{\epsilon_p-\epsilon_q,i}\right)^{1+1} \tilde{f}_{\epsilon_k-\epsilon_l,j} = \begin{cases} E^{\boxbslash}_{p,q}(-2uv^{-1}u) & \text{if } p=k \text{ and } q=l, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\left[E^{\boxbslash}_{p,q}(u),\ E^{\boxbslash}_{p,q}(-2uv^{-1}u)\right] = 0$, we get that $\left(\operatorname{ad} \tilde{e}_{\epsilon_p-\epsilon_q,i}\right)^{2+1} \tilde{f}_{\epsilon_k-\epsilon_l,j} = 0$.

E. $$\begin{aligned}
\left[\tilde{f}_{\epsilon_p-\epsilon_q,i}, \tilde{e}_{\epsilon_k-\epsilon_l,j}\right] &= \left[E^{\boxbslash}_{q,p}(u^{-1}),\ E^{\boxbslash}_{k,l}(v)\right] \\
&= \delta_{p,k} E^{\boxbslash}_{q,l}(u^{-1}v) + \delta_{l,q} E^{\boxbslash}_{k,p}(-vu^{-1}) \\
&= \begin{cases} E^{\boxbslash}_{q,l}(u^{-1}\, v) & \text{if } p=k \text{ but } q \neq l, \\ E^{\boxbslash}_{k,p}(-v\, u^{-1}) & \text{if } p \neq k \text{ but } q = l, \\ E^{\boxbslash}_{p,p}(-v\, u^{-1}) + E^{\boxbslash}_{q,q}(u^{-1}\, v) & \text{if } p=k \text{ and } q=l, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

- If $p=k$ but $q \neq l$, then $\left[E^{\boxbslash}_{q,p}(u^{-1}),\ E^{\boxbslash}_{q,l}(u^{-1}v)\right] = 0$ because $p \neq q$ and $l \neq q$.

- If $p \neq k$ but $q = l$, then $\left[E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(u^{-1}),\, E_{k,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-v\,u^{-1})\right] = 0$ because $p \neq k$ and $p \neq q$.
- If $p = k$ and $q = l$, then

$$\left[E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(u^{-1}),\, E_{p,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-vu^{-1}) + E_{q,q}^{\substack{\blacksquare\square\\\square\blacksquare}}(u^{-1}v)\right] = E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-u^{-1}vu^{-1}) + E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-u^{-1}vu^{-1})$$
$$= E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-2u^{-1}vu^{-1}).$$

So

$$\left(\operatorname{ad} \tilde{f}_{\epsilon_p-\epsilon_q,i}\right)^{1+1} \tilde{e}_{\epsilon_k-\epsilon_l,j} = \begin{cases} E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-2u^{-1}vu^{-1}) & \text{if } p = k \text{ and } q = l, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\left[E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(u^{-1}),\, E_{q,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-2u^{-1}vu^{-1})\right] = 0$, we get

$$(\operatorname{ad} \tilde{f}_{\epsilon_p-\epsilon_q,i})^{2+1} \tilde{e}_{\epsilon_k-\epsilon_l,j} = 0.$$

F. $\left[\tilde{h}_{\epsilon_p-\epsilon_q,i},\, \tilde{e}_{\epsilon_k-\epsilon_l,j}\right] = \left[E_{p,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(1) + E_{q,q}^{\substack{\blacksquare\square\\\square\blacksquare}}(-1),\, E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v)\right]$

$$= \delta_{p,k} E_{p,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v) + \delta_{l,p} E_{k,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-v) + \delta_{q,k} E_{q,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(-v) + \delta_{l,q} E_{k,q}^{\substack{\blacksquare\square\\\square\blacksquare}}(v)$$
$$= \delta_{p,k} E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v) - \delta_{p,l} E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v) - \delta_{q,k} E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v) + \delta_{q,l} E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v)$$
$$= (\delta_{p,k} - \delta_{p,l} - \delta_{q,k} + \delta_{q,l}) E_{k,l}^{\substack{\blacksquare\square\\\square\blacksquare}}(v)$$
$$= A_{\epsilon_p-\epsilon_q,\epsilon_k-\epsilon_l} \tilde{e}_{\epsilon_k-\epsilon_l,j}.$$

G. $\left[\tilde{h}_{\epsilon_p-\epsilon_q,i},\, \tilde{f}_{\epsilon_k-\epsilon_l,j}\right]$

$$= \left[E_{p,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(1) + E_{q,q}^{\substack{\blacksquare\square\\\square\blacksquare}}(-1),\, E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1})\right]$$
$$= \delta_{p,l} E_{p,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1}) + \delta_{k,p} E_{l,p}^{\substack{\blacksquare\square\\\square\blacksquare}}(-v^{-1}) + \delta_{q,l} E_{q,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(-v^{-1}) + \delta_{k,q} E_{l,q}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1})$$
$$= \delta_{p,l} E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1}) - \delta_{p,k} E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1}) - \delta_{q,l} E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1}) + \delta_{q,k} E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1})$$
$$= -(\delta_{p,k} - \delta_{p,l} - \delta_{q,k} + \delta_{q,l}) E_{l,k}^{\substack{\blacksquare\square\\\square\blacksquare}}(v^{-1})$$
$$= -A_{\epsilon_p-\epsilon_q,\epsilon_k-\epsilon_l} \tilde{f}_{\epsilon_k-\epsilon_l,j} \qquad \square$$

Proposition 15. *The map* $\varphi : \mathfrak{gim}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$ *is a surjective Lie algebra homomorphism.*

Proof. Let $\mathfrak{B} = \mathrm{Im}(\varphi) \subseteq L$, where $L = \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$. We show that $\mathfrak{B} = L$ in a sequence of steps.

#1. Let $\mathfrak{g} = \left\{M \in \mathrm{M}_{2r+1}(\mathbb{C}) : M^t G = -GM\right\}$ and

$$\mathfrak{s} = \left\{M \in \mathrm{M}_{2r+1}(\mathbb{C}) : M^t G = GM, \mathrm{tr}(M) = 0\right\}.$$

Let $A=\{a\in\mathfrak{b}:\eta a=a\}$ and $B=\{b\in\mathfrak{b}:\eta b=-b\}$. If $0\neq a\in A$, then $a\mathfrak{g}$ is an irreducible $\mathfrak{g}$-submodule of $\mathfrak{B}$ with highest weight $\epsilon_1+\epsilon_2$. If $0\neq b\in\mathfrak{B}$, then $b\mathfrak{s}$ is an irreducible $\mathfrak{g}$-submodule of $\mathfrak{B}$ with highest weight $2\epsilon_1$. These $\mathfrak{g}$-modules are not isomorphic.

#2. $\mathfrak{B}$ is a subalgebra of L containing $\mathfrak{g}$, so $\mathfrak{B}$ is a $\mathfrak{g}$-submodule of L.

#3. For $1\leq p,q\leq 2r+1$, $p\neq q$, $p\neq 2r+2-q$, let

$$I_{pq}=\big\{x\in\mathfrak{b}:E_{pq}(x)-E_{2r+2-q,2r+2-p}(\eta x)\in\mathfrak{B}\big\}.$$

Notice that I_{pq} is a subspace of $\mathfrak{B}$.

#4. I_{pq} is invariant under η. Indeed, let $x\in I_{pq}$, in which case

$$X:=E_{pq}(x)-E_{2r+2-q,2r+2-p}(\eta x)\in\mathfrak{B}.$$

But $X=X_1+X_2$, where $X_1=\frac{1}{2}(x+\eta x)\big(E_{pq}(1)-E_{2r+2-q,2r+2-p}(1)\big)\in(x+\eta x)\mathfrak{g}$ and $X_2=\frac{1}{2}(x-\eta x)\big(E_{pq}(1)+E_{2r+2-q,2r+2-p}(1)\big)\in(x-\eta x)\mathfrak{s}$. Thus, by #1 and #2, $X_1,X_2\in\mathfrak{B}$. So $x+\eta x$, $x-\eta x\in I_{pq}$, which implies that $\eta x\in I_{pq}$.

#5. By #4, $I_{pq}=I_{pq}\cap A+I_{pq}\cap B$. But by #1 and #2, $I_{pq}\cap A$ and $I_{pq}\cap B$ are independent of p,q. So $I:=I_{pq}$ is independent of p,q.

#6. We have

$$\big[E_{12}(x)-E_{2r,2r+1}(\eta x),\,E_{23}(y)-E_{2r-1,2r}(\eta y)\big]=E_{13}(xy)-E_{2r-1,2r+1}\big((\eta y)(\eta x)\big).$$

So, by #5, I is a subalgebra of $\mathfrak{B}$, and, by #4, I is invariant under η.

#7. The action of φ on the generators of $\mathfrak{gim}(A^{[d]})$ tells us that I contains the elements $x_{\omega,i}$, $x_{\omega,i}^{-1}$, $y_{\theta,i}$, $y_{\theta,i}^{-1}$. So by #6, $I=\mathfrak{B}$.

#8. By #7, we have $A\mathfrak{g}+B\mathfrak{s}\subseteq\mathfrak{B}$. But since $C=\{0\}$, we have $\sum_{\alpha\in\Delta}L_\alpha\subseteq A\mathfrak{g}+B\mathfrak{s}$. So $\sum_{\alpha\in\Delta}L_\alpha\subseteq\mathfrak{B}$. Hence, since $\mathfrak{B}$ is a subalgebra of L, $\mathfrak{B}=L$. ☐

Continuing our plan laid out on page 273, we next show that $\varphi:\mathfrak{gim}(A^{[d]})\to\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)$ is a graded homomorphism and that it induces a map from $\mathfrak{im}(A^{[d]})$ to $\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)$.

We saw in Sections 3 and 4, respectively, that $\mathfrak{gim}(A^{[d]})$ and $\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)$ are both Γ-graded Lie algebras, where

$$\Gamma=\bigoplus_{\mu\in\Delta}\mathbb{Z}\alpha_\mu.$$

The map $\varphi:\mathfrak{gim}(A^{[d]})\to\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)$ is engineered so that, for all $\alpha\in\Gamma$,

$$\varphi\big(\mathfrak{gim}(A^{[d]})_\alpha\big)\subset\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)_\alpha.$$

That is, the following result holds by design.

Proposition 16. *The map $\varphi:\mathfrak{gim}(A^{[d]})\to\mathrm{so}_{2r+1}(\mathfrak{b},\eta,C,\chi)$ is also a graded homomorphism.*

Moreover, since $\mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)_\gamma = 0$ for $\gamma \notin \Delta \cup \{0\}$, we get that the radical $\mathfrak{r}$ of $\mathfrak{gim}(A^{[d]})$ lies in the kernel of φ.

Proposition 17. *There exists a surjective, graded Lie algebra homomorphism*

$$\phi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$$

given by $\phi(u + \mathfrak{r}) = \varphi(u)$ for any $u + \mathfrak{r} \in \mathfrak{im}(A^{[d]})$, where $u \in \mathfrak{gim}(A^{[d]})$.

We now turn to centrality. Let $\mathrm{so}(\mathfrak{a})$ and $\mathrm{so}(\mathfrak{b})$ be shorthand for $\mathrm{so}_{2r+1}(\mathfrak{a}, \eta, C, \chi)$ and $\mathrm{so}_{2r+1}(\mathfrak{b}, \eta, C, \chi)$, respectively.

Since the elements of $\mathfrak{a}$ satisfy the defining relations of $\mathfrak{b}$, by universality, there exists a surjective associative algebra homomorphism $g : \mathfrak{b} \to \mathfrak{a}$. In particular, $g(x_{\omega,i}) = a \in \mathfrak{a}$ if $\psi(e_{\omega,i}) = E_{pq}(a) - E_{2r+2-q,2r+2-p}(\eta a)$, and $g(y_{\theta,i}) = b \in \mathfrak{a}$ if $\psi(e_{\theta,i}) = E_{pq}(b) - E_{2r+2-q,2r+2-p}(\eta b)$. This algebra homomorphism respects the involution and induces a surjective Lie algebra homomorphism

$$\widetilde{g} : \mathrm{so}(\mathfrak{b}) \to \mathrm{so}(\mathfrak{a}) \quad \text{such that } \widetilde{g}\phi = \psi.$$

Hence $\ker\phi \subset \ker\psi \subset \mathfrak{z}\big(\mathfrak{im}(A^{[d]})\big)$, where $\mathfrak{z}\big(\mathfrak{im}(A^{[d]})\big)$ denotes the center of $\mathfrak{im}(A^{[d]})$. Thus $\ker\phi \subset \mathfrak{z}\big(\mathfrak{im}(A^{[d]})\big)$, implying the following result:

Proposition 18. *The map $\phi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}(\mathfrak{b})$ is a central homomorphism.*

We also know that $\psi : \mathfrak{im}(A^{[d]}) \to \mathrm{so}(\mathfrak{a})$ is a universal central extension: so there exists a Lie algebra homomorphism $\widetilde{f} : \mathrm{so}(\mathfrak{a}) \to \mathrm{so}(\mathfrak{b})$ such that $\widetilde{f}\psi = \phi$. Since ψ is surjective, the generators of $\mathfrak{a}$ are of the form a, a^{-1}, $\eta(a)$, where a is the element in $\mathfrak{a}$ corresponding to the image $\psi(e_{\lambda,i})$ of the i-th copy of a long root λ in Δ_B which was adjoined.

Since $\widetilde{f}\psi = \phi$, the map $\widetilde{f}$ induces an associative algebra homomorphism $f : \mathfrak{a} \to \mathfrak{b}$ given by

$$f(a) = \begin{cases} x_{\omega,i} & \begin{array}{l}\text{if } a \text{ is the element in } \mathfrak{a} \text{ corresponding to} \\ \text{the image } \psi(e_{\omega,i}) = E_{pq}(a) - E_{2r+2-q,2r+2-p}(\eta a) \\ \text{for some } 1 \le p, q \le 2r+1 \text{ with } p \ne q,\ p \ne 2r+2-q \\ \text{and } \omega \in \Omega,\ 1 \le i \le N_\omega, \end{array} \\ y_{\theta,i} & \begin{array}{l}\text{if } a \text{ is the element in } \mathfrak{a} \text{ corresponding to} \\ \text{the image } \psi\left(e_{\theta,i}\right) = E_{pq}(a) - E_{2r+2-q,2r+2-p}(\eta a) \\ \text{for some } 1 \le p, q \le 2r+1 \text{ with } p \ne q,\ p \ne 2r+2-q \\ \text{and } \theta \in \Theta,\ 1 \le i \le N_\theta. \end{array} \end{cases}$$

We define $f(\eta a)$ to be $\eta f(a)$ so that f preserves the involution.

But then $g \circ f = \mathrm{id}_{\mathfrak{a}}$ and $f \circ g = \mathrm{id}_{\mathfrak{b}}$, that is, $\mathfrak{a} \cong \mathfrak{b}$ as associative algebras.

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SANDEEP BHARGAVA
MATHEMATICS DEPARTMENT, SCHOOL OF LIBERAL ARTS AND SCIENCES
HUMBER INSTITUTE OF TECHNOLOGY AND ADVANCED LEARNING
205 HUMBER COLLEGE BOULEVARD
TORONTO, ON M9W 5L7
CANADA
sandeep.bhargava@humber.ca

YUN GAO
DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
4700 KEELE STREET
TORONTO, ON M3J 1P3
CANADA
ygao@yorku.ca

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