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Gunningham (2012) calculated all spin Hurwitz numbers in terms of combinatorics of the Sergeev algebra. Here we use a spin curve degeneration to obtain a recursion formula for degree-three spin Hurwitz numbers.

Let D be a complex curve of genus h and N be a theta characteristic on D, that is, $N^2 = K_D$. The pair (D, N) is called a *spin curve* of genus h with parity $p \equiv h^0(N) \pmod 2$. For $i = 1, \ldots, k$, let $m^i = (m^i_1, \ldots, m^i_{\ell_i})$ be an odd partition of $d > 0$, namely, all components m^i_j are odd. Fix k points $q^1, \ldots, q^k$ in D and consider degree-d maps $f : C \to D$ from possibly disconnected domains C of Euler characteristic χ that are ramified only over the fixed points q^i with ramification data m^i. Observe that the Riemann–Hurwitz formula shows

$$2d(1-h) - \chi + \sum_{i=1}^{k} (\ell(m^i) - d) = 0, \tag{0-1}$$

where $\ell(m^i) = \ell_i$ is the length of m^i. By the Hurwitz formula, the twisted line bundle

$$L_f = f^*N \otimes \mathcal{O}\Big(\sum_{i,j} \tfrac{1}{2}(m^i_j - 1)x^i_j\Big) \tag{0-2}$$

is a theta characteristic on C where $f^{-1}(q^i) = \{x^i_j\}_{1 \le j \le \ell_i}$ and f has multiplicity m^i_j at x^i_j. We define the parity $p(f)$ of a map f by

$$p(f) \equiv h^0(L_f) \pmod 2. \tag{0-3}$$

Given odd partitions $m^1, \ldots, m^k$ of d, the spin Hurwitz number of genus h and parity p is defined as a (weighted) sum of (ramified) covers f satisfying (0-1) with sign determined by the parity $p(f)$:

$$H^{h,p}_{m^1,\ldots,m^k} = \sum_f \frac{(-1)^{p(f)}}{|\mathrm{Aut}(f)|} \tag{0-4}$$

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Eskin, Okounkov, and Pandharipande [Eskin et al. 2008] calculated the genus $h = 1$ and odd parity spin Hurwitz numbers in terms of characters of the Sergeev group. Gunningham [2012] calculated all spin Hurwitz numbers in terms of combinatorics of the Sergeev algebra.

The trivial partition (1^d) of d is a partition whose components are all 1. If $m^k = (1^d)$, f has no ramification points over the fixed point q^k and hence we have

$$H^{h,p}_{m^1,\dots,m^{k-1},(1^d)} = H^{h,p}_{m^1,\dots,m^{k-1}}. \tag{0-5}$$

When all partitions $m^i = (1^d)$, denote the spin Hurwitz numbers (0-4) by $H^{h,p}_d$. These are dimension-zero local GW invariants $GT^{\mathrm{loc},h,p}_d$ of spin curve (D, N) that give all dimension-zero GW invariants of Kähler surfaces with a smooth canonical divisor; see [Kiem and Li 2007; 2011; Lee and Parker 2007; Maulik and Pandharipande 2008]. For notational simplicity, we set $H^{h,p}_{(3)^0} = H^{h,p}_3$ and for $k \geq 1$ write

$$H^{h,p}_{(3)^k}$$

for the spin Hurwitz numbers $H^{h,p}_{(3),\dots,(3)}$ with the same k partitions (3). Since there are two odd partitions (1^3) and (3) of $d = 3$, by (0-5) it suffices to compute $H^{h,p}_{(3)^k}$ for $k \geq 0$. The aim of this paper is to use a spin curve degeneration to obtain the following recursion formula.

Theorem 0.1. *If $h = h_1 + h_2$ and $p \equiv p_1 + p_2 \pmod 2$, then, for $k_1 + k_2 = k$,*

$$H^{h,p}_{(3)^k} = 3!H^{h_1,p_1}_{(3)^{k_1}} \cdot H^{h_2,p_2}_{(3)^{k_2}} + 3H^{h_1,p_1}_{(3)^{k_1+1}} \cdot H^{h_2,p_2}_{(3)^{k_2+1}}. \tag{0-6}$$

One can use Theorem 0.1 and the result of [Eskin et al. 2008] to explicitly compute the spin Hurwitz numbers of degree $d = 3$. In Proposition 7.1, we show that

$$H^{h,\pm}_{(3)^k} = 3^{2h-2}[(-1)^k 2^{k+h-1} \pm 1], \tag{0-7}$$

where $+$ and $-$ denote the even and odd parities. When the degree d is 1 or 2, the dimension-zero local GW invariants are given by the formulas

$$GT^{\mathrm{loc},h,\pm}_1 = \pm 1 \quad \text{and} \quad GT^{\mathrm{loc},h,\pm}_2 = \pm 2^{h-1};$$

see Lemma 2.6 of [Lee 2013]. Since $GT^{\mathrm{loc},h,p}_d = H^{h,p}_d$ as mentioned above, formula (0-7) shows

$$GT^{\mathrm{loc},h,\pm}_3 = 3^{2h-2}(2^{h-1} \pm 1).$$

This calculation is, in fact, the main motivation for the paper.

In Section 1, we express the degree-d spin Hurwitz numbers (0-4) in terms of relative GW moduli spaces. We can then apply a degeneration method for a family of curves $\mathfrak{D} \to \Delta$ where the central fiber D_0 is a nodal curve and the general fiber

D_λ ($\lambda \neq 0$) is a smooth curve. Section 2 describes the relative moduli space $\mathcal{M}_0$ of maps f into the nodal curve D_0. In Section 3, we show that the union over $\lambda \in \Delta$ of relative moduli spaces $\mathcal{M}_\lambda$ of maps into D_λ consists of connected components $\mathfrak{X}_{m,f} \to \Delta$ containing $f \in \mathcal{M}_0$. Here m is the ramification data of f over nodes of D_0 such that $d - \ell(m)$ is even.

The (ordinary) Hurwitz numbers are sums of (ramified) maps modulo automorphism without sign. One can easily obtain a recursion formula for Hurwitz numbers by counting maps in the general fiber of $\mathfrak{X}_{m,f} \to \Delta$. For spin Hurwitz numbers, one needs to calculate parities of maps induced from a fixed spin structure on the family of curves $\mathfrak{D}$.

The novelty of our approach is to apply a Schiffer variation for the parity calculation. The space $\mathfrak{X}_{m,f}$ is, in general, not smooth. In Section 4, we construct a smooth model for $\mathfrak{X}_{m,f}$ by Schiffer variation. In Section 5, we use the smooth model to twist the pullback of the spin structure on $\mathfrak{D}$. When the degree d equals 3, the partition m is odd, either (1^3) or (3). In this case, a suitable twisting immediately yields a required parity calculation. We prove Theorem 0.1 in Section 6 and formula (0-7) in Section 7.

For higher degree $d \geq 4$, the partition m may not be odd! A new parity calculation is needed. In [Lee and Parker 2012], we generalized the recursion formula (0-6) for higher-degree spin Hurwitz numbers by employing additional geometric analysis arguments for parity calculations.

1. Dimension zero relative GW moduli spaces

In this section, we express the spin Hurwitz numbers (0-4) in terms of dimension-zero relative GW moduli spaces. We follow the definitions of [Ionel and Parker 2003] for the relative GW theory.

Let D be a smooth curve of genus h and let $V = \{q^1, \dots, q^k\}$ be a fixed set of points on D. Given partitions $m^1, \dots, m^k$ of d, a degree-d holomorphic map $f : C \to D$ from a possibly disconnected curve C is called *V-regular* with contact vectors $m^1, \dots, m^k$ if $f^{-1}(V)$ consists of $\sum \ell(m^i)$ contact marked points x^i_j ($1 \leq j \leq \ell(m^i)$) with $f(x^i_j) = q^i$ such that f has ramification index (or multiplicity) m^i_j at x^i_j. Two V-regular maps $(f, C; \{x^i_j\})$ and $(\tilde{f}, \widetilde{C}; \{\tilde{x}^i_j\})$ are equivalent if they are isomorphic, that is, there is a biholomorphism $\sigma : C \to \widetilde{C}$ with $\tilde{f} \circ \sigma = f$ and $\sigma(x^i_j) = \tilde{x}^i_j$ for all i, j. The relative moduli space

$$\mathcal{M}^V_{\chi, m^1, \dots, m^k}(D, d) \tag{1-1}$$

consists of equivalence classes of V-regular maps $(f, C; \{x^i_j\})$ with the Euler characteristic $\chi(C) = \chi$ and with contact vectors $m^1, \dots, m^k$. Since no confusion can arise, we regard a point in the space (1-1) as a V-regular map $(f, C; \{x^i_j\})$. For simplicity, we often write a V-regular map $(f, C; \{x^i_j\})$ simply as f.

The (formal) complex dimension of the space (1-1) is given by the left side of the Riemann–Hurwitz formula (0-1):

$$2d(1-h) - \chi - \sum_{i=1}^{k}(d - \ell(m^i)). \tag{1-2}$$

Suppose this dimension is zero. Then, for each V-regular map $(f, C; \{x^i_j\})$ in (1-1), forgetting the contact marked points x^i_j gives a (ramified) cover f that is ramified only over fixed points q^i and satisfies (0-1). The automorphism group $\mathrm{Aut}(f)$ of a (ramified) cover f consists of automorphisms $\sigma \in \mathrm{Aut}(C)$ with $f \circ \sigma = f$. The automorphism group $\mathrm{Aut}(f, V)$ of a V-regular map $(f, C; \{x^i_j\})$ consists of automorphisms $\sigma \in \mathrm{Aut}(f)$ with $\sigma(x^i_j) = x^i_j$ for all i, j.

For a partition m of d, let $\mathrm{Aut}(m)$ be the subgroup of symmetric group $S_{\ell(m)}$ permuting equal parts of the partition m.

Lemma 1.1. *Let $m^1, \dots, m^k$ be as above and suppose the dimension (1-2) is zero.*

(a) *If $m^i = (1^d)$ for some $1 \le i \le k$, $\mathrm{Aut}(f, V)$ is trivial for all f in (1-1).*

(b) *If $m^1, \dots, m^k$ are all odd partitions,*

$$H^{h,p}_{m^1,\dots,m^k} = \frac{1}{\prod_{i=1}^k |\mathrm{Aut}(m^i)|} \sum \frac{(-1)^{p(f)}}{|\mathrm{Aut}(f, V)|}$$

where the sum is over all f in (1-1) and $p(f)$ is the parity (0-3).

Proof. Let $(f, C; \{x^i_j\})$ be a V-regular map in (1-1) and $\sigma \in \mathrm{Aut}(f, V)$. If $m^i = (1^d)$, the set of branch points B of f is a subset of $V \setminus \{q^i\}$ and the restriction of σ to $C \setminus f^{-1}(B)$ is a covering transformation that fixes contact marked points $x^i_1, \dots, x^i_d$. Noting $f^{-1}(B)$ is finite, we conclude that σ is an identity map on C. This proves (a).

As mentioned above, forgetting contact marked points x^i_j gives a (ramified) cover f satisfying (0-1). Conversely, given a (ramified) cover f satisfying (0-1), one can mark a point over q^i with ramification index m^i_j as a contact marked point x^i_j. Such marking gives V-regular maps $(f, C; \{x^i_j\})$ in $\prod_{i=1}^k |\mathrm{Aut}(m^i)|$ ways. Observe that $(f, C; \{x^i_j\})$ and $(f, C; \{\sigma(x^i_j)\})$ are isomorphic for each $\sigma \in \mathrm{Aut}(f)$ and that $\mathrm{Aut}(f, V)$ is a normal subgroup of $\mathrm{Aut}(f)$. Consequently, the quotient group $G = \mathrm{Aut}(f)/\,\mathrm{Aut}(f, V)$ acts freely on the set of V-regular maps $(f, C; \{x^i_j\})$ obtained by the (ramified) cover f. Its orbits give $\prod_{i=1}^k |\mathrm{Aut}(m^i)|/|G|$ points (that is, equivalence classes of V-regular maps) in the space (1-1), each of which has the same automorphism group $\mathrm{Aut}(f, V)$. Now (b) follows from counting maps with the parity of map modulo automorphisms. □

2. Maps into a nodal curve

Let $D_0 = D_1 \cup E \cup D_2$ be a connected nodal curve of (arithmetic) genus h with two nodes p^1 and p^2 such that, for $i = 1, 2$, $E = \mathbb{P}^1$ meets D_i at node p^i and D_i has genus h_i with $h_1 + h_2 = h$. In this section, we consider maps into D_0 that are relevant to our subsequent discussion.

Below, we fix d, h, χ, and odd partitions $m^1, \dots, m^k$ of d so that the Riemann–Hurwitz formula (0-1) holds, or equivalently, the dimension formula (1-2) is zero. For each partition m of d, consider the product space

$$\mathcal{P}_m = \mathcal{M}^{V_1}_{\chi_1,(1^d),m^1,\dots,m^{k_1},m}(D_1, d) \times \mathcal{M}^{V_0}_{\chi_0,m,(1^d),m}(E, d) \times \mathcal{M}^{V_2}_{\chi_2,m,m^{k_1+1},\dots,m^k,(1^d)}(D_2, d)$$

where

$$V_1 = \{q^{k+1}, q^1, \dots, q^{k_1}, p^1\}, \quad V_0 = \{p^1, q^{k+2}, p^2\}, \quad V_2 = \{p^2, q^{k_1+1}, \dots, q^k, q^{k+3}\}$$

and

$$\chi_1 + \chi_0 + \chi_2 - 4\ell(m) = \chi. \tag{2-1}$$

For simplicity, let $\mathcal{M}^1_m$, $\mathcal{M}^0_m$, and $\mathcal{M}^2_m$ denote the first, second, and third factors of $\mathcal{P}_m$.

Lemma 2.1. *If $\mathcal{P}_m \neq \varnothing$, the spaces $\mathcal{M}^1_m$, $\mathcal{M}^0_m$, and $\mathcal{M}^2_m$ have dimension zero. Consequently, $\chi_0 = 2\ell(m)$ and $d - \ell(m)$ is even.*

Proof. Each $\mathcal{M}^i_m$ $(0 \le i \le 2)$ has nonnegative dimension by the Riemann–Hurwitz formula. The formula (2-1) and our assumption that the dimension (1-2) is zero thus imply that each $\mathcal{M}^i_m$ has dimension zero. The dimension formulas for $\mathcal{M}^0_m$ and $\mathcal{M}^i_m$ $(i = 1, 2)$ then show that $\chi_0 = 2\ell(m)$ and $d - \ell(m)$ is even because $d - \ell(m^i) = \sum(m^i_j - 1)$ is even for all $1 \le i \le k$. ☐

Let $|A|$ denote the cardinality of a set A.

Lemma 2.2.
$$|\mathcal{M}^0_m| = \frac{d!\,|\mathrm{Aut}(m)|}{\prod m_j}.$$

Proof. Let $f \in \mathcal{M}^0_m$. Since $\chi_0 = 2\ell(m)$, the domain of f is a disjoint union of smooth rational curves E_j for $1 \le j \le \ell(m)$, and each restriction $f_j = f|_{E_j}$ has exactly one contact marked point over p^i $(i = 1, 2)$ with multiplicity m_j, so f_j has degree m_j.

Consequently, forgetting contact marked points of maps in $\mathcal{M}^0_m$ gives exactly one map (as a cover) with automorphism group of order $|\mathrm{Aut}(m)| \prod m_j$. Here the factor $|\mathrm{Aut}(m)|$ appears because we can relabel maps f_j in $|\mathrm{Aut}(m)|$ ways and the factor $\prod m_j$ appears because each restriction map f_j (as a cover) has an automorphism group of order m_j. We then argue as in the proof of Lemma 1.1. ☐

For each $(f_1, f_0, f_2) \in \mathcal{P}_m$, by identifying contact marked points over $p^i \in D^i \cap E$ $(i = 1, 2)$, one can glue the domains of f_i and f_0 to obtain a map $f : C \to D_0$ with $\chi(C) = \chi$. For notational convenience, we often write the glued map f as $f = (f_1, f_0, f_2)$. Denote by

$$\mathcal{M}_{m,0} \tag{2-2}$$

the space of such glued maps $f = (f_1, f_0, f_2)$. Contact marked points are labeled, but nodal points of C are not labeled. Thus, we have the following.

Lemma 2.3. *$\mathcal{P}_m$ is a cover of $\mathcal{M}_{m,0}$ of degree* $|\mathrm{Aut}(m)|^2$.

3. Limiting and gluing

Following [Ionel and Parker 2004], this section describes limiting and gluing arguments under a degeneration of target curves. Let $D_0 = D_1 \cup E \cup D_2$ be the nodal curve with fixed points $q^1, \ldots, q^{k+3}$ as in Section 2. In Section 4, we construct a family of curves together with $k+3$ sections:

$$\begin{array}{c} \mathcal{D} \\ Q^i \Big(\Big\downarrow \rho \\ \Delta \end{array} \tag{3-1}$$

Here the total space $\mathcal{D}$ is a smooth complex surface, $\Delta \subset \mathbb{C}$ is a disk with parameter λ, the central fiber is D_0, the general fiber D_λ $(\lambda \neq 0)$ is a smooth curve of genus h, and $Q^i(0) = q^i$ for $1 \le i \le k+3$. By Gromov's convergence theorem, a sequence of holomorphic maps into D_λ with $\lambda \to 0$ has a map into D_0 as a limit. For notational simplicity, for $\lambda \neq 0$ we set

$$\mathcal{M}_\lambda = \mathcal{M}^{V_\lambda}_{\chi, m^1, \ldots, m^{k+3}}(D_\lambda, d), \quad \text{where } V_\lambda = \{Q^1(\lambda), \ldots, Q^{k+3}(\lambda)\}, \tag{3-2}$$

and denote the set of limits of sequences of maps in $\mathcal{M}_\lambda$ as $\lambda \to 0$ by

$$\lim_{\lambda \to 0} \mathcal{M}_\lambda. \tag{3-3}$$

Lemma 3.1 shows that limit maps in (3-3) lie in the union of spaces (2-2), namely,

$$\lim_{\lambda \to 0} \mathcal{M}_\lambda \subset \bigcup_m \mathcal{M}_{m,0} \tag{3-4}$$

where the union is over all partitions m of d with $d - \ell(m)$ even.

Conversely, by the gluing theorem of [Ionel and Parker 2004], the domain of each map in $\mathcal{M}_{m,0}$ can be smoothed to produce maps in $\mathcal{M}_\lambda$ for small $|\lambda|$. Shrinking Δ if necessary, for $\lambda \in \Delta$, one can assign to each $f_\lambda \in \mathcal{M}_\lambda$ a partition m of d by (3-4). Let $\mathcal{M}_{m,\lambda}$ be the set of all pairs (f_λ, m). For each $f \in \mathcal{M}_{m,0}$, let

$$\mathcal{X}_{m,f} \to \Delta \tag{3-5}$$

be the connected component of $\bigcup_{\lambda\in\Delta}\mathcal{M}_{m,\lambda}\to\Delta$ that contains f, and let

$$\mathfrak{X}_{m,f,\lambda} \tag{3-6}$$

denote the fiber of (3-5) over $\lambda\in\Delta$. It follows that, for $\lambda\neq 0$,

$$\mathcal{M}_\lambda=\bigsqcup_{f\in\mathcal{M}_{m,0}}\mathfrak{X}_{m,f,\lambda}. \tag{3-7}$$

For $f=(f_1,f_0,f_2)\in\mathcal{M}_{m,0}$ where $m=(m_1,\dots,m_\ell)$, let y^i_j be the node mapped to p^i at which f_i and f_0 have multiplicity m_j. The gluing theorem shows that one can smooth each node y^i_j in m_j ways to produce $\bigl(\prod m_j\bigr)^2$ maps in $\mathfrak{X}_{m,f,\lambda}$, so

$$|\mathfrak{X}_{m,f,\lambda}|=\bigl(\textstyle\prod m_j\bigr)^2\quad(\lambda\neq 0). \tag{3-8}$$

In order to prove (3-4), we use the following fact on stable maps. An irreducible component of a stable holomorphic map f is a ghost component if its image is a point. Write the domain of f as $C^g\cup C$ where C^g is a connected curve whose irreducible components are all ghost components. Then the stability of f implies that

$$\chi(C^g)-\ell^g-n\leq -1 \tag{3-9}$$

where $\ell^g=|C^g\cap C|$ and n is the number of marked points on C^g.

Lemma 3.1. *Let $\mathcal{M}_r$ and $\mathcal{M}_{m,0}$ be as above. Then we have*

$$\lim_{\lambda\to 0}\mathcal{M}_\lambda\subset\bigcup_m\mathcal{M}_{m,0}$$

where the union is over all partitions m of d with $d-\ell(m)$ even.

Proof. Let f be a limit map in (3-3). The domain C of f can be written as

$$C=C_1\cup C_0\cup C_2\cup\Bigl(\bigcup_{i=1}^{k+3}C^g_i\Bigr)\cup C^g\cup\widetilde{C}^g \tag{3-10}$$

where C_0 maps to E, C_1 and C_2 map to D_1 and D_2, C^g_i is the union of all ghost components over q^i, where $i=1,\dots,k+3$, C^g is the union of all ghost components over points in $D_0\setminus(V_1\cup V_0\cup V_2)$, and $\widetilde{C}^g$ is the union of all ghost components over $\{p^1,p^2\}$. Let $f_j=f|_{C_j}$ for $j=0,1,2$. Observe that f_j is V_j-regular because C_j has no ghost components. Let $\widehat{m}^i$ be a contact vector over q^i, $\widetilde{m}^1$ and $\widetilde{m}^2$ be contact vectors of f_1 and f_2 over p^1 and p^2, and $\widetilde{m}^{0;1}$ and $\widetilde{m}^{0;2}$ be contact vectors of f_0 over p^1 and p^2. The Riemann–Hurwitz formulas for f_0, f_1, and f_2 give

$$\sum_{j=0}^{2}\chi(C_j)\leq 2d(1-h)+\sum_{i=1}^{k+3}(\ell(\widehat{m}^i)-d)+\sum_{i=1}^{2}(\ell(\widetilde{m}^i)+\ell(\widetilde{m}^{0;i})). \tag{3-11}$$

For $i = 1, \dots, k+3$, let $\ell_i = |C_1 \cup C_0 \cup C_2 \cap C_i^g|$ and let n_i be the number of marked points on C_i^g. Since all marked points are limits of marked points, we have

$$\ell(\widehat{m}^i) = \ell(m^i) - n_i + \ell_i. \tag{3-12}$$

For $j = 0, 1, 2$, let $\tilde{\ell}_j = |C_j \cap \widetilde{C}^g|$. Counting the number of nodes mapped to p^1 and p^2 shows

$$\sum_{i=1}^{2} (\ell(\widetilde{m}^i) - \tilde{\ell}_i) = \sum_{i=1}^{2} |C_i \cap C_0| = \sum_{i=1}^{2} \ell(\widetilde{m}^{0;i}) - \tilde{\ell}_0. \tag{3-13}$$

Let $\ell^g = |C_1 \cup C_0 \cup C_2 \cap C^g|$. Since $\chi(C) = \chi$, by (3-10) and (3-13) we have

$$\begin{aligned}\chi = \sum_{j=0}^{2} \chi(C_j) + \sum_{i=1}^{k+3} (\chi(C_i^g) - 2\ell_i) + \chi(C^g) - 2\ell^g \\ + \chi(\widetilde{C}^g) - \tilde{\ell} - \sum_{i=1}^{2} (\ell(\widetilde{m}^i) + \ell(\widetilde{m}^{0;i})),\end{aligned} \tag{3-14}$$

where $\tilde{\ell} = \tilde{\ell}_0 + \tilde{\ell}_1 + \tilde{\ell}_2$. By our assumption that formula (0-1) holds, it follows from (3-11), (3-12), and (3-14) that

$$\chi \leq \chi + \sum_{i=1}^{k+3} (\chi(C_i^g) - \ell_i - n_i) + \chi(C^g) - 2\ell^g + \chi(\widetilde{C}^g) - \tilde{\ell}. \tag{3-15}$$

Noting that C^g and $\widetilde{C}^g$ have no marked points, by (3-9) and (3-15), we conclude that the domain C of f has no ghost components. Consequently,

- f_j is V_j-regular for $j = 0, 1, 2$,
- $\widetilde{m}^i = \widetilde{m}^{0;i}$ for $i = 1, 2$ (see Lemma 3.3 of [Ionel and Parker 2004]) and $\widehat{m}^i = m^i$ for $i = 1, \dots, k+3$.

In particular, the equality in (3-11) holds; otherwise we have a strict inequality in (3-15). So, we have $\chi(C_0) = \ell(\widetilde{m}^1) + \ell(\widetilde{m}^2)$. But $\chi(C_0) \leq 2\min\{\ell(\widetilde{m}^1), \ell(\widetilde{m}^2)\}$. It follows that

- C_0 has $\ell(\widetilde{m}^1) = \ell(\widetilde{m}^2)$ connected components E_j with $\chi(E_j) = 2$ for all j,
- $\widetilde{m}^1_j = \deg(f_0|_{E_j}) = \widetilde{m}^2_j$ for all j, that is, $\widetilde{m}^1 = \widetilde{m}^2$.

It follows that the Euler characteristics of C_0, C_1, and C_2 satisfy (2-1) by (3-14). Therefore, $f \in \mathcal{M}_{m,0}$ for $m = \widetilde{m}^1 = \widetilde{m}^2$ and $d - \ell(m)$ is even by Lemma 2.1. □

4. Smooth model by Schiffer variation

A *Schiffer variation* of a nodal curve (compare [Arbarello et al. 2011, p. 184]) is obtained by gluing deformations $uv = \lambda$ near nodes with the trivial deformation away from nodes. In this section, we use the method of Schiffer variation to construct a smooth model for the space $\mathfrak{X}_{m,f}$ in (3-5), which has several branches intersecting at f unless m is trivial.

In this section, we fix an odd partition $m = (n^\ell)$, that is, $m = (m_1, \dots, m_\ell)$ with

$$m_1 = \cdots = m_\ell = n, \quad \text{where } n = d/\ell \text{ is odd.} \tag{4-1}$$

Let $f = (f_1, f_0, f_2)$ be a map in $\mathcal{M}_{m,0}$ in (2-2). As described in Section 2, the central fiber of $\rho : \mathfrak{D} \to \Delta$ is the nodal curve $D_0 = D_1 \cup E \cup D_2$ with two nodes $p^1 \in D_1 \cap E$ and $p^2 \in D_2 \cap E$ where $E = \mathbb{P}^1$. The domain of f is a nodal curve

$$C = C_1 \cup C_0 \cup C_2, \quad \text{where } C_0 = \bigcup_{j=1}^{\ell} E_\ell,$$

with 2ℓ nodes, such that, for $i = 1, 2$ and $j = 1, \dots, \ell$,

- $f^{-1}(p^i)$ consists of the ℓ nodes $y^i_j \in C_i \cap E_j$,
- C_i is smooth and $f|_{C_i} = f_i$ has ramification index $m_j = n$ at the node y^i_j,
- $E_j = \mathbb{P}^1$ and $f|_{E_j} = f_0|_{E_j} : E_j \to E$ has ramification index $m_j = n$ at the node y^i_j.

The following is the main result of this section.

Proposition 4.1. *Let f be as above. Then, for each vector $\zeta = (\zeta^1_1, \zeta^2_1, \dots, \zeta^1_\ell, \zeta^2_\ell)$, where ζ^i_j is an n^{th} root of unity, there are a family of curves $\varphi_\zeta : \mathscr{C}_\zeta \to \Delta$, with smooth total space $\mathscr{C}_\zeta$, over a disk Δ (with parameter s) and a holomorphic map $\mathscr{F}_\zeta : \mathscr{C}_\zeta \to \mathfrak{D}$ satisfying:*

(a) *the central fiber $C_{\zeta,0} = C$ and the restriction map $\mathscr{F}_\zeta|_C = f$;*

(b) *the general fiber $C_{\zeta,s}$ $(s \neq 0)$ is smooth and, for $\lambda = s^n \neq 0$,*

$$\bigcup_\zeta \{f_{\zeta,s}\} = \mathfrak{X}_{m,f,\lambda}, \tag{4-2}$$

where the union is over all ζ, $f_{\zeta,s} = \mathscr{F}_\zeta|_{C_{\zeta,s}}$ and $\mathfrak{X}_{m,f,\lambda}$ is the space (3-6).

Proof. The proof consists of four steps.

Step 1. We first show how to construct the family of curves $\rho : \mathfrak{D} \to \Delta$ with $k+3$ sections. For $i = 1, 2$, a neighborhood of the node $p^i \in D_i \cap E$ can be regarded as the union $U^i \cup V^i$ of the two disks

$$U^i = \{u^i \in \mathbb{C} : |u^i| < 1\} \subset D_i \quad \text{and} \quad V^i = \{v^i \in \mathbb{C} : |v^i| < 1\} \subset E$$

with their origins identified. We may assume that the fixed points $q^1, \dots, q^{k+3}$ in D_0 described in (2-1) lie outside these sets. Consider the regions

$$A^i = \left\{(u^i, v^i, \lambda) \in U^i \times V^i \times \Delta : u^i v^i = \lambda\right\},$$
$$B = \bigcup_{i=1}^{2} G^i \cup \left[\left(D_0 \setminus \bigcup_{i=1}^{2}(U^i \cup V^i)\right) \times \Delta\right],$$

where

$$G^i = \left\{(u^i, \lambda) \in U^i \times \Delta : |u^i| > \sqrt{|\lambda|}\right\} \cup \left\{(v^i, \lambda) \in V^i \times \Delta : |v^i| > \sqrt{|\lambda|}\right\}.$$

We obtain a smooth complex surface $\mathscr{D}$ by gluing A^1, A^2, and B_0 using the maps

$$G^i \to A^i \quad \text{defined by } (u^i, \lambda) \to \left(u^i, \frac{\lambda}{u^i}, \lambda\right) \text{ and } (v^i, \lambda) \to \left(\frac{\lambda}{v^i}, v^i, \lambda\right). \tag{4-3}$$

Let $\rho : \mathscr{D} \to \Delta$ be the projection to the last factor and define $k+3$ sections Q^i of ρ by

$$Q^i(\lambda) = (q^i, \lambda).$$

Step 2. We can similarly construct a family of curves over a 2ℓ-dimensional polydisk:

$$\varphi_{2\ell} : \mathscr{X} \to \Delta_{2\ell} = \{t = (t_1^1, t_1^2, \ldots, t_\ell^1, t_\ell^2) \in \mathbb{C}^{2\ell} : |t_j^i| < 1\}. \tag{4-4}$$

For each node $y_j^i \in C_i \cap E_j$, choose a neighborhood obtained from two disks

$$U_j^i = \{u_j^i \in \mathbb{C} : |u_j^i| < 1\} \subset C_i \quad \text{and} \quad V_j^i = \{v_j^i \in \mathbb{C} : |v_j^i| < 1\} \subset E_j$$

by identifying the origins. Consider the regions

$$A_j^i = \left\{(u_j^i, u_j^i, t) \in U_j^i \times V_j^i \times \Delta_{2\ell} : u_j^i v_j^i = t_j^i\right\},$$
$$B_{2\ell} = \bigcup_{i,j} G_j^i \cup \left[\left(C \setminus \bigcup_{i,j}(U_j^i \cup V_j^i)\right) \times \Delta_{2\ell}\right],$$

where

$$G_j^i = \left\{(u_j^i, t) \in U_j^i \times \Delta_{2\ell} : |u_j^i| > \sqrt{|t_j^i|}\right\} \cup \left\{(v_j^i, t) \in V_j^i \times \Delta_{2\ell} : |v_j^i| > \sqrt{|t_j^i|}\right\}.$$

We can then obtain a smooth complex manifold $\mathscr{X}$ of dimension $2\ell+1$ by gluing $\bigcup A_j^i$ and $B_{2\ell}$ with the maps

$$G_j^i \to A_j^i \quad \text{defined by } (u_j^i, t) \to \left(u_j^i, \frac{t_j^i}{u_j^i}, t\right) \text{ and } (v_j^i, t) \to \left(\frac{t_j^i}{v_j^i}, v_j^i, t\right). \tag{4-5}$$

Let $\varphi_{2\ell} : \mathscr{X} \to \Delta$ be the projection to the factor t.

Step 3. Since f_i and $f_0|_{E_j}$ have ramification index $m_j = n$ at y_j^i, we may assume (after coordinates change) that on U_j^i and V_j^i the map f can be written as

$$U_j^i \to U^i \text{ by } u_j^i \to (u_j^i)^n \quad \text{and} \quad V_j^i \to V^i \text{ by } v_j^i \to (v_j^i)^n. \tag{4-6}$$

For each i, j, define a map

$$G^i_j \to G^i \quad \text{by } (u^i_j, t) \to ((u^i_j)^n, (t^i_j)^n) \text{ and } (u^i_j, t) \to ((v^i_j)^n, (t^i_j)^n). \tag{4-7}$$

On the other hand, for each i, j, we have a map

$$A^i_j \to A^i \quad \text{defined by } (u^i_j, v^i_j, t) \to ((u^i_j)^n, (v^i_j)^n, (t^i_j)^n). \tag{4-8}$$

These two maps (4-7) and (4-8) are glued together under the maps (4-3) and (4-5). The glued map extends to a holomorphic map $f_t : \mathscr{X}_t \to D_\lambda$ if and only if

$$(t^1_1)^n = (t^2_1)^n = \cdots = (t^1_\ell)^n = (t^2_\ell)^n = \lambda. \tag{4-9}$$

There are $n^{2\ell}$ solutions t of (4-9) and the extension map f_t is given by

$$(x, t) \to (f(x), \lambda) \quad \text{on } \mathscr{X}_t - \bigcup A^i_j.$$

Step 4. For each vector $\zeta = (\zeta^1_1, \zeta^2_1, \dots, \zeta^1_\ell, \zeta^2_\ell)$, where each ζ^i_j is an n^{th} root of unity, define

$$\delta_\zeta : \Delta \to \Delta_{2\ell} \text{ by } s \to (\zeta^1_1 s, \zeta^2_1 s, \zeta^1_2 s, \zeta^2_2 s, \dots, \zeta^1_\ell s, \zeta^2_\ell s).$$

The pullback $\delta^*_\zeta \mathscr{X}$ gives a family of curves:

$$\begin{array}{ccc} \mathscr{C}_\zeta = \delta^*_\zeta \mathscr{X} & \longrightarrow & \mathscr{X} \\ \downarrow \varphi_\zeta & & \downarrow \varphi_{2\ell} \\ \Delta & \xrightarrow{\delta_\zeta} & \Delta_{2\ell} \end{array} \tag{4-10}$$

The central fiber is $C_{\zeta,0} = C$ and the general fiber $C_{\zeta,s}$ $(s \neq 0)$ is smooth. A neighborhood of the node y^i_j of C in $\mathscr{C}_\zeta$ can be viewed as

$$\hat{A}^i_j = \{(u^i_j, v^i_j, s) \in \mathbb{C}^3 : |u^i_j| < 1, |v^i_j| < 1, u^i_j v^i_j = \zeta^i_j s\}. \tag{4-11}$$

It follows that the total space $\mathscr{C}_\zeta$ is a complex smooth surface. Noting $\delta_\zeta(s)$ is a solution of (4-9) for $\lambda = s^n$, we obtain a holomorphic map $\mathscr{F}_\zeta : \mathscr{C}_\zeta \to \mathscr{D}$ given by

$$\begin{aligned} (u^i_j, v^i_j, s) &\to ((u^i_j)^n, (v^i_j)^n, s^n) && \text{on } \hat{A}^i_j, \\ (x, s) &\to (f(x), s^n) && \text{on } \mathscr{C}_\zeta - \bigcup \hat{A}^i_j. \end{aligned} \tag{4-12}$$

Since the restriction $\mathscr{F}_\zeta|_C = f$ by (4-6) and (4-12), it remains to show (4-2). By our choice of fixed points q^i on D_0, each contact marked point x^i_j of f lies in $\mathscr{C}_\zeta - \bigcup \hat{A}^i_j$. Thus, by (4-12), the pullback $\mathscr{F}^*_\zeta Q^i$ of the section Q^i of ρ gives a section X^i_j of φ_ζ given by $X^i_j(s) = (x^i_j, s)$. After marking the points $X^i_j(s)$ in $C_{\zeta,s}$, the restriction map

$$f_{\zeta,s} = \mathscr{F}_\zeta|_{C_{\zeta,s}} : C_{\zeta,s} \to D_\lambda, \quad \text{where } \lambda = s^n \neq 0,$$

has contact marked points $X^i_j(s)$ over $Q^i(\lambda)$ with multiplicity m^i_j. This means that $f_{\zeta,s}$ lies in the space $\mathcal{M}_\lambda$ in (3-2) for $\lambda = s^n$. Therefore, noting that $f_{\zeta,s} \to f$ as $s \to 0$ and that $|\mathfrak{X}_{m,f,\lambda}| = n^{2\ell}$ by (3-8), we conclude (4-2). □

5. Spin structure and parity

The aim of this section is to use a spin structure on a family of nodal curves [Cornalba 1989] to show the parity calculation in Proposition 5.4. Twisting a bundle as in (5-6) is a key idea for parity calculation.

We first introduce a spin structure on a family of nodal curves that is relevant to our discussion. We refer to [Cornalba 1989] for the definition of spin structure and more details. The relative dualizing sheaf ω_ρ of the family of curves $\rho : \mathfrak{D} \to \Delta$ in (3-1) is the canonical bundle $K_{\mathfrak{D}}$ on the total space $\mathfrak{D}$, since $\mathfrak{D}$ is smooth and K_Δ is trivial. For each $\lambda \neq 0$, the restriction $K_{\mathfrak{D}}|_{D_\lambda}$ is the canonical bundle K_{D_λ} on D_λ, and the restriction $K_{\mathfrak{D}}|_{D_0}$ is the dualizing sheaf ω_{D_0} of the nodal curve $D_0 = D_1 \cup E \cup D_2$. As described in Section 4, D_0 is locally given by $u^i v^i = 0$ near each node p^i in $D_i \cap E$ for $i = 1, 2$. Then the local generators of ω_{D_0} are du^i/u^i and dv^i/v^i with a relation $du^i/u^i + dv^i/v^i = 0$; see [Harris and Morrison 1998, p. 82]. This implies the restriction $\omega_{D_0}|_{D_i} = K_{D_i} \otimes \mathcal{O}(p^i)$. On the other hand, $1/u^i$ is a local defining function for the divisor $-E$ on $\mathfrak{D}$ near p^i. By restricting $1/u^i$ to D_i, one can see that $\mathcal{O}(-E)|_{D_i} = \mathcal{O}(-p^i)$. Consequently, for $i = 1, 2$,

$$K_{\mathfrak{D}}|_{D_i} \otimes \mathcal{O}(-E)|_{D_i} = \omega_{D_0}|_{D_i} \otimes \mathcal{O}(-p^i) = K_{D_i}. \tag{5-1}$$

From Cornalba's construction [1989, p. 570], there are a line bundle $\mathcal{N} \to \mathfrak{D}$ and a homomorphism $\Phi : \mathcal{N}^2 \to \omega_\rho = K_{\mathfrak{D}}$ satisfying the following.

- Φ vanishes identically on the exceptional component E and $\mathcal{N}|_E = \mathcal{O}_E(1)$.
- Since $\Phi|_E \equiv 0$, there is an induced homomorphism $\hat{\Phi} : \mathcal{N}^2 \to K_{\mathfrak{D}} \otimes \mathcal{O}(-E)$ such that Φ is the composition of $\hat{\Phi}$ with tensoring with η:

$$\Phi : \mathcal{N}^2 \xrightarrow{\hat{\Phi}} K_{\mathfrak{D}} \otimes \mathcal{O}(-E) \xrightarrow{\otimes \eta} K_{\mathfrak{D}}, \tag{5-2}$$

  where η is a section of $\mathcal{O}(E)$ with zero divisor E. Then, for $i = 1, 2$, the restriction
  $$\hat{\Phi}|_{D_i} : (\mathcal{N}|_{D_i})^2 \to K_{\mathfrak{D}}|_{D_i} \otimes \mathcal{O}(-E)|_{D_i} = K_{D_i}$$
  is an isomorphism so that the restriction $N_i = \mathcal{N}|_{D_i}$ is a theta characteristic on D_i.
- For each $\lambda \neq 0$, the restriction $\Phi|_{D_\lambda} : (\mathcal{N}|_{D_\lambda})^2 \to K_{D_\lambda}$ is an isomorphism so that the restriction $N_\lambda = \mathcal{N}|_{D_\lambda}$ is a theta characteristic on D_λ.

The pair $(\mathcal{N}, \Phi)$ is a spin structure on $\rho : \mathfrak{D} \to \Delta$ and the restriction $\mathcal{N}|_{D_0}$ is a theta characteristic on the nodal curve D_0.

Remark 5.1. Atiyah [1971] and Mumford [1971] showed that the parity of a theta characteristic on a smooth curve is a deformation invariant. Cornalba [1989, Page 580] used the homomorphism Φ to extend Mumford's proof to the case of spin structure on a family of nodal curves. Thus, if p_1, p_2, and p are the parities of N_1, N_2, and N_λ ($\lambda \neq 0$), we have

$$p \equiv p_1 + p_2 \pmod 2.$$

Let $\varphi_\zeta : \mathscr{C}_\zeta \to \Delta$ be the family of curves in Proposition 4.1. Recall that the central fiber of φ_ζ is $C = C_1 \cup C_0 \cup C_2$, where $C_0 = \bigsqcup_j E_j$ is a disjoint union of ℓ exceptional components E_j and $C_i \cap E_j = \{y_j^i\}$ for $i = 1, 2$ and $1 \le j \le \ell$. Similarly as for (5-1), by restricting local defining functions, we have

$$\mathbb{O}(\pm C_0)|_{C_i} = \mathbb{O}\Big(\pm \sum_j y_j^i\Big) \quad (i = 1, 2) \quad \text{and} \quad \mathbb{O}(\pm C_0)|_{C_{\zeta,s}} = \mathbb{O} \quad (s \neq 0). \tag{5-3}$$

Since any fiber of φ_ζ is a principal divisor on $\mathscr{C}_\zeta$, $\mathbb{O}(C) = \mathbb{O}$ and hence $\mathbb{O}(C_0) = \mathbb{O}(-C_1 - C_2)$. We also have

$$\mathbb{O}(\pm C_0)|_{E_j} = \mathbb{O}(\mp(C_1 + C_2))|_{E_j} = \mathbb{O}(\mp(y_j^1 + y_j^2)) = \mathbb{O}(\mp 2)(1 \le j \le \ell). \tag{5-4}$$

Let $f = (f_1, f_0, f_2)$ and $\mathscr{F}_\zeta : \mathscr{C}_\zeta \to \mathscr{D}$ be the maps in Proposition 4.1. The ramification divisor $R_{\mathscr{F}_\zeta}$ of $\mathscr{F}_\zeta$ has local defining functions given by the Jacobian of $\mathscr{F}_\zeta$, so (4-12) shows

$$R_{\mathscr{F}_\zeta} = \mathbb{O}(X_\zeta + (n-1)C) = \mathbb{O}(X_\zeta), \tag{5-5}$$

where $X_\zeta = \sum_{i,j} (m_j^i - 1) X_j^i$ and X_j^i is the section of φ_ζ defined in (4-12). Note that

(i) the ramification divisor of $f_i = \mathscr{F}_\zeta|_{C_i}$ ($i = 1, 2$) is $R_{f_i} = X_\zeta|_{C_i} + \sum_j (n-1) y_j^i$;

(ii) the ramification divisor of $f_{\zeta,s} = \mathscr{F}_\zeta|_{C_{\zeta,s}}$ ($s \neq 0$) is $R_{f_{\zeta,s}} = X_\zeta|_{C_{\zeta,s}}$.

Now, noting n is odd, we twist the pullback bundle $\mathscr{F}_\zeta^* \mathscr{N}$ by setting

$$\mathscr{L}_\zeta = \mathscr{F}_\zeta^* \mathscr{N} \otimes \mathbb{O}\Big(\tfrac{1}{2} X_\zeta + \frac{(n-1)}{2} C_0\Big). \tag{5-6}$$

The lemma below shows that the twisted line $\mathscr{L}_\zeta$ restricts to a theta characteristic on each fiber of φ_ζ, including the central fiber C.

Lemma 5.2. *Let $\mathscr{L}_\zeta$ be as above. Then:*

(a) $\mathscr{L}_\zeta|_{E_j} = \mathbb{O}(1)$ *for* $1 \le j \le \ell$.

(b) $\mathscr{L}_\zeta|_{C_1} = L_{f_1}$, $\mathscr{L}_\zeta|_{C_2} = L_{f_2}$ *and* $\mathscr{L}_\zeta|_{C_{\zeta,s}} = L_{f_{\zeta,s}}$ *for* $s \neq 0$, *where* L_{f_1}, L_{f_2}, $L_{f_{\zeta,s}}$ *are the theta characteristics on* C_1, C_2, $C_{\zeta,s}$ *defined by* (0-2).

Proof. Part (a) follows from (5-4) and the fact that each restriction map $\mathcal{F}_\zeta|_{E_j}$ has degree n. Part (b) follows from (5-3), (i), and (ii). □

Observe that the relative dualizing sheaf ω_{φ_ζ} is the canonical bundle $K_{\mathscr{C}_\zeta}$ since $\mathscr{C}_\zeta$ is smooth. The Hurwitz formula and (5-5) thus imply that

$$\omega_{\varphi_\zeta} = K_{\mathscr{C}_\zeta} = \mathcal{F}_\zeta^* K_{\mathfrak{D}} \otimes \mathbb{O}(X_\zeta). \tag{5-7}$$

Define a homomorphism

$$\begin{aligned} \hat{\Psi}_\zeta : \mathscr{L}_\zeta^2 = \mathcal{F}_\zeta^* \mathcal{N}^2 \otimes \mathbb{O}(X_\zeta + (n-1)C_0) \\ \to \mathcal{F}_\zeta^*(K_{\mathfrak{D}} \otimes \mathbb{O}(-E)) \otimes \mathbb{O}(X_\zeta + (n-1)C_0) \end{aligned} \tag{5-8}$$

by $\hat{\Psi}_\zeta = \mathcal{F}_\zeta^* \hat{\Phi} \otimes \mathrm{Id}$, where $\hat{\Phi}$ is the induced homomorphism in (5-2). Noting that $\mathbb{O}(C) = \mathbb{O}$ and $\mathbb{O}(D_0) = \mathbb{O}$, by (4-12), we have

$$\mathcal{F}_\zeta^* \mathbb{O}(-E) = \mathcal{F}_\zeta^* \mathbb{O}(D_1 + D_2) = \mathbb{O}(n(C_1 + C_2)) = \mathbb{O}(-nC_0).$$

Together with (5-7), this implies that the right side of (5-8) is $K_{\mathscr{C}_\zeta} \otimes \mathbb{O}(-C_0)$. Now define a homomorphism $\Psi_\zeta : \mathscr{L}_\zeta^2 \to K_{\mathscr{C}_\zeta}$ to be the composition

$$\Psi_\zeta : \mathscr{L}_\zeta^2 \xrightarrow{\hat{\Psi}_\zeta} K_{\mathscr{C}_\zeta} \otimes \mathbb{O}(-C_0) \xrightarrow{\otimes \xi} K_{\mathscr{C}_\zeta}, \tag{5-9}$$

where ξ is a section of $\mathbb{O}(C_0)$ with zero divisor C_0.

Lemma 5.3. *$(\mathscr{L}_\zeta, \Psi_\zeta)$ is a spin structure on $\varphi_\zeta : \mathscr{C}_\zeta \to \Delta$.*

Proof. First, $\mathscr{L}_\zeta|_E = \mathbb{O}(1)$ by Lemma 5.2(a) and Ψ_ζ vanishes identically on each exceptional component E_j, since $\xi = 0$ on $C_0 = \bigsqcup_j E_j$. Second, since $\hat{\Phi}|_{D_i}$ is an isomorphism, (5-3) and (i) show that, for $i = 1, 2$, the restriction

$$\hat{\Psi}|_{C_i} = f_i^*(\hat{\Phi}|_{D_i}) \otimes \mathrm{Id} : (\mathscr{L}_\zeta|_{C_i})^2 = f_i^* N_i^2 \otimes \mathbb{O}(R_{f_i}) \to f_i^* K_{D_i} \otimes \mathbb{O}(R_{f_i}) = K_{C_i}$$

is an isomorphism. Lastly, let $\lambda = s^n \neq 0$. Since $\Phi|_{D_\lambda}$ is an isomorphism, so is $\hat{\Phi}|_{D_\lambda}$. Thus, by (5-3), (ii), and the facts $K_{\mathfrak{D}}|_{D_\lambda} = K_{D_\lambda}$ and $\mathbb{O}(-E)|_{D_\lambda} = \mathbb{O}$, the restriction

$$\hat{\Psi}_\zeta|_{C_{\zeta,s}} = f_{\zeta,s}^* \hat{\Phi}|_{D_\lambda} \otimes \mathrm{Id} : (\mathscr{L}_\zeta|_{C_{\zeta,s}})^2 = f_{\zeta,s}^* N_\lambda^2 \otimes \mathbb{O}(R_{f_{\zeta,s}}) \to f_{\zeta,s}^* K_{D_\lambda} \otimes \mathbb{O}(R_{f_{\zeta,s}}) = K_{C_{\zeta,s}}$$

is an isomorphism. This implies that the restriction

$$\Psi_\zeta|_{C_{\zeta,s}} : (\mathscr{L}_\zeta|_{C_{\zeta,s}})^2 \to K_{C_\zeta}|_{C_{\zeta,s}} = K_{C_\zeta,s}$$

is also an isomorphism. Therefore, we conclude that $(\mathscr{L}_\zeta, \Psi_\zeta)$ is a spin structure on φ_ζ. □

The following is a key fact for the proof of Theorem 0.1.

Proposition 5.4. *Let $f=(f_1, f_0, f_2)$ and $f_{\zeta,s}$ be maps in Proposition 4.1. Then, for all $s\neq 0$,*

$$p(f_{\zeta,s})\equiv p(f_1)+p(f_2)\pmod 2. \tag{5-10}$$

Proof. Since $(\mathscr{L}_\zeta, \Psi_\zeta)$ is a spin structure on φ_ζ, Cornalba's proof, mentioned in Remark 5.1, shows that, for all $s\neq 0$,

$$h^0(\mathscr{L}_\zeta|_{C_{\zeta,s}})\equiv h^0(\mathscr{L}_\zeta|_{C_1})+h^0(\mathscr{L}_\zeta|_{C_2})\pmod 2.$$

This and Lemma 5.2(b) prove (5-10). □

6. Proof of Theorem 0.1

Proof. Fix a spin structure $(\mathcal{N}, \Phi)$ on $\rho:\mathfrak{D}\to\Delta$ given in Section 5. Consider the space $\mathcal{M}_{m,0}$ in (2-2) where m is a partition of $d=3$. In this case, by Lemma 2.1, either $m=(1^3)$ or $m=(3)$. Note that both of them satisfy (4-1). Fix $\lambda\neq 0$ and let $f=(f_1, f_0, f_2)$ be a map in $\mathcal{M}_{m,0}$. Then (4-2) and (5-10) show that, for all $f_\mu\in\mathfrak{X}_{m,f,\lambda}$,

$$p(f_\mu)\equiv p(f_1)+p(f_2)\pmod 2. \tag{6-1}$$

Lemma 1.1 and (3-7) show that

$$\begin{aligned} H^{h,p}_{(3)^k} &= H^{h,p}_{(3)^k,(1^3)^3} \\ &= \frac{1}{(3!)^3}\Bigg(\sum_{f\in\mathcal{M}_{(1^3),0}}\ \sum_{f_\mu\in\mathfrak{X}_{(1^3),f,\lambda}}(-1)^{p(f_\mu)}+\sum_{f\in\mathcal{M}_{(3),0}}\ \sum_{f_\mu\in\mathfrak{X}_{(3),f,\lambda}}(-1)^{p(f_\mu)}\Bigg). \end{aligned} \tag{6-2}$$

By (3-8) and (6-1), (6-2) becomes

$$H^{h,p}_{(3)^k}=\sum_{f=(f_1,f_0,f_2)\in\mathcal{M}_{(1^3),0}}\frac{(-1)^{p(f_1)+p(f_2)}}{(3!)^3}+\sum_{f=(f_1,f_0,f_2)\in\mathcal{M}_{(3),0}}\frac{3^2(-1)^{p(f_1)+p(f_2)}}{(3!)^3}. \tag{6-3}$$

It then follows from Lemma 2.3 and (6-3) that

$$\begin{aligned} H^{h,p}_{(3)^k} &= \sum_{(f_1,f_0,f_2)\in\mathcal{P}_{(1^3)}}\frac{(-1)^{p(f_1)+p(f_2)}}{(3!)^5}+\sum_{(f_1,f_0,f_2)\in\mathcal{P}_{(3)}}\frac{3^2(-1)^{p(f_1)+p(f_2)}}{(3!)^3} \\ &= \frac{1}{(3!)^3}\sum_{f_1\in\mathcal{M}^1_{(1^3)}}(-1)^{p(f_1)}\sum_{f_2\in\mathcal{M}^2_{(1^3)}}(-1)^{p(f_2)}+\frac{3}{(3!)^2}\sum_{f_1\in\mathcal{M}^1_{(3)}}(-1)^{p(f_1)}\sum_{f_2\in\mathcal{M}^2_{(3)}}(-1)^{p(f_2)} \\ &= 3!H^{h_1,p_1}_{(3)^{k_1}}\cdot H^{h_2,p_2}_{(3)^{k_2}}+3H^{h_1,p_1}_{(3)^{k_1+1}}\cdot H^{h_2,p_2}_{(3)^{k_2+1}}; \end{aligned}$$

the second equality follows from Lemma 2.2 and the last from Lemma 1.1. □

7. Calculation

Proposition 7.1. $H^{h,\pm}_{(3)^k} = 3^{2h-2}[(-1)^k 2^{k+h-1} \pm 1]$.

Proof. The proof consists of four steps.

Step 1. We first show the following facts which we use in the computation below.

Lemma 7.2. (a) $H^{0,+}_{(3)^0} = H^{0,+}_3 = \frac{1}{3!}$, (b) $H^{0,+}_{(3)^3} = -\frac{1}{3}$, (c) $H^{1,+}_{(3)^0} = H^{1,+}_3 = 2$.

Proof. Consider the dimension-zero space $\mathcal{M}^V_\chi(\mathbb{P}^1, 3)$ where $V = \varnothing$. The Euler characteristic $\chi = 6$ by (0-1), and hence the space contains only one map $f : C \to \mathbb{P}^1$ where C is a disjoint union of three rational curves and $|\mathrm{Aut}(f)| = 3!$. This shows (a). Let (f, C) be a map in the dimension-zero space $\mathcal{M}^V_{\chi,(3),(3),(3)}(\mathbb{P}^1, 3)$. Then C is a connected curve of genus one and the theta characteristic L_f on C defined by (0-2) is

$$L_f = \mathcal{O}(-2x_1 + x_2 + x_3) = \mathcal{O}(x_1 - 2x_2 + x_3) = \mathcal{O}(x_1 + x_2 - 2x_3),$$

where x_1, x_2, and x_3 are ramification points of f. This implies $L^3_f = \mathcal{O}$, and hence $L_f = \mathcal{O}$ because $L^2_f = L^3_f = \mathcal{O}$. We have $p(f) = 1$. Therefore,

$$H^{0,+}_{(3)^3} = -H^0_{(3)^3} = -\tfrac{1}{3},$$

where $H^0_{(3)^3}$ denotes the (ordinary) Hurwitz number, which is calculated by using the character formula; see [Okounkov and Pandharipande 2006, (0.10)]. By Proposition 9.2 of [Lee and Parker 2007], the spin Hurwitz numbers $H^{h,p}_d$ are the dimension-zero local invariants of spin curve that count maps from possibly disconnected domains. Let $GW^{h,p}_d$ denote the dimension-zero local invariants of spin curve that count maps from connected domains. Then $H^{h,p}_d$ and $GW^{h,p}_d$ are related as follows:

$$1 + \sum_{d>0} H^{h,p}_d t^d = \exp\biggl(\sum_{d>0} GW^{h,p}_d t^d\biggr).$$

Now (c) follows from $GW^{1,+}_1 = 1$, $GW^{1,+}_2 = 1/2$, and $GW^{1,+}_3 = 4/3$; see Section 10 of [Lee and Parker 2007]. □

Step 2. In this step, we compute $H^{1,-}_{(3)^k}$. For a spin curve of genus one with trivial theta characteristic. It follows from formula (3.12) of [Eskin et al. 2008] that

$$H^{1,-}_{(3)^k} = 2^{-k}[(\boldsymbol{f}_{(3)}(21))^k - (\boldsymbol{f}_{(3)}(3))^k]. \tag{7-1}$$

Here the *central character* $\boldsymbol{f}_{(3)}$ can be written as

$$\boldsymbol{f}_{(3)} = \tfrac{1}{3}\boldsymbol{p}_3 + a_2\boldsymbol{p}^2_1 + a_1\boldsymbol{p}_1 + a_0$$

for some $a_i \in \mathbb{Q}$ $(0 \le i \le 2)$, and the *supersymmetric functions* $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ are defined by

$$\boldsymbol{p}_1(m) = d - \tfrac{1}{24} \quad \text{and} \quad \boldsymbol{p}_3(m) = \sum_j m_j^3 - \tfrac{1}{240},$$

where $m = (m_1, \ldots, m_\ell)$ is a partition of d. For $k = 0, 1$, (7-1) shows

$$H^{1,-}_{(3)^0} = 0 \quad \text{and} \quad H^{1,-}_{(3)} = -3. \tag{7-2}$$

Lemma 7.2(b), (7-2), and formula (0-6) give $H^{1,-}_{(3)^2} = 3H^{1,-}_{(3)} \cdot H^{0,+}_{(3)^3} = 3$. Together with (7-1) and (7-2), this yields $\boldsymbol{f}_{(3)}(21) = -4$ and $\boldsymbol{f}_{(3)}(3) = 2$. From this and (7-1) we have, for $k \geq 0$,

$$H^{1,-}_{(3)^k} = (-1)^k 2^k - 1. \tag{7-3}$$

Step 3. In this step, we compute $H^{h,+}_{(3)^k}$ for $h = 0, 1$. For $k \geq 1$, (7-2) and formula (0-6) give $H^{1,-}_{(3)^{k-1}} = 3H^{1,-}_{(3)} \cdot H^{0,+}_{(3)^k} = -3^2 H^{0,+}_{(3)^k}$. Combining this with Lemma 7.2(a) we obtain, for $k \geq 0$,

$$H^{0,+}_{(3)^k} = -\frac{1}{3^2}((-1)^{k-1} 2^{k-1} - 1). \tag{7-4}$$

Lemma 7.2(c), (7-3), (7-4), and formula (0-6) show

$$\begin{aligned}
H^{2,+}_{(3)^0} &= 3! H^{1,-}_{(3)^0} \cdot H^{1,-}_{(3)^0} + 3H^{1,-}_{(3)} \cdot H^{1,-}_{(3)} = 27, \\
H^{2,+}_{(3)} &= 3! H^{1,-}_{(3)^0} \cdot H^{1,-}_{(3)} + 3H^{1,-}_{(3)} \cdot H^{1,-}_{(3)^2} = -27, \\
H^{2,+}_{(3)^0} &= 3! H^{1,+}_{(3)^0} \cdot H^{1,+}_{(3)^0} + 3H^{1,+}_{(3)} \cdot H^{1,+}_{(3)} = 24 + 3H^{1,+}_{(3)} \cdot H^{1,+}_{(3)}, \\
H^{2,+}_{(3)} &= 3! H^{1,+}_{(3)^0} \cdot H^{1,+}_{(3)} + 3H^{1,+}_{(3)} \cdot H^{1,+}_{(3)^2} = 12H^{1,+}_{(3)} + 3H^{1,+}_{(3)} \cdot H^{1,+}_{(3)^2}, \\
H^{1,+}_{(3)^2} &= 3! H^{1,+}_{(3)^0} \cdot H^{0,+}_{(3)^2} + 3H^{1,+}_{(3)} \cdot H^{0,+}_{(3)^3} = 4 - H^{1,+}_{(3)}.
\end{aligned}$$

It follows that $H^{1,+}_{(3)} = -1$. Hence, Lemma 7.2(c), (7-4), and formula (0-6) give

$$H^{1,+}_{(3)^k} = 3! H^{1,+}_{(3)^0} \cdot H^{0,+}_{(3)^k} + 3H^{1,+}_{(3)} \cdot H^{0,+}_{(3)^{k+1}} = (-1)^k 2^k + 1. \tag{7-5}$$

Step 4. It remains to compute $H^{h,p}_{(3)^k}$ for $h \geq 2$. The formula (0-6) gives

$$H^{h,p}_{(3)^k} = 3! H^{h-1,p}_{(3)^0} \cdot H^{1,+}_{(3)^k} + 3H^{h-1,p}_{(3)} \cdot H^{1,+}_{(3)^{k+1}}.$$

From this, we can deduce that, for $h \geq 2$,

$$\begin{aligned}
\begin{pmatrix} H^{h,p}_{(3)^k} \\ H^{h,p}_{(3)^{k+1}} \end{pmatrix} &= \begin{pmatrix} 3!H^{1,+}_{(3)^k} & 3H^{1,+}_{(3)^{k+1}} \\ 3!H^{1,+}_{(3)^{k+1}} & 3H^{1,+}_{(3)^{k+2}} \end{pmatrix} \begin{pmatrix} H^{h-1,p}_{(3)^0} \\ H^{h-1,p}_{(3)} \end{pmatrix} \\
&= \begin{pmatrix} 3!H^{1,+}_{(3)^k} & 3H^{1,+}_{(3)^{k+1}} \\ 3!H^{1,+}_{(3)^{k+1}} & 3H^{1,+}_{(3)^{k+2}} \end{pmatrix} \begin{pmatrix} 3!H^{1,+}_{(3)^0} & 3H^{1,+}_{(3)} \\ 3!H^{1,+}_{(3)} & 3H^{1,+}_{(3)^2} \end{pmatrix}^{h-2} \begin{pmatrix} H^{1,p}_{(3)^0} \\ H^{1,p}_{(3)} \end{pmatrix}.
\end{aligned} \tag{7-6}$$

Therefore, (7-3), (7-5), and (7-6) complete the proof. □

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