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For a compact 3-manifold N with connected nonempty boundary, let Γ be an admissible trivalent graph in ∂N that decomposes ∂N into a set of disks. As an extension of small covers, from a $(\mathbb{Z}_2)^3$-coloring λ on $\partial N - \Gamma$, one can get a closed 3-manifold M_λ that admits a locally standard $(\mathbb{Z}_2)^3$-action.

Suppose N is irreducible and atoroidal: say, a handlebody. We give a combinatorial necessary and sufficient condition for a $(\mathbb{Z}_2)^3$-colorable pair (N, Γ) to admit a right-angled hyperbolic structure, which naturally induces a hyperbolic structure on M_λ.

1. Introduction

In this note, we study polyhedral hyperbolic 3-manifolds admitting $(\mathbb{Z}_2)^3$-colorings on their connected boundaries, which correspond to closed hyperbolic 3-manifolds admitting locally standard $(\mathbb{Z}_2)^3$-actions.

$(\mathbb{Z}_2)^3$-colorings and locally standard $(\mathbb{Z}_2)^3$-actions. Small covers, or Coxeter orbifolds, were studied in [Davis and Januszkiewicz 1991]. They are a class of manifolds which admit locally standard $(\mathbb{Z}_2)^n$-actions, such that the orbit spaces are n-dimensional simple polyhedra. The algebraic and topological properties of a small cover are closely related to the combinatorics of the orbit polyhedron and the coloring on its boundary. For example, the (mod 2) Betti number β_i of a small cover M agrees with h_i, where $h = (h_0, h_1, \dots, h_n)$ is the h-vector of the polyhedron.

Those manifolds admitting locally standard $(\mathbb{Z}_2)^n$-actions form a wider class than small covers. In this paper, we focus on the 3-dimensional case.

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A standard representation of the $(\mathbb{Z}_2)^3$-action on $\mathbb{R}^3$ is the natural action defined by

$$e_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \tag{1-1}$$

$$e_2 : (x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3), \tag{1-2}$$

$$e_3 : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3). \tag{1-3}$$

The actions e_1, e_2 and e_3 generate the group $(\mathbb{Z}_2)^3$. This action fixes the origin of $\mathbb{R}^3$ such that its orbit space is the positive cone

$$\mathbb{R}^3_{\geq 0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i \geq 0\}.$$

Definition 1.1. An effective $(\mathbb{Z}_2)^3$-action on a 3-dimensional closed manifold M is said to be *locally standard* if it locally looks like the standard representation of $(\mathbb{Z}_2)^3$-action on $\mathbb{R}^3$. More precisely, if for each point x in M, there is a $(\mathbb{Z}_2)^3$-invariant neighborhood U_x of x such that U_x is equivariantly homeomorphic to an invariant open subset of the standard $(\mathbb{Z}_2)^3$-action on $\mathbb{R}^3$.

The orbit space of a locally standard $(\mathbb{Z}_2)^3$-action on a 3-dimensional closed manifold M is a compact manifold N with corners. In other words, it is a 3-dimensional compact manifold N with a graph Γ on ∂N. The graph Γ on ∂N induces a cell decomposition on ∂N. The vertices of Γ are the image of fixed points of the $(\mathbb{Z}_2)^3$-action, the (open) edges of Γ are the image of fixed points of subgroups $(\mathbb{Z}_2)^2 < (\mathbb{Z}_2)^3$ and (open) components of $\partial N - \Gamma$ are the image of fixed points of subgroups $\mathbb{Z}_2 < (\mathbb{Z}_2)^3$.

Definition 1.2. Let N be a 3-dimensional manifold with nonempty boundary, and Γ a trivalent graph in ∂N that gives a cell decomposition of ∂N. A $(\mathbb{Z}_2)^3$*-coloring* is a map $\lambda : \partial N - \Gamma \to (\mathbb{Z}_2)^3 - 0$ such that $\lambda(f_1)$, $\lambda(f_2)$ and $\lambda(f_3)$ generate $(\mathbb{Z}_2)^3$ for each triple of faces f_1, f_2 and f_3 sharing a common vertex.

Associated to a locally standard $(\mathbb{Z}_2)^3$-action on M, there is a canonical $(\mathbb{Z}_2)^3$-coloring λ on $\partial N - \Gamma$ which colors each face $f \in \partial N - \Gamma$ by the element $e \in (\mathbb{Z}_2)^3$ that fixes f. For an i-dimensional cell f in the cell decomposition, $i = 0, 1, 2$, we have a group $G_f \cong (\mathbb{Z}_2)^{3-i}$ which is generated by the colorings in the faces which are adjacent to f. The locally standard $(\mathbb{Z}_2)^3$-action on M induces a principal $(\mathbb{Z}_2)^3$-bundle over N.

Conversely, by Lemma 3.1 of [Lü and Masuda 2009], from a $(\mathbb{Z}_2)^3$-coloring λ on $(\partial N, \Gamma)$ and a principal $(\mathbb{Z}_2)^3$-bundle over N, we can get a unique closed 3-manifold M. In particular, from a $(\mathbb{Z}_2)^3$-coloring λ and the trivial principal $(\mathbb{Z}_2)^3$-bundle over N, we get a 3-manifold M_λ which depends only on the coloring λ. By this we take eight copies of N, $N \times \{\alpha\}$ for each $\alpha \in (\mathbb{Z}_2)^3$, and construct a quotient space M_λ under the following gluing rule:

$$(x,\alpha_1)\sim(y,\alpha_2)\Leftrightarrow\begin{cases}x=y,\ \alpha_1=\alpha_2, & \text{if } x \text{ lies in the interior of } N,\\ x=y,\ \alpha_1\alpha_2^{-1}\in G_f, & \text{if } x \text{ lies in a cell } f.\end{cases}\tag{1-4}$$

Then it is easy to see M_λ is a closed 3-manifold. In this paper, we only consider closed 3-manifolds associated to $(\mathbb{Z}_2)^3$-colorings and trivial principal $(\mathbb{Z}_2)^3$-bundles over N.

A simple example is that if we consider a coloring of the four faces of a tetrahedron by $e_1, e_2, e_3, e_1+e_2+e_3$, respectively, then from the above construction, we get the closed orientable 3-manifold RP^3. A tetrahedron admits a unique right-angled spherical structure, and the spherical structures on eight copies of the tetrahedron, when glued together, give rise to the unique spherical structure on RP^3. This point of view is applied in this paper.

There are many works on manifolds with locally standard $(\mathbb{Z}_2)^3$-actions. For example, 3-dimensional small covers are studied in [Lü 2009; Lü and Yu 2011]. Six operations on small covers were defined in [Lü and Yu 2011], which topologically behave well, such that every 3-dimensional small cover is obtained from the two simple small covers RP^3 and $S^1\times RP^2$ by a sequence of these operations. It should be noted that the operations in [Lü and Yu 2011] give many disks in a simple convex polygon P, which intersects the 1-skeleton of P in at most four points but which is not vertex-linking or edge-linking. So the preimage of these disks are essential spheres or essential tori in the small cover M in general, and hence M does not admit a geometric structure [Scott 1983].

Polyhedral hyperbolic 3-manifolds. Andreev [1971] (see also [Roeder et al. 2007]) gives a complete characterization of compact hyperbolic polyhedra in dimension 3 with nonobtuse angles. The boundary of a compact hyperbolic polyhedron inherits a natural cell decomposition. The 1-skeleton of the cell decomposition is a graph Γ on the boundary of the 3-ball, and a dihedral angle is also given on each edge of Γ from the hyperbolic structure. Andreev's theorem is given in terms of a set of conditions on the dihedral angles. Besides its beauty, Andreev's theorem is also essential in the proof of Thurston's geometrization theorem for Haken 3-manifolds. The natural question is, given a cell decomposition of the boundary of the 3-ball, and a weight $\alpha_e\in(0,\pi)$ attached to each edge e of the cell decomposition, whether there is a compact hyperbolic polyhedron in $\mathbb{H}^3$ realizing this cell decomposition whose dihedral angles coincide with the attached weights. This question is still open now.

A clever approach for working with compact hyperbolic polyhedra having arbitrary dihedral angles is to express necessary and sufficient conditions for existence of a given polyhedron in terms of its polar dual in the de Sitter space; see [Hodgson and Rivin 1993]. For a generalization of Andreev's result to ideal and hyper-ideal hyperbolic polyhedra, see [Rivin 1996; Bao and Bonahon 2002]. Hyperbolic structures on topologically more complicated 3-manifold N with boundary are also studied. See [Schlenker 2002; 2003; 2005; 2006, Fillastre and Izmestiev 2009; 2011; Guéritaud 2009].

Suppose N is a compact 3-manifold with connected and nonempty boundary. In this note, we consider the right-angled hyperbolic structures on N with compressible boundary.

Given a graph Γ in ∂N, we call it *admissible* if the lift $\tilde{\Gamma}$ of Γ in the universal cover of N, say $\tilde{N}$, gives a cell decomposition of $\partial \tilde{N}$ such that each of its 2-cells has a closure homeomorphic to a disk, and each pair of such two disks shares at most one edge in $\tilde{\Gamma}$. A *right-angled hyperbolic realization* of (N, Γ) is a complete compact hyperbolic manifold N^* with right-angled polyhedral boundary (i.e., modeled on the orthogonal intersection of two half-spaces with totally geodesic boundaries in H^3 and having finite volume), endowed with a homeomorphism to N that sends the nonsmooth points of N^* precisely to the points of Γ. The nonsmooth points of N^* will be called the *singular locus* of this structure. From the homeomorphism between N^* and N, Γ is also called the *singular locus* for this hyperbolic realization. We will call such a structure on N^* a *hyperbolic structure with right-angled polyhedral boundary* on N. Hence these kinds of hyperbolic structures on N look locally like compact convex right-angled hyperbolic polyhedra in H^3.

Similar to all results above, it is interesting to give a kind of characterization of hyperbolic structures with right-angled polyhedral boundary on N. Since all the dihedral angles are right-angled, an easy argument shows that the graph Γ defined above must be trivalent.

It is well known that most 3-manifolds are hyperbolic 3-manifolds [Thurston 1982]. So it is interesting to consider locally standard $(\mathbb{Z}_2)^3$-actions on closed hyperbolic 3-manifolds. It is natural to ask which closed hyperbolic 3-manifold admits a locally standard $(\mathbb{Z}_2)^3$-action. The orbit space of a locally standard $(\mathbb{Z}_2)^3$-action is a compact manifold N with a coloring λ in ∂N. If (N, Γ) admits a right-angled hyperbolic structure, then it is easy to see that M is hyperbolic. A pair (N, Γ) admits a unique right-angled hyperbolic structure. However, it may admit many different colorings. Each coloring, together with a principle $(\mathbb{Z}_2)^3$-bundle over N, gives a manifold with a locally standard $(\mathbb{Z}_2)^3$-action. So these give many different hyperbolic manifolds of the same volume. [Inoue 2008] gives a very clear description of right-angled hyperbolic polyhedra from this point of view.

The most interesting case is that N is a handlebody, or the simplest one, a 3-ball.

Main result. Suppose Γ is an admissible graph in ∂N. For a vertex v of Γ, we take a small closed regular neighborhood B of v in N, then B intersects $N - \operatorname{int} B$ in a disk D_v. We call D_v a *vertex-linking disk*. It intersects Γ in three points. The preimage of a vertex-linking disk in M_λ is a sphere which bounds a 3-ball in M_λ for a $(\mathbb{Z}_2)^3$-coloring λ. For an edge e of Γ, we also take a small closed neighborhood B of e in N; then B intersects $N - \operatorname{int} B$ in a disk D_e. We call D_e an *edge-linking disk*. It intersects Γ in four points. The preimage of an edge-linking disk in M_λ is a torus (or a Klein bottle) which bounds a solid torus (or a solid Klein bottle) in M_λ. We say a properly embedded disk D in the 3-manifold N intersects Γ *efficiently* if ∂D and Γ are in general position and there is no bigon in $\partial N - (\partial D \cup \Gamma)$. In this note we always assume a disk D intersects with Γ efficiently.

Our main result is the following:

Theorem 1.3. *Let N be an irreducible, atoroidal and compact 3-manifold with connected nonempty boundary, and Γ be an admissible trivalent graph in ∂N which gives a cell decomposition of ∂N, such that $(\partial N, \Gamma)$ admits a $(\mathbb{Z}_2)^3$-coloring. Then (N, Γ) realizes a right-angled hyperbolic structure if and only if every properly embedded disk D in N has $|D \cap \Gamma| \geq 5$, except when D is a vertex-linking disk or an edge-linking disk. Moreover, the realization is unique up to isometry.*

Remark 1.4. In practice, much attention has been paid to the right-angled hyperbolic structures on handlebodies. They are irreducible, atoroidal and compact 3-manifolds with connected nonempty boundaries. So Theorem 1.3 can be applied to the handlebody case.

Remark 1.5. There are two canonical ways to study polyhedral hyperbolic structure on 3-manifold M: Alexandrov's method and the variational method; see, for example, [Fillastre and Izmestiev 2011]. Our approach in this note uses the doubling trick. A $(\mathbb{Z}_2)^3$-coloring helps us find a closed 3-manifold on which we can apply the geometrization theorem.

2. Preliminaries

If (N, Γ) admits a right-angled hyperbolic structure, then Γ is admissible, and each of its 2-dimensional faces is a right-angled hyperbolic n-polygon. So $n \geq 5$.

Definition 2.1. Let Γ^* be the dual graph of Γ in ∂N. A *k-circuit* is a simple closed curve C in Γ^* consisting of k successive edges of Γ^* which is contractible in ∂N. A circuit is *elementary* if it bounds a disk D in ∂N and there is exactly one vertex V of Γ that lies in D. A k-circuit is *prismatic* if the endpoints of all the edges of Γ which intersect C are distinct.

Obviously, there is a one-to-one correspondence between edges of Γ and those of Γ^*.

Lemma 2.2. *Suppose Γ is admissible. If C is a 3-circuit which is not prismatic, then C is isotopic to the boundary of a vertex-linking disk. If Γ contains no prismatic 3-circuit, and C is a 4-circuit which is not prismatic, then C is isotopic to the boundary of an edge-linking disk.*

Proof. The proof is similar to the proofs of Lemmas 1.2 and 1.3 of [Roeder et al. 2007], by which we have that, if C is a nonprismatic 3-circuit, then it is an elementary circuit. So it is isotopic to the boundary of a vertex-linking disk. If Γ contains no prismatic 3-circuit, then every nonprismatic 4-circuit C separates off exactly two vertices of Γ from the remaining vertices of Γ, which in turn implies that C is isotopic to the boundary of an edge-linking disk. Actually, the authors of [Roeder et al. 2007] proved this for any graph in S^2. Since Γ is admissible, their arguments can be extended verbatim in the general case. □

We give a proposition on the orientability of a 3-manifold M_λ with a locally standard $(\mathbb{Z}_2)^3$-action and trivial principal $(\mathbb{Z}_2)^3$-bundle.

Proposition 2.3. *Suppose N is a compact orientable connected 3-manifold with connected boundary. Then, for a $(\mathbb{Z}_2)^3$-coloring λ on $(\partial N, \Gamma)$, M_λ is orientable if and only if there is a basis $\{e_1, e_2, e_3\}$ of $(\mathbb{Z}_2)^3$, such that the image of λ is contained in $\{e_1, e_2, e_3, e_1+e_2+e_3\}$.*

Proof. For small covers, this proposition has been proved in Theorem 1.7 of [Nakayama and Nishimura 2005]. Recall that M_λ is determined by the coloring λ and the trivial principal $(\mathbb{Z}_2)^3$-bundle over N. So M_λ is obtained by gluing eight copies of (N, λ), and M_λ is orientable if and only if $H_3(M_\lambda, \mathbb{Z}) = \mathbb{Z}$. To calculate $H_3(M_\lambda, \mathbb{Z})$, we only need to consider the 3-cells and 2-cells in a cellular decomposition of M_λ, which is induced by a cellular decomposition of (N, Γ). Note that ∂N is connected, so the arguments of the proof of Theorem 1.7 of [Nakayama and Nishimura 2005] hold in our case word-by-word. □

3. Proof of Theorem 1.3

Proof of the necessity part of Theorem 1.3. Suppose (N, Γ) realizes a right-angled hyperbolic structure. If $D \subset N$ is a properly embedded disk which intersects Γ efficiently, and is not vertex-linking or edge-linking, then by Gauss–Bonnet theorem, we have $|D \cap \Gamma| \geq 5$. So the necessity part of Theorem 1.3 follows. □

Proof of the sufficiency part of Theorem 1.3 in the case that M_λ is orientable. Recall that a closed orientable 3-manifold M is *irreducible* if every embedded 2-sphere S in M bounds a 3-ball; otherwise M is *reducible*. An embedded 2-sphere S which does not bound a 3-ball in M is called *essential*. A closed irreducible orientable 3-manifold M is *atoroidal* if every embedded torus T in M bounds a solid torus; otherwise M is *toroidal*. An embedded torus T which does not bound a solid torus is *essential* in M. See [Hempel 1976] or [Jaco 1980].

We need the equivariant sphere theorem of Meeks, Simon, and Yau [Meeks et al. 1982], but the reformulation by Dunwoody [1985] is more convenient for us.

Theorem 3.1. *Let G be a finite group that acts on a closed orientable 3-manifold M by homeomorphisms. Suppose M has a G-equivariant triangulation. If there exists an essential 2-sphere S in M, then there exists an essential 2-sphere S_1 in M which is in general position with respect to the triangulation, such that $g(S_1) = S_1$ or $g(S_1) \cap S_1 = \varnothing$ for every $g \in G$.*

We also need the following equivariant torus theorem; see [Freedman et al. 1983; Jaco and Shalen 1979; Johannson 1979].

Theorem 3.2. *Let G be a finite group which acts on a closed 3-manifold M by homeomorphisms. Suppose M is irreducible, orientable and contains an essential torus. Then either M is Seifert-fibered, or M contains a G-equivariant essential torus.*

First, we show the following lemmas.

Lemma 3.3. *M_λ is irreducible.*

Proof. We give a triangulation $\mathcal{T}$ of N, such that the graph Γ is contained in the 1-skeleton of $\mathcal{T}$. So the triangulation $\mathcal{T}$ induces a triangulation of M_λ.

If M_λ is reducible, then by the equivariant sphere theorem, there is a $(\mathbb{Z}_2)^3$-equivariant sphere S which is essential in M_λ. We denote $S \cap N \times \{1\}$ by A, which is a compact surface with nonempty boundary if $A \neq \varnothing$. We may assume $A \neq \varnothing$, otherwise we can use the $(\mathbb{Z}_2)^3$-action to find another $(\mathbb{Z}_2)^3$-invariant sphere S' which has nonempty intersection with $N \times \{1\}$. Since A is obtained from S by the $(\mathbb{Z}_2)^3$-action and S is connected, A is connected.

Since S is in general position with respect the triangulation of M_λ, A is in general position with respect to the triangulation of ∂N, in particular, with respect to Γ. So there is a cell decomposition of ∂A: for each face f of $\partial N - \Gamma$, $f \cap A$ is an edge in ∂A. Moreover, the coloring on $\partial N - \Gamma$ now induces a coloring on ∂A, which we denote by λ_A, and S is obtained from copies of A by the gluing rule from λ_A.

By Definition 1.2, the colorings on any two adjacent edges of ∂A are different. So we have a subgroup G of $(\mathbb{Z}_2)^3$ which has index 1 or 2 in $(\mathbb{Z}_2)^3$, such that for any $g \in G$ we have $g(S) = S$, and for any $h \in (\mathbb{Z}_2)^3 - G$ we have $h(S) \cap S = \varnothing$.

In other words, S is obtained by gluing 4 or 8 copies of A, and the edges in ∂A contribute a 4-valence graph in S. So $\chi(S) = m(\chi(A) - E/2 + E/4) = 2$, where E is the number of edges in ∂A, and $m = 4$ or 8. If $m = 4$, then $E = 2$ and $\chi(A) = 1$. So A is a disk with ∂A consisting of 2 edges. This is impossible by the assumption in Theorem 1.3. If $m = 8$, then $E = 3$ and $\chi(A) = 1$. So A is a disk with ∂A consists of 3 edges. Moreover, we have that $\partial A \cap f$ is connected for each face f. Suppose otherwise; i.e., suppose that $\partial A \cap f$ consists of at least two arcs. We have an edge e of Γ which intersects A such that the two sides of e both are in the face f. Then when we lift ∂N to the universal cover $\tilde{N}$. The closure of the lifting $\tilde{f}$ of the face f

is not a disk, contradicting the assumption that Γ is admissible. So ∂A is a 3-circuit. Thus, by the assumption and Lemma 2.2, A is a vertex-linking disk in N. The preimage of a vertex-linking disk in M_λ is a sphere which bounds a 3-ball in M_λ. This contradicts the assumption that S is essential in M_λ. So M_λ is irreducible. □

Lemma 3.4. *If M_λ is a toroidal Seifert manifold, then there is an essential torus in M_λ which is $(\mathbb{Z}_2)^3$-equivariant.*

Proof. Suppose e_1, e_2 and e_3 are three orientation-reversing involutions which generate the $(\mathbb{Z}_2)^3$-action. Since M_λ admits orientation-reversing involutions, according to Theorems 8.2 and 8.5 of [Neumann and Raymond 1978], M_λ is Seifert-fibered with Euler number 0; i.e., M_λ contains horizontal incompressible surfaces which are transversal to each fiber. In other words, M_λ is a surface bundle over S^1 with horizontal incompressible surfaces as surface fibers. We already proved in Lemma 3.3 that M_λ is irreducible, so the Euler characteristic of the base orbifold of M_λ is negative or zero. Thus M_λ admits the geometries $H^2 \times \mathbb{R}$ or E^3. We refer the readers to [Scott 1983] for the details about these two geometries.

For each $i = 1, 2, 3$, $\mathrm{Fix}(e_i)$ contains no nonorientable closed surfaces since the nonorientable closed surfaces are one-sided in M_λ. According to [Meeks and Scott 1986], for each $i = 1, 2, 3$, e_i is isotopic to an isometry. So $\mathrm{Fix}(e_i)$ consists of some totally geodesic, and hence incompressible, closed surfaces in M_λ.

If M_λ admits the $H^2 \times \mathbb{R}$ geometry, then it has unique Seifert fibration structure. So each homeomorphism sends regular fibers to regular fibers. Then, among $\mathrm{Fix}(e_1)$, $\mathrm{Fix}(e_2)$ and $\mathrm{Fix}(e_3)$, at least two of them, say $\mathrm{Fix}(e_1)$ and $\mathrm{Fix}(e_2)$, consist of vertical essential tori, and e_3 keeps each regular fiber invariant and reverses its orientation. By the definition of $(\mathbb{Z}_2)^3$-action, $\mathrm{Fix}(e_1) \cap \mathrm{Fix}(e_2) \neq \varnothing$, and $\mathrm{Fix}(e_1)$ intersects $\mathrm{Fix}(e_2)$ transversely. Choose a torus component T in $\mathrm{Fix}(e_1)$ which intersects $\mathrm{Fix}(e_2)$ nontrivially. For any point $p \in T$, we have $e_1e_2(p) = e_2e_1(p) = e_2(p)$. So $e_2(T) \subset \mathrm{Fix}(e_1)$. Thus we have that either $e_2(T) = T$ or $e_2(T) \cap T = \varnothing$. By the assumption $T \cap \mathrm{Fix}(e_2) \neq \varnothing$, we have $e_2(T) = T$. Moreover, $e_3(T) = T$. Hence T is invariant by e_1, e_2 and e_3, and hence is invariant by each element of the group $(\mathbb{Z}_2)^3$. So it is an essential torus which is $(\mathbb{Z}_2)^3$-equivariant.

If M_λ admits the E^3 geometry, then according to Theorems 8.2 and 8.5 of [Neumann and Raymond 1978], it is either the 3-torus T^3 or the Seifert manifold with invariant $\{0; (2, 1), (2, -1), (2, 1), (2, -1)\}$. For the former case, we choose a Seifert fibration structure which is fibred by all circles isotopic to the circles in $\mathrm{Fix}(e_1) \cap \mathrm{Fix}(e_2)$. Then we can apply the same argument as above to obtain a $(\mathbb{Z}_2)^3$-equivariant essential torus. For the latter case, we fix the Seifert fibration structure given before, and then all horizontal incompressible surfaces in M_λ are isotopic essential tori. This is because M_λ has a unique structure of surface bundles over S^1, since its first Betti number is 1; see [Thurston 1986]. So we can still assume that both $\mathrm{Fix}(e_1)$ and $\mathrm{Fix}(e_2)$ consist of vertical essential tori, and e_3 keeps each regular fiber invariant and reverses its orientation. The same argument in the previous paragraph still applies, and the same conclusion still holds. □

Lemma 3.5. *M_λ is atoroidal.*

Proof. By Theorem 3.2 and Lemma 3.4, if M_λ is toroidal, then there is a $(\mathbb{Z}_2)^3$-equivariant essential torus $T \subset M_\lambda$.

Similar to the sphere case in Lemma 3.3, we also give a triangulation $\mathcal{T}$ of N. We denote $T \cap N \times \{1\}$ by A, which is nonempty, and is a compact connected surface with nonempty boundary. Also, there is a cell decomposition of ∂A induced from the triangulation of N.

Similar to the argument in the sphere case in Lemma 3.3, we have $\chi(T) = m(\chi(A) - E/2 + E/4) = 0$, where E is the number of edges in ∂A, and m is an integer. So $E = 4$ and $\chi(A) = 1$. Thus A is a disk with ∂A consists of 4 edges. Moreover, $\partial A \cap f$ is connected for each face f. Suppose otherwise; i.e., suppose that $\partial A \cap f$ consists of at least two arcs. When we lift ∂N to the universal cover $\tilde{N}$, two of the four edges forming ∂A belong to the same face. If these two edges are adjacent in ∂A, then by the same argument as in the proof of Lemma 3.3, we obtain a contradiction. If these two edges are not adjacent in ∂A, then the lift of these two arcs in the universal cover are identified. So in the universal cover, there are two disks which share two distinct edges, contradicting the assumption that Γ is admissible. Thus ∂A is a 4-circuit. Therefore, by the assumption and Lemma 2.2, A is an edge-linking disk. The preimage of an edge-linking disk in M_λ is a torus (or a Klein bottle) which bounds a solid torus (or a solid Klein bottle, which is impossible since we assume M_λ is orientable in this subsection), so it is not essential. This contradicts the assumption that T is essential. So M_λ is atoroidal. □

Lemma 3.6. *M_λ is not a Seifert manifold.*

Proof. Suppose M_λ is an orientable Seifert manifold with orientable base orbifold, and M_λ is neither a lens space nor S^3. Here the lens spaces don't include S^3 or $S^2 \times S^1$. By Theorem 8.2 of [Neumann and Raymond 1978] and its proof, if there is an orientation-reversing involution on M_λ, then the Seifert invariant of M_λ is $\{g; (a_1, b_1), (a_1, -b_1), (a_2, b_2), (a_2, -b_2), \ldots, (a_t, b_t), (a_t, -b_t)\}$, where g is the genus of the base orbifold. Since M_λ is atoroidal, we have $g = 0$ and $t = 1$, and hence M_λ is a lens space. This is a contradiction.

Suppose M_λ is an orientable Seifert manifold with nonorientable base orbifold, and M_λ is not a lens space. By Theorem 8.5 of [Neumann and Raymond 1978] and its proof, if there is an orientation-reversing involution on M_λ, then the Seifert invariant of M_λ is $\{k; (a_1, b_1), (a_1, -b_1), (a_2, b_2), (a_2, -b_2), \ldots, (a_t, b_t), (a_t, -b_t)\}$, where k is the genus of the nonorientable base orbifold. If $t \geq 1$, then M_λ cannot be atoroidal. If $t = 0$, then M_λ is either reducible or toroidal. In both cases, we arrive at contradictions.

Suppose M_λ is a lens space. By the main result in [Kwun 1970], among all lens spaces, only RP^3 admits orientation-reversing involutions. Moreover, RP^3 admits exactly one orientation-reversing involution up to isotopies, and the set of fixed points of this involution is an RP^2, which has Euler characteristic 1. However, according to the definition of locally standard $(\mathbb{Z}_2)^3$-action, for any nontrivial element $e \in (\mathbb{Z}_2)^3$, its fixed point set $\mathrm{Fix}(e)$ is a union of k-polygons ($k \geq 5$ by our assumption), and each vertex in $\mathrm{Fix}(e)$ is adjacent to 4 edges. Let v be the number of vertices in $\mathrm{Fix}(e)$. Then the number of edges in $\mathrm{Fix}(e)$ is $2v$, and the number of faces of $\mathrm{Fix}(e)$ is less than or equal to $4v/5$. So the Euler characteristic of $\mathrm{Fix}(e)$ is negative — a contradiction.

Suppose M_λ is the 3-sphere S^3. From the fact that the orientation-preserving mapping class group of S^3 is trivial, we know S^3 admits exactly one orientation-reversing involution up to isotopy, and the set of fixed points of this involution is an S^2, which has Euler characteristic 2. Then similar to the argument in the previous paragraph, we get a contradiction. So the lemma follows. □

By Lemmas 3.3, 3.5 and 3.6, M_λ is a closed, irreducible, and atoroidal manifold which is not Seifert-fibered. So by Perelman's proof of Thurston's geometrization theorem (see [Cao and Zhu 2006; Bessières et al. 2010; Kleiner and Lott 2008; Morgan and Tian 2007]), M_λ is a hyperbolic 3-manifold. By [Dinkelbach and Leeb 2009], every smooth action of a finite group on a hyperbolic 3-manifold is conjugate to an isometric action. Since each $e \in (\mathbb{Z}_2)^3$ is conjugate to an isometric involution, its fixed point set is a totally geodesic surface in M_λ. Since $(\mathbb{Z}_2)^3$ is an Abelian group, by elementary arguments for the isometric group of hyperbolic 3-space H^3, all these totally geodesic surfaces intersect orthogonally. So the hyperbolic structure on M_λ induces a hyperbolic structure on (N, Γ). Conversely, each right-angled hyperbolic structure on (N, Γ) induces a hyperbolic structure on M_λ. By Mostow's rigidity theorem [1973], there is only one hyperbolic structure on M_λ. So the right-angled realization of (N, Γ) is unique. This ends the proof of Theorem 1.3 in the case that M_λ is orientable. □

Proof of the sufficiency part of Theorem 1.3 in the case that M_λ is nonorientable. Let $\pi : \tilde{M}_\lambda \to M_\lambda$ be the orientable double cover of M_λ, and τ be the covering transformation of $\tilde{M}_\lambda$. Note that τ is orientation-reversing. By the lifting theorem, for each i, e_i lifts to an action, say $\tilde{e}_i$, on $\tilde{M}_\lambda$ such that $\tilde{e}_i(x_0) = x_0$, where $x_0 \in \tilde{M}_\lambda$ projects to a vertex of Γ in ∂N.

We show that $\tilde{e}_i$ and $\tilde{e}_j$ commute, for $1 \leq i, j \leq 3$. It is easy to verify that $\tilde{e}_i\tilde{e}_j$ is the lift of e_ie_j, and $\tilde{e}_j\tilde{e}_i$ is the lift of e_je_i. Since $e_ie_j = e_je_i$, and $\tilde{e}_i\tilde{e}_j(x_0) = \tilde{e}_j\tilde{e}_i(x_0)$, by the unique lifting property, $\tilde{e}_i\tilde{e}_j = \tilde{e}_j\tilde{e}_i$. We also show that τ and $\tilde{e}_i$ commute, for $1 \le i \le 3$. It is easy to verify that both $\tau\tilde{e}_i$ and $\tilde{e}_i\tau$ are lifts of e_i. So either $\tau\tilde{e}_i = \tilde{e}_i\tau$ or $\tau\tau\tilde{e}_i = \tilde{e}_i\tau$. The latter is $\tilde{e}_i = \tilde{e}_i\tau$ in fact, which is impossible. So $\tau\tilde{e}_i = \tilde{e}_i\tau$. Therefore we have an action of $(\mathbb{Z}_2)^4$ on $\tilde{M}_\lambda$.

If $\tilde{M}_\lambda$ is a toroidal Seifert manifold, then by Lemma 3.4, there is an essential vertical torus T in $\tilde{M}_\lambda$ which is fixed by $\tilde{e}_1$, and is invariant by $\tilde{e}_i$, for $i = 2, 3$. For any point $p \in T$, we have $\tilde{e}_1\tau(p) = \tau\tilde{e}_1(p) = \tau(p)$. So $\tau(T) \subset \mathrm{Fix}(\tilde{e}_1)$. Hence either $\tau(T) = T$ or $\tau(T) \cap T = \varnothing$. It is straightforward to verify that T is $(\mathbb{Z}_2)^4$-equivariant.

Therefore, similar to the previous subsection, we can prove that $\tilde{M}_\lambda$ is irreducible and atoroidal. Moreover, if $\tilde{M}_\lambda$ is an atoroidal Seifert manifold, then it must be S^3 or RP^3. The action of τ on $\tilde{M}_\lambda$ has no fixed points. However, as stated in the previous subsection, any orientation-reversing involution on S^3 or RP^3 must have fixed points. We arrive at a contradiction.

So $\tilde{M}_\lambda$ is hyperbolic. Similar to the arguments in the previous subsection, (N, Γ) admits a unique right-angled hyperbolic structure. □

4. Examples

In this section we give three examples.

Example 4.1. The simplest way to construct a handlebody which admits right-angled hyperbolic structure is from the Löbell polyhedron $L(n)$ for $n \ge 5$ (see, for example, [Inoue 2008]). A Löbell polyhedron $L(n)$ admits a right-angled hyperbolic structure. Gluing two opposite n-gon faces of $L(n)$, we get a solid torus admitting right-angled hyperbolic structures, and whose boundary consists of $2n$ octagons.

For instance, from $L(5)$, which is a dodecahedron, we can get three solid tori, according to the twisting angle of gluing. All these solid tori satisfy Theorem 1.3. It is easy to see that they admit $(\mathbb{Z}_2)^3$-colorings, but don't admit one which satisfies the orientability criterion in Proposition 2.3.

This kind of right-angled hyperbolic solid tori are "simple", by which we mean we can obtain a right-angled hyperbolic polyhedron by cutting along a totally geodesic right-angled n-polygon P from the solid tori, where P intersects the boundaries of the solid tori orthogonally.

Example 4.2. A hexagonal tessellation of $\mathbb{R}^2$ with a coloring is shown in Figure 1. We assume that the diameter of a hexagon is 1. We take a $\mathbb{Z}^2$-action on $\mathbb{R}^2$, such that its fundamental domain is a rectangle R whose vertical edges have length 4.5, and whose horizontal edges have length $3\sqrt{3}$. So there are six hexagons in each horizontal layer and each vertical layer. Gluing the boundaries of R, we get a torus T.

We can show that any solid torus bounded by T with coloring shown in Figure 1 satisfies the orientability criterion in Proposition 2.3 as well as the assumption of Theorem 1.3. So it admits a right-angled hyperbolic structure.

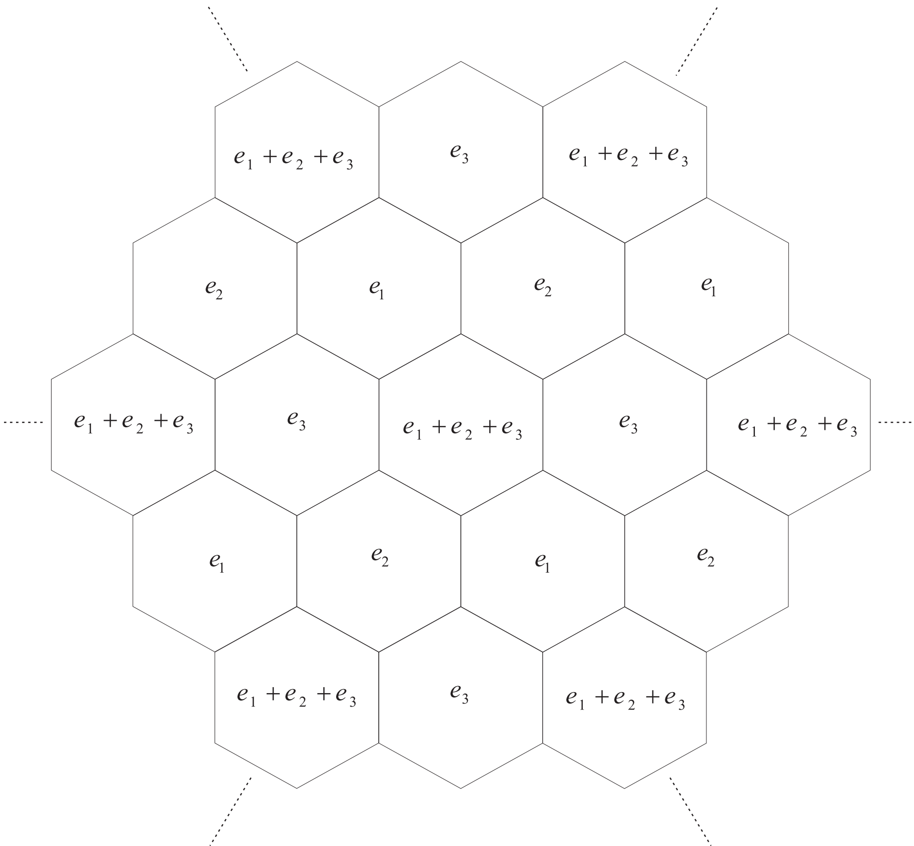

Figure 1

We fix a homeomorphism from T to the boundary of a solid torus J, so it is natural to ask whether the pair (J, Γ) admits a right-angled hyperbolic structure.

It is easy to see that any essential simple closed curve C in this T intersects Γ in at least five points, and any curve C which bounds a disk D in T intersects Γ in at least five points, unless that D is a vertex-linking disk or an edge-linking disk. So for any solid torus J which is bounded by T, (J, Γ) realizes a right-angled hyperbolic structure.

If the boundary of the unique essential disk in the solid torus J is the image of a horizontal line, then the hyperbolic solid torus can be decomposed into three copies of the Löbell polyhedron $L(6)$ along three totally geodesic right-angled hexagons in the solid torus. The same claim holds if the boundary of the unique essential disk in the solid torus is the image of the straight lines which have angles $\pi/3$ or $2\pi/3$ with the horizontal lines.

Except in these three cases, the right-angled hyperbolic structure cannot be obtained by gluing two faces of a right-angled hyperbolic polyhedron by an isometry,

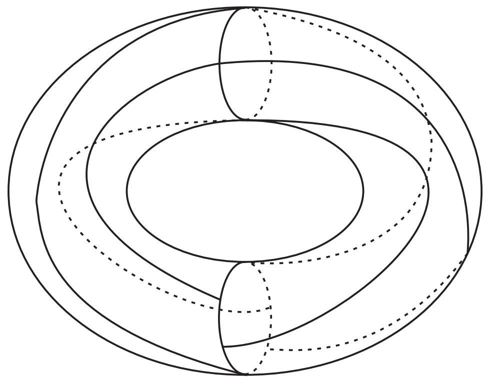

Figure 2

so it is not "simple". Suppose otherwise; then the totally geodesic right-angled k-polygon P which decomposes J is in general position with Γ, and so some faces of ∂J must be decomposed into a set of right-angled hyperbolic n-polygons by ∂P. Note that $n \geq 5$, so if ∂P enters a face f of $\partial J - \Gamma$, then it exits f from the opposite edge of f from where it enters. It is easy to see that ∂P is the image of the lines in $\mathbb{R}^2$ which have angles 0, $\pi/3$ or $2\pi/3$ with the horizontal lines.

Example 4.3. The graph Γ decomposes the torus illustrated in Figure 2 [Chen 2009] into three hexagons, say f_1, f_2 and f_3. We color f_i by $e_i \in (\mathbb{Z}_2)^3$ for $i = 1, 2, 3$. There are two sets of disks in Theorem 1.3. The first one consists of boundary parallel disks. The second one consists of essential disks, i.e., not boundary parallel.

For any embedding of (T^2, Γ) of Figure 2 into a solid torus J, the boundary parallel disks satisfy the assumption of Theorem 1.3. So if we embed (T^2, Γ) into a solid torus J by a map f so that the unique essential disk D (up to isotopy) intersects Γ in at least 5 points, then by Theorem 1.3, we get a right-angled hyperbolic structure on $(J, f(\Gamma))$. Note that for a fixed embedding of $\Gamma \to T^2$, there are only finitely many isotopy classes of simple closed curves which intersect Γ in at most 4 points.

In general, if the pair $(\partial N, \Gamma)$ admits a $(\mathbb{Z}_2)^3$-coloring and ∂N has genus at least one, then it may admit many colorings for a re-embedding of Γ into ∂N. This in turn induces many closed 3-manifolds from locally standard $(\mathbb{Z}_2)^3$-actions.

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