Pacific Journal of Mathematics

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Volume 263 No. 2 June 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 263, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.263.475

AN ANALOGUE TO THE WITT IDENTITY

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We solve combinatorial and algebraic problems associated with a multivariate identity first considered by Sherman, which he called an analog to the Witt identity. We extend previous results obtained for the univariate case.

1. Introduction

S. Sherman [1962] considered the formal identity in the indeterminates $z_1, \ldots, z_n$,

$$\prod_{m_1,\ldots,m_R \geq 0} (1 + z_1^{m_1} \cdots z_R^{m_R})^{N_+} (1 - z_1^{m_1} \cdots z_R^{m_R})^{N_-} = \prod_{j=1}^{R} (1 + z_j)^2, \tag{1-1}$$

where N_+ and N_- are the number of distinct classes of equivalence of nonperiodic closed paths with positive and negative signs, respectively, which traverse without backtracking m_i times edge i, $i = 1, \ldots, R$, of a graph G_R with $R > 1$ edges forming loops counterclockwise oriented and hooked to a single vertex, $\sum m_i \geq 1$.

Sherman [1962] refers to (1-1) as *an analog to the Witt identity*. The reason will become clear soon. The *Sherman identity*, as we call it, is a special nontrivial case of another identity called the *Feynman identity*, first conjectured by Richard Feynman. This identity relates the Euler polynomial of a graph to a formal product over the classes of equivalence of closed nonperiodic paths with no backtracking in the graph, and it is an important ingredient in a combinatorial formulation of the Ising model in two dimensions, much studied in physics. The Feynman identity was proved for planar and toroidal graphs by Sherman [1960], and in great generality by M. Loebl [2004] and D. Cimasoni [2010].

Sherman compared (1-1) with the multivariate Witt identity [Witt 1937]:

$$\prod_{m_1,\ldots,m_R \geq 0} (1 - z_1^{m_1} \cdots z_R^{m_R})^{\mathcal{M}(m_1,\ldots,m_R)} = 1 - \sum_{i=1}^{R} z_i, \tag{1-2}$$

$$\mathcal{M}(m_1, \ldots, m_R) = \sum_{g \mid m_1,\ldots,m_R} \frac{\mu(g)}{g} \frac{(N/g)!}{(N/g)(m_1/g)! \cdots (m_R/g)!} \tag{1-3}$$

MSC2010: primary 05C30; secondary 05C25, 05C38.
Keywords: Sherman identity, paths counting, (generalized) Witt formula, free Lie algebras.

where $N = m_1 + \cdots + m_R > 0$, μ is the Möbius function defined by the rules

(a) $\mu(+1) = +1$,

(b) $\mu(g) = 0$, for $g = p_1^{e_1} \cdots p_q^{e_q}$, $p_1, \ldots, p_q$ primes, and any $e_i > 1$,

(c) $\mu(p_1 \cdots p_q) = (-1)^q$.

The summation runs over all the common divisors of $m_1, \ldots, m_R$.

Originally, the Witt identity appeared associated with Lie algebras. In this context the formula gives the dimensions of the homogeneous subspaces of a finitely generated free Lie algebra L. If $L(m_1, \ldots, m_R)$ is the subspace of L generated by all homogeneous elements of multidegree $(m_1, \ldots, m_R)$, then $\dim L = \mathcal{M}$. However, formula (1-3) has many applications in combinatorics as well [Moree 2005]. Especially relevant is that $\mathcal{M}$ can be interpreted as the number of equivalence classes of closed nonperiodic paths which traverse counterclockwise the edges of G_R, the same graph associated to the Sherman identity (1-1). This property is stated in [Sherman 1962] without a proof, but this combinatorial interpretation of the Witt formula can be reinterpreted as a coloring problem of a necklace with N beads with colors chosen out of a set of R colors such that the colored beads form a nonperiodic configuration. In other words, $\mathcal{M}(m_1, \ldots, m_R)$ is the number of nonperiodic colored necklaces composed of m_i occurrences of the color i, $i = 1, \ldots, R$.

Sherman [1962] called attention to this association of identities (1-1) and (1-2) to paths in the same graph, which motivated him to consider the problem of finding a relation between (1-1) and Lie algebras. To interpret (1-1) in algebraic terms means to relate the exponents $N_\pm$ to some Lie algebraic data.

An investigation of Sherman's problem was initiated in [da Costa 1997; da Costa and Variane 2005] and a solution obtained for the univariate case of identity (1-1). In the present paper we solve the problem in the multivariate formal case, which requires important improvements. The counting method developed in [da Costa 1997; da Costa and Variane 2005] is based on a sign formula for a path given in terms of data encoded in the word representation for the path. It played a crucial role in getting formulas for $N_\pm$ in the univariate case. However, the counting method based on this sign formula is complicated. In the present paper we make improvements in the counting method in order to apply it to the multivariate case without depending too much on the sign formula. The formula is used here only to prove a simple lemma.

S-J. Kang and M-H. Kim [1999] derived dimension formulas for the homogeneous spaces of general free graded Lie algebras. We use some of their results to solve Sherman's problem. At the same time our results give a combinatorial realization for some of theirs in terms of paths in a graph.

The paper is organized as follows. In Section 2, we recall the word representation of a path and some basic definitions. We prove a basic lemma about the distribution of signs in the set of words of a given length. In Section 3, we compute formulas for the number of equivalence classes of closed nonperiodic paths of given length. The first of these generalizes Witt's formula in the sense that it counts paths that traverse the edges of the graph without backtracking. The other formulas give the exponents in Sherman's identity (1-1). We also interpret these formulas in terms of a coloring problem. Sherman's problem, that is, to give an algebraic meaning to the exponents in (1-1) is solved in Section 4.

2. Preliminaries

A path in G_R is an ordered sequence of the edges which does not necessarily respect their orientation. A path is closed and subjected to the constraint that it never goes immediately backwards over a previous edge.

Given $G_r \subseteq G_R$, denote by $i_1, \dots, i_r$ an enumeration of the edges of G_r in increasing order. A closed path of length $N \geq r$ in G_r is best represented by a word of the form

$$D_{j_1}^{e_{j_1}} D_{j_2}^{e_{j_2}} \cdots D_{j_l}^{e_{j_l}} \tag{2-1}$$

where $l = r, r+1, \dots, N$, $j_k \in \{i_1, \dots, i_r\}$, $j_k \neq j_{k+1}$, $j_l \neq j_1$, and

$$\sum_{k=1}^{l} |e_{j_k}| = N.$$

All edges of G_r are traversed by a path such that each i_k appears at least once in the sequence $S_l = (j_1, j_2, \dots, j_l)$. The order in which the symbols $D_j^{e_j}$ appear in the word indicates the edges traversed by p and in which order. If the sign of e_j is positive, the path traverses edge j exactly e_j times following the edge's orientation; if negative, it traverses the edge $|e_j|$ times in the opposite direction.

A word is called *periodic* if it equals

$$(D_{j_1}^{e_{j_1}} D_{j_2}^{e_{j_2}} \cdots D_{j_\alpha}^{e_{j_\alpha}})^g$$

for some $g > 1$. The number g is called the *period* of the word if the word in parentheses is nonperiodic.

Permuting the symbols $D_j^{e_j}$ in (2-1) cyclically, one gets l words that represent the same closed path. (For example, $D_1^{-2} D_2^{+1} D_1^{+1} D_2^{+3}$ is a cyclic permutation of $D_2^{+1} D_1^{+1} D_2^{-3} D_1^{-2}$.) Words obtained from one another by a cyclic permutation are taken to be equivalent for this reason. Although the word (2-1) and its inverse

$$D_{j_l}^{-e_{j_l}} \cdots D_{j_1}^{-e_{j_1}}$$

also represent the same path, they are not taken as equivalent here. This is the reason for the exponent 2 on the right side of (1-1), also present in [Sherman 1962].

In Section 3 we consider signed paths. The sign of a path is given by the formula

$$\mathrm{sign}(p) = (-1)^{1+n(p)}, \tag{2-2}$$

where $n(p)$ is the number of integral revolutions of the tangent vector of p. From this definition it follows that if $p = (h)^g$ is a periodic path with odd period g, then $\mathrm{sign}(p) = \mathrm{sign}(h)$. If g is even, $\mathrm{sign}(p) = -1$. The sign of a path can be computed from its word representation (2-1) using the formula [da Costa and Variane 2005]

$$(-1)^{N+l+T+s+1}, \tag{2-3}$$

where T is the number of subsequences in the decomposition of S_l into subsequences (see [da Costa and Variane 2005] for the definition and an example of a decomposition) and s is the number of negative exponents in (2-1). It follows from the previous sign formulas that periodic words with even period have negative sign.

The following lemma is important in the proof of several results in Section 3. It was assumed in [da Costa 1997; da Costa and Variane 2005] without a proof.

Lemma 2.1. *Given $G_r \subseteq G_R$, consider all paths that traverse each edge of G_r at least once (no backtracking allowed) and the set of all representative words (periodic or not, cyclic permutations and inversions included) of fixed length $N \geq r > 1$. Then half of the words have positive sign and the other half have negative sign.*

Proof. It suffices to consider the subset of words associated to a fixed sequence $S_l = (j_1, j_2, \ldots, j_l)$. For this sequence the numbers N, l, and T are fixed. The words with these numbers have signs which depend only on $s \in \{0, 1, 2, \ldots, l\}$. For $N + l + T$ even, the sign of a word is $(-1)^{s+1}$. If $l = 2k$, then, for each odd value of s, there are

$$\binom{2k}{s}$$

words with positive sign. Summing over the odd values of s, we get the total number of 2^{2k-1} words with positive sign. Summing over the even values of s, we get the same number of words with negative sign. If $l = 2k+1$, a similar counting gives 2^{2k} words with positive (negative) signs. The case $N + l + T$ odd is analogous. □

3. Counting paths in G_r

Fix a subgraph $G_r \subseteq G_R$. Given distinct edges $i_1, \ldots, i_r$ in G_r and positive integers $m_{i_1}, \ldots, m_{i_r}$, with $m_{i_1} + \cdots + m_{i_r} = N > r$, let $\theta_\pm(m_{i_1}, \ldots, m_{i_r})$ be the number of equivalence classes of closed nonperiodic paths of length N with $\pm$ signs that traverse each edge i_j exactly m_{i_j} times, for $j = 1, \ldots, r$, with no backtracking, and traverse the edges in $G_R \backslash G_r$ zero times. In this section we derive formulas for $\theta := \theta_+ + \theta_-$ and $\theta_\pm$. Notice that $\theta_\pm$ is just another name for the exponents $N_\pm$ in (1-1) showing only the nonzero entries in $N_\pm$.

Firstly, we compute θ. In the case $r = 1$, a path with $m_i > 1$ is periodic. The nonperiodic ones are two, the path with length $N = 1$ and its inversion so that $\theta(m_i) = 0$ if $m_i > 1$ and $\theta(m_i) = 2$ if $m_i = 1$. In other cases, θ is given as follows.

Theorem 3.1. *For $r = 2$, define*

$$\mathcal{F}\left(\frac{m_{i_1}}{g}, \frac{m_{i_2}}{g}\right) = \sum_{a=1}^{M/g} \frac{2^{2a}}{a} \binom{m_{i_1}/g - 1}{a - 1} \binom{m_{i_2}/g - 1}{a - 1}, \tag{3-1}$$

where $M = \min\{m_{i_1}, m_{i_2}\}$ and, if $r \geq 3$,

$$\mathcal{F}\left(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\right) = \sum_{a=r}^{N/g} \frac{2^a}{a} \sum_{\{S_a\}} \prod_{c=1}^{r} \binom{m_{i_c}/g - 1}{t_{i_c} - 1} \tag{3-2}$$

where $\{S_a\}$ is the set of sequences $(j_1, \dots, j_a)$ such that $j_k \in \{i_1, \dots, i_r\}$ and $j_k \neq j_{k+1}$, $j_a \neq j_1$. Number t_{i_c} counts how many times edge i_c occurs in a sequence S_a. Use is made of the convention that the combination symbol in (3-2) *is zero whenever $t_{i_c} > m_{i_c}/g$. Then*

$$\theta(m_{i_1}, \dots, m_{i_r}) = \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}\left(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\right). \tag{3-3}$$

The summation is over all the common divisors g of $m_{i_1}, \dots, m_{i_r}$, and $\mu(g)$ is the Möbius function.

Proof. The number $\mathcal{K}(l, m_{i_1}, \dots, m_{i_r})$ of words that have the same values of $m_{i_1}, \dots, m_{i_r}$ and $l \in \{r, r+1, \dots, N\}$ is given by

$$\mathcal{K}(l, m_{i_1}, \dots, m_{i_r}) = 2^l \sum_{\{S_l\}} \prod_{c=1}^{r} \binom{m_{i_c} - 1}{n_{i_c} - 1}.$$

Let's explain this formula a bit. The number n_{i_c} counts the number of occurrences of edge i_c in a sequence $S_l = (j_1, \dots, j_l)$. The combination symbol counts the number of unrestricted partitions of m_{i_c} into n_{i_c} nonzero positive parts [Andrews 1976]; thus the product times 2^l (there are 2^l ways of assigning $\pm$ signs to the exponents in (2-1)) gives the total number of words representing paths traversing each edge i_j of $G_r \subseteq G_R$ exactly m_{i_j} times in all possible ways. Then we sum over all sequences S_l with the convention that a combination symbol equals zero when $m < n$.

In the set of $\mathcal{K}(l, m_{i_1}, \dots, m_{i_r})$ words, there is the subset of nonperiodic words plus their cyclic permutations and inversions, and the subset of periodic words, if any, whose periods are the common divisors of $l, m_{i_1}, \dots, m_{i_r}$ plus their cyclic permutations and inversions. Denote by $\overline{\mathcal{K}(l, m_{i_1}, \dots, m_{i_r})}$ the number of elements in the former set. The words with period g are of the form

$$(D_{k_1}^{e_{k_1}} D_{k_2}^{e_{k_2}} \cdots D_{k_\alpha}^{e_{k_\alpha}})^g$$

where $\alpha = l/g$ and $D_{k_1}^{e_{k_1}} D_{k_2}^{e_{k_2}} \cdots D_{k_\alpha}^{e_{k_\alpha}}$ is nonperiodic so that the number of periodic words with period g plus their cyclic permutations and inversions is given by $\overline{\mathcal{K}(l/g, m_{i_1}/g, \ldots, m_{i_r}/g)}$. Therefore,

$$\mathcal{K}(l, m_{i_1}, \ldots, m_{i_r}) = \sum_{g|l,k,m_{i_1},\ldots,m_{i_r}} \overline{\mathcal{K}\left(\frac{l}{g}, \frac{m_{i_1}}{g}, \ldots, \frac{m_{i_r}}{g}\right)}.$$

The summation is over all the common divisors g of $l, m_{i_1}, \ldots, m_{i_r}$.

Applying the Möbius inversion formula [Apostol 1976], it follows that

$$\overline{\mathcal{K}(l, m_{i_1}, \ldots, m_{i_r})} = \sum_{g|(l,m_{i_1},\ldots,m_{i_r})} \mu(g)\mathcal{K}\left(\frac{l}{g}, \frac{m_{i_1}}{g}, \ldots, \frac{m_{i_r}}{g}\right). \tag{3-4}$$

where μ is the Möbius function. To eliminate cyclic permutations divide (3-4) by l. Summing over all possible values of l one gets

$$\theta(m_{i_1}, \ldots, m_{i_r}) = \sum_{l=r}^{N} \frac{\overline{\mathcal{K}(l, m_{i_1}, \ldots, m_{i_r})}}{l}. \tag{3-5}$$

Upon substitution of (3-4) into (3-5) one gets, for the case $r \geq 3$,

$$\theta(m_{i_1}, \ldots, m_{i_r}) = \sum_{l=r}^{N} \frac{1}{l} \sum_{g|(l,m_{i_1},\ldots,m_{i_r})} \mu(g) 2^{l/g} \sum_{\{S_{l/g}\}} \prod_{c=1}^{r} \binom{m_{i_c}/g - 1}{n_{i_c}/g - 1}.$$

Proceed now as follows. For a given common divisor g of $m_{i_1}, \ldots, m_{i_r}$, sum over all values of l which are multiples of g. Then sum over all possible divisors of $m_{i_1}, \ldots, m_{i_r}$. Write $l = ag$, and $n = tg$. If $r \geq 3$, one has $r/g \leq a \leq N/g$, but, unless $g = 1$, it is not admissible to have $a < r$, because all r edges of the graph should be traversed. For this reason, $r \leq a \leq N/g$. Result (3-2) follows. If $r = 2$, l is even and, for each l, only sequences of the form $(i_1, i_2, \ldots, i_1, i_2)$ with $n_{i_1} = n_{i_2} = l/2$ are possible. Put $l = 2a$, $a = 1, 2, \ldots, M = \min\{m_1, m_2\}$ to get (3-1). □

Example 1. From (3-1), we have

$$\begin{aligned}
&\mathcal{F}(1,1) = \mathcal{F}(1,2) = \mathcal{F}(2,1) = \mathcal{F}(1,3) = \mathcal{F}(3,1) = 4,\\
&\mathcal{F}(2,2) = 12,\\
&\mathcal{F}(1,4) = \mathcal{F}(4,1) = \mathcal{F}(1,5) = \mathcal{F}(5,1) = 4,\\
&\mathcal{F}(2,3) = \mathcal{F}(3,2) = 20,\\
&\mathcal{F}(2,4) = \mathcal{F}(4,2) = 28,\\
&\mathcal{F}(3,3) = 172/3.
\end{aligned}$$

From (3-3),

$$\begin{aligned}\theta(1,1) &= \theta(1,2) = \theta(2,1) = \theta(1,3)\\ &= \theta(3,1) = \theta(1,4) = \theta(4,1) = \theta(1,5) = \theta(5,1) = 4,\\ \theta(2,2) &= 10, \quad \theta(2,3) = \theta(3,2) = 20, \quad \theta(3,3) = 56.\end{aligned}$$

Example 2. From (3-2),

$$\begin{aligned}&\mathcal{F}(1,1,1) = 16,\\ &\mathcal{F}(1,1,2) = \mathcal{F}(1,2,1) = \mathcal{F}(2,1,1) = 32,\\ &\mathcal{F}(1,2,2) = \mathcal{F}(2,1,2) = \mathcal{F}(2,2,1) = 112,\\ &\mathcal{F}(1,1,3) = \mathcal{F}(1,3,1) = \mathcal{F}(3,1,1) = 48,\\ &\mathcal{F}(1,1,4) = \mathcal{F}(1,4,1) = \mathcal{F}(4,1,1) = 64,\\ &\mathcal{F}(1,2,3) = \mathcal{F}(3,1,2) = \mathcal{F}(2,3,1) = \mathcal{F}(3,2,1) = \mathcal{F}(1,3,2) = \mathcal{F}(2,1,3) = 256,\\ &\mathcal{F}(2,2,2) = 1056.\end{aligned}$$

From (3-3),

$$\begin{aligned}&\theta(1,1,1) = 16,\\ &\theta(1,1,2) = \theta(2,1,1) = \theta(1,2,1) = 32,\\ &\theta(1,2,2) = \theta(2,1,2) = \theta(2,2,1) = 112,\\ &\theta(1,1,3) = \theta(3,1,1) = \theta(1,3,1) = 48,\\ &\theta(1,1,4) = \theta(4,1,1) = \theta(1,4,1) = 64,\\ &\theta(1,2,3) = \theta(3,1,2) = \theta(2,3,1) = \theta(3,2,1) = \theta(1,3,2) = \theta(2,1,3) = 256,\\ &\theta(2,2,2) = 1048.\end{aligned}$$

Remark. (a) Notice that θ, and likewise the Witt formula, is given in terms of the Möbius function. However, formula (3-3) counts closed nonperiodic paths traversing the edges of G_R in all directions (without backtracking) and, in that sense, generalizes the Witt formula. Also, our formula has an algebraic meaning of a dimension. See Section 4.

(b) If $m_{i_1}, \dots, m_{i_r}$ are coprime, $\mathcal{F} = \theta$. Otherwise, $\mathcal{F}$ can be rational. For instance, $\mathcal{F}(3,3) = 172/3$. But $\mathcal{F}' := N\mathcal{F}$, $N = m_{i_1} + \cdots + m_{i_r}$, is always a positive integer which counts the number of words of length N. For example, in the case $N = 4$, $m_1 = m_2 = 2$, $\mathcal{F}' = 48$. The words are

$$D_1^{\pm 2} D_2^{\pm 2}, \quad D_1^{-1} D_2^{+1} D_1^{+1} D_2^{+1}, \quad D_1^{+1} D_2^{-1} D_1^{+1} D_2^{+1}, \quad D_1^{-1} D_2^{-1} D_1^{+1} D_2^{+1},$$
$$D_1^{-1} D_2^{+1} D_1^{+1} D_2^{-1}, \quad D_1^{-1} D_2^{-1} D_1^{-1} D_2^{+1}, \quad D_1^{-1} D_2^{-1} D_1^{+1} D_2^{-1},$$

plus four cyclic permutations for each of them, and the four periodic words $(D_1^{\pm 1} D_2^{\pm 1})^2$ plus two cyclic permutations for each.

In terms of $\mathcal{F}'$,

$$\theta(m_{i_1}, \ldots, m_{i_r}) = \frac{1}{N} \sum_{g | m_{i_1}, \ldots, m_{i_r}} \mu(g) \mathcal{F}' \left(\frac{m_{i_1}}{g}, \ldots, \frac{m_{i_r}}{g} \right).$$

Although the Möbius function is negative for some divisors g, the right hand side is nevertheless always a positive number because $\mathcal{F}'(m_{i_1}/g, \ldots, m_{i_r}/g)$ counts words in a subset of the words counted by $\mathcal{F}'(m_{i_1}, \ldots, m_{i_r})$.

(c) Given a circular necklace with N beads, consider the problem of counting inequivalent nonperiodic colorings of these beads with $2r$ colors $\{c_i, \bar{c}_i\}$, $i = 1, \ldots, r$, with m_i occurrences of the index i, $N = \sum m_i$, with the restriction that no two colors c_i and $\bar{c}_i$ (same index) occur adjacent in a coloring. Now, consider an oriented graph with r loops hooked to a single vertex. Each loop edge corresponds to a color c_i. A nonperiodic closed nonbacktracking path of length N in the graph corresponds to a coloring, and a color $\bar{c}_i$ corresponds to an edge being traversed in the opposite orientation. The presence of a single vertex in the graph reflects the fact that adjacent to a bead with, say, color c_i, any other with distinct index may follow. The number of inequivalent colorings is given by θ.

As a basic test of our counting ideas, we prove Sherman's statement [1962] relating the Witt formula to paths in G_R.

Proposition 3.2. *Relative to graph G_R, formula* (1-2) *gives the number $\mathcal{M}$ of equivalence classes of closed nonperiodic paths of length $N > 0$ that traverse each edge i counterclockwise $m_i \geq 0$ times* ($i = 1, 2, \ldots, R$), *where $m_1 + \cdots + m_R = N$.*

Proof. Denote by $m_{i_1}, \ldots, m_{i_r}$, $r \leq R$, the nonzero entries in $\mathcal{M}(m_1, \ldots, m_R)$, which we call $\mathcal{M}_r(m_{i_1}, \ldots, m_{i_r})$. Words representing counterclockwise paths have positive exponents so that the factors 2^{2a} and 2^a in formulas (3-1) and (3-2) are not needed. Hence

$$\mathcal{M}_r(m_{i_1}, \ldots, m_{i_r}) = \sum_{g | m_{i_1}, \ldots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}_c \left(\frac{m_{i_1}}{g}, \ldots, \frac{m_{i_r}}{g} \right) \tag{3-6}$$

where

$$\mathcal{F}_c \left(\frac{m_{i_1}}{g}, \frac{m_{i_2}}{g} \right) = \sum_{a=1}^{M/g} \frac{1}{a} \binom{m_{i_1}/g - 1}{a-1} \binom{m_{i_2}/g - 1}{a-1} \quad \text{if } r = 2, \tag{3-7}$$

with $M = \min\{m_{i_1}, m_{i_2}\}$, and

$$\mathcal{F}_c\left(\frac{m_{i_1}}{g},\dots,\frac{m_{i_r}}{g}\right)=\sum_{a=r}^{N/g}\frac{1}{a}\sum_{\{S_a\}}\prod_{c=1}^{r}\binom{m_{i_c}/g-1}{t_{i_c}-1}\quad \text{if } r\geq 3. \tag{3-8}$$

In the case $r=2$ suppose $m_{i_1}\leq m_{i_2}$. Using formula (A-3) from the Appendix (with $l=2$), it follows that

$$\begin{aligned}\sum_{a=1}^{m_{i_1}/g}\frac{1}{a}\binom{m_{i_1}/g-1}{a-1}\binom{m_{i_2}/g-1}{a-1}&=\frac{g}{m_{i_2}}\binom{m_{i_1}/g+m_{i_2}/g-1}{m_{i_1}/g}\\&=\frac{(N/g)!}{(N/g)(m_{i_1}/g)!(m_{i_2}/g)!}.\end{aligned}$$

Similarly if $m_{i_2}\leq m_{i_1}$. In the case $r\geq 3$ define

$$I=\sum_{\substack{m_i>0\\ m_{i_1}+\cdots+m_{i_r}=N}}\mathcal{F}_c\left(\frac{m_{i_1}}{g},\dots,\frac{m_{i_r}}{g}\right). \tag{3-9}$$

Upon substituting (3-8) into (3-9) and exchanging the summation symbols, we get

$$I=\sum_{a=r}^{N/g}\frac{1}{a}\sum_{\{S_a\}}\sum_{\substack{m_i>0\\ m_{i_1}+\cdots+m_{i_r}=N}}\prod_{c=1}^{r}\binom{\frac{m_{i_c}}{g}-1}{t_{i_c}-1}.$$

Applying Lemma A.2,

$$I=\sum_{a=r}^{N/g}\frac{1}{a}\sum_{\{S_a\}}\binom{N/g-1}{a-1}=\sum_{a=r}^{N/g}\frac{1}{a}\binom{N/g-1}{a-1}rw_r(a)$$

where

$$rw(a)=\sum_{j=1}^{r}(-1)^{r+j}\binom{r}{j}(j-1)^a+(-1)^{a+r}$$

is the number of sequences in $\{S_a\}$ [da Costa and Variane 2005]. Using that

$$\sum_{a=r}^{N/g}\frac{1}{a}\binom{N/g-1}{a-1}(j-1)^a=\frac{g}{N}(j^{N/g}-1)$$

and

$$\sum_{a=r}^{N/g}\frac{1}{a}\binom{N/g-1}{a-1}(-1)^{a+r}=(-1)^{r+1}\frac{g}{N},$$

we get

$$I=\frac{g}{N}\sum_{j=1}^{r}(-1)^{r+j}\binom{r}{j}j^{N/g}. \tag{3-10}$$

The Stirling numbers $S(N/g, r)$ of the second kind are given by [Chen and Koh 1992]

$$S\Big(\frac{N}{g}, r\Big) = \frac{1}{r!}\sum_{k=0}^{r}(-1)^k\binom{r}{k}(r-k)^{N/g} = \frac{1}{r!}\sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}j^{N/g} \tag{3-11}$$

so that

$$I = r!\frac{g}{N}S\Big(\frac{N}{g}, r\Big). \tag{3-12}$$

Stirling numbers have the property that

$$\sum_{\substack{m_i>0 \\ m_{i_1}+\cdots+m_{i_r}=N}} \frac{(N/g)!}{(m_{i_1}/g)!\cdots(m_{i_r}/g)!} = r!S\Big(\frac{N}{g}, r\Big). \tag{3-13}$$

Comparing relations (3-12), (3-13), and (3-9),

$$\mathcal{F}_c\Big(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\Big) = \frac{g}{N}\frac{(N/g)!}{(m_1/g)!\cdots(m_r/g)!}. \tag{3-14}$$

Upon substitution of (3-14) into (3-6), the result follows.

In the following we compute formulas for θ_+ and θ_-.

Theorem 3.3. *Suppose any of the following conditions is satisfied*:

(a) $N = m_{i_1} + \cdots + m_{i_r} < 2r$.

(b) $m_{i_1}, \dots, m_{i_r}$ *are coprime.*

(c) $m_{i_1}, \dots, m_{i_r}$ *are neither all odd nor all even.*

(d) $m_{i_1}, \dots, m_{i_r}$ *are all odd.*

Then

$$\theta_-(m_{i_1}, \dots, m_{i_r}) = \theta_+(m_{i_1}, \dots, m_{i_r}). \tag{3-15}$$

Proof. The proof is similar to that of [da Costa 1997, Theorem 1] and uses Lemma 2.1. □

Theorem 3.4. *The number* $\theta_+(m_{i_1}, \dots, m_{i_r})$ *is given by*

$$\theta_+(m_{i_1}, \dots, m_{i_r}) = \sum_{\text{odd } g|m_{i_1},\dots,m_{i_r}} \frac{\mu(g)}{g}\mathcal{G}\Big(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\Big) \tag{3-16}$$

where the summation is over all the common odd divisors of $m_{i_1}, \dots, m_{i_r}$, *and* $\mathcal{G} = \mathcal{F}/2$ *with* $\mathcal{F}$ *as in* (3-1) *and* (3-2). *If* $m_{i_1}, \dots, m_{i_r}$ *are all even numbers, then*

$$\theta_-(m_{i_1}, \dots, m_{i_r}) = \theta_+(m_{i_1}, \dots, m_{i_r}) - \theta_+\Big(\frac{m_{i_1}}{2}, \dots, \frac{m_{i_r}}{2}\Big). \tag{3-17}$$

Proof. First, suppose that all common divisors of $m_{i_1}, \dots, m_{i_r}$ are odd numbers. In this case,

$$\theta(m_{i_1}, \dots, m_{i_r}) = \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}\left(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\right).$$

Since $\theta = \theta_+ + \theta_-$ and $\theta_+ = \theta_-$ (Theorem 3.3) it follows that $\theta = 2\theta_+$, hence

$$\theta_+ = \tfrac{1}{2} \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}\left(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\right). \tag{3-18}$$

If the numbers $m_{i_1}, \dots, m_{i_r}$ are all even, θ_+ is again given by (3-18) because, in this case, the m_i's have common divisors which are even numbers, but since periodic words with even period have negative sign, only the odd divisors are relevant to getting θ_+. The reason one should have the factor $\frac{1}{2}$ is that, by Lemma 2.1, when one considers the set of all possible words representing paths of a given length which traverse the edges of G_r $m_{i_1}, \dots, m_{i_r}$ times, half of them have positive sign and the other half have negative sign. To account for the positive half, one needs the factor $\frac{1}{2}$. Let's now compute θ_- in the even case. Write

$$\begin{aligned}
\theta &= \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F} + \sum_{\text{even } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F} \\
&= \tfrac{1}{2} \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F} + \tfrac{1}{2} \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F} + \sum_{\text{even } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F} \\
&= 2\theta_+ + \sum_{\text{even } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}.
\end{aligned}$$

Using that $\theta = \theta_+ + \theta_-$, we obtain

$$\theta_- = \theta_+ + \sum_{\text{even } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{F}.$$

Now the relevant even divisors are $\{2n\}$ where n are the odd common divisors of $\{m_i\}$. For the other possible divisors, if any, use that $\mu(2^j n) = 0$, $j \geq 2$. Using the equality $\mu(2n) = -\mu(n)$, we see that the summation over the even divisors is equal to $-\theta_+\big(m_{i_1}/2, \dots, m_{i_r}/2\big)$, proving the result. $\square$

Remark. Like θ, the numbers $\theta_\pm$ can be interpreted as the number of inequivalent nonperiodic colorings of a circular necklace with N beads. However, now these colorings are classified as positive or negative according to formula (2-3). It is positive (negative) if the number $N + l + T + s$ is odd (even). In this case, s is the number $\bar{c}$ of colors present in a coloring. Interpret T in terms of the color indices.

Definition. Let $s_1, \dots, s_r$ be arbitrary positive integers. Let the number $\mathcal{P}$ be defined as follows. If $s_1, \dots, s_r$ are all even numbers,

$$\mathcal{P}(s_1, \dots, s_r) = \sum_{\text{even } g \mid s_1, \dots, s_r} \frac{\mu(g)}{g} \mathcal{G}\left(\frac{s_1}{g}, \dots, \frac{s_r}{g}\right), \tag{3-19}$$

otherwise, $\mathcal{P}(s_1, \dots, s_r) = 0$. Also, define

$$\mathcal{H}(s_1, \dots, s_r) = \begin{cases} \mathcal{G}(s_1, \dots, s_r) & \text{if } s_1, \dots, s_r \text{ not all even,} \\ \mathcal{G}(s_1, \dots, s_r) - \displaystyle\sum_{k \mid s_1, \dots, s_r} \frac{1}{k} \mathcal{P}\left(\frac{s_1}{k}, \dots, \frac{s_r}{k}\right) & \text{otherwise.} \end{cases} \tag{3-20}$$

Lemma 3.5. $$\mathcal{P} = \sum_{g \mid s_1, \dots, s_r} \frac{\mu(g)}{g} (\mathcal{G} - \mathcal{H}).$$

Proof. From the definition, $\mathcal{G} = \mathcal{H}$ if $s_1, \dots, s_r$ are not all even. Otherwise,

$$\mathcal{G} - \mathcal{H} = \sum_{g \mid s_1, \dots, s_r} \frac{1}{g} \mathcal{P}\left(\frac{s_1}{g}, \dots, \frac{s_r}{g}\right). \tag{3-21}$$

Now apply Lemma A.1 to get the result. □

Theorem 3.6. $$\theta_+(m_{i_1}, \dots, m_{i_r}) = \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{H}\left(\frac{m_{i_1}}{g}, \dots, \frac{m_{i_r}}{g}\right).$$

Proof. When $m_{i_1}, \dots, m_{i_r}$ are not all even, their odd divisors are the only possible common divisors. In this case, $\mathcal{P} = 0$ and

$$\theta_+ = \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{H}, \tag{3-22}$$

with $\mathcal{H} = \mathcal{G}$. If $m_{i_1}, \dots, m_{i_r}$ are all even, the sum over odd divisors of $m_{i_1}, \dots, m_{i_r}$ can be expressed as

$$\begin{aligned} \theta_+ &= \sum_{\text{odd } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{G} \\ &= \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{G} - \sum_{\text{even } g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{G} = \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{G} - \mathcal{P} \\ &= \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{G} - \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} (\mathcal{G} - \mathcal{H}) = \sum_{g \mid m_{i_1}, \dots, m_{i_r}} \frac{\mu(g)}{g} \mathcal{H}. \end{aligned}$$ □

Example 3.

$$\theta_\pm(1,1) = \theta_\pm(1,2) = \theta_\pm(2,1) = \theta_\pm(1,3) = \theta_\pm(3,1) = \theta_\pm(1,4) = \theta_\pm(4,1) = \theta_\pm(1,5) = \theta_\pm(5,1) = 2,$$

$$\theta_+(2,2) = 6, \quad \theta_-(2,2) = 4, \quad \theta_\pm(2,3) = \theta_\pm(3,2) = 10, \quad \theta_+(2,4) = 14,$$

$$\theta_-(2,4) = 12, \quad \theta_+(4,2) = 14, \quad \theta_-(4,2) = 12, \quad \theta_\pm(3,3) = 28.$$

Example 4.

$$\theta_\pm(1,1,1) = 8,$$
$$\theta_\pm(1,1,2) = \theta_\pm(2,1,1) = \theta_\pm(1,2,1) = 16,$$
$$\theta_\pm(1,2,2) = \theta_\pm(2,1,2) = \theta_\pm(2,2,1) = 56,$$
$$\theta_\pm(1,1,3) = \theta_\pm(3,1,1) = \theta_\pm(1,3,1) = 24,$$
$$\theta_\pm(1,1,4) = \theta_\pm(4,1,1) = \theta_\pm(1,4,1) = 32,$$
$$\theta_\pm(1,2,3) = \theta_\pm(3,1,2) = \theta_\pm(2,3,1) = \theta_\pm(3,2,1) = \theta_\pm(1,3,2) = \theta_\pm(2,1,3) = 128,$$
$$\theta_+(2,2,2) = 524,$$
$$\theta_-(2,2,2) = 516.$$

4. Sherman identity and Lie algebras

In this section we relate our previous results with Lie algebras and solve Sherman's problem. The solution is provided by the following proposition.

Proposition 4.1 [Kang and Kim 1999]. *Let* $V = \bigoplus_{(k_1,\dots,k_r)\in\mathbb{Z}^r_{>0}} V_{(k_1,\dots,k_r)}$ *be a* $\mathbb{Z}^r_{>0}$*-graded vector space over* $\mathbb{C}$ *with* $\dim V_{(k_1,\dots,k_r)} = d(k_1, \dots, k_r) < \infty$, *for all* $(k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$, *and let*

$$L = \bigoplus_{(k_1,\dots,k_r)\in\mathbb{Z}^r_{>0}} L_{(k_1,\dots,k_r)}$$

be the free Lie algebra generated by V. *Then the dimensions of the subspaces* $L_{(k_1,\dots,k_r)}$ *are given by*

$$\dim L_{(k_1,\dots,k_r)} = \sum_{g|(k_1,\dots,k_r)} \frac{\mu(g)}{g} \mathscr{W}\left(\frac{k_1}{g}, \dots, \frac{k_r}{g}\right) \tag{4-1}$$

where summation is over all common divisors g *of* $k_1, \dots, k_r$ *and* $\mathscr{W}$ *is given by*

$$\mathscr{W}(k_1, \dots, k_r) = \sum_{s\in T(k_1,\dots,k_r)} \frac{(|s|-1)!}{s!} \prod_{i_1,\dots,i_r=1}^{\infty} d(i_1, \dots, i_r)^{s_{i_1,\dots,i_r}}. \tag{4-2}$$

The exponents $s_{i_1,\dots,i_r}$ *are the components of* $s \in T$,

$$T(k_1,\dots,k_r) = \{s = (s_{i_1,\dots,i_r}) | s_{i_1,\dots,i_r} \in \mathscr{Z}_{\geq 0}, \sum_{i_1,\dots,i_r=1}^{\infty} s_{i_1,\dots,i_r}(i_1,\dots,i_r) = (k_1,\dots,k_r)\}, \tag{4-3}$$

and

$$|s| = \sum_{i_1,\dots,i_r=1}^{\infty} s_{i_1,\dots,i_r}, \quad s! = \prod_{i_1,\dots,i_r=1}^{\infty} s_{i_1,\dots,i_r}!. \tag{4-4}$$

Moreover, *the numbers* $\dim L_{(k_1,\dots,k_r)}$ *satisfy*

$$\prod_{k_1,\dots,k_r=1}^{\infty} (1 - z_1^{k_1} \cdots z_r^{k_r})^{\dim L_{(k_1,\dots,k_r)}} = 1 - f(z_1,\dots,z_r) \tag{4-5}$$

where

$$f(z_1,\dots,z_r) := \sum_{k_1,\dots,k_r=1}^{\infty} d(k_1,\dots,k_r) z_1^{k_1} \cdots z_r^{k_r}. \tag{4-6}$$

This function is associated with the generating function of the $\mathscr{W}$*'s*,

$$g(z_1,\dots,z_r) := \sum_{k_1,\dots,k_r=1}^{\infty} \mathscr{W}(k_1,\dots,k_r) z_1^{k_1} \cdots z_r^{k_r}, \tag{4-7}$$

by the relation

$$e^{-g} = 1 - f. \tag{4-8}$$

□

Identity (4-5) is a consequence of the famous *Poincaré–Birkhoff–Witt theorem* for the free Lie algebra. Computation of the formal logarithm of the left-hand side of (4-5) and its expansion gives that the infinite product equals the exponential in (4-8). Raise both members of (4-5) to the power -1, compute the formal logarithm of both members, and expand them. Identification of the coefficients of the same order, definition (4-2), and application of the Möbius inversion give (4-1). See [Kang and Kim 1999] for details. In [da Costa and Variane 2005], (4-1) is called the *generalized Witt formula*, $\mathscr{W}$ is called the *Witt partition function*, and (4-5) the *generalized Witt identity*.

Formulas (3-3) and (3-20) have exactly the form of (4-1) with corresponding *Witt partition functions* given by $\mathscr{F}$ and $\mathscr{H}$, respectively, so we interpret θ and θ_+ as giving the dimensions of the homogeneous spaces of graded Lie algebras. In each case, the algebra is generated by a graded vector space whose dimensions can be computed recursively from (4-2) as a function of the Witt partition function.

However, a general formula can be obtained from (4-8) using (4-6) as the formal Taylor expansion of $1 - e^{-g}$. This gives

$$d(k_1, \dots, k_r) = \frac{1}{k_1! \cdots k_r!} \frac{\partial^{|k|}}{\partial z_1^{k_1} \cdots \partial z_r^{k_r}} (1 - e^{-g})|_{z_1=\cdots=z_r=0} \tag{4-9}$$

with

$$g(z_1, \dots, z_r) := \sum_{k_1,\dots,k_r=1}^{\infty} \mathcal{W}(k_1, \dots, k_r) z^{k_1} \cdots z^{k_r} \tag{4-10}$$

and $\mathcal{W} = \mathcal{F}, \mathcal{H}$ given by (3-1), (3-2), and (3-20). Furthermore, $\dim L_{(k_1,\dots,k_r)} = \theta, \theta_+$ given by (3-3), 3.6 satisfy the *generalized Witt identity* (4-5) with the corresponding dimensions given by (4-9). In fact, an explicit formula for (4-9) can be derived:

Theorem 4.2. *The numbers $d(k_1, \dots, k_r)$ are given by the formula*

$$d(k_1, \dots, k_r) = \sum_{\lambda=1}^{|k|} (-1)^{\lambda+1} \sum_{p(\lambda,k)} \prod_{i=1}^{q} \frac{[\mathcal{W}(l_{i1}, \dots, l_{ir})]^{a_i}}{a_i!}, \tag{4-11}$$

where $|k| = k_1 + \cdots + k_r$, $q = -1 + \prod_{i=1}^{r}(k_i + 1)$, $p_{\lambda,k}$ is the set of all $a_i \in \{0, 1, 2, \dots\}$ such that $\sum_{i=1}^{q} a_i = \lambda$, $\sum_{i=1}^{q} a_i l_{ij} = k_j$, and the vectors $l_i = (l_{i1}, \dots, l_{ir})$, l_{ij} satisfying $0 \le l_{ij} \le k_j$, $\forall j = 1, \dots, r$, $\forall i = 1, \dots, q$ and $\sum_{j=1}^{r} l_{ij} > 0$. Set $\mathcal{W}(l_i) = 0$ if $l_{ij} = 0$ for some j; otherwise, $\mathcal{W}$ is the Witt partition function.

Proof. A generalization of Faà di Bruno's relation [Constantine and Savits 1996; Savits 2006] gives a formula for the $|k|$-th derivative of the exponential of a function $g(z_1, \dots, z_r)$. From this formula and (4-9), (4-11) follows. □

Example 5. We compute $d(2, 2)$ explicitly. In this case, $k_1 = k_2 = 2$, $|k| = 4$, $q = 8$. The possible vectors $l \le (2, 2)$ are $l_1 = (0, 1)$, $l_2 = (1, 0)$, $l_3 = (1, 1)$, $l_4 = (0, 2)$, $l_5 = (2, 0)$, $l_6 = (2, 1)$, $l_7 = (1, 2)$, and $l_8 = (2, 2)$. Next we give the values of $a_1, \dots, a_8 \ge 0$ satisfying

$$\sum_{i=1}^{8} a_i = \lambda, \quad \sum_{i=1}^{8} a_i l_i = (2, 2).$$

Define the vector $a = (a_1, \dots, a_8)$. The possible a's for each λ are

for $\lambda = 1$, $\quad a = (0, \dots, 0, 1)$;

for $\lambda = 2$, $\quad a = (0, 1, 0, 0, 0, 0, 1, 0),\ (0, 0, 2, 0, 0, 0, 0, 0),$
$(0, 0, 0, 1, 1, 0, 0, 0),\ (1, 0, 0, 0, 0, 1, 0, 0)$;

for $\lambda = 3$, $\quad a = (0, 2, 0, 1, 0, 0, 0, 0),\ (2, 0, 0, 0, 1, 0, 0, 0),\ (1, 1, 1, 0, 0, 0, 0, 0)$;

for $\lambda = 4$, $\quad a = (2, 2, 0, 0, 0, 0, 0, 0)$.

We get

$$d(2,2) = \mathcal{W}(2,2) - \tfrac{1}{2}\mathcal{W}(1,1)^2.$$

The dimensions up to $d(3,3)$ are

$$\begin{aligned}
N&=2, && d(1,1)=\mathcal{W}(1,1),\\
N&=3, && d(1,2)=\mathcal{W}(1,2), \quad d(2,1)=\mathcal{W}(2,1),\\
N&=4, && d(1,3)=\mathcal{W}(1,3), \quad d(3,1)=\mathcal{W}(3,1),\\
& && d(2,2)=\mathcal{W}(2,2)-\tfrac{1}{2}\mathcal{W}(1,1)^2,\\
N&=5, && d(1,4)=\mathcal{W}(1,4), \quad d(4,1)=\mathcal{W}(4,1),\\
& && d(2,3)=\mathcal{W}(2,3)-\mathcal{W}(1,1)\mathcal{W}(1,2),\\
& && d(3,2)=\mathcal{W}(3,2)-\mathcal{W}(1,1)\mathcal{W}(2,1),\\
N&=6, && d(1,5)=\mathcal{W}(1,5), \quad d(5,1)=\mathcal{W}(5,1),\\
& && d(2,4)=\mathcal{W}(2,4)-\mathcal{W}(1,1)\mathcal{W}(1,3)-\tfrac{1}{2}\mathcal{W}(1,2)^2,\\
& && d(4,2)=\mathcal{W}(4,2)-\mathcal{W}(1,1)\mathcal{W}(3,1)-\tfrac{1}{2}\mathcal{W}(2,1)^2\\
& && d(3,3)=\mathcal{W}(3,3)-\mathcal{W}(1,1)\mathcal{W}(2,2)-\mathcal{W}(1,2)\mathcal{W}(2,1)+\tfrac{1}{6}\mathcal{W}(1,1)^3.
\end{aligned}$$

For $r=3$, the dimensions up to $d(2,2,2)$ are

$$\begin{aligned}
N&=3, && d(1,1,1)=\mathcal{W}(1,1,1),\\
N&=4, && d(1,1,2)=\mathcal{W}(1,1,2), \quad d(1,2,1)=\mathcal{W}(1,2,1), \quad d(2,1,1)=\mathcal{W}(2,1,1),\\
N&=5, && d(1,2,2)=\mathcal{W}(1,2,2), \quad d(2,1,2)=\mathcal{W}(2,1,2), \quad d(2,2,1)=\mathcal{W}(2,2,1),\\
& && d(1,1,3)=\mathcal{W}(1,1,3), \quad d(1,3,1)=\mathcal{W}(1,3,1), \quad d(3,1,1)=\mathcal{W}(3,1,1),\\
N&=6, && d(1,1,4)=\mathcal{W}(1,1,4), \quad d(1,4,1)=\mathcal{W}(1,4,1), \quad d(4,1,1)=\mathcal{W}(4,1,1),\\
& && d(1,2,3)=\mathcal{W}(1,2,3), \quad d(3,1,2)=\mathcal{W}(3,1,2), \quad d(2,3,1)=\mathcal{W}(2,3,1),\\
& && d(3,2,1)=\mathcal{W}(3,2,1), \quad d(1,3,2)=\mathcal{W}(1,3,2), \quad d(2,1,3)=\mathcal{W}(2,1,3),\\
& && d(2,2,2)=\mathcal{W}(2,2,2)-\tfrac{1}{2}\mathcal{W}^2(1,1,1).
\end{aligned}$$

Example 6. Relative to θ with $\mathcal{W}=\mathcal{G}$ and applying data from previous examples, for the case $r=2$, we find the dimensions

$$\begin{aligned}
d(1,1)&=d(1,2)=d(2,1)=d(1,3)=d(3,1)=d(1,4)=d(4,1)=d(2,3)\\
&=d(3,2)=d(1,5)=d(5,1)=d(2,2)=d(2,4)=d(4,2)=d(3,3)=4.
\end{aligned}$$

In the case $r = 3$, the dimensions are

$d(1,1,1) = 8$,

$d(1,1,2) = d(2,1,1) = d(1,2,1) = 16, \quad d(1,2,2) = d(2,1,2) = d(2,2,1) = 56$,

$d(1,1,3) = d(3,1,1) = d(1,3,1) = 24, \quad d(1,1,4) = d(4,1,1) = d(1,4,1) = 32$,

$d(1,2,3) = d(3,1,2) = d(2,3,1) = d(3,2,1) = d(1,3,2) = d(2,1,3) = 128$,

$d(2,2,2) = 496$.

Example 7. Relative to θ_+ with $\mathcal{W} = \mathcal{H}$, we find for the case $r = 2$

$$\begin{aligned} d(1,1) &= d(1,2) = d(2,1) = d(1,3) = d(3,1) = d(1,4) = d(4,1) = d(2,3) \\ &= d(3,2) = d(1,5) = d(5,1) = 2, \\ d(2,2) &= 5, \quad d(2,4) = d(4,2) = 9, \quad d(3,3) = 28, \end{aligned}$$

and for $r = 3$

$d(1,1,1) = 8$,

$d(1,1,2) = d(2,1,1) = d(1,2,1) = 16, \quad d(1,2,2) = d(2,1,2) = d(2,2,1) = 56$,

$d(1,1,3) = d(3,1,1) = d(1,3,1) = 24, \quad d(1,1,4) = d(4,1,1) = d(1,4,1) = 32$,

$d(1,2,3) = d(3,1,2) = d(2,3,1) = d(3,2,1) = d(1,3,2) = d(2,1,3) = 128$,

$d(2,2,2) = 504$.

Remark. In spite of the negative terms in the formulas for the dimensions, they give positive results. To understand why, consider, for example, the case

$$d(2,2) = \mathcal{W}(2,2) - \tfrac{1}{2}\mathcal{W}(1,1)^2$$

with $\mathcal{W}(a,b) = \mathcal{F}' = (a+b)\mathcal{F}$. So $d(2,2)$ is four times the result in example 6. In the set of words counted by $\mathcal{F}'(2,2) = 48$ there is a subset whose elements are words that are obtained by gluing together the words in the set counted by $\mathcal{W}(1,1) = 8$. The gluing produces an overcounting which is corrected by the one half factor. So $d(2,2)$ is positive. The same argument can be used to get positivity for the other formulas.

Theorem 4.3. *For each $G_r \subseteq G_R$, we have*

$$\prod_{m_{i_1},\dots,m_{i_r}=1}^{\infty} (1 + z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_+} = e^{-g(z_{i_1}^2,\dots,z_{i_r}^2) + g(z_{i_1},\dots,z_{i_r})}, \tag{4-12}$$

and

$$\prod_{m_{i_1},\dots,m_{i_r}=1}^{\infty} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_-} = e^{+g(z_{i_1}^2,\dots,z_{i_r}^2) - g(z_{i_1},\dots,z_{i_r})}. \tag{4-13}$$

Proof. To prove (4-12), multiply and divide its left-hand side by

$$\prod_{m_{i_1},\dots,m_{i_r}=1}^{\infty} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_+}$$

and use (4-8). To get (4-13), write

$$\prod_{m_{i_1},\dots,m_{i_r}=1}^{\infty} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_-} = \prod_{N=r}^{\infty} \prod_{\substack{m_i>0 \\ m_{i_1}+\cdots+m_{i_r}=N}} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_-}.$$

Decompose the product over N into three products, namely, one over all $N < 2r$, one over all even $N \geq 2r$, and another over all odd $N > 2r$. Then apply Theorems 3.3 and 3.4 and formula (3-17). □

Theorem 4.4. $$\prod_{m_{i_1},\dots,m_{i_r}=1}^{\infty} (1 + z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_+} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_-} = 1.$$

Proof. Multiply (4-12) and (4-13). □

The left side of (1-1) equals

$$\prod_{j=1}^{R} (1 + z_j)^2 \prod_{r=2}^{R} \prod_{G_r} \prod_{m_{i_1},\dots,m_{i_r}>0} (1 + z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_+} (1 - z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}})^{\theta_-}.$$

The Sherman identity now follows from Theorem 4.4.

Appendix

Lemma A.1. *If*

$$g(n_1, \dots, n_k) = \sum_{d|n_1,\dots,n_k} \frac{\mu(d)}{d} f\left(\frac{n_1}{d}, \dots, \frac{n_k}{d}\right), \tag{A-1}$$

then

$$f(n_1, \dots, n_k) = \sum_{d|n_1,\dots,n_k} \frac{1}{d} g\left(\frac{n_1}{d}, \dots, \frac{n_1}{d}\right). \tag{A-2}$$

Proof. Set $G(n_1, \dots, n_k) := (n_1 + \cdots + n_k) g(n_1, \dots, n_k)$ and

$$F\left(\frac{n_1}{d}, \dots, \frac{n_k}{d}\right) := \left(\frac{n_1}{d} + \cdots + \frac{n_k}{d}\right) f\left(\frac{n_1}{d}, \dots, \frac{n_k}{d}\right).$$

Then (A-1) can be expressed in the form

$$G(n_1, \dots, n_k) = \sum_{d|n_1,\dots,n_k} \mu(d) F\left(\frac{n_1}{d}, \dots, \frac{n_1}{d}\right).$$

Möbius inversion gives

$$F(n_1,\dots,n_k)=\sum_{d\mid n_1,\dots,n_k} G\Big(\frac{n_1}{d},\dots,\frac{n_k}{d}\Big).$$

Therefore,

$$(n_1+\cdots+n_k)f(n_1,\dots,n_k)=\sum_{d\mid n_1,\dots,n_k}\Big(\frac{n_1}{d}+\cdots+\frac{n_k}{d}\Big)g\Big(\frac{n_1}{d},\dots,\frac{n_k}{d}\Big). \qquad \square$$

The converse is also true.

Lemma A.2. *Let* $N\geq\alpha=n_1+\cdots+n_l$, $n_1,\dots,n_l$, $n_i>0$, *be a partition of* α. *Then*

$$\sum_{\sum_{i=1}^{l}k_i=N}\prod_{i=1}^{l}\binom{k_i-1}{n_i-1}=\binom{N-1}{\alpha-1} \tag{A-3}$$

with the convention that a bracket in the left side is zero whenever $k_i<n_i$.

Proof. Using

$$\frac{q^\alpha}{(1-q)^\alpha}=\sum_{N=\alpha}^{\infty}\binom{N-1}{\alpha-1}q^N,$$

it follows that

$$\frac{q^\alpha}{(1-q)^\alpha}=\prod_{i=1}^{l}\frac{q^{n_i}}{(1-q)^{n_i}}=\prod_{i=1}^{l}\sum_{k_i=n_i}^{\infty}\binom{k_i-1}{n_i-1}q^{k_i}=\sum_{N=\alpha}^{\infty}\sum_{\substack{k_i\geq n_i\\ \sum_{i=1}^{l}k_i=N}}\prod_{i=1}^{l}\binom{k_i-1}{n_i-1}q^N.$$

Comparison with the previous expression and the convention gives the result. $\square$

Acknowledgements

We thank Professor Peter Moree (Max Planck Institute for Mathematics, Bonn) and Professor Thomas Ward (University of East Anglia, UK) for email correspondence regarding the positivity of the Möbius inversion formula.

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Received May 5, 2012.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 263 No. 2 June 2013

0030-8730(201306)263:2;1-5