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ON THE CLASSIFICATION OF STABLE SOLUTIONS TO BIHARMONIC PROBLEMS IN LARGE DIMENSIONS

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We give a new bound on the exponent for nonexistence of stable solutions to the biharmonic problem $\Delta^2 u = u^p$ in $\mathbb{R}^n$, where $u > 0$, $p > 1$, and $n \geq 20$.

1. Introduction

Of concern is the biharmonic equation

$$\Delta^2 u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^n \tag{1-1}$$

where $n \geq 5$ and $p > 1$. Set

$$\Lambda_u(\varphi) := \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx - p \int_{\mathbb{R}^n} u^{p-1}\varphi^2 dx \quad \text{for all } \varphi \in H^2(\mathbb{R}^n). \tag{1-2}$$

The Morse index $\operatorname{ind}(u)$ of a classical solution to (1-1) is defined as the maximal dimension of all subspaces of $H^2(\mathbb{R}^n)$ such that $\Lambda_u(\varphi) < 0$ in $H^2(\mathbb{R}^n) \setminus \{0\}$. We say u is a stable solution to (1-1) if $\Lambda_u(\varphi) \geq 0$ for any test function $\varphi \in H^2(\mathbb{R}^n)$; that is, if the Morse index is zero.

In the first part of the paper, we obtain the following classification result on stable solutions of (1-1).

Theorem 1.1. *Let $n \geq 20$ and $1 < p < 1 + \frac{8p^*}{n-4}$. Then* (1-1) *has no stable solutions.*

Here p^* stands for the smallest real root greater than $\frac{n-4}{n-8}$ of the algebraic equation

$$\begin{aligned}512(2-n)x^6 &+ 4(n^3-60n^2+670n-1344)x^5 - 2(13n^3-424n^2+3064n-5408)x^4 \\ &+ 2(27n^3-572n^2+3264n-5440)x^3 - (49n^3-772n^2+3776n-5888)x^2 \\ &+ 4(5n^3-66n^2+288n-416)x - 3(n^3-12n^2+48n-64) = 0.\end{aligned}$$

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Some remarks are in order. Let us recall that for the second-order problem

$$\Delta u + u^p = 0 \quad u > 0 \text{ in } \mathbb{R}^n, \ \ p > 1, \tag{1-3}$$

Farina gave a complete classification of all finite Morse index solutions. The main result of [Farina 2007] is that no stable solution exists to (1-3) if either $n \le 10$, $p > 1$ or $n \ge 11$, $p < p_{JL}$. Here p_{JL} denotes the Joseph–Lundgren exponent [Gui et al. 1992]. On the other hand, a stable radial solution exists for $p \ge p_{JL}$.

For the fourth-order case, the nonexistence of positive solutions to (1-1) is shown if $p < \frac{n+4}{n-4}$, and all entire solutions are classified if $p = \frac{n+4}{n-4}$. See [Lin 1998; Wei and Xu 1999]. When $p > \frac{n+4}{n-4}$, radially symmetric solutions to (1-1) are completely classified in [Ferrero et al. 2009; Gazzola and Grunau 2006; Guo and Wei 2010]. The radial solutions are shown to be stable if and only if $p \ge p'_{JL}$ and $n \ge 13$, where p'_{JL} stands for the corresponding Joseph–Lundgren exponent (see [Ferrero et al. 2009; Gazzola and Grunau 2006]). In the general nonradial case, Wei and Ye [Wei and Ye 2010] showed the nonexistence of stable or finite Morse index solutions when either $n \le 8$, $p > 1$ or $n \ge 9$, $p \le \frac{n}{n-8}$. In dimensions $n \ge 9$, a perturbation argument is used to show the nonexistence of stable solutions for $p < \frac{n}{n-8} + \varepsilon_n$ for some $\varepsilon_n > 0$. However, no explicit value of ε_n was given. The proof of Wei and Ye [2010] follows an earlier idea of Cowan, Esposito and Ghoussoub [2010] in which a similar problem in a bounded domain was studied. Theorem 1.1 gives an explicit value on ε_n for $n \ge 20$.

In the second-order case, the proof of Farina uses basically the Moser iterations: namely multiply (1-3) by the power of u, like u^q, $q > 1$. Moser iteration works because of the following simple identity

$$\int_{\mathbb{R}^n} u^q(-\Delta u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} |\nabla u^{\frac{q+1}{2}}|^2, \forall u \in C_0^1(\mathbb{R}^n).$$

In the fourth-order case, such equality does not hold, and in fact we have

$$\int_{\mathbb{R}^n} u^q(\Delta^2 u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} |\Delta u^{\frac{q+1}{2}}|^2 - q(q-1)^2 \int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4, \forall u \in C_0^2(\mathbb{R}^n).$$

The additional term $\int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4$ makes the Moser iteration argument difficult to use. Wei and Ye [2010] used instead the new test function $-\Delta u$ and showed that $\int_{\mathbb{R}^2} |\Delta u|^2$ is bounded. Thus the exponent $\frac{n}{n-8}$ is obtained. In this paper, we use the Moser iteration for the fourth-order problem and give a control on the term $\int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4$ (Lemma 2.3). As a result, we obtain a better exponent $\frac{n}{n-8} + \varepsilon_n$ where ε_n is explicitly given. As far as we know, this seems to be the first result for Moser iteration for a fourth-order problem.

In the second part of this paper, we show that the same idea can be used to establish the regularity of extremal solutions to

$$\begin{cases} \Delta^2 u = \lambda(u+1)^p, \ \lambda > 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-4}$$

where Ω is a smooth and bounded convex domain in $\mathbb{R}^n$.

For problem (1-4), it is known [Berchio and Gazzola 2005] that for $p > \frac{n+4}{n-4}$ there exists a critical value $\lambda^* > 0$ depending on $p > 1$ and Ω such that

- If $\lambda \in (0, \lambda^*)$, (1-4) has a minimal and classical solution which is stable;
- If $\lambda = \lambda^*$, a unique weak solution, called the extremal solution u^* exists for (1-4);
- No weak solution of (1-4) exists whenever $\lambda > \lambda^*$.

The regularity of the extremal solution of problem (1-4) at $\lambda = \lambda_*$ has been studied in [Cowan et al. 2010; Wei and Ye 2010], where it was shown that the extremal solution is bounded provided $n \le 8$ or $p < \frac{n}{n-8} + \varepsilon_n$, $n \ge 9$ (ε_n very small). Here, we also give a explicit bound for the exponent p in large dimensions and our second result is the following.

Theorem 1.2. *The extremal solution u^* of* (1-4) *when $\lambda = \lambda^*$ is bounded provided that $n \ge 20$ and $1 < p < 1 + \frac{8p^*}{n-4}$, where p^* is defined as above.*

As $n \to +\infty$, the value ε_n is asymptotically $8\sqrt{8/3}/(n-8)^{3/2}$ and thus the upper bound for p has the expansion

$$1 + \frac{8}{n-8} + \frac{8\sqrt{8/3}}{(n-8)^{3/2}} + O\left(\frac{1}{(n-8)^2}\right). \tag{1-5}$$

On the other hand, for radial solutions, the Joseph–Lundgren exponent [Gui et al. 1992] has the following asymptotic expansion

$$1 + \frac{8}{n-8} + \frac{16}{(n-8)^{3/2}} + O\left(\frac{1}{(n-8)^2}\right). \tag{1-6}$$

In this paper, we have only considered fourth-order problems with power-like nonlinearity. Other kinds of nonlinearity, such as exponential and negative powers, also appear in many applications; see [Cowan et al. 2010]. However, our technique here yields no improvements of results of that reference in the case of exponential and negative nonlinearities.

This paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 respectively in Section 2 and Section 3. Some technical inequalities are given in the Appendix.

2. Proof of Theorem 1.1

Lemma 2.1. *For any $\varphi \in C_0^4(\mathbb{R}^n)$ with $\varphi \geq 0$, any $\gamma > 1$ and $\varepsilon > 0$ an arbitrary small number, we have*

$$(2\text{-}1)\quad \int_{\mathbb{R}^n} (\Delta(u^\gamma \varphi^\gamma))^2 \leq \int_{\mathbb{R}^n} ((\Delta u^\gamma \varphi^\gamma)^2 + \varepsilon |\nabla u|^4 \varphi^{2\gamma} u^{2\gamma-4} + C u^{2\gamma} \|\nabla^4(\varphi^{2\gamma})\|),$$

$$(2\text{-}2)\quad \int_{\mathbb{R}^n} (\Delta(u^\gamma \varphi^\gamma))^2 \geq \int_{\mathbb{R}^n} ((\Delta u^\gamma \varphi^\gamma)^2 - \varepsilon |\nabla u|^4 \varphi^{2\gamma} u^{2\gamma-4} - C u^{2\gamma} \|\nabla^4(\varphi^{2\gamma})\|),$$

$$(2\text{-}3)\quad \int_{\mathbb{R}^n} ((u^\gamma)_{ij})^2 \varphi^{2\gamma} \leq \int_{\mathbb{R}^n} ((u^\gamma \varphi^\gamma)_{ij})^2 + \varepsilon \int_{\mathbb{R}^n} |\nabla u|^4 u^{2\gamma-4} \varphi^{2\gamma} + C \int_{\mathbb{R}^n} u^{2\gamma} \|\nabla^4(\varphi^{2\gamma})\|,$$

where C is a positive number that only depends on γ and ε, and $\|\nabla^4(\varphi^{2\gamma})\|$ is defined by

$$\|\nabla^4(\varphi^{2\gamma})\|^2 = \varphi^{-2\gamma} |\nabla \varphi^\gamma|^4 + |\varphi^\gamma (\Delta^2 \varphi^\gamma)| + |\nabla^2 \varphi^\gamma|^2.$$

In the following, unless said otherwise, the constant C always denotes a positive number which may change term by term but only depends on γ, ε.

Proof. Since φ is compactly supported, we can use integration by parts without considering the boundary terms. First, by direct calculation, we get

$$(2\text{-}4)\quad (\Delta(u^\gamma \varphi^\gamma))^2 = [(\Delta u^\gamma)\varphi^\gamma]^2 + 4\nabla u^\gamma \nabla \varphi^\gamma \Delta \varphi^\gamma u^\gamma + 4\nabla u^\gamma \nabla \varphi^\gamma \Delta u^\gamma \varphi^\gamma + 4(\nabla u^\gamma \nabla \varphi^\gamma)^2 + 2\Delta u^\gamma u^\gamma \Delta \varphi^\gamma \varphi^\gamma + u^{2\gamma} (\Delta \varphi^\gamma)^2.$$

We now need to deal with the third and fifth terms on the right side of this equality, up to the integration of both sides.

For the third term, we have

$$\int_{\mathbb{R}^n} \Delta u^\gamma \nabla u^\gamma \nabla \varphi^\gamma \varphi^\gamma = -\int_{\mathbb{R}^n} (u^\gamma)_i (u^\gamma)_{ij} (\varphi^\gamma)_j \varphi^\gamma - \int_{\mathbb{R}^n} (u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_{ij} \varphi^\gamma - \int_{\mathbb{R}^n} (u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_j (\varphi^\gamma)_i,$$

where $f_i = \partial f / \partial x_i$ and $f_{ij} = \partial^2 f / \partial x_j \partial x_i$. (Here and in the sequel, we use the Einstein summation convention, so for example $\partial_i (u_i u_j \varphi_j) = \sum_{1 \leq i,j \leq n} \partial_i (u_i u_j \varphi_j)$.) The first term on the right side of the previous equation can be estimated as

$$2\int_{\mathbb{R}^n} (u^\gamma)_i (u^\gamma)_{ij} (\varphi^\gamma)_j \varphi^\gamma = \int_{\mathbb{R}^n} \partial_j ((u^\gamma)_i (u^\gamma)_i (\varphi^\gamma)_j \varphi^\gamma) - \int_{\mathbb{R}^n} ((u^\gamma)_i)^2 (\varphi^\gamma)_{jj} \varphi^\gamma - \int_{\mathbb{R}^n} ((u^\gamma)_i)^2 (\varphi^\gamma)_j (\varphi^\gamma)_j.$$

Combining these two equalities, we get

$$\begin{aligned}2\int_{\mathbb{R}^n}\Delta u^\gamma\nabla u^\gamma\nabla\varphi^\gamma\varphi^\gamma = &-\int_{\mathbb{R}^n}\partial_j\big((u^\gamma)_i(u^\gamma)_i(\varphi^\gamma)_j\varphi^\gamma\big)\\ &-\int_{\mathbb{R}^n}2(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_{ij}\varphi^\gamma-\int_{\mathbb{R}^n}2(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_j(\varphi^\gamma)_i\\ &+\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_{jj}\varphi^\gamma+\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_j(\varphi^\gamma)_j.\end{aligned}$$

Rewriting this equality we have

$$\begin{aligned}4\int_{\mathbb{R}^n}\Delta u^\gamma\nabla u^\gamma\nabla\varphi^\gamma\varphi^\gamma = {}&2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2\Delta\varphi^\gamma\varphi^\gamma+2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2\\ &-4\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_{ij}\varphi^\gamma-4\int_{\mathbb{R}^n}\langle\nabla u^\gamma,\nabla\varphi^\gamma\rangle^2.\end{aligned}\tag{2-5}$$

For the fifth term on the right side of (2-4) we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Delta u^\gamma u^\gamma\Delta\varphi^\gamma\varphi^\gamma = {}&-\int_{\mathbb{R}^n}u^\gamma\langle\nabla u^\gamma,\nabla(\Delta\varphi^\gamma)\rangle\varphi^\gamma\\ &-\int_{\mathbb{R}^n}\langle\nabla u^\gamma,\nabla\varphi^\gamma\rangle u^\gamma\Delta\varphi^\gamma-\int_{\mathbb{R}^n}|\nabla u^\gamma|^2\Delta\varphi^\gamma\varphi^\gamma.\end{aligned}\tag{2-6}$$

Combining (2-4), (2-5) and (2-6), one obtains

$$\begin{aligned}&\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2-\int_{\mathbb{R}^n}(\Delta u^\gamma)^2\varphi^{2\gamma}\\ &\quad=2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2-4\int_{\mathbb{R}^n}\varphi^\gamma(\nabla^2\varphi^\gamma(\nabla u^\gamma,\nabla u^\gamma))\\ &\qquad+\int_{\mathbb{R}^n}u^{2\gamma}\varphi^\gamma\Delta^2(\varphi^\gamma)-2\int_{\mathbb{R}^n}u^{2\gamma}(\Delta\varphi^\gamma)^2.\end{aligned}\tag{2-7}$$

Now by the Young equality, for any $\varepsilon>0$, there exists a constant $C=C(\gamma,\varepsilon)$ such that

$$|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2\le\frac{\varepsilon}{4}|\nabla u^\gamma|^4u^{-2\gamma}\varphi^{2\gamma}+C|\nabla\varphi^\gamma|^4u^{2\gamma}\varphi^{-2\gamma}$$

and

$$|\varphi^\gamma(\nabla^2\varphi^\gamma(\nabla u^\gamma,\nabla u^\gamma))|\le\frac{\varepsilon}{8}|\nabla u^\gamma|^4u^{-2\gamma}\varphi^{2\gamma}+Cu^{2\gamma}|\nabla^2\varphi^\gamma|^2.$$

Thus by (2-7), together with the two estimates above, one gets

$$\left|\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2-\int_{\mathbb{R}^n}(\Delta u^\gamma)^2\varphi^{2\gamma}\right|\le\varepsilon\int_{\mathbb{R}^n}|\nabla u^\gamma|^4u^{-2\gamma}\varphi^{2\gamma}+6C\int_{\mathbb{R}^n}u^{2\gamma}\|\nabla^4\varphi^\gamma\|^2.$$

The estimates (2-1) and (2-2) follow from this easily.

Next we observe that $|\nabla^2 u^\gamma|^2\varphi^{2\gamma} = \big[\frac{1}{2}\Delta|\nabla u^\gamma|^2 - \langle \nabla u^\gamma, \nabla\Delta u^\gamma\rangle\big]\varphi^{2\gamma}$. Thus up to the integration by parts, with the help of (2-5) and the estimates we just proved, the estimate (2-3) also follows by noticing the identity $\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2 = \int_{\mathbb{R}^n}|\nabla^2(u^\gamma\varphi^\gamma)|^2$. The proof of Lemma 2.1 is thus completed. □

Let us return to the equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n. \tag{2-8}$$

Fix $q = 2\gamma - 1 > 0$ and $\gamma > 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Multiplying (2-8) by $u^q\varphi^{2\gamma}$ and integration by parts, we obtain

$$\int_{\mathbb{R}^n} \Delta u \Delta(u^q\varphi^{2\gamma}) = \int_{\mathbb{R}^n} u^{p+q}\varphi^{2\gamma}. \tag{2-9}$$

For the left side of (2-9), we have:

Lemma 2.2. *For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, for any $\varepsilon > 0$ and γ with q defined above, there exists a positive constant C depends on γ, ε such that*

$$\int_{\mathbb{R}^n} \frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma}) \geq \int_{\mathbb{R}^n}(\Delta u^\gamma\varphi^\gamma)^2 - \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| - \int_{\mathbb{R}^n}(\gamma^2(\gamma-1)^2+\varepsilon)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}. \tag{2-10}$$

Proof. First, by direct computations, we obtain

$$\begin{aligned}\Delta u\Delta(u^{2\gamma-1}\varphi^{2\gamma}) &= \Delta u\big((2\gamma-1)u^{2\gamma-2}\Delta u\varphi^{2\gamma} + 2(2\gamma-1)u^{2\gamma-2}\nabla u\nabla(\varphi^{2\gamma}) \\ &\quad + (2\gamma-1)(2\gamma-2)u^{2\gamma-3}|\nabla u|^2\varphi^{2\gamma} + u^{2\gamma-1}\Delta\varphi^{2\gamma}\big), \\ (\Delta u^\gamma\varphi^\gamma)^2 &= \gamma^2u^{2\gamma-2}(\Delta u)^2\varphi^{2\gamma} + \gamma^2(\gamma-1)^2u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} \\ &\quad + 2(\gamma-1)\gamma^2u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma}.\end{aligned}$$

Combining these two identities, we get

$$\frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma}) = (\Delta u^\gamma\varphi^\gamma)^2 + 2\gamma^2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma} + \frac{\gamma^2}{q}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma} - \gamma^2(\gamma-1)^2u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}. \tag{2-11}$$

For the term $u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma}$, we have

$$\begin{aligned}u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma} &= \partial_i(u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_j) - (2\gamma-2)u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j \\ &\quad - u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j - u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}.\end{aligned}$$

We can regroup the term $u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j$ as

$$2u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j=\partial_j(u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_j)-(2\gamma-2)u^{2\gamma-3}u_j(u_i)^2(\varphi^{2\gamma})_j \\ -u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj}.$$

Therefore we get

$$2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma}=2\partial_i(u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_j)-\partial_j(u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_j) \\ -(2\gamma-2)u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j+u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj} \\ -2u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}. \tag{2-12}$$

For the last three terms on the right side of (2-12), applying Young's inequality, we get

$$\begin{aligned}|u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j|&\le\frac{\varepsilon}{6\gamma^2(\gamma-1)}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\\ |u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj}|&\le\frac{\varepsilon}{6\gamma^2}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\\ |u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}|&\le\frac{\varepsilon}{6\gamma^2}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.\end{aligned}$$

These three inequalities and (2-12) imply

$$\int_{\mathbb{R}^n}2\gamma^2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma}\ge-\frac{\varepsilon}{2}\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}-C\int_{\mathbb{R}^n}u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2-13}$$

Similarly we get

$$\int_{\mathbb{R}^n}\frac{\gamma^2}{q}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma}\ge-\frac{\varepsilon}{2}\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}-C\int_{\mathbb{R}^n}u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2-14}$$

Inequality (2-10) follows from (2-11), (2-13) and (2-14). □

As a result of (2-1) and (2-10), we have

$$\int_{\mathbb{R}^n}\frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma})\ge\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2-\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| \\ -\int_{\mathbb{R}^n}(\gamma^2(\gamma-1)^2+\varepsilon)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}. \tag{2-15}$$

Next we estimate the most difficult term, $\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}$, in (2-15). This is the key step in proving Theorem 1.1.

Lemma 2.3. *If u is the classical solution to the biharmonic equation* (2-8), *and φ is defined as above*, *then for any sufficiently small $\varepsilon>0$, we have the following* *inequality*

$$\left(\tfrac{1}{2}-\varepsilon\right)\int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} \le \frac{2}{\gamma^2}\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2+\int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| - \int_{\mathbb{R}^n}\frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}. \tag{2-16}$$

Proof. It is easy to see that

$$\int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} = \frac{1}{\gamma^4}\int_{\mathbb{R}^n} u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma}, \tag{2-17}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} & u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n} u^{-2\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\nabla u^\gamma\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n} -\nabla u^{-\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\Delta u^\gamma\varphi^{2\gamma}+\int_{\mathbb{R}^n} u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma\varphi^{2\gamma}\\ &\qquad + \int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\nabla\varphi^{2\gamma},\end{aligned} \tag{2-18}$$

where in the last step we used integration by parts. For the first term in the last part of this equality, we have

$$\int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\Delta u^\gamma\varphi^{2\gamma} = \gamma^3\int_{\mathbb{R}^n}((\gamma-1)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma}).$$

Substituting this into (2-18) and combining with (2-17), we obtain

$$\int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} = \int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma\varphi^{2\gamma} + \int_{\mathbb{R}^n} u^{2\gamma-3}(|\nabla u|^2)\Delta u\varphi^{2\gamma}+\int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}(|\nabla u^\gamma|^2)\nabla u^\gamma\nabla\varphi^{2\gamma}. \tag{2-19}$$

The first term on the right side of (2-19) can be estimated as

$$\begin{aligned} u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma &= 2u^{-\gamma}((u^\gamma)_{ij}(u^\gamma)_i(u^\gamma)_j)\\ &\le 2\gamma(u^\gamma)_{ij}(u^\gamma)_{ij}+\frac{u^{-2\gamma}}{2\gamma}(u^\gamma)_i(u^\gamma)_j(u^\gamma)_i(u^\gamma)_j\\ &= 2\gamma|\nabla^2u^\gamma|^2+\frac{u^{-2\gamma}}{2\gamma}|\nabla u^\gamma|^4.\end{aligned} \tag{2-20}$$

As a consequence, we have

$$\begin{aligned}
(2\text{-}21)\quad &\int_{\mathbb{R}^n} \frac{1}{\gamma^3} u^{-\gamma} \nabla(|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma} \\
&\le \int_{\mathbb{R}^n} \frac{2}{\gamma^2} |\nabla^2 u^\gamma|^2 \varphi^{2\gamma} + \int_{\mathbb{R}^n} \frac{1}{2\gamma^4} u^{-2\gamma} |\nabla u^\gamma|^4 \varphi^{2\gamma} \\
&\le \int_{\mathbb{R}^n} \frac{2}{\gamma^2} |\nabla^2 (u^\gamma \varphi^\gamma)|^2 + \int_{\mathbb{R}^n} C u^{2\gamma} \|\nabla^4(\varphi^{2\gamma})\| + \int_{\mathbb{R}^n} \frac{1+4\gamma^2\varepsilon}{2\gamma^4} u^{-2\gamma} |\nabla u^\gamma|^4 \varphi^{2\gamma} \\
&= \int_{\mathbb{R}^n} \frac{2}{\gamma^2} (\Delta(u^\gamma \varphi^\gamma))^2 + \int_{\mathbb{R}^n} C u^{2\gamma} \|\nabla^4(\varphi^{2\gamma})\| + \int_{\mathbb{R}^n} \frac{1+4\gamma^2\varepsilon}{2\gamma^4} u^{-2\gamma} |\nabla u^\gamma|^4 \varphi^{2\gamma},
\end{aligned}$$

where we used (2-3) in the last step.

For the second term on the right side of (2-19), applying estimate (2.3) from [Wei and Ye 2010], that is, $(\Delta u)^2 \ge \frac{2}{p+1} u^{p+1}$, and the fact that $\Delta u < 0$ from Theorem 3.1 in [Wei and Xu 1999] or Theorem 2.1 in [Xu 2000], we have

$$\begin{aligned}
(2\text{-}22)\quad \int_{\mathbb{R}^n} u^{2\gamma-3}(|\nabla u|^2) \Delta u \varphi^{2\gamma} &\le -\int_{\mathbb{R}^n} \sqrt{\frac{2}{p+1}}\, u^{2\gamma-3+\frac{p+1}{2}} (|\nabla u|^2) \varphi^{2\gamma} \\
&= \int_{\mathbb{R}^n} \frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{p+1}{2}} u^{2\gamma-2+\frac{p+1}{2}} \Delta u \varphi^{2\gamma} \\
&\quad + \int_{\mathbb{R}^n} \frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{p+1}{2}} u^{2\gamma-2+\frac{p+1}{2}} \nabla u \nabla \varphi^{2\gamma}.
\end{aligned}$$

Using the inequality $-\Delta u \ge \sqrt{\frac{2}{p+1}}\, u^{\frac{p+1}{2}}$, we get

$$(2\text{-}23)\quad \int_{\mathbb{R}^n} \frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{2}{p+1}} u^{2\gamma-2+\frac{p+1}{2}} \Delta u \varphi^{2\gamma} \le -\int_{\mathbb{R}^n} \frac{\frac{2}{p+1}}{2\gamma-2+\frac{p+1}{2}} u^{2\gamma+p-1} \varphi^{2\gamma}.$$

On the other hand, for the second term on the right side of (2-22), we have

$$\begin{aligned}
(2\text{-}24)\quad \int_{\mathbb{R}^n} u^{2\gamma-2+\frac{p+1}{2}} \nabla u \nabla \varphi^{2\gamma} &= -\int_{\mathbb{R}^n} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^{2\gamma} \\
&= -\int_{\{x \mid \Delta\varphi^{2\gamma} > 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^{2\gamma} \\
&\quad - \int_{\{x \mid \Delta\varphi^{2\gamma} \le 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^{2\gamma},
\end{aligned}$$

where the first equality follows from integration by parts and $L = 2\gamma - 1 + \frac{p+1}{2}$. As for the first term on the last part of (2-24), using the inequality

$$\Delta u \le -\sqrt{\frac{2}{p+1}}\, u^{\frac{p+1}{2}} < 0,$$

we have

$$\frac{\sqrt{\frac{p+1}{2}}}{L}\int_{\{x\mid\Delta\varphi^{2\gamma}>0\}} u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma} \le -\int_{\{x\mid\Delta\varphi^{2\gamma}>0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}. \tag{2-25}$$

Similarly to the proof of Lemma 2.1, it is easy to get

$$\left|\int_{\{x\mid\Delta\varphi^{2\gamma}>0\}}\frac{\sqrt{\frac{p+1}{2}}}{L}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma}\right| \le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.$$

From this and (2-25), we have

$$\left|\int_{\{x\mid\Delta\varphi^{2\gamma}>0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\right| \le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.$$

Similarly, we also obtain

$$\left|\int_{\{x\mid\Delta\varphi^{2\gamma}\le 0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\right| \le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.$$

From the last two inequalities and (2-24), we have

$$\left|\int_{\mathbb{R}^n}u^{2\gamma-2+\frac{p+1}{2}}\nabla u\nabla\varphi^{2\gamma}\right| \le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2-26}$$

Combining (2-22), (2-23) and (2-26), we get the inequality

$$\begin{aligned}\int_{\mathbb{R}^n}u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma} &\le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| \\ &\quad -\int_{\mathbb{R}^n}\frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}.\end{aligned} \tag{2-27}$$

Finally, we apply Young's inequality to the third term on the right side of (2-19), and get

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}(|\nabla u^\gamma|^2)\nabla u^\gamma\nabla\varphi^{2\gamma} &= \int_{\mathbb{R}^n}u^{2\gamma-3}|\nabla u|^2\nabla u\nabla(\varphi^{2\gamma}) \\ &\le \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.\end{aligned} \tag{2-28}$$

By (2-19), (2-21), (2-27) and (2-28), we finally obtain

$$\begin{aligned}\left(\tfrac{1}{2}-\varepsilon\right)\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} &\le \frac{2}{\gamma^2}\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| \\ &\quad -\int_{\mathbb{R}^n}\frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}. \quad \square\end{aligned}$$

By (2-9), (2-15) and (2-16), since the number ε is arbitrary small in those three places, we have, for $\delta > 0$ sufficiently small,

$$\begin{aligned}(2\text{-}29)\quad &\int_{\mathbb{R}^n} \big(1-4(\gamma-1)^2-\delta\big)(\Delta(u^\gamma\varphi^\gamma))^2 \\ &-\int_{\mathbb{R}^n}\left(\frac{\gamma^2}{2\gamma-1}-\frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)}\right)u^{p+2\gamma-1}\varphi^{2\gamma} \le \int_{\mathbb{R}^n} C_\delta u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\end{aligned}$$

where C_δ is a positive constant that depends on δ only. Here, we need to require $1-4(\gamma-1)^2 > 0$, since we have assumed that $\gamma > 1$ in Lemma 2.1. So γ is required be in $\big(1, \frac{3}{2}\big)$. If we can choose δ small enough to make $1-4(\gamma-1)^2-\delta$ positive, by the stability property of function u, we obtain

$$(2\text{-}30)\qquad \int_{\mathbb{R}^n}(E-p\delta)u^{p+q}\varphi^{2\gamma} \le \int_{\mathbb{R}^n} C_\delta u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,$$

where E is defined to be

$$(2\text{-}31)\qquad E = p(1-4(\gamma-1)^2) - \frac{\gamma^2}{q} + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)}.$$

Now we take $\varphi = \eta^m$ with m sufficiently large, and choose η a cut-off function satisfying $0 \le \eta \le 1$, $\eta = 1$ for $|x| < R$ and $\eta = 0$ for $|x| > 2R$. By Young's inequality again, we have

$$\begin{aligned}(2\text{-}32)\quad \int_{\mathbb{R}^n} u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| &\le C_\delta R^{-4}\int_{\mathbb{R}^n} u^{2\gamma}\eta^{2\gamma m-4} \\ &\le C_{\delta,\varepsilon}R^{-\frac{4}{1-\theta}}\int_{\mathbb{R}^n} u^2\eta^{2\gamma m-\frac{4}{1-\theta}} + \varepsilon C_\delta\int_{\mathbb{R}^n} u^{2\gamma+p-1}\eta^{2\gamma m},\end{aligned}$$

where $C_{\delta,\varepsilon}$ is a positive constant depends on δ and ε, and θ is a number such that $2(1-\theta)+(2\gamma+p-1)\theta = 2\gamma$, so that $0<\theta<1$ for $2<2\gamma<2\gamma+p-1$. By (2-30) and (2-32), we get

$$(2\text{-}33)\qquad (E-p\delta-\varepsilon C_\delta)\int_{\mathbb{R}^n} u^{p+2\gamma-1}\eta^{2\gamma m} \le C_{\delta,\varepsilon}R^{-\frac{4}{1-\theta}}\int_{\mathbb{R}^n} u^2\eta^{2\gamma m-\frac{4}{1-\theta}}.$$

Since θ is strictly less than 1 and will be fixed for given γ, p, we can choose m sufficiently large to make $2\gamma m - \frac{4}{1-\theta} > 0$. On the other hand, if $E > 0$, we can find small δ and then small ε, such that $E-p\delta-\varepsilon C_\delta > 0$. Therefore, by the definition of function η and (2-33), we obtain

$$(2\text{-}34)\qquad (E-p\delta-\varepsilon C_\delta)\int_{B_R} u^{p+2\gamma-1} \le C_{\delta,\varepsilon}R^{-\frac{4}{1-\theta}}\int_{B_{2R}} u^2.$$

By (2.10) of [Wei and Ye 2010], we have $\displaystyle\int_{B_{2R}} u^2 \le CR^{n-\frac{8}{p-1}}$, as a result, the left side of (2-34) is less equal than $C_{\delta,\varepsilon} R^{n-\frac{8}{p-1}-\frac{4}{1-\theta}}$, which tends to 0 as R tends to ∞, provided the power $n-\frac{8}{p-1}-\frac{4}{1-\theta}$ is negative, which is equivalent to $(p+2\gamma-1)>(p-1)\frac{n}{4}$ according to the definition of θ. So, if $(p+2\gamma-1)>(p-1)\frac{n}{4}$ and $E-p\delta-C_\delta\varepsilon>0$, we have $u\equiv 0$.

Thus, we have proved the nonexistence of stable solution to (2-8) if p satisfies the condition $(p+2\gamma-1)>(p-1)\frac{n}{4}$ and $E>0$ (for δ, ε are arbitrary small). By Lemma A.2 in the Appendix, the power p can be in the interval $(\frac{n}{n-8}, 1+\frac{8p^*}{n-4})$. Combining with Theorem 1.1 of [Wei and Ye 2010], we have proved Theorem 1.1, that is, for any $1<p<1+\frac{8p^*}{n-4}$, $n\geq 20$, (2-8) has no stable solution.

3. Proof of Theorem 1.2

In proving Theorem 1.2, it is enough to consider stable solutions u_λ to (1-4), since $u^*=\lim_{\lambda\to\lambda^*} u_\lambda$. Now we give a uniform bound for the stable solutions to (1-4) when $0<d<\lambda<\lambda^*$, where d is a fixed positive constant from $(0,\lambda^*)$.

First, we need to analyze the solution near the boundary. Specifically, we need the regularity of the stable solutions of the equation

$$\begin{cases} \Delta^2 u=\lambda(u+1)^p,\ \ \lambda>0 & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

near the boundary (as well as their derivatives; see remark after the next theorem).

Theorem 3.1. *Let Ω be a bounded, smooth, and convex domain. There exists a constant C (independent of λ, u) and small positive number ε, such that for stable solutions u to* (3-1) *we have*

$$u(x)<C \quad \textit{for all } x\in\Omega_\varepsilon:=\{z\in\Omega:\ d(z,\partial\Omega)<\varepsilon\}. \tag{3-2}$$

Proof. This result is well known. See [Guo and Wei 2009]. For the sake of completeness, we include a proof here. By Lemma 3.5 of [Cowan et al. 2010], we see that there exists a constant C independent of λ, u, such that

$$\int_\Omega (1+u)^p\,dx\leq C. \tag{3-3}$$

We write (3-1) as

$$\begin{cases} \Delta u+v=0, & \text{in } \Omega,\\ \Delta v+\lambda(1+u)^p=0, & \text{in } \Omega,\\ u=v=0, & \text{in } \partial\Omega. \end{cases}$$

If we set $f_1(u,v)=v$, $f_2(u,v)=\lambda(u+1)^p$, we see that $\partial f_1/\partial v=1>0$ and $\partial f_2/\partial u=\lambda p(u+1)^{p-1}>0$. Therefore, the convexity of Ω, Lemma 5.1 of [Troy 1981], and the moving plane method near $\partial\Omega$ (as in the appendix of [Guo and

Webb 2002]) imply that there exist $t_0 > 0$ and α which depends only on the domain Ω, such that $u(x - t\nu)$ and $v(x - t\nu)$ are nondecreasing for $t \in [0, t_0]$, $\quad \nu \in \mathbb{R}^n$ satisfying $|\nu| = 1$ and $(\nu, n(x)) \geq \alpha$ and $x \in \partial\Omega$. Therefore, we can find $\rho, \varepsilon > 0$ such that for any $x \in \Omega_\varepsilon := \{z \in \Omega : \ d(z, \partial\Omega) < \varepsilon\}$ there exists a fixed-sized cone Γ_x (with x as its vertex) with

- $\operatorname{meas}(\Gamma_x) \geq \rho$,
- $\Gamma_x \subset \{z \in \Omega : \ d(z, \partial\Omega) < 2\varepsilon\}$, and
- $u(y) \geq u(x)$ for any $y \in \Gamma_x$.

Then, for any $x \in \Omega_\varepsilon$, we have

$$(1 + u(x))^p \leq \frac{1}{\operatorname{meas}(\Gamma_x)} \int_{\Gamma_x} (1 + u)^p \leq \frac{1}{\rho} \int_\Omega (1 + u)^p \leq C.$$

This implies that $(1 + u(x))^p \leq C$, therefore $u(x) \leq C$. □

Remark. By classical elliptic regularity theory, $u(x)$ and its derivatives up to fourth order are bounded on the boundary by a constant independent of u. See [Wei 1996] for more details.

We now turn to the proof of Theorem 1.2 proper, using the ideas of Section 2. Multiplying (1-4) by $(u + 1)^q$ and integrating by parts, we have

$$\int_\Omega \lambda (u+1)^{p+q} = \int_\Omega \Delta^2 u (u+1)^q = \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial n} + \int_\Omega \Delta(u+1)\Delta(u+1)^q. \tag{3-4}$$

Setting $v = u + 1$, by direct calculation, we get

$$\begin{aligned}\int_\Omega (\Delta v^\gamma)^2 = \int_\Omega \gamma^2 v^{2\gamma-2} (\Delta v)^2 + \int_\Omega \gamma^2 (\gamma - 1)^2 v^{2\gamma-4} |\nabla v|^4 \\ + 2 \int_\Omega \gamma^2 (\gamma - 1) v^{2\gamma-3} \Delta v |\nabla v|^2,\end{aligned}$$

$$\int_\Omega \Delta v \Delta v^q = \int_\Omega q (\Delta v)^2 v^{q-1} + \int_\Omega q(q-1) |\nabla v|^2 \Delta v v^{q-2}.$$

From these two equalities and (3-4) we obtain

$$\int_\Omega \left(\frac{q}{\gamma^2} (\Delta v^\gamma)^2 - q(\gamma - 1)^2 |\nabla v|^4 v^{2\gamma-4} \right) + \int_{\partial\Omega} \frac{\partial(\Delta v)}{\partial n} = \int_\Omega \lambda v^{p+q}. \tag{3-5}$$

For the second term in (3-5), we have

$$\begin{aligned}(3\text{-}6)\quad &\int_\Omega |\nabla v|^4 v^{2\gamma-4}\\ &= \frac{1}{\gamma^4}\int_\Omega v^{-2\gamma}|\nabla v^\gamma|^4 = \frac{1}{\gamma^4}\int_\Omega |\nabla v^\gamma|^2\nabla v^\gamma(-\nabla v^{-\gamma})\\ &= \frac{1}{\gamma^4}\int_\Omega\Big(-\nabla\frac{|\nabla v^\gamma|^2\nabla v^\gamma}{v^\gamma}+\frac{\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma}{v^\gamma}+\frac{|\nabla v^\gamma|^2\Delta v^\gamma}{v^\gamma}\Big)\\ &= \frac{1}{\gamma^4}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma+|\nabla v^\gamma|^2\Delta v^\gamma-\frac{1}{\gamma}\int_{\partial\Omega} v^{2\gamma-3}|\nabla v|^2\frac{\partial v}{\partial n}.\end{aligned}$$

A simple calculation yields

$$(3\text{-}7)\quad \frac{1}{\gamma^4}\int_\Omega v^{-\gamma}|\nabla v^\gamma|^2\Delta v^\gamma = \frac{\gamma-1}{\gamma}\int_\Omega v^{2\gamma-4}|\nabla v|^4+\frac{1}{\gamma}\int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v.$$

Substituting (3-7) into (3-6), we get

$$\begin{aligned}(3\text{-}8)\quad &\int_\Omega |\nabla v|^4 v^{2\gamma-4}\\ &= \int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v+\frac{1}{\gamma^3}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma-\int_{\partial\Omega}|\nabla v|^2\frac{\partial v}{\partial n}.\end{aligned}$$

We now estimate the second term on the right side of (3-8). From the proof of Lemma 2.3, together with the identity $\frac{1}{2}\Delta|\nabla v^\gamma|^2 = |\nabla^2 v^\gamma|^2+\langle\nabla\Delta v^\gamma,\nabla v^\gamma\rangle$, we have

$$\begin{aligned}(3\text{-}9)\quad \frac{1}{\gamma^3}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma &\le \frac{1}{2}\int_\Omega |\nabla v|^4 v^{2\gamma-4}+\frac{2}{\gamma^2}\int_\Omega(\Delta v^\gamma)^2\\ &\quad+\frac{1}{\gamma^2}\int_{\partial\Omega}\frac{\partial|\nabla v^\gamma|^2}{\partial n}-\frac{2}{\gamma^2}\int_{\partial\Omega}(\Delta v^\gamma)\frac{\partial v^\gamma}{\partial n}.\end{aligned}$$

By (3-8) and (3-9), thanks to the convexity of the domain Ω, we get

$$\begin{aligned}(3\text{-}10)\quad &\frac{1}{2}\int_\Omega |\nabla v|^4 v^{2\gamma-4}\\ &\le \int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v+\frac{2}{\gamma^2}\int_\Omega(\Delta v^\gamma)^2-(2\gamma-1)\int_{\partial\Omega}|\nabla v|^2\frac{\partial v}{\partial n}.\end{aligned}$$

For the first term on the right side of (3-10), since $v = u+1$, we have $\Delta v = \Delta u < 0$ by maximal principle, and the inequality

$$(3\text{-}11)\quad \Delta v < -\sqrt{\frac{2\lambda}{p+1}}\,v^{\frac{p+1}{2}} < 0,$$

by Lemma 3.2 of [Cowan et al. 2010]. Thus

$$\int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v \le \int_\Omega -\sqrt{\frac{2\lambda}{p+1}}\,v^{2\gamma-3+\frac{p+1}{2}}|\nabla v|^2.$$

Moreover, we have

$$\int_\Omega -\sqrt{\frac{2\lambda}{p+1}}v^{2\gamma-3+\frac{p+1}{2}}|\nabla v|^2 = -\int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}\nabla(v^{2\gamma-2+\frac{p+1}{2}}\nabla v) + \int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}v^{2\gamma-2+\frac{p+1}{2}}\Delta v.$$

For the second term on the right, using (3-11) again, we have

$$\int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}v^{2\gamma-2+\frac{p+1}{2}}\Delta v \le -\int_\Omega \frac{\frac{2\lambda}{p+1}}{2\gamma-2+\frac{p+1}{2}}v^{2\gamma+p-1}.$$

Hence, we obtain

$$\int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v \le -\int_{\partial\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}\frac{\partial v}{\partial n} - \int_\Omega \frac{\frac{2\lambda}{p+1}}{2\gamma-2+\frac{p+1}{2}}v^{2\gamma+p-1}, \tag{3-12}$$

where we used $v|_{\partial\Omega} = u+1|_{\partial\Omega} = 1$, for the boundary term in (3-4), (3-10) and (3-12). By the remark after Theorem 3.1, we find that there exists a constant C (the constant C appeared now and later in this section is independent of u), such that

$$\int_{\partial\Omega}\left(|\nabla u|^2\left|\frac{\partial u}{\partial n}\right| + \left|\frac{\partial(\Delta u)}{\partial n}\right| + \left|\frac{\partial u}{\partial n}\right|\right) \le C. \tag{3-13}$$

Combining (3-5), (3-10), (3-12) and (3-13), we get

$$\big(1-4(\gamma-1)^2\big)\int_\Omega(\Delta(u+1)^\gamma)^2 + \left(\frac{8\lambda\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\lambda\gamma^2}{q}\right)\int_\Omega(u+1)^{p+q} \le C.$$

If $1-4(\gamma-1)^2 > 0$ and

$$p(1-4(\gamma-1)^2) + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\gamma^2}{q} > 0 \tag{3-14}$$

and u is a stable solution to (1-4), we have

$$\left(p(1-4(\gamma-1)^2) + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\gamma^2}{2\gamma-1}\right)\int_\Omega(u+1)^{p+q} \le \frac{C}{\lambda}.$$

This leads to $u+1 \in L^{p+q}$.

If $p+q > (p-1)n/4$, then classical regularity theory implies that $u \in L^\infty(\Omega)$.

Therefore we have established the bound of extremal solutions of (1-4) if (3-14) is satisfied and

$$p < \frac{8\gamma+n-4}{n-4}.$$

By Lemma A.2 and Theorem 3.8 of [Wei and Ye 2010], we have proved that the extremal solution u^*, the unique solution of (1-4) (where $\lambda = \lambda^*$), is bounded provided that one of these conditions hold:

(1) If $n \le 8$, then $p > 1$.

(2) If $9 \le n \le 19$, there exists $\varepsilon_n > 0$ such that for any $1 < p < \frac{n}{n-8} + \varepsilon_n$.

(3) If $n \ge 20$, then $1 < p < 1 + \frac{8p^*}{n-4}$, where p^* was defined immediately after Theorem 1.1.

Appendix

In this appendix, we study the inequalities

$$p(1-4(\gamma-1)^2) - \frac{\gamma^2}{2\gamma-1} + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)} > 0 \tag{A-1}$$

and

$$p < \frac{8\gamma+n-4}{n-4}. \tag{A-2}$$

In order to get a better range for the power p from (A-1) and (A-2), we must study the following equation obtained by letting $p = \frac{8\gamma+n-4}{n-4}$ in (A-1):

$$\frac{8\gamma+n-4}{n-4}\big(1-4(\gamma-1)^2\big) - \frac{\gamma^2}{2\gamma-1} + \frac{8\gamma^2(\gamma-1)^2}{\big(4\gamma-3+\frac{8\gamma+n-4}{n-4}\big)\big(\frac{8\gamma+n-4}{n-4}+1\big)} = 0. \tag{A-3}$$

We need only consider the behavior of (A-3) for $\gamma \in \big(1, \frac{3}{2}\big)$. Through tedious computations, we see that the equation at the bottom of page 495 is the simplified form of (A-3). As a consequence, they have same roots in $\big(1, \frac{3}{2}\big)$.

We denote the left side of (A-3) by $h(\gamma)$. Notice that if $\gamma = \frac{n-4}{n-8}$, then $p = \frac{n}{n-8}$ and $\gamma - 1 = \frac{4}{n-8}$. Hence

$$h\Big(\frac{n-4}{n-8}\Big) = \frac{8}{n-8}(n^4 - 18n^3 - 56n^2 + 384n - 512).$$

In fact, if $n = 20$, then $h\big(\frac{4}{3}\big) = 512 > 0$. On the other hand, it is also easy to see that $h(\frac{3}{2}) < 0$, while it is obvious that $\big(4\gamma - 3 + \frac{8\gamma+n-4}{n-4}\big)\big(\frac{8\gamma+n-4}{n-4} + 1\big) > 0$ and $(2\gamma-1) > 0$ when $\gamma \in \big(\frac{n-4}{n-8}, \frac{3}{2}\big)$. Therefore, by continuity, (A-3) possesses a root in $(\frac{n-4}{n-8}, \frac{3}{2})$. We denote the smallest root of (A-3) greater than $\frac{n-4}{n-8}$ by p^*. Once we pick out a γ from the interval $(\frac{n-4}{n-8}, p^*)$, $h(\gamma)$ is of course positive. By continuity, we can find a small positive number δ such that the inequality

$$p(1-4(\gamma-1)^2) - \frac{\gamma^2}{2\gamma-1} + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)} > 0$$

holds when $p \in \left(\frac{8\gamma+n-4}{n-4} - \delta, \frac{8\gamma+n-4}{n-4}\right)$. So, we conclude that when γ runs in the whole interval $\left(\frac{n-4}{n-8}, p^*\right)$, the power p can be in the whole interval $\left(\frac{n}{n-8}, 1 + \frac{8p^*}{n-4}\right)$. We summarize the result as follows:

Lemma A.2. *When $n \geq 20$, the range of p satisfying* (A-1) *and* (A-2) *equals* $\left(\frac{n}{n-8}, 1 + \frac{8p^*}{n-4}\right)$, *and this interval is not empty.*

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JUNCHENG WEI
DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
SHATIN, NT
HONG KONG
wei@math.cuhk.edu.hk

XINGWANG XU
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
BLOCK S17 (SOC1)
10 LOWER KENT RIDGE ROAD
SINGAPORE 119076
SINGAPORE
matxuxw@nus.edu.sg

WEN YANG
DEPARTMENT OF MATHEMTICS
CHINESE UNIVERSITY OF HONG KONG
SHATIN, NT
HONG KONG
wyang@math.cuhk.edu.hk

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