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We determine the first two derivatives of the classical Boltzmann–Shannon entropy of the conjugate heat equation on general evolving manifolds. Based on the second derivative of the Boltzmann–Shannon entropy, we construct Perelman's $\mathcal{F}$ and $\mathcal{W}$ entropy in abstract geometric flows. Monotonicity of the entropies holds when a technical condition is satisfied.

This condition is satisfied on static Riemannian manifolds with nonnegative Ricci curvature, for Hamilton's Ricci flow, List's extended Ricci flow, Müller's Ricci flow coupled with harmonic map flow and Lorentzian mean curvature flow when the ambient space has nonnegative sectional curvature.

Under the extra assumption that the lowest eigenvalue is differentiable along time, we derive an explicit formula for the evolution of the lowest eigenvalue of the Laplace–Beltrami operator with potential in the abstract setting.

1. Introduction

Geometric flows have been studied extensively. The idea is to evolve metrics in certain ways usually by heat-type equations to obtain better metrics on manifolds and thus to gain topological information of the manifolds. It is desirable to derive evolution equations in a general setting such that the formulas may be applied to various flows. For instance, very nice general approaches to get monotone quantities on evolving manifolds have been developed in [Ecker et al. 2008; Müller 2010].

We briefly introduce notation for an abstract geometric flow. Let M be an n-dimensional compact manifold. Assume that $\alpha(t, y)$ is a time-dependent symmetric two-tensor on M, and that $g(t)$ is a family of one parameter Riemannian metrics evolving along the flow equation

$$\frac{\partial g}{\partial t} = -2\alpha, \quad t \in (0, T), \tag{1-1}$$

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where T is some fixed positive constant. Let $A := g^{ij}\alpha_{ij}$ be the trace of α with respect to $g(t)$.

Classical quantities on static manifolds have nice applications on evolving manifolds by certain natural modifications. The Boltzmann–Shannon entropy is such a quantity for the heat equation. Formally, the conjugate of the heat operator $\partial/\partial t - \Delta$ on space-time is $-\partial/\partial t - \Delta + A$. As Perelman [2002] showed, on evolving manifolds it is natural to work with the entropy for the conjugate heat equation. We will derive the first two derivatives of Boltzmann–Shannon entropy for the conjugate heat equation, and based on that we define Perelman's $\mathcal{F}$ and $\mathcal{W}$ entropy in the framework of abstract geometric flows.

Other classical quantities on static Riemannian manifolds are the eigenvalues of the Laplace–Beltrami operator Δ. When the metric evolves, it is natural to include a potential function. Perelman [2002] shows that the lowest eigenvalue of $-\Delta + R/4$ is monotone nondecreasing along the Ricci flow. Furthermore by deriving explicit formula of the derivative, Cao [2007; 2008] shows that the monotonicity holds for the lowest eigenvalue of $-\Delta + cR$ for any $c \geq \frac{1}{4}$; see also [Li 2007].

Reto Müller [2010] derived formulas for the reduced volume in abstract geometric flows. His formulation is very general and thus can be applied to different flows. He shows that the reduced volume is monotone when a technical assumption holds, which is satisfied for static manifolds with positive Ricci curvature, Hamilton's Ricci flow, List's extended Ricci flow, Müller's Ricci flow coupled with harmonic map flow and Lorentzian mean curvature flow when the ambient manifold has nonnegative sectional curvature. This allows him to establish new monotonicity formulas for these flows.

One of our purposes in this paper is to show that the same phenomena as for reduced volume holds for entropy and eigenvalues.

Notation and main results. Throughout the paper, M will be a compact manifold without boundary. Along the flow equation (1-1) the Riemannian volume dy of M evolves by

$$\frac{\partial}{\partial t}\, dy = -A\, dy$$

and A satisfies

$$\frac{\partial A}{\partial t} = 2|\alpha|^2 + g^{ij}\frac{\partial \alpha_{ij}}{\partial t},$$

where $|\alpha|^2 = g^{ij}g^{kl}\alpha_{ik}\alpha_{jl}$. To simplify the notation, we let $\beta_{ij} := \partial\alpha_{ij}/\partial t$ and $B := g^{ij}\beta_{ij}$, so that

$$\frac{\partial A}{\partial t} = 2|\alpha|^2 + B. \tag{1-2}$$

In particular, $A = R$ and $B = \Delta R$ under the Ricci flow.

For any time-dependent vector field V on M we define

$$\Theta_{g,\alpha}(V) := (\mathrm{Rc}-\alpha)(V,V) + \langle \nabla A - 2\,\mathrm{Div}(\alpha), V\rangle + \tfrac{1}{2}(B-\Delta A), \tag{1-3}$$

where Rc is the Ricci tensor and Div the divergence operator: $\mathrm{Div}(\alpha)_k = g^{ij}\nabla_i \alpha_{jk}$. In the rest of this paper we omit the subscripts of $\Theta_{g,\alpha}(V)$ and denote it by $\Theta(V)$.

The quantity $\Theta(V)$ appears as an error term in our main results. In the expression of $\Theta(V)$, the Rc term is caused by the Bochner's formula. This explains technically why our results are particularly useful for the Ricci flow and its various modifications. Müller [2010] introduced the quantity $\mathfrak{D}$. In our notation his definition reads as

$$\mathfrak{D}(V) = \partial_t A - \Delta A - 2|\alpha_{ij}|^2 + 4\nabla_i \alpha_{ij} V_j - 2\nabla_j A V_j + 2R_{ij}V_iV_j - 2\alpha_{ij}V_iV_j.$$

Note that $\mathfrak{D}$ and Θ are essentially the same; indeed $\mathfrak{D}(V) = 2\Theta(-V)$. Müller [2010] further explained that $\mathfrak{D}$ is the difference between two differential Harnack-type quantities for the tensor α.

Let $u(t,y)$ be a nonnegative solution to the conjugate heat equation

$$\frac{\partial u(t,y)}{\partial t} = -\Delta u(t,y) + A(t,y)\,u(t,y), \quad t\in(0,T),\ y\in M, \tag{1-4}$$

where Δ is the Laplace–Beltrami operator calculated with respect to the evolving metric $g(t)$. Note that $\int_M u(t,y)\,dy$ remains constant along the flow, and without loss of generality we assume this constant to be 1.

The classical Boltzmann–Shannon entropy functional is defined by

$$\mathscr{E}(t) = \int_M u(t,y)\log u(t,y)\,dy. \tag{1-5}$$

If $\Theta(V)\geq 0$ for all V, we will show that $\mathscr{E}$ is convex. Based on this observation we construct Perelman's $\mathscr{F}$ and $\mathscr{W}$ entropy in abstract geometric flows. We then derive the explicit evolution equations of the entropies along the conjugate heat equation, and show that they are monotone if $\Theta\geq 0$. We thus present a unified formula of various $\mathscr{W}$ entropies established by various authors for different flows (including the static case); see [Feldman et al. 2005; Li 2007; List 2008; Müller 2012; Ni 2004b; 2004a; Perelman 2002].

We show indeed that the generalized entropy $\mathscr{F}_k$ ($k\geq 1$), see Definition 4.1 below, is monotone under the additional assumption $B-\Delta A\geq 0$, which is satisfied by all previously mentioned flows. The study of the $\mathscr{F}_k$ entropy leads to a simpler argument to rule out nontrivial steady breathers.

The eigenvalues and eigenfunctions of the Laplace–Beltrami operator with potential cA where c is a constant, satisfy

$$\lambda(t) f(t,y) = -\Delta f(t,y) + cA(t,y) f(t,y). \tag{1-6}$$

Let $\lambda(t)$ be the lowest eigenvalue. We shall determine the derivative of $\lambda(t)$. A remarkable fact is that the derivative $\lambda'(t)$ does not depend on the time derivative of the corresponding eigenfunction; this allows to establish a formula for $\lambda'(t)$ not requiring knowledge of the eigenfunction evolution. We will prove eigenvalue monotonicity and apply it to rule out nontrivial steady and expanding breathers in various flows.

2. The first two derivatives of the Boltzmann–Shannon entropy

Theorem 2.1. *Suppose that $(M, g(t))$ is a solution to the abstract geometric flow (1-1), and that $u(t, y)$ is a positive solution to the conjugate heat equation (1-4), normalized by $\int_M u(t, y)\, dy = 1$. The first two derivatives of $\mathscr{E}(t)$ are given by*

$$\mathscr{E}'(t) = \int_M (|\nabla \log u|^2 + A)u\, dy, \tag{2-7}$$

$$\mathscr{E}''(t) = \int_M 2\big(|\alpha - \nabla\nabla \log u|^2 + \Theta(\nabla \log u)\big)u\, dy. \tag{2-8}$$

In particular, if Θ is nonnegative then $\mathscr{E}(t)$ is convex in time.

Proof. Since M is closed we can integrate by parts freely. Direct calculations show that

$$\begin{aligned}\mathscr{E}'(t) &= \int_M (u_t \log u + u_t - Au \log u)\, dy \\ &= \int_M \big((-\Delta u + Au)\log u - \Delta u + Au - Au \log u\big)\, dy \\ &= \int_M (-\Delta u \log u + Au)\, dy = \int_M (|\nabla \log u|^2 + A)u\, dy,\end{aligned}$$

and

$$\begin{aligned}\mathscr{E}''(t) &= \int_M \left(\frac{\partial(|\nabla \log u|^2 + A)}{\partial t}u + (|\nabla \log u|^2 + A)\frac{\partial u}{\partial t} - (|\nabla \log u|^2 + A)uA\right) dy \\ &= \int_M \left(\Big(2\alpha(\nabla \log u, \nabla \log u) + 2\Big\langle \nabla \frac{u_t}{u}, \nabla \log u\Big\rangle + 2|\alpha|^2 + B\Big)u \right. \\ &\qquad \left. + (|\nabla \log u|^2 + A)(-\Delta u + Au) - (|\nabla \log u|^2 + A)uA\right) dy \\ &= \int_M \left(\Big(2\alpha(\nabla \log u, \nabla \log u) + 2\Big\langle \nabla\Big(-\frac{\Delta u}{u} + A\Big), \nabla \log u\Big\rangle\Big)u \right. \\ &\qquad \left. + (2|\alpha|^2 + B)u - (|\nabla \log u|^2 + A)\Delta u\right) dy \\ &= \int_M \left(2u\alpha(\nabla \log u, \nabla \log u) - 2\Big\langle \nabla\Big(\frac{\Delta u}{u}\Big), \nabla u\Big\rangle + 2\langle \nabla A, \nabla u\rangle \right. \\ &\qquad \left. + 2u|\alpha|^2 + Bu - \Delta(|\nabla \log u|^2)u - \Delta Au\right) dy.\end{aligned}$$

Plugging in $\Delta \log u = \Delta u/u - |\nabla \log u|^2$ and

$$\Delta(|\nabla \log u|^2) = 2|\nabla\nabla \log u|^2 + 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) + 2\langle \nabla \log u, \nabla(\Delta \log u)\rangle,$$

we have

$$\begin{aligned}\mathscr{E}''(t) &= \int_M \big(2u(|\alpha|^2+|\nabla\nabla \log u|^2)+2u(\alpha+\mathrm{Rc})(\nabla \log u,\nabla \log u)+Bu-3\Delta Au\big)\,dy\\ &= \int_M 2\Big(u\,|\alpha - \nabla\nabla \log u|^2 + 4u\langle \alpha, \nabla\nabla \log u\rangle\\ &\qquad + 2u(\alpha+\mathrm{Rc})(\nabla \log u, \nabla \log u) + (B-\Delta A)u + 2\langle \nabla A, \nabla u\rangle\Big)\,dy.\end{aligned}$$

By observing that

$$\mathrm{Div}(u\alpha(\nabla \log u)) = \alpha(\nabla \log u, \nabla u) + u\,\mathrm{Div}(\alpha)(\nabla \log u) + u\langle \alpha, \nabla\nabla \log u\rangle,$$

and by the divergence theorem, we get

$$\begin{aligned}\mathscr{E}''(t) = \int_M \Big(&2u|\alpha - \nabla\nabla \log u|^2 + 2u(\mathrm{Rc}-\alpha)(\nabla \log u, \nabla \log u)\\ &+ (B-\Delta A)u + \langle 2\nabla A - 4\,\mathrm{Div}(\alpha), \nabla u\rangle\Big)\,dy,\end{aligned}$$

which is exactly (2-8). □

3. Examples where Θ and $B - \Delta A$ are nonnegative

We next list some examples where Θ and $B - \Delta A$ are nonnegative. Calculations on the Ricci flow and extended Ricci flow are carried out in detail. For other examples we list values of Θ and $B - \Delta A$, and for details we refer to [Müller 2010]. This section is organized in the same way as the corresponding section there. Recall that

$$\Theta(V) = (\mathrm{Rc}-\alpha)(V, V) + \langle \nabla A - 2\,\mathrm{Div}(\alpha), V\rangle + \tfrac{1}{2}(B - \Delta A).$$

Riemannian manifolds. In the case of a static metric we have $\alpha = 0$ and hence

$$\Theta(V) = \mathrm{Rc}(V, V), \quad B - \Delta A = 0. \tag{3-9}$$

Thus Θ is nonnegative if M has nonnegative Ricci curvature.

Hamilton's Ricci flow. In the case of Ricci flow where $\alpha = \mathrm{Rc}$, we have $A = R$. The evolution equation $\partial R/\partial t = 2|\mathrm{Rc}|^2 + \Delta R$ gives

$$B = \frac{\partial A}{\partial t} - 2|\alpha|^2 = \Delta R.$$

Notice that $\nabla R = 2\,\mathrm{Div}(\mathrm{Rc})$ by the second Bianchi identity; we thus get

$$\Theta(V) = 0, \quad B - \Delta A = 0. \tag{3-10}$$

List's extended Ricci flow. Bernhard List [2008] introduced an extended Ricci flow system, namely

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc} + 2a_n \nabla v \otimes \nabla v, \tag{3-11}$$

where v is a solution to the heat equation $\partial v/\partial t = \Delta v$ and a_n a positive constant depending only on the dimension n of the manifold. It turns out that one can exhibit List's flow as a Ricci–DeTurck flow in one higher dimension. This connection has been observed by Jun-Fang Li according to [Akbar and Woolgar 2009]. The extended Ricci flow is interesting by itself since its stationary points are solutions to the vacuum Einstein equations, and it is desirable to work on this flow directly.

In our notation for the extended Ricci flow, $\alpha = \mathrm{Rc} - a_n\, dv \otimes dv$ and $A = R - a_n|\nabla v|^2$, which gives

$$\nabla A = \nabla R - 2a_n \nabla\nabla v(\nabla v, \cdot\,).$$

Since $\mathrm{Div}(dv \otimes dv)_k = g^{ij}\nabla_i(\nabla_j v \nabla_k v) = (\Delta v)\nabla_k v + g^{ij}\nabla_j v \nabla_i \nabla_k v$, we have

$$\mathrm{Div}\,\alpha = \mathrm{Div}\,\mathrm{Rc} - a_n\,\mathrm{Div}(dv \otimes dv) = \tfrac{1}{2}\nabla R - a_n(\Delta v \nabla v + \nabla\nabla v(\nabla v, \cdot\,)).$$

Thus we find

$$\nabla A - 2\,\mathrm{Div}(\alpha) = 2a_n \Delta v \nabla v. \tag{3-12}$$

The evolution equation of α is given by (cf. [List 2008])

$$\beta_{ij} = \frac{\partial \alpha_{ij}}{\partial t} = \Delta\alpha_{ij} - R_{ip}\alpha_{pj} - R_{jp}\alpha_{pi} + 2R_{ipqj}\alpha_{pq} + 2a_n \Delta v \nabla_i \nabla_j v.$$

(Note that in our notation $R_{ij} = g^{pq} R_{ipqj}$, while many authors, including List, write $R_{ij} = -g^{pq} R_{ipqj}$.) Hence we have $B = \Delta A + 2a_n(\Delta v)^2$ and

$$B - \Delta A = 2a_n(\Delta v)^2. \tag{3-13}$$

Plugging in our formula for Θ we arrive at

$$\Theta(V) = a_n\langle \nabla v, V\rangle^2 + 2a_n \Delta v \langle \nabla v, V\rangle + a_n(\Delta v)^2 = a_n(\langle \nabla v, V\rangle + \Delta v)^2.$$

Müller's Ricci flow coupled with harmonic map flow. The Ricci flow coupled with an harmonic map flow was introduced in [Müller 2012] as a generalization of the extended Ricci flow. Suppose that (N, γ) is a further closed static Riemannian manifold, $a(t)$ a nonnegative function depending only on time, and $\varphi(t)\colon M \to N$ a family of 1-parameter smooth maps. Then $(g(t), \varphi(t))$ is called a solution to Müller's Ricci flow coupled with harmonic map flow with coupling function $a(t)$, if it satisfies

$$\begin{cases} \partial g/\partial t = -2\mathrm{Rc} + 2a(t)\,\nabla\varphi \otimes \nabla\varphi, \\ \partial \varphi/\partial t = \tau_g \varphi, \end{cases} \tag{3-14}$$

where τ_g denotes the tension field of the map φ with respect to the evolving metric $g(t)$ and $\nabla\varphi \otimes \nabla\varphi \equiv \varphi^*\gamma$ the pullback of the metric γ on N via the map φ.

Recall that $\mathcal{D}(V) = 2\Theta(-V)$; we have, as in [Müller 2010],

$$B - \Delta A = 2a\,|\tau_g\varphi|^2 - a'|\nabla\varphi|^2, \quad \Theta(V) = a\,|\tau_g\varphi + \nabla_V\varphi|^2 - \tfrac{1}{2}a'|\nabla\varphi|^2. \tag{3-15}$$

Thus both $B - \Delta A$ and Θ are nonnegative as long as $a(t)$ is nonincreasing in time.

Lorentzian mean curvature flow when the ambient space has nonnegative sectional curvature. Let L^{n+1} be a Lorentzian manifold, and $M(t)$ be a family of space-like hypersurfaces of L. Denote by ν the future-oriented time-like unit normal vector of M, by h_{ij} the second fundamental form, and by H its mean curvature. Let $F(t, y)$ be the position function of M in L. The Lorentzian mean curvature flow is then defined by

$$\frac{\partial F}{\partial t} = H\nu. \tag{3-16}$$

The induced metric $g(t)$ of $M(t)$ satisfies $\partial_t g = 2Hh_{ij}$. We have

$$\begin{aligned} B - \Delta A &= 2H^2|h|^2 + 2|\nabla H|^2 + 2H^2\,\overline{\mathrm{Rc}}(\nu, \nu), \\ \Theta(V) &= |\nabla H + h(V, \cdot)|^2 + \overline{\mathrm{Rc}}(H\nu + V, H\nu + V) + \overline{\mathrm{Rm}}(V, \nu, \nu, V), \end{aligned} \tag{3-17}$$

where $\overline{\mathrm{Rc}}$ and $\overline{\mathrm{Rm}}$ denote the Ricci and Riemann curvature tensors of L^{n+1}. Both $B - \Delta A$ and Θ are obviously nonnegative when the sectional curvature of L^{n+1} is nonnegative.

4. Perelman's $\mathcal{F}_k$ functional in abstract geometric flows

We proved the following. If $(M, g(t))$ is a solution to the abstract flow equation (1-1) and u a positive solution to the conjugate heat equation (1-4) then

$$\frac{d}{dt}\int_M (|\nabla \log u|^2 + A)u\,dy = \int_M 2\big(|\alpha - \nabla\nabla \log u|^2 + \Theta(\nabla \log u)\big)u\,dy. \tag{4-18}$$

We note that

$$\begin{aligned} \frac{d}{dt}\int_M Au\,dy &= \int_M \left(\frac{\partial A}{\partial t}u + A\frac{\partial u}{\partial t} - A^2u\right)dy \\ &= \int_M \big((2|\alpha|^2 + B)u + A(-\Delta u + Au) - A^2u\big)\,dy \\ &= \int_M 2\big(|\alpha|^2 + \tfrac{1}{2}(B - \Delta A)\big)u\,dy. \end{aligned} \tag{4-19}$$

Let $\phi := -\log u$; then

$$\frac{\partial \phi}{\partial t} = -\Delta\phi + |\nabla\phi|^2 - A, \tag{4-20}$$

with constraint $\int_M e^{-\phi}\, dy = 1$. We rewrite (4-18) in the more familiar form following Perelman's notation:

$$\frac{d}{dt}\int_M (|\nabla\phi|^2 + A)e^{-\phi}\, dy = \int_M 2\big(|\alpha + \nabla\nabla\phi|^2 + \Theta(-\nabla\phi)\big)e^{-\phi}\, dy. \tag{4-21}$$

Definition 4.1. For any $\phi \in C^\infty(M)$ with $\int_M e^{-\phi}\, dy = 1$ and any constant k we define Perelman's $\mathcal{F}_k$-functional for abstract geometric flows by

$$\mathcal{F}_k(g, \phi) = \int_M (|\nabla\phi|^2 + kA)e^{-\phi}\, dy. \tag{4-22}$$

When $k = 1$ we simply denote $\mathcal{F}_1$ by $\mathcal{F}$.

For Perelman's $\mathcal{F}_k$-functional in an abstract geometric flow we have:

Theorem 4.2. *If g is a solution of the abstract geometric flow equation (1-1) and ϕ a solution to (4-20) then we have*

$$\frac{d}{dt}\mathcal{F}_k = \int_M 2\Big(|\alpha + \nabla\nabla\phi|^2 + (k-1)|\alpha|^2 + \Theta(-\nabla\phi) + \frac{k-1}{2}(B - \Delta A)\Big) \cdot e^{-\phi}\, dy. \tag{4-23}$$

Thus for $k > 1$, $\mathcal{F}_k$ is monotone nondecreasing as long as $B - \Delta A$ and Θ are nonnegative. Moreover the monotonicity is strict unless

$$\alpha = 0, \quad \phi = \text{constant}, \quad B - \Delta A = 0.$$

For $k = 1$ we have

$$\frac{d}{dt}\mathcal{F} = \int_M 2\big(|\alpha + \nabla\nabla\phi|^2 + \Theta(-\nabla\phi)\big)e^{-\phi}\, dy. \tag{4-24}$$

In particular, $\mathcal{F}$ is monotone nondecreasing when $\Theta \geq 0$, and monotonicity is strict unless

$$\alpha + \nabla\nabla\phi = 0, \quad \Theta(-\nabla\phi) = 0.$$

Proof. Since

$$\mathcal{F}_k(g, \phi) = \int_M (|\nabla\phi|^2 + A)e^{-\phi}\, dy + (k-1)\int_M Ae^{-\phi}\, dy,$$

and by (4-21) and (4-19) we immediately get formula (4-23).

Furthermore for $k > 1$, the functional $\mathcal{F}_k$ is monotone nondecreasing as long as $B - \Delta A$ and Θ are nonnegative. When $d/dt\mathcal{F}_k = 0$, each term on the right side of (4-23) has to be identically zero. In particular we have

$$\alpha + \nabla\nabla\phi = 0, \quad \alpha = 0,$$

which further implies $\Delta\phi = 0$ on the closed manifold M, and thus ϕ has to be a constant. Now $\Theta(-\nabla\phi) = \Theta(0) = (B - \Delta A)/2$ and $B - \Delta A = 0$.

When $k = 1$ the statement in the theorem is obvious. □

The advantage of $\mathcal{F}_k$ over $\mathcal{F}$ is that when $k > 1$, extra terms in $\mathcal{F}_k'$ can tell more about the manifold M. Li [2007] has studied $\mathcal{F}_k$ in the Ricci flow. We state an analogous application of $\mathcal{F}_k$ to rule out nontrivial steady breathers in abstract geometric flows.

Recall that a breather of a geometric flow is a periodic solution changing only by diffeomorphism and rescaling. A solution $(M, g(t))$ is called a breather if there are a diffeomorphism $\eta\colon M \to M$, a positive constant c and times $t_1 < t_2$ such that

$$g(t_2) = c\,\eta^* g(t_1), \quad \alpha(t_2) = \eta^*\alpha(t_1). \tag{4-25}$$

When $c < 1$, $c = 1$ or $c > 1$, the breather is called shrinking, steady or expanding, respectively.

We now apply monotonicity of $\mathcal{F}_k$ to rule out nontrivial steady breathers of abstract geometric flows.

Corollary 4.3. *Suppose that $(M, g(t))$ is a steady breather to an abstract geometric flow (1-1). Suppose that $\Theta \geq 0$ and $B - \Delta A \geq 0$. Then $B - \Delta A = 0$ and the steady breather is α-flat, namely $\alpha = 0$.*

Proof. The arguments are standard and follow from Perelman's proof [2002] of the no steady breather theorem for the Ricci flow. We follow [Kleiner and Lott 2008] and only sketch the proof. Define

$$\lambda(t) = \inf\left\{\mathcal{F}_k(g,\phi)\colon \int_M e^{-\phi}\,dy = 1,\ \phi \in C^\infty(M)\right\}. \tag{4-26}$$

Since we are on a steady breather we have $\lambda(t_1) = \lambda(t_2)$. Let $\bar\phi(t_2)$ be a minimizer of $\lambda(t_2)$. Solve the conjugate heat equation backwards with end value $e^{-\bar\phi(t_2)}$. Denote the solution by $u(t)$. Let $\phi(t) = -\log u(t)$ then $\phi(t)$ satisfies the constraint

$$\int_M e^{-\phi}\,dy = 1,$$

and $\mathcal{F}_k(g(t), \phi(t))$ is monotone nondecreasing as its derivative is nonnegative when $e^{-\phi(t)}$ is a solution to the conjugate heat equation. Thus we have

$$\lambda(t_1) \leq \mathcal{F}_k(g(t_1), \phi(t_1)) \leq \mathcal{F}_k(g(t_2), \bar\phi(t_2)) = \lambda(t_2). \tag{4-27}$$

Since on a breather $\lambda(t_1) = \lambda(t_2)$, we get

$$\mathcal{F}_k(g(t_1), \phi(t_1)) = \mathcal{F}_k(g(t_2), \phi(t_2)),$$

and in particular $\mathcal{F}_k'(g(t), \phi(t)) = 0$ when $t \in [t_1, t_2]$. Now we apply Theorem 4.2 to conclude that $\alpha = 0$ and $B - \Delta A = 0$ on M when $t \in [t_1, t_2]$. □

Remark 4.4. From (4-26) we know that λ is the lowest eigenvalue of $-\Delta + (k/4)A$. Thus, by Theorem 4.2, under the assumptions that $B - \Delta A \geq 0$ and $\Theta \geq 0$, the lowest eigenvalue of $-\Delta + (k/4)A$ is monotone in t when $k \geq 1$. An explicit formula for the derivative of the lowest eigenvalue will be given in Section 7 under the assumption that λ is differentiable along time.

5. Construction of Perelman's $\mathcal{W}$ entropy

We have noted that Perelman's $\mathcal{F}$-functional is the derivative of $\mathcal{E}$, whose stationary points are steady solitons. The purpose of this section is to construct functionals corresponding to the shrinking solitons. Our construction is just completing squares of $\mathcal{E}''$ (or $\mathcal{F}'$ by Perelman's notation). Monotonicity of $\mathcal{W}$ holds in the flows mentioned in Section 3.

We rewrite the second derivative of $\mathcal{E}(t)$ in order to fit the shrinking soliton equation simply by completing squares:

$$\begin{aligned}
\mathcal{E}''(t) &= \int_M 2\big(|\alpha - \nabla\nabla \log u|^2 + \Theta(\nabla \log u)\big) u \, dy \\
&= \int_M \bigg(2u \left|\alpha - \nabla\nabla \log u - \frac{1}{2(T-t)} g\right|^2 + \frac{2u}{T-t}(A - \Delta \log u) \\
&\qquad - \frac{2nu}{4(T-t)^2} + 2u\Theta(\nabla \log u)\bigg) dy \\
&= \int_M 2\bigg(\left|\alpha - \nabla\nabla \log u - \frac{1}{2(T-t)} g\right|^2 + \Theta(\nabla \log u)\bigg) u \, dy \\
&\qquad + \frac{2}{T-t}\mathcal{E}'(t) - \frac{n}{2(T-t)^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_M 2\bigg(\left|\alpha - \nabla\nabla \log u - \frac{1}{2(T-t)} g\right|^2 + \Theta(\nabla \log u)\bigg) u \, dy \\
&\qquad = \mathcal{E}''(t) - \frac{2}{T-t}\mathcal{E}'(t) + \frac{n}{2(T-t)^2} \\
&\qquad = \frac{1}{T-t}\frac{d}{dt}\bigg((T-t)\mathcal{E}' - \mathcal{E} - \frac{n}{2}\log(T-t)\bigg).
\end{aligned}$$

Now in terms of

$$\mathcal{W} := (T-t)\mathcal{E}' - \mathcal{E} - \frac{n}{2}\log(T-t) - \frac{n}{2}\log(4\pi) - n,$$

we have proved that

$$\frac{d}{dt}\mathcal{W} = (T-t)\int_M 2\left(\left|\alpha - \nabla\nabla\log u - \frac{1}{2(T-t)}g\right|^2 + \Theta(\nabla\log u)\right)u\,dy. \tag{5-28}$$

Following Perelman, we let

$$\tau := T-t, \quad \phi := -\log((4\pi\tau)^{n/2}u),$$

and introduce the following definition.

Definition 5.1. For a solution (M, g) to the abstract geometric flow equation (1-1) and for $\phi \in C^\infty(M)$, let Perelman's $\mathcal{W}$-entropy be defined as

$$\mathcal{W}(g,\phi,t) = \int_M \left(\tau(|\nabla\phi|^2 + A) + \phi - n\right)(4\pi\tau)^{-n/2}e^{-\phi}\,dy. \tag{5-29}$$

We can rewrite (5-28) in the following way.

Theorem 5.2. *Let (M, g) be a solution to the abstract geometric flow equation (1-1). If ϕ satisfies*

$$\frac{\partial\phi}{\partial t} = -\Delta\phi + |\nabla\phi|^2 - A + \frac{n}{2\tau} \quad \text{and} \quad \int_M (4\pi\tau)^{-n/2}e^{-\phi}\,dy = 1,$$

then

$$\frac{d}{dt}\mathcal{W} = \int_M 2\tau\left(\left|\alpha + \nabla\nabla\phi - \frac{1}{2\tau}g\right|^2 + \Theta(-\nabla\phi)\right)(4\pi\tau)^{-n/2}e^{-\phi}\,dy.$$

If $\Theta \geq 0$ then $\mathcal{W}$ is monotone nondecreasing, and the monotonicity is strict unless

$$\alpha + \nabla\nabla\phi - \frac{1}{2\tau}g = 0, \quad \Theta(-\nabla\phi) = 0.$$

The monotonicity of $\mathcal{W}$ can be applied to rule out nontrivial shrinking breathers in abstract flows with $\Theta \geq 0$. The arguments are almost identical to the Ricci flow case. We omit details.

6. Expander entropy $\mathcal{W}_+$

Feldman, Ilmanen, and Ni [2005] established expander entropy $\mathcal{W}_+$ for Ricci flow, and there has been a very nice explanation of their motivation in [Feldman et al. 2005]. We attempt to explain formally why $\mathcal{W}_+$ should be the way as they defined it. In short, the signs in $\mathcal{W}$ and $\mathcal{W}_+$ are caused by antiderivatives of $1/(t-T)$ depending on the situation whether $t > T$ or $t < T$.

We now carry out the details. Note that $t > T$ on expanders and that

$$\begin{aligned}\mathscr{E}''(t) &= \int_M 2\left(|\alpha - \nabla\nabla \log u|^2 + \Theta(\nabla \log u)\right) u\, dy \\ &= \int_M \left(2u\left|\alpha - \nabla\nabla \log u + \frac{1}{2(t-T)}g\right|^2 - \frac{2u}{t-T}(A - \Delta \log u)\right. \\ &\qquad \left. - \frac{2nu}{4(t-T)^2} + 2u\Theta(\nabla \log u)\right) dy \\ &= \int_M 2\left(\left|\alpha - \nabla\nabla \log u + \frac{1}{2(t-T)}g\right|^2 + \Theta(\nabla \log u)\right) u\, dy \\ &\qquad - \frac{2}{t-T}\mathscr{E}'(t) - \frac{n}{2(t-T)^2};\end{aligned}$$

moreover

$$\begin{aligned}&\int_M 2\left(\left|\alpha - \nabla\nabla \log u + \frac{1}{2(t-T)}g\right|^2 + \Theta(\nabla \log u)\right) u\, dy \\ &\qquad = \mathscr{E}''(t) + \frac{2}{t-T}\mathscr{E}'(t) + \frac{n}{2(t-T)^2} \\ &\qquad = \frac{1}{t-T}\frac{d}{dt}\left((t-T)\mathscr{E}' + \mathscr{E} + \frac{n}{2}\log(t-T)\right).\end{aligned}$$

The calculations suggest to define

$$\mathscr{W}_+ := (t-T)\mathscr{E}' + \mathscr{E} + \frac{n}{2}\log(t-T) + \frac{n}{2}\log(4\pi) + n$$

which is the definition of expander entropy in [Feldman et al. 2005] in the case of Ricci flow. One has

$$\frac{d\mathscr{W}_+}{dt} = (t-T)\int_M 2\left(\left|\alpha - \nabla\nabla \log u + \frac{1}{2(t-T)}g\right|^2 + \Theta(\nabla \log u)\right) u\, dy.$$

This again may be rewritten following [Feldman et al. 2005] in terms of

$$\sigma := t - T, \quad \phi_+ := -\log((4\pi\sigma)^{n/2}u).$$

Definition 6.1. For a solution (M, g) to the abstract geometric flow equation (1-1) and $\phi_+ \in C^\infty(M)$ one defines Perelman's entropy for expanders by

$$\mathscr{W}_+(g, \phi_+, t) = \int_M \left(\sigma\left(|\nabla\phi_+|^2 + A\right) - \phi_+ + n\right)(4\pi\sigma)^{-n/2}e^{-\phi_+}\, dy. \tag{6-30}$$

Theorem 6.2. *Let (M, g) be a solution to the abstract geometric flow equation (1-1). Assume that ϕ_+ satisfies*

$$\frac{\partial \phi_+}{\partial t} = -\Delta \phi_+ + |\nabla \phi_+|^2 - A - \frac{n}{2(t-T)} \quad \text{and} \quad \int_M (4\pi\sigma)^{-n/2} e^{-\phi_+}\, dy = 1.$$

Then

$$\frac{d\mathcal{W}_+}{dt} = \int_M 2\sigma \left(\left| \alpha + \nabla\nabla\phi_+ + \frac{1}{2(t-T)} g \right|^2 + \Theta(-\nabla\phi_+) \right) (4\pi\sigma)^{-n/2} e^{-\phi_+}\, dy.$$

Furthermore, if $\Theta \geq 0$ then $\mathcal{W}_+$ is monotone nondecreasing, and monotonicity is strict unless

$$\alpha + \nabla\nabla\phi_+ + \frac{1}{2(t-T)} g = 0, \quad \Theta(-\nabla\phi_+) = 0.$$

Remark 6.3. The constants $\pm\left(\frac{n}{2}\log(4\pi) + n\right)$ in the definition of $\mathcal{W}$ and $\mathcal{W}_+$ are for normalization purposes.

7. Evolution equation of the lowest eigenvalue

In this section, assuming that the lowest eigenvalue $\lambda(t)$ is differentiable along t, we derive an explicit formula for its derivative in terms of its normalized eigenfunction. Although monotonicity of $\mathcal{F}_k$ in Theorem 4.2 is sufficient for our geometric applications, an explicit formula which holds at points where λ is differentiable, may be of independent interest. Time derivatives of the eigenfunction do not appear in the formula.

In the literature, for instance [Kleiner and Lott 2008, Section 7], it has been stated that smooth dependence on time of the lowest eigenvalue and the corresponding eigenfunction follows from perturbation theory as presented in [Reed and Simon 1978, Chapter XII]. However it is not immediately clear how perturbation theory is applied to our context, where the operator depends only smoothly, but not analytically on t.

Lemma 7.1. *Assume that M is a closed manifold and let $\psi \in C^\infty(M)$. Let λ be the lowest eigenvalue of $-\Delta + \psi$ and f a positive eigenfunction corresponding to λ, so that $\lambda f = -\Delta f + \psi f$. Then*

$$\int_M \psi\, \Delta f^2\, dy = \int_M 2\left(|\nabla\nabla \log f|^2 + \mathrm{Rc}(\nabla \log f, \nabla \log f) \right) f^2\, dy. \tag{7-31}$$

Proof. We have $\psi f = \lambda f + \Delta f$ and

$$\begin{aligned}\psi \Delta f^2 &= 2\psi f \Delta f + 2\psi |\nabla f|^2 \\ &= 2(\lambda f + \Delta f)\Delta f + 2(\lambda f + \Delta f)\frac{|\nabla f|^2}{f} \\ &= \lambda(2f\Delta f + 2|\nabla f|^2) + 2(\Delta f)^2 + 2\,\frac{\Delta f |\nabla f|^2}{f} \\ &= \lambda \Delta f^2 + 2(\Delta f)^2 + 2\,\frac{\Delta f |\nabla f|^2}{f}.\end{aligned}$$

We observe that

$$\begin{aligned}\int_M \psi \Delta f^2\, dy &= \int_M \left(2(\Delta f)^2 + 2\,\frac{\Delta f |\nabla f|^2}{f}\right) dy \\ &= \int_M \left(-2\,\langle \nabla f, \nabla(\Delta f)\rangle - 2\left\langle \nabla f, \nabla\Big(\frac{|\nabla f|^2}{f}\Big)\right\rangle\right) dy.\end{aligned} \tag{7-32}$$

Now we calculate the two terms on the right side of (7-32). For the first term we have by Bochner's formula

$$-2\langle \nabla f, \nabla(\Delta f)\rangle = 2|\nabla\nabla f|^2 + 2\,\mathrm{Rc}(\nabla f, \nabla f) - \Delta(|\nabla f|^2). \tag{7-33}$$

The second term can be written as

$$\begin{aligned}\left\langle \nabla f, \nabla\Big(\frac{|\nabla f|^2}{f}\Big)\right\rangle &= \langle \nabla f, \nabla(f|\nabla \log f|^2)\rangle \\ &= \langle \nabla f, \nabla f|\nabla \log f|^2 + 2f\nabla\nabla \log f(\nabla \log f, \cdot)\rangle \\ &= f^2|\nabla \log f|^4 + 2f^2\nabla\nabla \log f(\nabla \log f, \nabla \log f) \\ &= |\nabla\nabla f|^2 - f^2|\nabla\nabla \log f|^2,\end{aligned} \tag{7-34}$$

where in the last equality we used that

$$\nabla\nabla \log f = \frac{\nabla\nabla f}{f} - \frac{\nabla f \otimes \nabla f}{f^2} = \frac{\nabla\nabla f}{f} - \nabla \log f \otimes \nabla \log f,$$

and moreover

$$\begin{aligned}|\nabla\nabla f|^2 &= f^2|\nabla\nabla \log f + \nabla \log f \otimes \nabla \log f|^2 \\ &= f^2|\nabla\nabla \log f|^2 + 2f^2\nabla\nabla \log f(\nabla \log f, \nabla \log f) + f^2|\nabla \log f|^4.\end{aligned}$$

Plugging (7-33) and (7-34) into (7-32), we get

$$\int_M \psi \Delta f^2\, dy = \int_M \big(2f^2|\nabla\nabla \log f|^2 + 2\,\mathrm{Rc}(\nabla f, \nabla f)\big)\, dy. \qquad \square$$

Let $\lambda(t)$ be the lowest eigenvalue of $-\Delta + cA$, where c is a constant; indeed

$$\lambda(t) = \inf\left\{\int_M |\nabla\phi|^2 + cA\phi^2\, dy : \int_M \phi^2\, dy = 1,\ \phi \in C^\infty(M)\right\}. \tag{7-35}$$

Let $f(t, \cdot)$ be the corresponding positive eigenfunction normalized by

$$\int_M f^2(t, y)\, dy = 1.$$

Theorem 7.2. *At all times t_0 when the function $t \mapsto \lambda(t)$ is differentiable we have*

$$\begin{aligned}&\lambda'(t_0) \\ &= \frac{1}{2}\int_M \Big(|\alpha - 2\nabla\nabla \log f|^2 + (4c-1)|\alpha|^2 + \Theta(2\nabla \log f) + \frac{4c-1}{2}(B - \Delta A)\Big) f^2\, dy.\end{aligned} \tag{7-36}$$

In particular, for $c = \frac{1}{4}$ we have

$$\lambda' = \frac{1}{2}\int_M \big(|\alpha - 2\nabla\nabla \log f|^2 + \Theta(2\nabla \log f)\big) f^2\, dy. \tag{7-37}$$

Proof. Fix $t_0 \in (0, T)$ where the function $t \mapsto \lambda(t)$ is differentiable, and let $\varphi : (0, T) \times M \to \mathbb{R}_{>0}$ be a smooth function such that

(1) $\int_M \varphi(t, y)^2\, dy = 1$ for all $t \in (0, T)$, and

(2) $\varphi(t_0, \cdot) = f(t_0, \cdot)$.

For instance $\varphi(t)$ may be chosen as $f(t_0)\sqrt{dy(g(t_0))/dy(g(t))}$, where $dy(g(t))$ is the volume form with respect to the metric $g(t)$. Let

$$\mu(t) := \int_M \big(|\nabla\varphi(t, y)|^2 + cA(t, y)\varphi(t, y)^2\big)\, dy. \tag{7-38}$$

Then $\mu(t)$ is a smooth function by definition. The trick to work with $\mu(t)$ rather than $\lambda(t)$ allows to bypass time derivatives of the eigenfunction $f(t, \cdot)$. Note that $\mu(t) \geq \lambda(t)$ for all $t \in (0, T)$, and $\mu(t_0) = \lambda(t_0)$, so that

$$\lambda'(t_0) = \mu'(t_0).$$

Differentiation of (7-38) gives

$$\begin{aligned}
\mu' &= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) + 2\langle\nabla\varphi',\nabla\varphi\rangle + cA'\varphi^2 + 2cA\varphi\varphi' - (|\nabla\varphi|^2 + cA\varphi^2)A\big)\,dy \\
&= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) - 2\varphi'\Delta\varphi + cA'\varphi^2 + 2cA\varphi\varphi' + \varphi\langle\nabla A,\nabla\varphi\rangle + A\varphi\Delta\varphi \\
&\qquad\qquad - cA^2\varphi^2\big)\,dy \\
&= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) + cA'\varphi^2 + \varphi\langle\nabla A,\nabla\varphi\rangle\big)\,dy + \lambda\int_M (2\varphi'\varphi - A\varphi^2)\,dy \\
&= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) + cA'\varphi^2 + \varphi\langle\nabla A,\nabla\varphi\rangle\big)\,dy \\
&= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) + c(2|\alpha|^2 + B)\varphi^2 - \tfrac{1}{2}A\Delta\varphi^2\big)\,dy \\
&= \int_M \big(2\alpha(\nabla\varphi,\nabla\varphi) + 2c|\alpha|^2\varphi^2 + c(B-\Delta A)\varphi^2 + cA\Delta\varphi^2 - \tfrac{1}{2}A\Delta\varphi^2\big)\,dy,
\end{aligned}$$

where in the fourth equality we used that $\int_M (2\varphi\varphi' - A\varphi^2)\,dy = 0$ (which is due to the normalization of φ).

Noting that

$$\begin{aligned}
\operatorname{Div}(\varphi\alpha(\nabla\varphi,\cdot)) &= \alpha(\nabla\varphi,\nabla\varphi) + \varphi\operatorname{Div}(\alpha)(\nabla\varphi) + \varphi\langle\alpha,\nabla\nabla\varphi\rangle \\
&= 2\alpha(\nabla\varphi,\nabla\varphi) + \varphi\operatorname{Div}(\alpha)(\nabla\varphi) + \varphi^2\langle\alpha,\nabla\nabla\log\varphi\rangle,
\end{aligned}$$

and by the divergence theorem, we have

$$\begin{aligned}
&\int_M 2\alpha(\nabla\varphi,\nabla\varphi)\,dy \\
&\qquad = \int_M 4\alpha(\nabla\varphi,\nabla\varphi) - 2\alpha(\nabla\varphi,\nabla\varphi)\,dy \\
&\qquad = \int_M -2\varphi\operatorname{Div}(\alpha)(\nabla\varphi) - 2\varphi^2\langle\alpha,\nabla\nabla\log\varphi\rangle - 2\alpha(\nabla\varphi,\nabla\varphi)\,dy.
\end{aligned} \tag{7-39}$$

In (7-31) let $\psi = cA$, we get

$$\int_M cA\Delta\varphi^2\,dy = \int_M 2\varphi^2|\nabla\nabla\log\varphi|^2 + 2\operatorname{Rc}(\nabla\varphi,\nabla\varphi)\,dy. \tag{7-40}$$

Plugging (7-39) and (7-40) into the equation for μ' we obtain

$$\begin{aligned}\mu' &= \int_M \big(-2\varphi \operatorname{Div}(\alpha)(\nabla\varphi) - 2\varphi^2\langle\alpha, \nabla\nabla\log\varphi\rangle - 2\alpha(\nabla\varphi,\nabla\varphi) + 2c|\alpha|^2\varphi^2 \\ &\qquad + c(B-\Delta A)\varphi^2 + 2\varphi^2|\nabla\nabla\log\varphi|^2 + 2\operatorname{Rc}(\nabla\varphi,\nabla\varphi) - \tfrac{1}{2}A\Delta\varphi^2\big)\,dy \\ &= \int_M \Big(\big(2|\nabla\nabla\log\varphi|^2 - 2\langle\alpha,\nabla\nabla\log\varphi\rangle + \tfrac{1}{2}|\alpha|^2 + \big(2c-\tfrac{1}{2}\big)|\alpha|^2\big)\varphi^2 \\ &\qquad + (2(\operatorname{Rc}-\alpha)(\nabla\log\varphi,\nabla\log\varphi) + \langle\nabla A - 2\operatorname{Div}(\alpha),\nabla\log\varphi\rangle)\varphi^2 \\ &\qquad + c(B-\Delta A)\varphi^2\Big)\,dy \\ &= \int_M \Big(\tfrac{1}{2}|\alpha - 2\nabla\nabla\log\varphi|^2 + \big(2c-\tfrac{1}{2}\big)|\alpha|^2 + \tfrac{1}{2}\Theta(2\nabla\log\varphi) + \big(c-\tfrac{1}{4}\big)(B-\Delta A)\Big) \\ &\qquad \cdot\varphi^2\,dy,\end{aligned}$$

so that

$$\begin{aligned}\lambda'(t_0) &= \mu'(t_0) \\ &= \frac{1}{2}\int_M \Big(|\alpha - 2\nabla\nabla\log f|^2 + (4c-1)|\alpha|^2 + \Theta(2\nabla\log f) + \frac{4c-1}{2}(B-\Delta A)\Big) \\ &\qquad \cdot f^2\,dy,\end{aligned}$$

as claimed. □

Let us compare Theorem 4.2 with Theorem 7.2, and (4-23) with (7-36). Let $\phi = -2\log f$; then (7-36) can be rewritten as

$$\lambda' = \frac{1}{2}\int_M \Big(|\alpha + \nabla\nabla\phi|^2 + (4c-1)|\alpha|^2 + \Theta(-\nabla\phi) + \frac{4c-1}{2}(B-\Delta A)\Big)e^{-\phi}\,dy.$$

Letting $k = 4c$, we see that the two evolution equations are formally proportional. We note that in (4-23) the exponential $e^{-\phi}$ is a normalized solution to the conjugate heat equation, while $e^{-\phi/2}$ in the preceding integrand is the normalized eigenfunction of $\lambda(t)$.

8. Eigenvalue monotonicity in various flows

In this section we list explicit formulas of the eigenvalue evolution in different flows. The constant c is assumed to be no less than $\frac{1}{4}$.

Hamilton's Ricci flow. In the case of Ricci flow, monotonicity of the lowest eigenvalue of $-\Delta + cR$ for $c \geq \frac{1}{4}$ and its applications has been established by Cao [2007; 2008], as mentioned in the introduction. See also [Li 2007]. Plugging

$$\alpha = \operatorname{Rc}, \quad \Theta = 0, \quad B - \Delta A = 0$$

into (7-36) we get Cao's formula [2008] for the Ricci flow:

$$\lambda'(t) = \int_M \tfrac{1}{2}\big(|\mathrm{Rc} - 2\nabla\nabla \log f|^2 + (4c-1)|\mathrm{Rc}|^2\big) f^2\, dy. \tag{8-41}$$

This can be applied to show that every steady breather in the Ricci flow is Ricci flat.

List's extended Ricci flow. We work out the details in the extended Ricci flow.

Corollary 8.1. *Assume that $(M, g(t))$ is a solution to the extended Ricci flow equation, and that $\lambda(t)$ is the lowest eigenvalue of*

$$-\Delta + c(R - a_n|\nabla v|^2), \tag{8-42}$$

then we have

$$\lambda'(t) = \int_M \Big(\tfrac{1}{2}|\mathrm{Rc} - a_n \nabla v \otimes \nabla v - 2\nabla\nabla \log f|^2 + \big(2c - \tfrac{1}{2}\big)|\mathrm{Rc} - a_n \nabla v \otimes \nabla v|^2 + \frac{a_n}{2}\big((\Delta v - 2\langle \nabla v, \nabla \log f\rangle)^2 + (4c-1)(\Delta v)^2\big)\Big) f^2\, dy.$$

In particular, a steady breather of the extended Ricci flow is trivial in the sense that

$$\mathrm{Rc} = 0, \quad v \equiv \text{constant}.$$

Proof. The formula for $\lambda'(t)$ is a direct plug-in. When $(M, g(t))$ is a steady breather, there are times $t_1 < t_2$ such that $\lambda(t_1) = \lambda(t_2)$ for any $c > \frac{1}{4}$. In particular we have $\Delta v = 0$ on the closed manifold M, thus v is constant, and moreover M is Ricci flat by $\mathrm{Rc} - a_n \nabla v \otimes \nabla v = 0$. □

Müller's Ricci flow coupled with harmonic map flow. We already used $\mathcal{F}_k$ to rule out nontrivial steady breathers. Using eigenvalue monotonicity, one does not need to solve the conjugate heat equation. The lowest eigenvalue of

$$-\Delta + c(R - a(t)|\nabla \varphi|^2)$$

is nondecreasing along the flow. The conclusions remain the same as in Corollary 4.3.

Lorentzian mean curvature flow when the ambient space has nonnegative sectional curvature. When M evolves along the Lorentzian mean curvature flow (3-16), the lowest eigenvalue of

$$-\Delta - cH^2$$

is nondecreasing provided sectional curvature of the ambient space is nonnegative.

9. Normalized eigenvalue and no expanding breathers theorem

The eigenvalue of $-\Delta + cA$ is not scale invariant. Suppose that α is invariant under scaling which is true in all of our examples. If we rescale a Riemannian metric g to εg by a positive constant ε, then

$$-\Delta_{\varepsilon g} + cA_{\varepsilon g} = \varepsilon^{-1}(-\Delta_g + cA_g),$$

and for the lowest eigenvalue we get $\lambda_{\varepsilon g} = \varepsilon^{-1}\lambda_g$. Thus the (nonnormalized) lowest eigenvalue only works in the steady case. Following [Perelman 2002] we define the scale invariant eigenvalue by

$$\bar{\lambda}_g := \lambda_g V_g^{2/n}, \tag{9-43}$$

where V denotes the volume of M.

In the following for simplicity of calculations we let $c = \frac{1}{4}$.

Proposition 9.1. *Suppose that $(M, g(t))$ is a solution to the abstract geometric flow (1-1) with α being scale invariant. Assume that Θ is nonnegative. Let $\lambda(t)$ be the lowest eigenvalue of $-\Delta + A/4$. Then whenever $\bar{\lambda}(t) \le 0$ one has $\bar{\lambda}'(t) \ge 0$.*

Proof. Recall that by (7-35) and choosing $\phi(t, y) = V^{-1/2}$ we have

$$\lambda(t) \le \frac{1}{4V}\int_M A\, dy.$$

When $\bar{\lambda}(t) \le 0$ we obtain

$$\begin{aligned}
\bar{\lambda}'(t) &= \lambda'(t)V^{2/n} + \frac{2\lambda}{n}V^{n/2-1}\int_M (-A)\, dy \\
&\ge V^{n/2}\left(\lambda'(t) - \frac{8\lambda^2(t)}{n}\right) \\
&\ge \frac{V^{n/2}}{2}\left(\int_M \left(|\alpha - 2\nabla\nabla \log f|^2 + \Theta(2\nabla \log f)\right) f^2\, dy - \frac{16\lambda^2(t)}{n}\right),
\end{aligned}$$

where f is the normalized positive eigenfunction corresponding to λ.

We observe that

$$|\alpha - 2\nabla\nabla \log f|^2 = \left|\alpha - 2\nabla\nabla \log f - \frac{1}{n}(A - 2\Delta \log f)g\right|^2 + \frac{1}{n}(A - 2\Delta \log f)^2.$$

Recall that f is the normalized eigenfunction and by Hölder's inequality we obtain

$$\begin{aligned}\int_M (A-2\Delta\log f)^2 f^2\,dy &= \int_M (A-2\Delta\log f)^2 f^2\,dy \int_M f^2\,dy \\ &\geq \left(\int_M (A-2\Delta\log f) f\cdot f\,dy\right)^2 \\ &= \left(\int_M Af^2+4|\nabla f|^2\,dy\right)^2 \\ &= 16\,\lambda^2(t).\end{aligned} \tag{9-44}$$

Finally we have $\bar{\lambda}'(t)\geq 0$. □

If $\lambda(t)\leq 0$ we have in fact derived the inequality

$$\begin{aligned}\bar{\lambda}'(t) \geq{}& \frac{V^{2/n}}{2}\left(\int_M\left(\left|\alpha-2\nabla\nabla\log f-\frac{1}{n}(A-2\Delta\log f)g\right|^2+\Theta(2\nabla\log f)\right)f^2\,dy\right) \\ &+\frac{V^{2/n}}{2n}\left(\int_M (A-2\Delta\log f)^2 f^2\,dy-\left(\int_M (A-2\Delta\log f) f\cdot f\,dy\right)^2 dy\right).\end{aligned} \tag{9-45}$$

Now we may use (9-45) to rule out nontrivial expanding breathers.

Theorem 9.2. *Suppose that $(M, g(t))$ is a solution to the abstract geometric flow equation (1-1) with α being scale invariant. Assume that Θ is nonnegative. If $(M, g(t))$ is an expanding breather for $t_1 < t_2$, then it has to be a gradient soliton on (t_1, t_2) in the sense that*

$$\alpha-2\nabla\nabla\log f-\frac{4\lambda}{n}g=0$$

where f is the positive normalized eigenfunction corresponding to $\lambda(t)$. Moreover one has

$$\Theta(2\nabla\log f)=0.$$

Proof. Since $\bar{\lambda}$ is invariant under diffeomorphism and rescaling, we have $\bar{\lambda}(t_1)=\bar{\lambda}(t_2)$. Since $V(t_1)<V(t_2)$ there must be a time $t_0\in(t_1,t_2)$ such that $V'(t_0)\geq 0$. Hence

$$\begin{aligned}\lambda(t_0) &\leq \frac{1}{4V(t_0)}\int_M A(t_0)\,dy \\ &= -\frac{1}{4V(t_0)}V'(t_0) \\ &\leq 0.\end{aligned}$$

Proposition 9.1 then implies $\bar{\lambda}(t_1) \le \bar{\lambda}(t_0) \le 0$. Thus, on the whole interval $[t_1, t_2]$, the function $\bar{\lambda}(t)$ is nonpositive increasing and equals at the end points. This means that the right side of (9-45) vanishes. In particular, the second line of (9-45) being zero means that equality holds in Hölder's inequality (9-44). Thus $A - 2\Delta \log f$ must be a spatial constant which is $4\lambda(t)$ because f is a normalized eigenfunction corresponding to $\lambda(t)$. The vanishing of the first line of (9-45) means that

$$\alpha - 2\nabla\nabla \log f - \frac{4\lambda}{n} g = 0, \quad \Theta(2\nabla \log f) = 0. \qquad \square$$

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HONGXIN GUO
SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE
WENZHOU UNIVERSITY
CHASHAN UNIVERSITY TOWN
WENZHOU, ZHEJIANG 325035
CHINA

guo@wzu.edu.cn

ROBERT PHILIPOWSKI
MATHEMATICS RESEARCH UNIT
UNIVERSITY OF LUXEMBOURG
6, RUE RICHARD COUDENHOVE-KALERGI
L-1359 LUXEMBOURG
LUXEMBOURG

robert.philipowski@uni.lu

ANTON THALMAIER
MATHEMATICS RESEARCH UNIT
UNIVERSITY OF LUXEMBOURG
6, RUE RICHARD COUDENHOVE-KALERGI
L-1359 LUXEMBOURG
LUXEMBOURG

anton.thalmaier@uni.lu
http://math.uni.lu/thalmaier/

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