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HARMONIC MAPS ON DOMAINS WITH PIECEWISE LIPSCHITZ CONTINUOUS METRICS

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We study harmonic maps $(\Omega, g) \to (N, h)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain divided into two pieces, the Riemannian metric g is Lipschitz in each piece, and (N, h) is a closed Riemannian submanifold of $\mathbb{R}^k$. We prove the partial regularity of stationary harmonic maps, and the global Lipschitz and piecewise $C^{1,\alpha}$-regularity of weakly harmonic maps from (Ω, g) to manifolds (N, h) that support convex distance square functions.

1. Introduction

Throughout this paper we assume that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$ is a bounded domain of $\mathbb{R}^n$ decomposed into two subdomains Ω^+ and Ω^- by a $C^{1,1}$-hypersurface Γ, and that g is a piecewise Lipschitz metric on Ω, satisfying $g \in C^{0,1}(\Omega^+) \cap C^{0,1}(\Omega^-)$ and discontinuous at every $x \in \Gamma$. For example, let $\Omega = B_1 \subset \mathbb{R}^n$ be the unit ball, $\Gamma = B_1 \cap \{x = (x', 0) \in \mathbb{R}^n\}$, and

$$\bar{g}(x) = \begin{cases} g_0 & \text{if } x \in B_1^+ = \{x^n > 0\} \cap B_1, \\ kg_0 & \text{if } x \in B_1^- = \{x^n < 0\} \cap B_1, \end{cases}$$

where g_0 is the standard metric on $\mathbb{R}^n$ and k $(\neq 1)$ is a positive constant. Let $(N, h) \hookrightarrow \mathbb{R}^k$ be an l-dimensional, smooth compact Riemannian manifold without boundary, isometrically embedded in the Euclidean space $\mathbb{R}^k$.

Motivated by the recent studies on elliptic systems arising from composite materials (see [Li and Nirenberg 2003]) and the periodic homogenization theory in calculus of variations (see [Avellaneda and Lin 1987] and [Lin and Yan 2003]), we are interested in the regularity issue of harmonic maps from (Ω, g) to (N, h).

In order to describe the problem, let's recall some notations. Throughout this paper, we use the Einstein convention for summation. For the metric $g = g_{ij}\, dx^i\, dx^j$, let $(g^{ij}) = (g_{ij})^{-1}$, and $dv_g = \sqrt{g}\, dx (= \sqrt{\det(g_{ij})}\, dx)$ be the volume form of g. For $1 < p < +\infty$, define the Sobolev space

$$W^{1,p}(\Omega, N) = \left\{ u : \Omega \to \mathbb{R}^k \;\middle|\; u(x) \in N \text{ a.e. } x \in \Omega,\ E_p(u, g) = \int_\Omega |\nabla u|_g^p\, dv_g < \infty \right\},$$

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where

$$|\nabla u|_g^2 \equiv g^{ij}\left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\right\rangle$$

is the energy density of u with respect to g, and $\langle\,\cdot\,,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^k$. Denote $W^{1,2}(\Omega, N)$ by $H^1(\Omega, N)$. Now let's recall the definition of stationary harmonic maps.

Definition 1.1. A map $u \in H^1(\Omega, N)$ is called a (weakly) harmonic map if it is a critical point of $E_2(\,\cdot\,, g)$, i.e., if u satisfies

$$\Delta_g u + A(u)(\nabla u, \nabla u)_g = 0 \quad \text{in } \Omega \tag{1-1}$$

in the sense of distributions. Here

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}\right)$$

is the Laplace–Beltrami operator on (Ω, g), $A(\,\cdot\,)(\,\cdot\,,\cdot\,)$ is the second fundamental form of $(N, h) \hookrightarrow \mathbb{R}^k$, and $A(u)(\nabla u, \nabla u)_g = g^{ij}A(u)(\partial u/\partial x_i, \partial u/\partial x_j)$.

Definition 1.2. A (weakly) harmonic map $u \in H^1(\Omega, N)$ is called a stationary harmonic map if, in addition, it is a critical point of $E_2(\,\cdot\,, g)$ with respect to the following domain variations:

$$\frac{d}{dt}\bigg|_{t=0}\int_\Omega \left|\nabla u^t\right|_g^2 \, dv_g = 0, \quad \text{with } u^t(x) = u\big(F_t(x)\big), \tag{1-2}$$

where $F(t, x) := F_t(x) \in C^1\big([-\delta, \delta], C^1(\Omega, \Omega)\big)$, for some small $\delta > 0$, is a family of diffeomorphisms that satisfies

$$\begin{cases} F_0(x) = x & \text{for } x \in \Omega, \\ F_t(x) = x & \text{for } (x, t) \in \partial\Omega \times [-\delta, \delta], \\ F_t\big(\overline{\Omega^\pm}\big) \subset \overline{\Omega^\pm} & \text{for } t \in [-\delta, \delta]. \end{cases} \tag{1-3}$$

In particular, $F_t(\Gamma) \subset \Gamma$ for $0 \le t \le \delta$.

It is readily seen that any minimizing harmonic map from (Ω, g) to (N, h) is a stationary harmonic map. Definition 1.2 implies that a stationary harmonic map on (Ω, g) is a stationary harmonic map on $(\Omega^\pm, g)$. Since $g \in C^{0,1}(\Omega^\pm)$, we can see that u satisfies an energy monotonicity inequality on $\Omega^\pm$. We will show in Section 2 that a stationary harmonic map on (Ω, g) also satisfies an energy monotonicity inequality in Γ under the condition (1-4) below.

The first result is concerned with the (partial) Lipschitz and (partial) piecewise $C^{1,\alpha}$-regularity of stationary harmonic maps. In this context, we are able to extend the well-known partial regularity theorem of stationary harmonic maps on domains with smooth metrics, due to Hélein [2002], Evans [1991], and Bethuel [1993]. More precisely:

Theorem 1.1. *Let $u \in H^1(\Omega, N)$ be a stationary harmonic map on (Ω, g). Suppose that g satisfies the following jump condition on Γ for $n \geq 3$: for any $x \in \Gamma$, there exists a positive constant $k(x) \neq 1$ such that*

$$\lim_{\substack{y\in\Omega^+\\ y\to x}} g(y) = k(x) \lim_{\substack{y\in\Omega^-\\ y\to x}} g(y). \tag{1-4}$$

There exists a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, such that $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega \setminus \Sigma, N)$, and for some $0 < \alpha < 1$, $u \in C^{1,\alpha}_{\mathrm{loc}}\big((\Omega^+ \cup \Gamma) \setminus \Sigma, N\big) \cap C^{1,\alpha}_{\mathrm{loc}}\big((\Omega^- \cup \Gamma) \setminus \Sigma, N\big)$.

The jump condition is needed for both energy monotonicity inequalities for u and the piecewise $C^{1,\alpha}$-regularity of u.

We point out that in dimension $n = 2$, since the energy monotonicity inequality automatically holds for H^1-maps, Theorem 1.1 holds for any weakly harmonic map from domains of piecewise $C^{0,1}$-metrics, i.e., any weakly harmonic map on domains with piecewise Lipschitz continuous metrics satisfying (1-4) is both Lipschitz continuous and piecewise $C^{1,\alpha}$ for some $0 < \alpha < 1$.

Weakly harmonic maps from domains with smooth metrics into Riemannian manifolds may not enjoy partial regularity properties in dimensions $n \geq 3$; see [Rivière 1995]. Here we consider weakly harmonic maps on domains with piecewise Lipschitz continuous metrics into a Riemannian manifold (N, h), on which $d_N^2(\cdot, p)$ is convex for $p \in N$. Such Riemannian manifolds N include those with nonpositive sectional curvatures and geodesic convex balls in Riemannian manifolds. In particular, we extend the classical regularity theorems on harmonic maps on domains with smooth metrics, due to [Eells and Sampson 1964] and [Hildebrandt et al. 1977].

Theorem 1.2. *Let g satisfy the conditions of Theorem 1.1. Assume that on the universal cover $(\widetilde{N}, \tilde{h})$ of (N, h),[1] the square of distance function $d_{\widetilde{N}}^2(\cdot, p)$ is convex for any $p \in \widetilde{N}$. If $u \in H^1(\Omega, N)$ is a weakly harmonic map, then $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, N)$, and for some $0 < \alpha < 1$, $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega^+ \cup \Gamma, N) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega^- \cup \Gamma, N)$.*

The idea for the proof of Theorem 1.1 is motivated in [Evans 1991] and [Bethuel 1993]. However, there are several new technical difficulties:

(i) Establishing an almost energy monotonicity inequality for stationary harmonic maps in (Ω, g). This is achieved by observing that an exact monotonicity inequality holds at any $x \in \Gamma$, see Section 2 below.

[1] Here the covering map $\Pi : \widetilde{N} \to N$ is a Riemannian submersion.

(ii) Establishing a Hodge decomposition in $L^p(B, \mathbb{R}^n)$, for any $1 < p < +\infty$, on a ball $B = B_r(0)$, equipped with certain piecewise continuous metrics g. More precisely, we need to show that the solution of

$$\begin{cases} \dfrac{\partial}{\partial x_i}\Big(a_{ij}\dfrac{\partial v}{\partial x_j}\Big) = \operatorname{div} f & \text{in } B, \\ v = 0 & \text{on } \partial B \end{cases}$$

enjoys a $W^{1,p}$-estimate: for any $1 < p < +\infty$,

$$\|\nabla v\|_{L^p(B)} \le C\|f\|_{L^p(B)}$$

provided that $(a_{ij}) \in C(\overline{B^\pm}) \cap C(B^\delta)$ for some $\delta > 0$, is uniformly elliptic, but is discontinuous on $\partial B^+ \setminus B^\delta$, where $B^\delta = \{x \in B : \operatorname{dist}(x, \partial B) \le \delta\}$. This follows from a recent theorem in [Byun and Wang 2010; Dong and Kim 2010]; see also [Dong and Kim 2011a; 2011b] and Section 3 below.

(iii) Employing the moving frame method to establish the decay estimate in suitable Morrey spaces under a smallness condition, analogous to [Ishizuka and Wang 2008]. To obtain Lipschitz and piecewise $C^{1,\alpha}$-regularity, we compare the harmonic map system with an elliptic system with piecewise constant coefficients and perform a hole-filling argument, similar to [Giaquinta and Hildebrandt 1982].

The paper is organized as follows. In Section 2, we derive an almost energy monotonicity inequality. In Section 3, we show the global $W^{1,p}$ $(1 < p < \infty)$ estimate for elliptic systems with certain piecewise continuous coefficients, and a Hodge decomposition theorem. In Section 4, we adapt the moving frame method of [Hélein 2002] and [Bethuel 1993] to establish an ϵ-Hölder continuity. In Section 5, we establish both Lipschitz and piecewise $C^{1,\alpha}$ regularity for Hölder continuous harmonic maps. In Section 6, we consider harmonic maps into manifolds supporting convex distance square functions and prove Theorem 1.2.

2. Energy monotonicity inequality

This section is devoted to the derivation of energy monotonicity inequalities for stationary harmonic maps from (Ω, g) to (N, h).

Theorem 2.1. *Under the same assumptions as in Theorem 1.1, there exist $C > 0$ and $r_0 > 0$, depending only on Ω, Γ, and g, such that if $u \in W^{1,2}(\Omega, N)$ is a stationary harmonic map on (Ω, g), then for any $x_0 \in \Omega$, there holds*

$$s^{2-n}\int_{B_s(x_0)} |\nabla u|_g^2\, dv_g \le e^{Cr} r^{2-n}\int_{B_r(x_0)} |\nabla u|_g^2\, dv_g \tag{2-1}$$

for all $0 < s \le r \le \min\{r_0, \operatorname{dist}(x_0, \partial\Omega)\}$.

Since $g \in C^{0,1}(\Omega^{\pm})$, there are $C > 0$ and $r_0 > 0$ such that (2-1) holds for any $x_0 \in \Omega^{\pm}$ and $0 < s \le r \le \min\{r_0, \operatorname{dist}(x_0, \partial\Omega^{\pm})\}$; see [Hélein 2002]. In particular, (2-1) holds for any $x_0 \in \Omega \setminus \Gamma^{r_0}$ and $0 < s \le r \le \min\{r_0, \operatorname{dist}(x_0, \partial\Omega)\}$, where $\Gamma^{r_0} = \{x \in \Omega : \operatorname{dist}(x, \Gamma) \le r_0\}$ is the r_0-neighborhood of Γ. To show (2-1) for $x_0 \in \Gamma^{r_0}$, it suffices to consider the case $x_0 \in \Gamma$.

It follows from the assumption on Γ and g that there exists $r_0 > 0$ such that for any $x_0 \in \Gamma$ there exists a $C^{1,1}$-diffeomorphism $\Phi_0 : B_1 \to B_{r_1}(x_0)$, where $r_1 = \min\{r_0, \operatorname{dist}(x_0, \partial\Omega)\}$, such that

$$\begin{cases} \Phi_0(B_1^{\pm}) = \Omega^{\pm} \cap B_{r_1}(x_0), \\ \Phi_0(\Gamma_1) = \Gamma \cap B_{r_1}(x_0), \ \text{where } \Gamma_1 = \{x \in B_1 : x_n = 0\}. \end{cases}$$

Define $\tilde{u}(x) = u(\Phi_0(x))$ and $\tilde{g}(x) = \Phi_0^*(g)(x)$ for $x \in B_1$. Then it is readily seen that $\tilde{g}$ is piecewise $C^{0,1}$, with Γ as its discontinuity set, and satisfies (1-4) on Γ_1. (In fact, since

$$\Phi_0^*(g)_{ij}(x) = g_{kl}(\Phi_0(x)) \frac{\partial \Phi_0^k}{\partial x_i}(x) \frac{\partial \Phi_0^l}{\partial x_j}(x),$$

condition (1-4) implies that

$$\lim_{\substack{y \in \Omega^+ \\ y \to x}} \Phi_0^* g(y) = k(\Phi_0(x)) \lim_{\substack{y \in \Omega^- \\ y \to x}} \Phi_0^* g(y)$$

for any $x \in \Gamma_1$.) It is also easy to see that, if $u : (B_{r_1}(x_0), g) \to (N, h)$ is a stationary harmonic map, so $\tilde{u} : (B_1, \tilde{g}) \to (N, h)$.

Thus we may assume that $\Omega = B_1$, that g is a piecewise $C^{0,1}$-metric which satisfies (1-4) on the set of discontinuity Γ_1, and that $u : (B_1, g) \to (N, h)$ is a stationary harmonic map. It suffices to establish (2-1) in $B_{1/2}$. We first derive a stationarity identity for u.

Proposition 2.2. *Let $u \in W^{1,2}(B_1, N)$ be a stationary harmonic map on (B_1, g). Then*

$$\int_{B_1} \left(2g^{ij} \left\langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j} \right\rangle Y_i^k - |\nabla u|_g^2 \operatorname{div} Y \right) \sqrt{g}\, dx = \int_{B_1} \frac{\partial}{\partial x_k} \left(\sqrt{g} g^{ij} \right) Y^k \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle dx \tag{2-2}$$

for all $Y = (Y^1, \ldots, Y^{n-1}, Y^n) \in C_0^1(B_1, \mathbb{R}^n)$ satisfying

$$Y^n(x) \begin{cases} \ge 0 & \text{for } x^n > 0, \\ = 0 & \text{for } x^n = 0, \\ \le 0 & \text{for } x^n < 0, \end{cases} \tag{2-3}$$

where $Y_i^k = \partial Y^k / \partial x_i$ and $\operatorname{div} Y = \sum_{i=1}^n \partial Y^i / \partial x_i$.

Proof. Let $Y \in C_0^1(B_1, \mathbb{R}^n)$ satisfy (2-3). Then there exists $\delta > 0$ such that $F_t(x) = x + tY(x),\ t \in [-\delta, \delta]$, is a family of diffeomorphisms from B_1 to B_1 satisfying the condition (1-3). Hence

$$\begin{aligned}0 &= \frac{d}{dt}\Big|_{t=0} \int_{B_1} \big|\nabla u(F_t(x))\big|_g^2 \, dv_g \\ &= \frac{d}{dt}\Big|_{t=0} \bigg(\int_{B_1^+} \big|\nabla u(F_t(x))\big|_g^2 \, dv_g + \int_{B_1^-} \big|\nabla u(F_t(x))\big|_g^2 \, dv_g \bigg).\end{aligned}$$

Set $G_t = F_t^{-1}$, for $t \in [-\delta, \delta]$. Direct calculations yield

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} \int_{B_1^\pm} \big|\nabla(u(F_t(x))\big|_g^2 \, dv_g \\ &= \frac{d}{dt}\Big|_{t=0} \int_{B_1^\pm} \sqrt{g(x+tY(x))}\, g^{ij}(x+tY(x)) \left\langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_l} \right\rangle \\ &\qquad\qquad \times (x+tY(x))(\delta_{ki} + tY_i^k)(\delta_{lj} + tY_j^l)\, dx \\ &= \int_{B_1^\pm} \sqrt{g(x)}\, g^{ij}(x) \left\langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_l} \right\rangle (\delta_{ki} Y_j^l + \delta_{lj} Y_i^k)\, dx \\ &\qquad + \int_{B_1^\pm} \frac{d}{dt}\Big|_{t=0} \Big(g^{ij}(G_t(x)) \sqrt{g(G_t(x))}\, JG_t(x) \Big) \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle dx \\ &= \int_{B_1^\pm} \left(2g^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_l} \right\rangle Y_j^l - g^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle \operatorname{div} Y \right) \sqrt{g}\, dx \\ &\qquad - \int_{B_1^\pm} \frac{\partial}{\partial x_k} \big(\sqrt{g} g^{ij}\big) Y^k \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle dx,\end{aligned}$$

where we have used the equalities

$$\begin{cases} \dfrac{d}{dt}\Big|_{t=0} JG_t(x) = -\operatorname{div} Y, \\ \dfrac{d}{dt}\Big|_{t=0} G_t(x) = -Y(x), \\ \dfrac{d}{dt}\Big|_{t=0} \big(g^{ij}(G_t(x)) \sqrt{g(G_t(x))} \big) = -\dfrac{\partial}{\partial x_k} \big(\sqrt{g} g^{ij}\big) Y^k. \end{cases}$$

This completes the proof. □

Proposition 2.3. *Let $u \in W^{1,2}(B_1, N)$ be a stationary harmonic map on (B_1, g). There exists $C > 0$ such that*:

(i) *For any $x_0 = (x_0', x_0^n) \in B_{1/2} \setminus \Gamma_1$, there exists $0 < R_0 \le \min\{\frac{1}{4}, |x_0^n|\}$ such that*

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g \le e^{CR} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g \quad \text{if } 0 < r \le R < R_0. \tag{2-4}$$

(ii) *For any* $x_0 \in B_{1/2}$,

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g \le e^{CR} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g \quad \textit{if } 0 < r \le R \le \tfrac{1}{4}. \tag{2-5}$$

Proof. (i) By choosing $Y \in C_c^\infty(B_1^+, \mathbb{R}^n)$ or $Y \in C_c^\infty(B_1^-, \mathbb{R}^n)$, we conclude that u is a stationary harmonic map on (B_1^+, g) and (B_1^-, g). Hence the monotonicity inequality (2-4) holds; see [Hélein 2002].

(ii) Step 1. We first consider the case where $x_0 \in \Gamma_1$. Without loss of generality, we can assume that $x_0 = (0', 0)$. For $\epsilon > 0$ and $0 < r \le \frac{1}{2}$, let $Y_\epsilon(x) = x\eta_\epsilon(x)$, where $\eta_\epsilon(x) = \eta_\epsilon(|x|) \in C_0^\infty(B_1)$ satisfies

$$0 \le \eta_\epsilon \le 1, \quad \eta_\epsilon' \le 0, \quad |\eta_\epsilon'| \le \frac{2}{\epsilon}, \quad \eta_\epsilon(s) = \begin{cases} 1 & \text{for } 0 \le s \le r - \epsilon, \\ 0 & \text{for } s \ge r. \end{cases} \tag{2-6}$$

Then

$$(Y_\epsilon)_i^j = \delta_{ij}\eta_\epsilon(|x|) + \eta_\epsilon'(|x|)\frac{x^i x^j}{|x|}. \tag{2-7}$$

Substituting Y_ϵ into the right side of (2-2), and using

$$\left| \frac{\partial}{\partial x_k}\big(\sqrt{g} g^{ij}\big) \right| \le C \quad \text{for a.e. } x \in B_1 \setminus \Gamma_1,$$

we have

$$\begin{aligned} \left| \int_{B_1} \frac{\partial}{\partial x_k}\big(\sqrt{g} g^{ij}\big) Y_\epsilon^k \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle dx \right| &\le Cr \int_{B_r} |\nabla u|^2 \, dx \\ &\le Cr \int_{B_r} |\nabla u|_g^2 \, dv_g. \end{aligned} \tag{2-8}$$

Substituting (2-7) into the left side of (2-2), we obtain

$$\begin{aligned} \int_{B_1} &\left(2g^{ij} \left\langle \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k} \right\rangle (Y_\epsilon)_i^k - |\nabla u|_g^2 \operatorname{div} Y_\epsilon \right) \sqrt{g} \, dx \\ &= (2-n) \int_{B_1} |\nabla u|_g^2 \eta_\epsilon(|x|)\sqrt{g} \, dx - \int_{B_1} |\nabla u|_g^2 |x| \eta_\epsilon'(|x|)\sqrt{g} \, dx \\ &\quad + \int_{B_1} 2g^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k} \right\rangle \frac{x^k x^j}{|x|} \eta_\epsilon'(|x|)\sqrt{g} \, dx. \end{aligned} \tag{2-9}$$

Define $\bar{g}$ by

$$\bar{g}(x', x^n) = \begin{cases} \lim_{y \to 0,\ y^n \ge 0} g(y) & \text{if } x^n \ge 0, \\ \lim_{y \to 0,\ y^n < 0} g(y) & \text{if } x^n < 0. \end{cases}$$

Then we have

$$|g(x) - \bar{g}(x)| \le C|x| \quad \text{for all } x \in B_1. \tag{2-10}$$

Further, by (1-4) we can assume

$$\bar{g}(x) = \begin{cases} g_0 & \text{if } x^n \geq 0, \\ kg_0 & \text{if } x^n < 0 \quad (k \neq 1). \end{cases}$$

Hence we can write

$$\int_{B_1} 2g^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k} \right\rangle \frac{x^k x^j}{|x|} \eta'_\epsilon(|x|) \sqrt{g}\, dx = I_\epsilon + II_\epsilon. \tag{2-11}$$

where

$$I_\epsilon = 2 \int_{B_1} \bar{g}^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k} \right\rangle \frac{x^k x^j}{|x|} \eta'_\epsilon(|x|) \sqrt{g}\, dx,$$

$$II_\epsilon = 2 \int_{B_1} (g^{ij} - \bar{g}^{ij}) \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k} \right\rangle \frac{x^k x^j}{|x|} \eta'_\epsilon(|x|) \sqrt{g}\, dx.$$

Since

$$\bar{g}^{ij} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k} \right\rangle \frac{x^k x^j}{|x|} = \begin{cases} |x| |\partial u / \partial r|^2 & \text{if } x^n \geq 0, \\ (1/k) |x| |\partial u / \partial r|^2 & \text{if } x^n < 0 \end{cases}$$

is nonnegative in B_1 and $\eta'_\epsilon(|x|) \leq 0$, we have $I_\epsilon \leq 0$. For II_ϵ, by (2-10) we have

$$|II_\epsilon| \leq Cr^2 \int_{B_r} |\nabla u|_g^2 \, |\eta'_\epsilon|(|x|)\, dv_g. \tag{2-12}$$

Putting these estimates first into (2-11) and then into (2-9), and finally combining (2-9) and (2-8) with (2-2), we obtain, after taking ϵ to zero,

$$\begin{aligned} (2-n) \int_{B_r} |\nabla u|_g^2 \, dv_g + r \int_{\partial B_r} |\nabla u|_g^2 \sqrt{g}\, dH^{n-1} \\ \geq -C\left(r \int_{B_r} |\nabla u|_g^2 \, dv_g + r^2 \int_{\partial B_r} |\nabla u|_g^2 \sqrt{g} dH^{n-1} \right). \end{aligned} \tag{2-13}$$

It is not hard to see that (2-13) implies

$$\frac{d}{dr} \left(e^{Cr} r^{2-n} \int_{B_r} |\nabla u|_g^2 \, dv_g \right) \geq 0,$$

so that (2-5) holds when $x_0 \in B_{1/2}$.

Step 2. To show (2-5) in the general case, it suffices to consider $x_0 \in B_{1/2} \setminus \Gamma_1$ such that

$$|B_R(x_0) \cap B_1^+| > 0 \quad \text{and} \quad |B_R(x_0) \cap B_1^-| > 0.$$

For simplicity, assume $x_0 \in B_1^-$. We consider two cases:

Suppose $d(x_0, \Gamma_1) = |x_0^n| \geq \frac{1}{4} R$. Then:

- If $R \geq r \geq \frac{1}{4}R$, it is easy to see that

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g \leq 4^{n-2} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g.$$

- If $0 < r < \frac{1}{4}R$ ($\leq d(x_0, \Gamma_1)$), we have $B_{R/4}(x_0) \subset B_1^-$, so (2-4) implies

$$\begin{aligned} r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g &\leq e^{CR} \left(\frac{R}{4}\right)^{2-n} \int_{B_{R/4}(x_0)} |\nabla u|_g^2 \, dv_g \\ &\leq e^{CR} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g. \end{aligned}$$

Suppose instead that $d(x_0, \Gamma_1) = |x_0^n| < \frac{1}{4}R$. Then:

- If $R \geq r \geq \frac{1}{4}R$, then

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g \leq 4^{n-2} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g.$$

- If $0 < r \leq d(x_0, \Gamma_1) = |x_0^n| < \frac{1}{4}R$, then by setting $\bar{x}_0 = (x_0^1, \dots, x_0^{n-1}, 0)$ we have

$$B_r(x_0) \subset B_{|x_0^n|}(x_0) \subset B_{2|x_0^n|}(\bar{x}_0) \subset B_{R/2}(\bar{x}_0) \subset B_R(x_0),$$

so that (2-5) yields

$$\begin{aligned} r^{2-n} \int_{B_r(x_0)} |\nabla u|_g^2 \, dv_g &\leq |x_0^n|^{2-n} \int_{B_{|x_0^n|}(x_0)} |\nabla u|_g^2 \, dv_g \\ &\leq 2^{n-2} (2|x_0^n|)^{2-n} \int_{B_{2|x_0^n|}(\bar{x}_0)} |\nabla u|_g^2 \, dv_g \\ &\leq 2^{n-2} e^{CR} \left(\frac{R}{2}\right)^{2-n} \int_{B_{R/2}(\bar{x}_0)} |\nabla u|_g^2 \, dv_g \\ &\leq e^{CR} R^{2-n} \int_{B_R(x_0)} |\nabla u|_g^2 \, dv_g. \end{aligned}$$

- If $d(x_0, \Gamma_1)$ $(= |x_0^n|) \leq r < \frac{1}{4}R$, then we have

$$B_r(x_0) \subset B_{2r}(\bar{x}_0) \subset B_{R/2}(\bar{x}_0) \subset B_R(x_0),$$

so that (2-5) yields

$$
\begin{aligned}
r^{2-n}\int_{B_r(x_0)}|\nabla u|_g^2\,dv_g &\le 2^{n-2}(2r)^{2-n}\int_{B_{2r}(\bar{x}_0)}|\nabla u|_g^2\,dv_g\\
&\le 2^{n-2}e^{CR}\left(\frac{R}{2}\right)^{2-n}\int_{B_{R/2}(\bar{x}_0)}|\nabla u|_g^2\,dv_g\\
&\le e^{CR}R^{2-n}\int_{B_R(x_0)}|\nabla u|_g^2\,dv_g.
\end{aligned}
$$

Therefore (2-5) is proved in all cases. □

3. $W^{1,p}$-estimate for elliptic equations with piecewise continuous coefficients

In this section, we will provide the global $W^{1,p}$-estimate for elliptic equations with piecewise continuous coefficients. The proof is a slight modification of that of [Dong and Kim 2010] (see also [Dong and Kim 2011a; 2011b]) or [Byun and Wang 2010]. As a corollary, we will establish the Hodge decomposition theorem (Theorem 3.2) for piecewise continuous metrics g, a crucial ingredient to prove Theorem 1.1.

For a ball $B = B_r(0) \subset \mathbb{R}^n$, set $B^\epsilon = \{x \in B : \operatorname{dist}(x, \partial B) \le \epsilon\}$ for $\epsilon > 0$. Let $(a_{ij}(x))_{1\le i,j\le n}$ be bounded measurable, uniformly elliptic on B; i.e., there exist $0 < \lambda \le \Lambda < +\infty$ such that

$$
\lambda|\xi|^2 \le a_{ij}(x)\xi_i^\alpha\xi_\beta^j \le \Lambda|\xi|^2 \ \text{ a.e. } x \in B \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3-1}
$$

Theorem 3.1. *Assume (a_{ij}) satisfies* (3-1), *and there exists $\epsilon > 0$ such that $(a_{ij}) \in C\big(\overline{B^\pm}\big) \cap C(B^\epsilon)$ and is discontinuous on $\partial B^+ \setminus B^\epsilon$. For $p \in (1, +\infty)$, let $f \in L^p(B, \mathbb{R}^n)$. Then there exists a unique weak solution $v \in W_0^{1,p}(B, \mathbb{R}^n)$ to*

$$
\begin{cases}
\displaystyle\sum_{i,j}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial v}{\partial x_j}\Big) = \sum_i \frac{\partial f_i}{\partial x_i} & \textit{in } B,\\
u = 0 & \textit{on } \partial B,
\end{cases} \tag{3-2}
$$

that satisfies

$$
\|\nabla v\|_{L^p(B)} \le C\,\|f\|_{L^p(B)} \tag{3-3}
$$

for some $C > 0$ depending only on p and (a_{ij}).

Proof. By (3-1), we see that for any $\delta > 0$, there exists $R = R(\delta) > 0$ such that the coefficient function (a_{ij}) satisfies the (δ, R)-vanishing of codimension-one conditions (2.5) and (2.6) of [Byun and Wang 2010, p. 2562]; see also [Dong and Kim 2010; 2011a; 2011b]. In fact, we have

$$
\lim_{r\downarrow 0}\ \max_{x_0=(x_0',x_0^n)\in\overline{B}}\big\|a_{ij}(x',x^n) - a_{ij}(x_0',x^n)\big\|_{L^\infty(B_r((x_0',x_0^n)))} = 0.
$$

Therefore Theorem 3.1 follows directly from [Byun and Wang 2010, Theorem 2.2, p. 2653]. □

As an immediate consequence of Theorem 3.1, we have the following Hodge decomposition on B equipped with certain piecewise continuous metrics g.

Theorem 3.2. *Let $\bar{g}$ be a piecewise continuous metric on B such that $\bar{g}$ is continuous on $B^{\pm}$ and on B^{δ} for some $\delta > 0$, and is discontinuous on $\partial B^{+} \setminus B^{\delta}$. Then for any $p \in (1, +\infty)$ and $F = (F_1, \dots, F_n) \in L^p(B, \mathbb{R}^n)$, there exist $G \in W_0^{1,p}(B)$ and $H \in L^p(B, \mathbb{R}^n)$ such that*

$$F = \nabla G + H, \quad \operatorname{div}_{\bar{g}} H \left(:= \frac{1}{\sqrt{\bar{g}}} \frac{\partial}{\partial x_i} (\sqrt{\bar{g}} \bar{g}^{ij} H_j) \right) = 0 \quad \text{in } B. \tag{3-4}$$

Further, there exists $C = C(p, n, \bar{g}) > 0$ such that

$$\|\nabla G\|_{L^p(B)} + \|H\|_{L^p(B)} \le C \|F\|_{L^p(B)} . \tag{3-5}$$

Proof. For $1 \le i, j \le n$, set $a_{ij} = \sqrt{\bar{g}} \bar{g}^{ij}$ on B. Then (a_{ij}) satisfies the conditions of Theorem 3.1, so that there exists a unique solution $G \in W_0^{1,p}(B)$ to

$$\begin{cases} \dfrac{\partial}{\partial x_i} \left(\sqrt{\bar{g}} \bar{g}^{ij} \dfrac{\partial G}{\partial x_j} \right) = \dfrac{\partial}{\partial x_i} \big(\sqrt{\bar{g}} \bar{g}^{ij} F_j \big) & \text{in } B, \\ G = 0 & \text{on } \partial B, \end{cases} \tag{3-6}$$

and

$$\|\nabla G\|_{L^p(B)} \le C \left\| \sqrt{\bar{g}} \bar{g}^{ij} F_j \right\|_{L^p(B)} \le C \|F\|_{L^p(B)} .$$

Set $H = F - \nabla G$. Then we have

$$\operatorname{div}_{\bar{g}} H = \frac{1}{\sqrt{\bar{g}}} \frac{\partial}{\partial x_i} \left(\sqrt{\bar{g}} \bar{g}^{ij} \left(F_j - \frac{\partial G}{\partial x_j} \right) \right) = 0 \quad \text{on } B,$$

and

$$\|H\|_{L^p(B_{1/2})} \le \|F\|_{L^p(B_{1/2})} + \|\nabla G\|_{L^p(B)} \le C \|F\|_{L^p(B)} .$$

This completes the proof. □

4. Hölder continuity

In this section, we will prove that any stationary harmonic map on (B_1, g), with $g \in C^{0,1}(B_1^{\pm} \cup \Gamma_1)$, is Hölder continuous provided that $\int_{B_1} |\nabla u|_g^2 \, dv_g$ is sufficiently small. The idea is based on suitable modifications of the original argument in [Bethuel 1993] (see also [Ishizuka and Wang 2008]), thanks to both the energy monotonicity inequality and the Hodge decomposition theorem established in the previous two sections. More precisely:

Theorem 4.1. *There exist $\epsilon_0 > 0$ and $\alpha_0 \in (0, 1)$, depending only on n, g, such that if the metric $g \in C^{0,1}(B_1^{\pm} \cup \Gamma_1)$ satisfies the condition* (1-4) *on Γ_1, and $u \in W^{1,2}(B_1, N)$ is a stationary harmonic map satisfying*

$$r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|_g^2 \, dv_g \le \epsilon_0^2 \tag{4-1}$$

for some $x_0 \in B_{1/2}$ and $0 < r_0 \le \frac{1}{4}$, then $u \in C^{\alpha_0}(B_{r_0/2}(x_0), N)$, and

$$[u]_{C^{\alpha_0}(B_{r_0/2}(x_0))} \le C(r_0, \epsilon_0). \tag{4-2}$$

Proof of Theorem 4.1. The proof is based on suitable modifications of [Bethuel 1993; Ishizuka and Wang 2008]. First, observe that if $x_0 = (x_0', x_0^n) \in B^{\pm}$, it follows from the monotonicity inequality (2-5) that we may assume (4-1) holds for some $0 < r_0 < |x_0^n|$. Then the ϵ_0-regularity theorem in [Bethuel 1993] (see [Ishizuka and Wang 2008] for domains with $C^{0,1}$metrics) implies that for some $0 < \alpha_0 < 1$, $u \in C^{\alpha_0}(B_{r_0/2}(x_0))$ and (4-2) holds. Hence it suffices to consider the case $x_0 = (x_0', 0) \in \Gamma_{1/2}$. By translation and scaling, we may assume $x_0 = (0, 0)$ and proceed as follows.

Step 1. As in [Bethuel 1993; Hélein 2002; Ishizuka and Wang 2008], we assume that there exists an orthonormal frame on $u^*TN|_{B_1}$. For $0 < \theta < \frac{1}{2}$, to be determined later, let $\{e_\alpha\}_{\alpha=1}^l \subset W^{1,2}(B_{2\theta}, \mathbb{R}^k)$ be a Coulomb gauge orthonormal frame of $u^*TN|_{B_{2\theta}}$; that is,

$$\begin{gathered} \operatorname{div}_g(\langle \nabla e_\alpha, e_\beta \rangle) = 0 \quad \text{in } B_{2\theta} \quad (1 \le \alpha, \beta \le l), \\ \sum_{\alpha=1}^{l} \int_{B_{2\theta}} |\nabla e_\alpha|_g^2 \, dv_g \le C \int_{B_{2\theta}} |\nabla u|_g^2 \, dv_g. \end{gathered} \tag{4-3}$$

For $1 \le \alpha \le l$, consider $\langle \nabla\left((u - u_{2\theta})\eta\right), e_\alpha \rangle$, where $u_{2\theta} = \fint_{B_{2\theta}} u$ is the average of u on $B_{2\theta}$, and $\eta \in C_0^\infty(B_1)$ satisfies

$$0 \le \eta \le 1; \quad \eta = 1 \text{ in } B_\theta; \quad \eta = 0 \text{ outside } B_{7\theta/4}; \quad |\nabla \eta| \le \frac{2}{\theta}.$$

Define the metric $\tilde{g}$ on $B_{2\theta}$ by

$$\tilde{g}(x) = \eta(x) g(x) + (1 - \eta(x)) g_0(x), \quad x \in B_{2\theta}.$$

Then it is easy to see that

$$\tilde{g} \equiv g \text{ on } B_\theta; \quad \tilde{g} \equiv g_0 \text{ outside } B_{7\theta/4}; \quad \tilde{g} \in C(\overline{B_{2\theta}^{\pm}}) \cap C(B_{2\theta} \setminus B_{7\theta/4}).$$

In particular, $\tilde{g}$ satisfies the conditions of Theorem 3.2. Hence, by Theorem 3.2, for

$1 < p < n/(n-1)$, there exist $\phi_\alpha \in W_0^{1,p}(B_{2\theta})$ and $\psi_\alpha \in L^p(B_{2\theta})$ such that

$$\begin{aligned} &\langle \nabla((u-u_{2\theta})\eta), e_\alpha\rangle = \nabla\phi_\alpha + \psi_\alpha, \quad \mathrm{div}_{\tilde g}(\psi_\alpha) = 0 \quad \text{in } B_{2\theta},\\ &\|\nabla\phi_\alpha\|_{L^p(B_{2\theta})} + \|\psi_\alpha\|_{L^p(B_{2\theta})} \lesssim \|\nabla((u-u_{2\theta})\eta)\|_{L^p(B_{2\theta})} \lesssim \|\nabla u\|_{L^p(B_{2\theta})}.\end{aligned} \tag{4-4}$$

Since u satisfies the harmonic map equation (1-1), we have

$$\mathrm{div}_g\,(\langle \nabla u, e_\alpha\rangle) = g^{ij}\nabla_i u\langle \nabla_j e_\alpha, e_\beta\rangle\, e_\beta \quad \text{in } B_1. \tag{4-5}$$

Thus we obtain

$$\Delta_g \phi_\alpha = g^{ij}\nabla_i u\langle \nabla_j e_\alpha, e_\beta\rangle\, e_\beta \quad \text{in } B_\theta. \tag{4-6}$$

Decompose $\phi_\alpha = \phi_\alpha^{(1)} + \phi_\alpha^{(2)}$, where $\phi_\alpha^{(1)}$ solves

$$\begin{cases} \Delta_g \phi_\alpha^{(1)} = 0 & \text{in } B_\theta,\\ \phi_\alpha^{(1)} = \phi_\alpha & \text{on } \partial B_\theta, \end{cases} \tag{4-7}$$

and $\phi_\alpha^{(2)}$ solves

$$\begin{cases} \Delta_g \phi_\alpha^{(2)} = g^{ij}\nabla_i u\langle \nabla_j e_\alpha, e_\beta\rangle\, e_\beta & \text{in } B_\theta,\\ \phi_\alpha^{(2)} = 0 & \text{on } \partial B_\theta. \end{cases} \tag{4-8}$$

Step 2: Estimation of $\phi_\alpha^{(1)}$. We will need the Morrey space defined, for arbitrary $E \subset \mathbb{R}^n$, by

$$M^{p,p}(E) := \left\{ f : E \to \mathbb{R} \,\middle|\, \|f\|^p_{M^{p,p}(E)} := \sup_{B_r(x)\subset\mathbb{R}^n} \left\{ r^{p-n} \int_{B_r(x)\cap E} |f|^p\, dx \right\} < +\infty \right\}.$$

It is well-known (see [Gilbarg and Trudinger 1983]) that $\phi_\alpha^{(1)} \in C^{\alpha_0}(B_\theta)$ for some $\alpha_0 \in (0,\ 1)$, and for any $0 < r \le \theta/2$,

$$\left[\phi_\alpha^{(1)}\right]^p_{C^{\alpha_0}(B_{r/2})} \lesssim \theta^{p-n} \int_{B_\theta} |\nabla \phi_\alpha^{(1)}|^p\, dx \le C\theta^{p-n} \int_{B_{2\theta}} |\nabla u|^p\, dx, \tag{4-9}$$

and

$$(\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla\phi_\alpha^{(1)}|^p \le C\tau^{p\alpha_0} \|\nabla u\|_{M^{p,p}(B_1)} \quad \text{for all } 0 < \tau < 1, \tag{4-10}$$

Step 3: Estimation of $\phi_\alpha^{(2)}$. Denote by $\mathscr{H}^1(\mathbb{R}^n)$ the Hardy space on $\mathbb{R}^n$ and $BMO(E)$ the BMO space on E for any open set $E \subset \mathbb{R}^n$. By (4.13) of [Ishizuka and Wang 2008, p. 435], for $p' = p/(p-1) > n$, there exists $h \in W_0^{1,p'}(B_\theta)$, with $\|\nabla h\|_{L^{p'}(B_\theta)} = 1$, such that

$$\left\|\nabla \phi_\alpha^{(2)}\right\|_{L^p(B_\theta)} \le C \int_{B_\theta} \langle \nabla\phi_\alpha^{(2)}, \nabla h\rangle_g\, dv_g.$$

Using (4-8), (4-4), and the duality between $\mathcal{H}^1$ and BMO, we show that

$$
\begin{aligned}
(4\text{-}11)\qquad \bigl\|\nabla\phi_\alpha^{(2)}\bigr\|_{L^p(B_\theta)} &\le C\int_{B_\theta}\sqrt{g}g^{ij}\bigl\langle\nabla_i u,\langle\nabla_j e_\alpha, e_\beta\rangle\bigr\rangle(e_\beta h)\,dx\\
&= -C\int_{B_\theta}\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\nabla_i(e_\beta h)u\,dx\\
&\le C\bigl\|\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\nabla_i(e_\beta h)\bigr\|_{\mathcal{H}^1(\mathbb{R}^n)}[u]_{BMO(B_\theta)}\\
&\lesssim \|\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\|_{L^2(B_\theta)}\|\nabla(e_\beta h)\|_{L^2(B_\theta)}[u]_{BMO(B_\theta)}\\
&\lesssim \|\nabla u\|_{L^2(B_{2\theta})}\|\nabla u\|_{M^{p,p}(B_1)}\cdot\theta^{n/p-n/2}.
\end{aligned}
$$

(Here, to go from the third line to the fourth, we used that $h\in W_0^{1,p'}(B_\theta)$ and that $\operatorname{div}_g\langle\nabla e_\alpha, e_\beta\rangle$ vanishes in B_θ, so $\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\nabla_i(e_\beta h)\in\mathcal{H}^1(\mathbb{R}^n)$ and

$$
\bigl\|\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\nabla_i(e_\beta h)\bigr\|_{\mathcal{H}^1(\mathbb{R}^n)}\le C\,\bigl\|\sqrt{g}g^{ij}\langle\nabla_j e_\alpha, e_\beta\rangle\bigr\|_{L^2(B_\theta)}\|\nabla_i(e_\beta h)\|_{L^2(B_\theta)}.
$$

This last factor satisfies

$$
\|\nabla(e_\beta h)\|_{L^2(B_\theta)}\le\|\nabla e_\beta\|_{L^2(B_\theta)}\|h\|_{L^\infty(B_\theta)}+\|\nabla h\|_{L^p(B_\theta)}\theta^{n/p-n/2}\le C\theta^{n/p-n/2},
$$

since the Sobolev embedding implies (because $p'>n$) that $h\in C^{1-n/p'}(B_\theta)$ and

$$
\|h\|_{L^\infty(B_\theta)}\le C\theta^{1-n/p'}.
$$

Finally, the estimate $[u]_{BMO(B_\theta)}\le C\|\nabla u\|_{M^{p,p}(B_1)}$ is a consequence of the Poincaré inequality.)

Putting the estimates of $\phi_\alpha^{(1)}$ and $\phi_\alpha^{(2)}$ together, we obtain that, for all $0<\tau<1$,

$$
(4\text{-}12)\qquad \left((\tau\theta)^{p-n}\int_{B_{\tau\theta}}|\nabla\phi_\alpha|^p dx\right)^{1/p}\le C\bigl(\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0\bigr)\|\nabla u\|_{M^{p,p}(B_1)}.
$$

Step 4: Estimation of ψ_α. Since $\operatorname{div}_{\tilde g}(\psi_\alpha)=0$ on $B_{2\theta}$, we have

$$
\begin{aligned}
\int_{B_{2\theta}}|\psi_\alpha|_{\tilde g}^2\,dv_{\tilde g} &= \int_{B_{2\theta}}\langle\psi_\alpha+\nabla\phi_\alpha, \psi_\alpha\rangle_{\tilde g}\,dv_{\tilde g}\\
&= \int_{B_{2\theta}}\bigl\langle\langle\nabla((u-u_{2\theta})\eta), e_\alpha\rangle, \psi_\alpha\bigr\rangle_{\tilde g}\,dv_{\tilde g}\\
&= -\int_{B_{2\theta}}(u-u_{2\theta})\eta\langle\nabla e_\alpha, \psi_\alpha\rangle_{\tilde g}\,dv_{\tilde g}\\
&\lesssim \bigl\|\sqrt{\tilde g}\;\tilde g^{ij}\nabla_i e_\alpha\psi_\alpha^j\bigr\|_{\mathcal{H}^1}[(u-u_{2\theta})\eta]_{BMO}\\
&\lesssim \|\psi_\alpha\|_{L^2(B_{2\theta})}\|\nabla e_\alpha\|_{L^2(B_{2\theta})}[(u-u_{2\theta})\eta]_{BMO}\\
&\lesssim \|\nabla u\|_{L^2(B_{2\theta})}\|\psi_\alpha\|_{L^2(B_{2\theta})}\|\nabla u\|_{M^{p,p}(B_1)},
\end{aligned}
$$

where we have used the inequality

$$[(u-u_{2\theta})\eta]_{BMO} \le C\,[u]_{BMO(B_{2\theta})} \le C\,\|\nabla u\|_{M^{p,p}(B_1)}.$$

This, combined with Hölder's inequality, implies

$$\left(\theta^{p-n}\int_{B_\theta}|\psi_\alpha|^p\right)^{1/p} \le\ C\epsilon_0\,\|\nabla u\|_{M^{p,p}(B_1)}. \tag{4-13}$$

Step 5: Decay estimation of ∇u. Putting (4-12) and (4-13) together, we have that, for some $0<\alpha_0<1$,

$$\left((\tau\theta)^{p-n}\int_{B_{\tau\theta}}|\nabla u|^p\right)^{1/p} \le C(\epsilon_0+\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0)\|\nabla u\|_{M^{p,p}(B_1)} \tag{4-14}$$

for any $0<\tau<1$ and $0<\theta<\frac{1}{2}$. Now we claim that for some $\alpha_0\in(0,\ 1)$, we have

$$\|\nabla u\|_{M^{p,p}(B_{\tau/4})} \le C(\epsilon_0+\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0)\,\|\nabla u\|_{M^{p,n-p}(B_1)} \tag{4-15}$$

for all $0<\tau<1$. To show this, let $B_s(y)\subset B_{\tau/4}$. We divide into three cases:

(a) $y\in B_{\tau/4}\cap B^{\pm}$ and $s<|y^n|$. As remarked at the beginning of the proof, for some $0<\alpha_0<1$ we have

$$\begin{aligned}
\left(s^{p-n}\int_{B_s(y)}|\nabla u|^p\right)^{1/p} &\le C\left(\frac{s}{|y^n|}\right)^{\alpha_0}\left(|y^n|^{p-n}\int_{B_{|y^n|}(y)}|\nabla u|^p\right)^{1/p}\\
&\le C\left(\frac{s}{|y^n|}\right)^{\alpha_0}\left((2|y^n|)^{p-n}\int_{B_{2|y^n|}(y',0)}|\nabla u|^p\right)^{1/p}\\
&\le C\left(\left(\frac{\tau}{2}\right)^{p-n}\int_{B_{\tau/2}(y',0)}|\nabla u|^p\right)^{1/p}\ \text{(since } |y^n|\le\tau/4)\\
&\le C(\epsilon_0+\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0)\|\nabla u\|_{M^{p,p}(B_1)}\ \text{(by (4-14))}.
\end{aligned}$$

(b) $y\in B_{\tau/4}\cap B^{\pm}$ and $s\ge|y^n|$. Then $B_s(y)\subset B_{|y^n|+s}(y',0)\subset B_{2s}(y',0)$. Hence

$$\begin{aligned}
\left(s^{p-n}\int_{B_s(y)}|\nabla u|^p\right)^{1/p} &\le 2^{n/p-1}\left((2s)^{p-n}\int_{B_{2s}(y',0)}|\nabla u|^p\right)^{1/p}\\
&\le C(\epsilon_0+\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0)\,\|\nabla u\|_{M^{p,p}(B_1)}\ \text{(by (4-14))}.
\end{aligned}$$

(c) $y\in B_{\tau/4}\cap\Gamma_1$, i.e., $y^n=0$. Then it follows directly from (4-14) that

$$\left(s^{p-n}\int_{B_s(y)}|\nabla u|^p\right)^{1/p} \le C(\epsilon_0+\tau^{\alpha_0}+\tau^{1-n/p}\epsilon_0)\,\|\nabla u\|_{M^{p,p}(B_1)}.$$

Combining (a), (b) and (c) together and taking the supremum over all $B_s(y)\subset B_{\tau/4}$, we obtain (4-15).

It is clear that by first choosing τ and then ϵ sufficiently small, we can arrange that

$$\|\nabla u\|_{M^{p,p}(B_{\tau/4})} \le \tfrac{1}{2}\, \|\nabla u\|_{M^{p,p}(B_1)}\,.$$

Iterating this inequality finitely many times yields that there exists $\alpha_1 \in (0,\ 1)$ such that for any $x \in B_{1/4}$ and $0 < r \le \frac{1}{2}$, it holds

$$r^{p-n} \int_{B_r(x)} |\nabla u|^p\ dx \le C\, r^{p\alpha_1}\, \|\nabla u\|^p_{M^{p,p}(B_1)}\,.$$

This implies $u \in C^{\alpha_1}(B_{1/2})$ by Morrey's lemma. The proof is now completed. □

5. Lipschitz and piecewise $C^{1,\alpha}$-regularity

In this section, we will first establish Lipschitz and piecewise $C^{1,\alpha}$-regularity for stationary harmonic maps on domains with piecewise $C^{0,\,1}$-metrics, under a smallness condition of energy. Then we will prove Theorem 1.1.

Theorem 5.1. *There exist $\epsilon_0 > 0$ and $\beta_0 \in (0,\ 1)$, depending only on n and g, such that if the metric $g \in C^{0,1}(B_1^{\pm} \cup \Gamma_1)$ satisfies the condition* (1-4) *on Γ_1, and $u \in W^{1,2}(B_1, N)$ is a stationary harmonic map on (B_1, g) satisfying*

$$r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|_g^2\, dv_g \le \epsilon_0^2 \tag{5-1}$$

for some $x_0 \in B_{1/2}$ and $0 < r_0 \le \frac{1}{4}$, then $u \in C^{1,\beta_0}\big(B_{r_0/2}(x_0) \cap \overline{B^{\pm}}, N\big)$, and $u \in C^{0,1}\big(B_{r_0/2}(x_0), N\big)$.

Proof. The proof is based on the hole filling argument and the freezing coefficient method. It is divided into two steps.

<u>Step 1</u>: $u \in C^{\alpha}(B_{3r_0/4}(x_0), N)$ for any $0 < \alpha < 1$. To see this, first recall Theorem 4.1 implies that there exists $0 < \alpha_0 < \frac{2}{3}$ such that $u \in C^{\alpha_0}(B_{7r_0/8}(x_0))$ and for any $y \in B_{7r_0/8}(x_0)$, it holds

$$s^{2-n} \int_{B_s(y)} |\nabla u|^2\, dx \le C \left(\frac{s}{r}\right)^{2\alpha_0} r^{2-n} \int_{B_r(y)} |\nabla u|^2\, dx, \quad 0 < s \le r < \frac{r_0}{8}, \tag{5-2}$$

and

$$\operatorname{osc}_{B_r(y)} u \le C r^{\alpha_0}, \quad 0 < r < \frac{r_0}{8}. \tag{5-3}$$

For $y \in B_{7r_0/8}(x_0)$ and $0 < r < r_0/8$, let $v : B_r(y) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_g v = 0 & \text{in } B_r(y), \\ v = u & \text{on } \partial B_r(y). \end{cases} \tag{5-4}$$

By the maximum principle and (5-3), we then have

$$\operatorname{osc}_{B_r(y)} v \le \operatorname{osc}_{\partial B_r(y)} u \le C r^{\alpha_0}.$$

Moreover, since $g \in C^{0,1}(B_1^\pm \cup \Gamma_1)$, it follows from [Li and Nirenberg 2003, Theorem 1.1] that $v \in C^{0,1}\big(B_{r/2}(y), \mathbb{R}^k\big)$ and $v \in C^{1,\beta}\big(B_{r/2}(y) \cap \overline{B^\pm}, \mathbb{R}^k\big)$ for any $0 < \beta < 1$.

Multiplying (1-1) and (5-4) by $u - v$, subtracting one result from the other and integrating over $B_r(y)$, we obtain

$$\int_{B_r(y)} |\nabla(u - v)|^2 \, dx \lesssim \int_{B_r(y)} |\nabla u|^2 |u - v| \lesssim r^{n-2+3\alpha_0}.$$

This, combined with

$$\int_{B_{r/2}(y)} |\nabla v|^2 \, dx \le C \|\nabla v\|^2_{L^\infty(B_{r/2}(y))} r^n,$$

implies

$$\Big(\frac{r}{2}\Big)^{2-n} \int_{B_{r/2}(y)} |\nabla u|^2 \, dx \le C\big(\|\nabla v\|^2_{L^\infty(B_{r/2}(y))} \, r^2 + r^{3\alpha_0}\big) \le C r^{3\alpha_0}.$$

This, combined with Morrey's lemma, yields $u \in C^{3\alpha_0/2}(B_{7r_0/8}(x_0))$. Repeating this argument, we can show that $u \in C^\alpha(B_{3r_0/4}(x_0))$ for any $0 < \alpha < 1$, and

$$r^{2-n} \int_{B_r(y)} |\nabla u|^2 \, dx \le C r^{2\alpha} \quad \text{for all } y \in B_{3r_0/4}(x_0),\ 0 < r < \frac{r_0}{4}. \tag{5-5}$$

Step 2: There exists $0 < \beta_0 < 1$ such that $u \in C^{1,\beta_0}\big(B_{r_0/2}(x_0) \cap \overline{B^\pm}, N\big)$. There are two cases to consider:

Case I: $x_0 = (x_0', x_0^n) \in B_1^\pm$. We may assume $0 < r_0 < |x_0^n|$, so that $B_{r_0}(x_0) \subset B^\pm$. For $B_r(x) \subset B_{r_0}(x_0)$, let $v : B_r(x) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_g v = 0 & \text{in } B_r(x), \\ v = u & \text{on } \partial B_r(x). \end{cases} \tag{5-6}$$

Then by (5-5), for any $\frac{2}{3} < \alpha < 1$,

$$\int_{B_r(x)} |\nabla(u - v)|^2 \, dx \le C \int_{B_r(x)} |\nabla u|^2 |u - v| \, dx \le C \, r^{3\alpha+n-2}. \tag{5-7}$$

Also, since $g \in C^{0,1}(B_{r_0}(x_0))$, we have for any $0 < \beta < 1$ that $v \in C^{1,\beta}(B_{r/2}(x))$ and

$$⨍_{B_s(x)} |\nabla v - (\nabla v)_{B_s(x)}|^2 \, dx \le C \Big(\frac{s}{r}\Big)^{2\beta} ⨍_{B_r(x)} |\nabla u - (\nabla u)_{B_r(x)}|^2 \, dx, \tag{5-8}$$

for $< s \le r/2$. (Here $\fint_E f = \frac{1}{|E|}\int_E f\,dx$.) Note that (5-8) also holds trivially for $r/2 \le s \le r$. Combining (5-7) and (5-8) we obtain, for any $0 < \theta < 1$,

$$\begin{aligned}\fint_{B_{\theta r}(x)} \left|\nabla u - (\nabla u)_{B_{\theta r}(x)}\right|^2 dx &\le 2\biggl(\fint_{B_{\theta r}(x)} |\nabla u - \nabla v|^2\,dx + \fint_{B_{\theta r}(x)} \left|\nabla v - (\nabla v)_{B_{\theta r}(x)}\right|^2 dx\biggr)\\ &\le C\biggl(\theta^{2\beta}\fint_{B_r(x)} \left|\nabla u - (\nabla u)_{B_r(x)}\right|^2 dx + \theta^{-n} r^{3\alpha-2}\biggr).\end{aligned}$$

For $(3\alpha-2)/2 < \beta_0 < \beta$, let $0 < \theta_0 < 1$ be such that $C\theta_0^{2\beta} = \theta_0^{2\beta_0}$. Then

$$\fint_{B_{\theta_0 r}(x)} \left|\nabla u - (\nabla u)_{B_{\theta_0 r}(x)}\right|^2 dx \le \theta_0^{2\beta_0}\fint_{B_r(x)} \left|\nabla u - (\nabla u)_{B_r(x)}\right|^2 dx + Cr^{3\alpha-2}. \tag{5-9}$$

Iterating (5-9) m-times, $m \ge 1$, yields

$$\begin{aligned}\fint_{B_{\theta_0^m r}(x)} &\left|\nabla u - (\nabla u)_{B_{\theta_0 r}(x)}\right|^2 dx\\ &\le \left(\theta_0^m\right)^{2\beta_0}\fint_{B_r(x)} \left|\nabla u - (\nabla u)_{B_r(x)}\right|^2 dx + C(\theta_0^m r)^{3\alpha-2}\sum_{j=1}^m \theta_0^{j(2\beta_0-(3\alpha-2))}\\ &\le (\theta_0^m)^{3\alpha-2}\biggl(\fint_{B_r(x)} \left|\nabla u - (\nabla u)_{B_r(x)}\right|^2 dx + Cr^{3\alpha-2}\biggr).\end{aligned} \tag{5-10}$$

This clearly implies that $\nabla u \in C^{3\alpha/2-1}(B_{r_0}(x_0))$.

Case II: $x_0 = (x_0', 0) \in \Gamma_1$. For simplicity, we assume $x_0' = 0$. Define $\bar g$ on B_1 by

$$\bar g(x) = \begin{cases} \lim_{t\downarrow 0^+} g(0', t) & \text{if } x \in B_1^+\\ \lim_{t\uparrow 0^-} g(0', t) & \text{if } x \in B_1^-.\end{cases}$$

Then we have

$$|g(x) - \bar g(x)| \le C|x|, \quad x \in B_1. \tag{5-11}$$

Moreover, by suitable dilations and rotations of the coordinate system, (1-4) implies that there exists a positive constant $k \ne 1$ such that

$$\bar g(x) = (1 + (k-1)\chi_{B_1^-}(x))g_0, \quad x \in B_1,$$

where $\chi_{B_1^-}$ is the characteristic function of B_1^-.

For $0 < r < r_0/2$, let $v : B_r(0) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_{\bar g} v = 0 & \text{in } B_r(0),\\ v = u & \text{on } \partial B_r(0).\end{cases} \tag{5-12}$$

Then we have

$$\operatorname{osc}_{B_r(0)} v \le \operatorname{osc}_{B_r(0)} u \le C r^{\alpha}, \quad \int_{B_r(0)} |\nabla v|^2\,dx \le C \int_{B_r(0)} |\nabla u|^2 \le C r^{n-2+2\alpha}.$$

Multiplying (1-1) and (5-12) by $u - v$ and integrating over $B_r(0)$, we obtain

$$\begin{aligned}
\int_{B_r(0)} & |\nabla(u-v)|^2\,dx \\
&\le \int_{B_r(0)} g^{ij}(u-v)_i(u-v)_j\sqrt{g}\,dx \\
&\le C\int_{B_r(0)} |\nabla u|^2|u-v|\,dx + \int_{B_r(0)} |\sqrt{g}g^{ij} - \sqrt{\bar g}\bar g^{ij}|\,|v_i|\,|(u-v)_j|\,dx \\
&\le C \operatorname{osc}_{B_r(0)} v \int_{B_r(0)} |\nabla u|^2\,dx + Cr^2\int_{B_r(0)} |\nabla v|^2 + \frac{1}{2}\int_{B_r(0)} |\nabla(u-v)|^2\,dx \\
&\le C r^{n-2+3\alpha} + C r^{n+\alpha} + \frac{1}{2}\int_{B_r(0)} |\nabla(u-v)|^2\,dx.
\end{aligned}$$

This implies

$$\int_{B_r(0)} |\nabla(u-v)|^2\,dx \le C r^{n-2+3\alpha}. \tag{5-13}$$

It is well-known that $v \in C^\infty\big(\overline{B_s^{\pm}(0)}\big)$ for any $0 < s < r$. In fact, (5-12) is equivalent to:

$$\frac{\partial}{\partial x_i}\left((1 + (k^{n/2}-1)\chi_{B_1^-})\frac{\partial v}{\partial x_i}\right) = 0 \quad \text{in } B_r(0), \tag{5-14}$$

we conclude

(i) $\partial v/\partial x_n$ satisfies the jump property on Γ_1:

$$\lim_{x_n \downarrow 0^+} \frac{\partial v}{\partial x_n}(x', x_n) = k^{n/2} \lim_{x_n \uparrow 0^-} \frac{\partial v}{\partial x_n}(x', x_n) \quad \text{for all } (x', 0) \in \Gamma_1 \cap B_r(0).$$

(ii) $\nabla^\alpha v \in C^0(B_r(0))$ for any multiindex $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$.

(iii) $\nabla v \in L^\infty(B_s(0))$ for any $0 < s < r$, and

$$\|\nabla v\|^2_{L^\infty(B_{r/2}(0))} \le C r^{2-n} \int_{B_r(0)} |\nabla u|^2. \tag{5-15}$$

For $f : B_r(0) \to \mathbb{R}^k$, set

$$\widetilde{D} f := \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}}, (1 + (k^{n/2}-1)\chi_{B_1^-})\frac{\partial f}{\partial x_n}\Big), \tag{5-16}$$

and denote by $(\widetilde{D}f)_s = \fint_{B_s(0)} \widetilde{D}f\,dx$ the average of $\widetilde{D}f$ over $B_s(0)$. Then, for any $0<\beta<1$,

$$\fint_{B_s(0)} \big|\widetilde{D}v-(\widetilde{D}v)_s\big|^2\,dx \le C\Big(\frac{s}{r}\Big)^{2\beta}\fint_{B_r(0)} \big|\widetilde{D}u-(\widetilde{D}u)_r\big|^2\,dx \quad \text{for all } 0<s\le r.$$

Combining this with (5-13) yields, for any $0<\theta<1$,

$$\fint_{B_{\theta r}(0)} \big|\widetilde{D}u-(\widetilde{D}u)_{\theta r}\big|^2\,dx \le C\theta^{2\beta}\fint_{B_r(0)} \big|\widetilde{D}u-(\widetilde{D}u)_r\big|^2\,dx + C\theta^{-n}r^{3\alpha-2}.$$

As in case I, iterations of this inequality yield, for any $0<s\le r$,

$$\fint_{B_s(0)} \big|\widetilde{D}u-(\widetilde{D}u)_s\big|^2\,dx \le C\Big(\frac{s}{r}\Big)^{3\alpha-2}\fint_{B_r(0)} \big|\widetilde{D}u-(\widetilde{D}u)_r\big|^2\,dx + Cs^{3\alpha-2}.$$

This, combined with case I, implies that for any $B_r(x)\subset B_{r_0}(x_0)$ and $0<s\le r$,

$$\fint_{B_s(x)} \big|\widetilde{D}u-(\widetilde{D}u)_{x,s}\big|^2\,dx \le C\Big(\frac{s}{r}\Big)^{3\alpha-2}\fint_{B_r(x)} \big|\widetilde{D}u-(\widetilde{D}u)_{x,r}\big|^2\,dx + Cs^{3\alpha-2},$$

where $(\widetilde{D}u)_{x,s}$ denotes the average of $\widetilde{D}u$ over $B_s(x)$. It is readily seen that the preceding inequality yields $u\in C^{1,3\alpha/2-1}(B_{r_0/2}(x_0)\cap\overline{B_1^{\pm}})$ and $u\in C^{0,1}(B_{r_0/2}(x_0))$. This completes the proof. $\square$

Proof of Theorem 1.1 . Define the singular set

$$\Sigma=\Big\{x\in\Omega:\ \varliminf_{r\to 0} r^{2-n}\int_{B_r(x)}|\nabla u|^2\,dx\ge\epsilon_0^2\Big\}.$$

Then by a covering argument we have $H^{n-2}(\Sigma)=0$; see [Evans and Gariepy 1992]. For any $x_0\in\Omega\setminus\Sigma$, there exists $0<r_0<\mathrm{dist}(x_0,\partial\Omega)$ such that

$$r_0^{2-n}\int_{B_{r_0}(x)}|\nabla u|^2\,dx\le\epsilon_0^2.$$

Hence by Theorems 2.1, 4.1, and 5.1, we have

$$u\in C^{1,\alpha}(B_{r_0/2}(x_0)\cap\overline{\Omega^{\pm}},N) \quad\text{and}\quad u\in C^{0,1}(B_{r_0/2}(x_0),N),$$

for some $0<\alpha<1$. In particular, we have

$$\lim_{r\downarrow 0} r^{2-n}\int_{B_r(x)}|\nabla u|^2\,dx=0 \quad\text{for all } x\in B_{r_0/2}(x_0),$$

so that $B_{r_0/2}(x_0)\cap\Sigma=\varnothing$, i.e., Σ is closed. This completes the proof. $\square$

6. Harmonic maps to manifolds supporting convex distance square functions

In this section, we consider weakly harmonic maps u from (Ω, g), with g the piecewise Lipschitz continuous metric as in Theorem 1.1, to (N, h), whose universal cover $(\widetilde{N}, \tilde{h})$ supports a convex distance square function $d^2_{\widetilde{N}}(\cdot\,, p)$ for any $p \in \widetilde{N}$. We will establish both the global Lipschitz continuity and piecewise $C^{1,\alpha}$-regularity for such harmonic maps u. This can be viewed as a generalization of the well-known regularity theorem of Eells and Sampson [1964] and Hildebrand, Kaul and Widman [Hildebrandt et al. 1977].

The crucial step is the following theorem on Hölder continuity.

Theorem 6.1. *Assume that the metric g is bounded measurable on Ω, i.e., there exist two constants $0 < \lambda < \Lambda < +\infty$ such that $\lambda \mathbb{I}_n \le g(x) \le \Lambda \mathbb{I}_n$ for a.e. $x \in \Omega$, and the universal cover $(\widetilde{N}, \tilde{h})$ of (N, h) supports a convex distance square function $d^2_{\widetilde{N}}(\cdot\,, p)$ for any $p \in \widetilde{N}$. If $u \in H^1(\Omega, N)$ is a weakly harmonic map, then there exists $\alpha \in (0,\ 1)$ such that $u \in C^\alpha(\Omega, N)$.*

Proof. Here we sketch a proof that is based on modifications of that in [Lin 1997]. Similar ideas have been used by Evans in his celebrated work [1982] and by Caffarelli [1982] for quasilinear systems under smallness conditions. First, by lifting $u : \Omega \to N$ to a harmonic map $\tilde{u} : \Omega \to \widetilde{N}$, we may assume $(N, h) = (\widetilde{N}, \tilde{h})$ and $d^2_N(\cdot\,, p)$ is convex on N for any $p \in N$.

We first claim that

$$\Delta_g d^2(u, p) \ge 0. \tag{6-1}$$

In fact, by the chain rule of harmonic maps (see [Jost 1991]), we have

$$\Delta_g d^2(u, p) = \nabla_u d^2(u, p)(\Delta_g u) + \nabla^2_u d^2(u, p)(\nabla u, \nabla u)_g.$$

Since $\Delta_g u \perp T_u N$, $\nabla_u d^2(u, p) \in T_u N$, the first term in the right side vanishes. By the convexity of d^2_N, the second term in the right side satisfies

$$\nabla^2_u d^2(u, p)(\nabla u, \nabla u)_g \ge 0.$$

Since $u \in H^1(\Omega, N)$, by suitably choosing $p \in N$ and applying Poincaré inequality and Harnack's inequality, (6-1) implies $u \in L^\infty_{\mathrm{loc}}(\Omega, N)$.

For a set $E \subset N$, let $\operatorname{diam}_N E$ denote the diameter of E with respect to the distance function $d_N(\cdot\,,\cdot)$. For any ball $B_r(x) \subset \Omega$, we want to show that $u \in C^\alpha(B_{r/2}(x))$ for some $0 < \alpha < 1$. To do it, set $C_r := \operatorname{diam}_N u(B_r(x))$. We may assume $C_r > 0$ (otherwise, u is constant on $B_r(x)$ and we are done). Now we want to show that there exists $0 < \delta_0 = \delta_0(N) \le \frac{1}{2}$ such that

$$\operatorname{diam}_N u(B_{\delta_0 r}(x)) \le \tfrac{1}{2} C_r. \tag{6-2}$$

Since $u_r(y) = u(x+ry) : B_1(0) \to N$ is a harmonic map $(B_1(0), g_r)$, with $g_r(y) = g(x+ry)$, we may, for simplicity, assume $x = 0$ and $r = 2$. For any $0 < \epsilon < \frac{1}{2}$, since $u(B_1) \subset N$ is a bounded set, there exists $m = m(\epsilon) \geq 1$ such that $u(B_1)$ is covered by m balls $B^1, \dots, B^m$ of radius ϵC_1.

Claim. *There exists sufficiently small $\epsilon > 0$ such that $u(B_{1/2})$ can be covered by at most $(m-1)$ balls among $B^1, \dots, B^m$.*

To see this, let $x_i \in B_1$ such that $B^i \subset B_{2\epsilon\ C_1}(p_i)$, $p_i = u(x_i)$, for $1 \leq i \leq m$. Let $1 \leq m' \leq m$ be the maximum number of points in $\{p_i\}_{i=1}^m$ such that the distance between any two of them is at least $C_1/32$. Thus the sets $B_{C_1/16}(p_i)$, for $1 \leq i \leq m'$, cover $u(B_1)$. For convenience, set $U_i = u^{-1}\big(B^N(p_i, C_1/16)\big)$, the notation $B^N(x, R)$ referring to the ball in N with center x and radius R. We will show that there exists $i_0 \in \{1, \dots, m'\}$ such that

$$\frac{1}{4}C_1^2 \leq \sup_{x \in B_2} d_N^2(u(x), p_{i_0}) \leq C_1^2, \tag{6-3}$$

and

$$H^n(U_{i_0} \cap B_{1/2}) \geq c_0, \tag{6-4}$$

for some universal constant $c_0 > 0$. Indeed, since $B_{1/2} \subset \bigcup_{i=1}^{m'} U_i$, we have

$$\sum_{i=1}^{m'} H^n(U_i \cap B_{1/2}) \geq H^n(B_{1/2}).$$

Hence there exists $i_0 \in \{1, \dots, m'\}$ such that

$$H^n(U_i \cap B_{1/2}) \geq c_0 := \frac{1}{m'} H^n(B_{1/2}).$$

This implies (6-4). Now (6-3) follows from the triangle inequality.

Next we define

$$f(x) := \sup_{z \in B_1} d_N^2(u(z), p_{i_0}) - d_N^2(u(x), p_{i_0}), \quad x \in B_1.$$

It is clear that $f \geq 0$ in B_1, and (6-1) implies $\Delta_g f \leq 0$ in B_1. By Moser's Harnack inequality, we have

$$\begin{aligned}
\inf_{B_{1/2}} f &\geq C \fint_{B_1} f \geq C \int_{B_{1/2}} f \geq C \int_{B_{1/2} \cap U_{i_0}} f \\
&\geq C\big(\sup_{B_1} d_N^2(u, p_{i_0}) - \sup_{B_1 \cap U_{i_0}} d_N^2(u, p_{i_0})\big) H^n\big(B_{1/2} U_{i_0}\big) \\
&\geq C\left(\tfrac{1}{4}C_1^2 - \tfrac{1}{256}C_1^2\right) c_0 =: \theta_0^2 C_1^2
\end{aligned}$$

for some universal constant $\theta_0 > 0$. This implies

$$\sup_{z\in B_1} d_N(u(z), p_{i_0}) - \sup_{z\in B_{1/2}} d_N(u(z), p_{i_0}) \geq \theta_0 C_1 = (1-\theta_0)C_1. \tag{6-5}$$

Now we argue that the claim follows from (6-5). For, otherwise, we would have $u(B_{1/2}) \cap B_{2\epsilon C_1}(p_j) \neq \varnothing$ for all $1 \leq j \leq m$. Let $z_0 \in B_1$ be such that

$$\epsilon C_1 + d_N(u(z_0), p_{i_0}) \geq \sup_{B_1} d_N(u(z), p_{i_0}).$$

Since $u(B_1) \subset \bigcup_{i=}^m B_{2\epsilon C_1}(p_i)$, there exists $p_{i_1} \in \{p_1, \ldots, p_m\}$ such that $u(z_0) \in B_{2\epsilon C_1}(p_{i_1})$. Since $u(B_{1/2}) \cap B_{2\epsilon C_1}(p_{i_1}) \neq \varnothing$, there exists $z_1 \in B_{1/2}$ such that $u(z_1) \in B_{2\epsilon C_1}(p_{i_1})$. Therefore we have $d_N(u(z_1), u(z_0)) \leq 2\epsilon C_1$. Therefore we have

$$\begin{aligned}\sup_{z\in B_1} d_N(u(z), p_{i_0}) - \sup_{z\in B_{1/2}} d_N(u(z), p_{i_0}) &\leq \epsilon C_1 + d_N(u(z_0), p_{i_0}) - d_N(u(z_1), p_{i_0})\\ &\leq \epsilon C_1 + d_N(u(z_0), u(z_1)) \leq 3\epsilon C_1.\end{aligned}$$

This contradicts (6-5) if $\epsilon > 0$ is chosen to be sufficiently small.

From this claim, we have either

(i) $\operatorname{diam}_N u(B_{1/2}) \leq \frac{1}{2}C_1$ — in which case (6-2) holds with $\delta_0 = \frac{1}{2}$ — or

(ii) $\operatorname{diam}_N u(B_{1/2}) > \frac{1}{2}C_1$.

Then we consider $v(x) = u(x/2) : B_1 \to N$ and conclude:

- v is a harmonic map on $(B_1, g_{1/2})$, with the metric $g_{1/2}(x) = g(x/2)$.
- $\frac{1}{2}C_1 < \operatorname{diam}_N v(B_1) \leq C_1$.
- $v(B_1)$ is covered by at most $m-1$ balls $B_1, \ldots, B^{m-1}$ of radius ϵC_1.

Thus the claim is applicable to v so that $u(B_{1/4}) = v(B_{1/2})$ can be covered by at most $m-2$ balls among $B^1, \ldots, B^{m-1}$.

If $\operatorname{diam}_N v(B_{1/2}) \leq \frac{1}{2}C_1$, we are done. Otherwise, we can repeat the above argument. It is clear that the process can at most be repeated m times, and the process will not be stopped at step $k_0 \leq m$ unless $\operatorname{diam}_N u(B_{2^{-k_0}}) \leq \frac{1}{2}C_1$. Thus (6-2) is proven.

It is readily seen that iteration of (6-2) implies Hölder continuity. □

Proof of Theorem 1.2. First, by Theorem 6.1 and the argument from Section 4, we can show that for some $0 < \alpha < 1$,

$$\int_{B_r(x)} |\nabla u|^2\, dx \leq C r^{n-2+2\alpha} \quad \text{for all } B_r(x) \subset \Omega.$$

Then we can follow the proof of (5-2) to show that $u \in C^{0,1}(\Omega) \cap C^{1,\alpha}(\Omega^{\pm} \cup \Gamma, N)$. □

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