

*Pacific  
Journal of  
Mathematics*

**MONIC REPRESENTATIONS AND  
GORENSTEIN-PROJECTIVE MODULES**

XIU-HUA LUO AND PU ZHANG

Volume 264 No. 1

July 2013

## MONIC REPRESENTATIONS AND GORENSTEIN-PROJECTIVE MODULES

XIU-HUA LUO AND PU ZHANG

*Dedicated to the memory of Hua Feng*

Given a finite-dimensional algebra  $A$  over a field  $k$ , and a finite acyclic quiver  $Q$ , let  $\Lambda = A \otimes_k kQ$ , where  $kQ$  is the path algebra of  $Q$  over  $k$ . Then the category  $\Lambda\text{-mod}$  of  $\Lambda$ -modules is equivalent to the category  $\text{Rep}(Q, A)$  of representations of  $Q$  over  $A$ . This yields the notion of monic representations of  $Q$  over  $A$ . We denote the full subcategory of  $\text{Rep}(Q, A)$  consisting of monic representations of  $Q$  over  $A$  by  $\text{Mon}(Q, A)$ . It is proved that  $\text{Mon}(Q, A)$  has Auslander–Reiten sequences.

The main result of this paper explicitly describes the Gorenstein-projective  $\Lambda$ -modules via the monic representations plus an extra condition. As a corollary, we prove the equivalence of three conditions:  $A$  is self-injective; Gorenstein-projective  $\Lambda$ -modules are exactly the monic representations of  $Q$  over  $A$ ;  $\text{Mon}(Q, A)$  is a Frobenius category.

### 1. Introduction

Let  $A$  be an Artin algebra, and  $A\text{-mod}$  the category of finitely generated left  $A$ -modules. A *complete  $A$ -projective resolution* is an exact sequence of finitely generated projective  $A$ -modules

$$P^\bullet = \dots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \dots$$

such that  $\text{Hom}_A(P^\bullet, A)$  is also exact. A module  $M \in A\text{-mod}$  is *Gorenstein-projective* if there exists a complete  $A$ -projective resolution  $P^\bullet$  such that  $M \cong \text{Ker } d^0$ . Let  $\mathcal{P}(A)$  be the full subcategory of  $A\text{-mod}$  of projective modules, and  $\mathcal{GP}(A)$  the full subcategory of  $A\text{-mod}$  of Gorenstein-projective modules. Then

$$\mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^\perp A = \{X \in A\text{-mod} \mid \text{Ext}_A^i(X, A) = 0 \text{ for all } i \geq 1\}.$$

---

The authors were supported by the NSF of China (11271251) and the Doctoral Fund of the Ministry of Education of China (20120073110058). Pu Zhang is the corresponding author.

*MSC2010:* primary 16G10; secondary 16E65, 16G50, 16G60.

*Keywords:* representations of a quiver over an algebra, monic representations, Gorenstein-projective modules.

It is clear that  $\mathcal{GP}(A) = A\text{-mod}$  if and only if  $A$  is self-injective. If  $A$  is of finite global dimension,  $\mathcal{GP}(A) = \mathcal{P}(A)$  (but the converse is *not* true); and if  $A$  is a Gorenstein algebra (that is,  $\text{inj.dim } {}_A A < \infty$  and  $\text{inj.dim } A_A < \infty$ ), then  $\mathcal{GP}(A) = {}^\perp A$  (but the converse is *not* true); see, for example, [Enochs and Jenda 2000, Corollary 11.5.3]. This class of modules enjoys more stable properties than the usual projective modules (see [Auslander and Bridger 1969], where it was called a module of  $G$ -dimension zero); it becomes a main ingredient in the relative homological algebra [Enochs and Jenda 1995; 2000] and in the representation theory of algebras (see [Auslander and Reiten 1991a; 1991b; Beligiannis 2005; Gao and Zhang 2010; Iyama et al. 2011], for example), and plays a central role in the Tate cohomology of algebras (see [Avramov and Martsinkovsky 2002; Buchweitz 1987], for example). An important feature is that  $\mathcal{GP}(A)$  is a Frobenius category with relative projective-injective objects being projective  $A$ -modules, and hence the stable category  $\underline{\mathcal{GP}}(A)$  of  $\mathcal{GP}(A)$  modulo  $\mathcal{P}(A)$  is a triangulated category. By [Buchweitz 1987; Happel 1991], the singularity category of a Gorenstein algebra  $A$  is triangle equivalent to  $\underline{\mathcal{GP}}(A)$ . Thus explicitly constructing all the Gorenstein-projective modules is a fundamental problem, and is useful to all of these applications.

On the other hand, the submodule category has been extensively studied by C. M. Ringel and M. Schmidmeier [2006; 2008a; 2008b]; see also [Simson 2007]. By [Kussin et al. 2012] it is also related to the singularity category; see also [Chen 2011]. It turns out that the category of the Gorenstein-projective modules is closely related to the submodule category (see [Li and Zhang 2010; Xiong and Zhang 2012]), or, in general, to the monomorphism category [Zhang 2011]. The present paper explores such a relation in a more general set-up.

Given a finite-dimensional algebra  $A$  over a field  $k$ , and a finite acyclic quiver  $Q$  (here “acyclic” means that  $Q$  has no oriented cycles), let

$$\Lambda = A \otimes_k kQ,$$

where  $kQ$  is the path algebra of  $Q$  over  $k$ . We call  $\Lambda$  the path algebra of a finite quiver  $Q$  over  $A$ . As in the case of  $A = k$ ,  $\Lambda\text{-mod}$  is equivalent to the category  $\text{Rep}(Q, A)$  of representations of  $Q$  over  $A$ . This interpretation permits us to introduce the so-called monic representations of  $Q$  over  $A$ . See Definition 2.2. Let  $\text{Mon}(Q, A)$  be the full subcategory of  $\text{Rep}(Q, A)$  consisting of monic representations of  $Q$  over  $A$ . Then  $\text{Mon}(Q, A)$  is a resolving, functorially finite subcategory of  $\text{Rep}(Q, A)$ , and hence has Auslander–Reiten sequences (see Theorem 3.1). The main result of this paper, Theorem 5.1, explicitly describes all the Gorenstein-projective  $\Lambda$ -modules, via the monic representations of  $Q$  over  $A$  plus an extra condition. We emphasize that here  $\Lambda$  is not necessarily Gorenstein. By our main result, if we know all the Gorenstein-projective  $A$ -modules, we know all the Gorenstein-projective  $\Lambda$ -modules, and, in this way, we give an inductive construction of the Gorenstein-projective modules.

The proof of Theorem 5.1 use induction on  $|Q_0|$  and a description of the Gorenstein-projective modules over the triangular extension of two algebras via a bimodule which is projective in both sides (Theorem 4.1). As a corollary, we see that  $A$  is self-injective if and only if  $\mathcal{GP}(\Lambda) = \text{Mon}(Q, A)$ , and if and only if  $\text{Mon}(Q, A)$  is a Frobenius category (Corollary 6.1). As another corollary, if  $Q$  has an arrow,  $\mathcal{P}(\Lambda) = \text{Mon}(Q, A)$  if and only if  $\Lambda$  is hereditary (Corollary 6.3).

**2. Monic representations of a quiver over an algebra**

Throughout this section  $k$  is a field,  $Q$  a finite quiver, and  $A$  a finite-dimensional  $k$ -algebra. We consider the path algebra  $AQ$  of  $Q$  over  $A$ , describe its module category, and introduce the concept of monic representations of  $Q$  over  $A$ . In Subsections 2A–2D,  $Q$  is not assumed to be acyclic if not otherwise stated.

**2A.** Given a finite quiver

$$Q = (Q_0, Q_1, s, e),$$

let  $\mathcal{P}$  be the set of paths of  $Q$ . We write the conjunction of paths from right to left. If  $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$  with  $\alpha_i \in Q_1, l \geq 1$ , and  $e(\alpha_i) = s(\alpha_{i+1})$  for  $1 \leq i \leq l - 1$ , we call  $l$  the length of  $p$  and denote it by  $l(p)$ , and define the starting vertex  $s(p) = s(\alpha_1)$  and the ending vertex  $e(p) = e(\alpha_l)$ . We denote a vertex  $i$  by  $e_i$ , and regard it as a path of length 0, with  $s(e_i) = i = e(e_i)$ . Let  $kQ$  be the path algebra of  $Q$  over  $k$ . It is well-known that the category  $kQ\text{-mod}$  of finite-dimensional  $kQ$ -modules is equivalent to the category  $\text{Rep}(Q, k)$  of finite-dimensional representations of  $Q$  over  $k$ ; see, for example, [Ringel 1984, p. 44].

**2B.** Let  $\Lambda = AQ$  be the free left  $A$ -module with basis  $\mathcal{P}$ . An element of  $AQ$  is written as a finite sum  $\sum_{p \in \mathcal{P}} a_p p$ , where  $a_p \in A$  and  $a_p = 0$  for all but finitely many  $p$ . Then  $\Lambda$  is a  $k$ -algebra, with multiplication bilinearly given by

$$(a_p p)(b_q q) = (a_p b_q)(pq),$$

where  $a_p b_q$  is the product in  $A$ , and  $pq$  is the product in  $kQ$ . We have isomorphisms  $\Lambda \cong A \otimes_k kQ \cong kQ \otimes_k A$  of  $k$ -algebras, and we call  $\Lambda = AQ$  the path algebra of  $Q$  over  $A$ .

For example, if  $Q = \bullet_n \rightarrow \cdots \rightarrow \bullet_1$ , the algebra  $\Lambda$  is given by the upper triangular matrix algebra of  $A$ :

$$T_n(A) = \begin{pmatrix} A & A & \cdots & A & A \\ 0 & A & \cdots & A & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A & A \\ 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

In general, if  $Q$  is acyclic and  $Q_0$  is labeled as  $1, \dots, n$  in such a way that  $j > i$  whenever there is an arrow  $\alpha : j \rightarrow i$  in  $Q_1$ , then

$$(2-1) \quad kQ \cong \begin{pmatrix} k & k^{m_{21}} & k^{m_{31}} & \dots & k^{m_{n1}} \\ 0 & k & k^{m_{32}} & \dots & k^{m_{n2}} \\ 0 & 0 & k & \dots & k^{m_{n3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & k \end{pmatrix}_{n \times n},$$

where  $m_{ji}$  is the number of paths from  $j$  to  $i$  and  $k^{m_{ji}}$  is the direct sum of  $m_{ji}$  copies of  $k$ , and hence

$$(2-2) \quad \Lambda \cong \begin{pmatrix} A & A^{m_{21}} & A^{m_{31}} & \dots & A^{m_{n1}} \\ 0 & A & A^{m_{32}} & \dots & A^{m_{n2}} \\ 0 & 0 & A & \dots & A^{m_{n3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A \end{pmatrix}_{n \times n}.$$

**2C.** By definition, a representation  $X$  of  $Q$  over  $A$  is a datum

$$X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1),$$

where  $X_i$  is an  $A$ -module for each  $i \in Q_0$  and  $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$  is an  $A$ -map for each  $\alpha \in Q_1$ . It is a *finite-dimensional representation* if each  $X_i$  is finite-dimensional. We call  $X_i$  the  $i$ -th branch of  $X$ . A morphism  $f$  from representation  $X$  to representation  $Y$  is a datum  $(f_i, i \in Q_0)$ , where  $f_i : X_i \rightarrow Y_i$  is an  $A$ -map for each  $i \in Q_0$ , such that, for each arrow  $\alpha : j \rightarrow i$ , the diagram

$$(2-3) \quad \begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ \downarrow X_\alpha & & \downarrow Y_\alpha \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

commutes. We call  $f_i$  the  $i$ -th branch of  $f$ . If  $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$  with  $\alpha_i \in Q_1$ ,  $l \geq 1$ , and  $e(\alpha_i) = s(\alpha_{i+1})$  for  $1 \leq i \leq l-1$ , we put  $X_p$  to be the  $A$ -map  $X_{\alpha_l} \cdots X_{\alpha_1}$ . Denote by  $\text{Rep}(Q, A)$  the category of finite-dimensional representations of  $Q$  over  $A$ . A morphism  $f = (f_i, i \in Q_0)$  in  $\text{Rep}(Q, A)$  is a monomorphism (epimorphism, isomorphism) if and only if  $f_i$  is injective (surjective, an isomorphism) for each  $i \in Q_0$ .

**Lemma 2.1.** *Let  $\Lambda$  be the path algebra of  $Q$  over  $A$ . Then we have an equivalence  $\Lambda\text{-mod} \cong \text{Rep}(Q, A)$  of categories.*

We omit the proof of Lemma 2.1, which is similar to the case of  $A = k$ ; see [Auslander et al. 1995, Theorem 1.5, p. 57; Ringel 1984, p. 44]. Throughout this paper we will identify a  $\Lambda$ -module with a representation of  $Q$  over  $A$ . Under this identification, a  $\Lambda$ -module  $X$  is a representation  $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  of  $Q$

over  $A$ , where  $X_i = (1e_i)X$ ,  $1$  is the identity of  $A$ , and the  $A$ -action on  $X_i$  is given by  $a(1e_i)x = (1e_i)(ae_i)x$  for all  $x \in X$  and  $a \in A$ ; and  $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$  is the  $A$ -map given by the left action by  $1\alpha \in \Lambda$ . On the other hand, a representation  $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  of  $Q$  over  $A$  is a  $\Lambda$ -module  $X = \bigoplus_{i \in Q_0} X_i$ , with the  $\Lambda$ -action on  $X$  given by

$$(ap)(x_i) = \begin{cases} 0 & \text{if } s(p) \neq i, \\ ax_i & \text{if } p = e_i, \\ aX_p(x_i) \in X_{e(p)} & \text{if } s(p) = i \text{ and } l(p) \geq 1, \end{cases}$$

for all  $a \in A$ ,  $p \in \mathcal{P}$ ,  $x_i \in X_i$ . Let  $f : X \rightarrow Y$  be a morphism in  $\text{Rep}(Q, A)$ . Then  $\text{Ker } f$  and  $\text{Coker } f$  can be explicitly written out. For example,  $\text{Coker } f = (\text{Coker } f_i, \tilde{Y}_\alpha, i \in Q_0, \alpha \in Q_1)$ , where, for each arrow  $\alpha : j \rightarrow i$ ,

$$\tilde{Y}_\alpha : \text{Coker } f_j \rightarrow \text{Coker } f_i$$

is the  $A$ -map induced by  $Y_\alpha$ ; see (2-3). A sequence of morphisms

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in  $\text{Rep}(Q, A)$  is exact if and only if each

$$0 \longrightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \longrightarrow 0$$

is exact in  $A\text{-mod}$ , for  $i \in Q_0$ .

In the following, if  $Q_0$  is labeled as  $1, \dots, n$ , we also write a representation  $X$  of  $Q$  over  $A$  as

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(X_\alpha, \alpha \in Q_1)},$$

and a morphism in  $\text{Rep}(Q, A)$  as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

**2D.** The following is a central notion of this paper.

**Definition 2.2.** A representation  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  of  $Q$  over  $A$  is a *monic representation*, or a monic  $\Lambda$ -module, if, for each  $i \in Q_0$ , the  $A$ -map

$$(X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i$$

is injective, or, equivalently, if the following two conditions are satisfied.

- (m1) For each  $\alpha \in Q_1$ , the map  $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$  is injective.
- (m2) For each  $i \in Q_0$ , there holds  $\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha = \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha$ .

Denote by  $\text{Mon}(Q, A)$  the full subcategory of  $\text{Rep}(Q, A)$  consisting of monic representations of  $Q$  over  $A$ . We call  $\text{Mon}(Q, A)$  *the monomorphism category of  $A$  over  $Q$* .

If  $Q$  is a quiver in which, for any vertex  $i$ , there is at most one arrow ending at  $i$ , condition (m2) vanishes. For example, if  $Q = \bullet \rightarrow \bullet$ , then  $\text{Mon}(Q, A)$  is called *the submodule category of  $A$  in [Ringel and Schmidmeier 2006; 2008a]*. If

$$Q = \underset{n}{\bullet} \rightarrow \cdots \rightarrow \underset{1}{\bullet},$$

$\text{Mon}(Q, A)$  is called *the filtered chain category of  $A$  in [Arnold 2000; Simson 2007]*.

**2E.** Let  $Q$  be a finite acyclic quiver,  $A$  a finite-dimensional algebra, and  $\Lambda = A \otimes_k kQ$ . Throughout this paper, we label the vertices of  $Q$  as  $1, 2, \dots, n$ , in such a way that if there is an arrow from  $j$  to  $i$ , then  $j > i$ . Denote by  $P(i)$  the indecomposable projective  $kQ$ -module at  $i \in Q_0$ . It is clear that  $P(i) \in \text{Mon}(Q, k)$ ; it follows that  $M \otimes_k P(i) \in \text{Mon}(Q, A)$  for  $M \in A\text{-mod}$ . Thus we have the functors

$$- \otimes_k P(i) : A\text{-mod} \rightarrow \text{Mon}(Q, A), \quad -_i : \text{Rep}(Q, A) \rightarrow A\text{-mod}$$

(by taking the  $i$ -th branch).

We also need the adjoint pair  $(- \otimes_k P(i), -_i)$ .

**Lemma 2.3.** *For each object  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1) \in \Lambda\text{-mod}$  and each  $A$ -module  $M$ , we have isomorphisms of abelian groups, which are natural in both positions*

$$(2-4) \quad \text{Hom}_\Lambda(M \otimes_k P(i), X) \cong \text{Hom}_A(M, X_i)$$

for all  $i \in Q_0$ .

*Proof.* For  $f = (f_j, j \in Q_0) \in \text{Hom}_\Lambda(M \otimes_k P(i), X)$ , we have  $f_i \in \text{Hom}_A(M, X_i)$ . Since  $M \otimes_k P(i) = (M \otimes_k e_j k Q e_i, \text{id}_M \otimes \alpha, j \in Q_0, \alpha \in Q_1)$ , it follows from the commutative diagram (2-3) that

$$(2-5) \quad f_j = \begin{cases} 0 & \text{if there are no paths from } i \text{ to } j, \\ m \otimes_k p \mapsto X_p f_i(m) & \text{if there is a path } p \text{ from } i \text{ to } j. \end{cases}$$

By (2-5) we see that  $f \mapsto f_i$  gives an injective map

$$\text{Hom}_\Lambda(M \otimes_k P(i), X) \rightarrow \text{Hom}_A(M, X_i).$$

This map is also surjective, since for a given  $f_i \in \text{Hom}_A(M, X_i)$ ,  $f = (f_j, j \in Q_0)$  given by (2-5) is indeed a morphism in  $\text{Rep}(Q, A)$  from  $M \otimes_k P(i)$  to  $X$ .  $\square$

- Proposition 2.4.** (i) *The indecomposable projective  $\Lambda$ -modules have the form  $P \otimes_k P(i)$ , where  $P$  is an indecomposable projective  $A$ -module, and  $P(i)$  is the indecomposable projective  $kQ$ -module at  $i \in Q_0$ .*
- (ii) *The indecomposable projective objects in  $\text{Mon}(Q, A)$  are exactly the indecomposable projective  $\Lambda$ -modules.*
- (iii) *If  $I$  is an indecomposable injective  $A$ -module and  $P(i)$  is the indecomposable projective  $kQ$ -module at  $i \in Q_0$ ,  $I \otimes_k P(i)$  is an indecomposable injective object in  $\text{Mon}(Q, A)$ .*

*Proof.* (i) As a direct summand of the regular  $\Lambda$ -module  ${}_{\Lambda}\Lambda$ , we see that  $P \otimes_k P(i)$  is a projective  $\Lambda$ -module, and each projective  $\Lambda$ -module has this form. By (2-4) we have

$$\text{End}_{\Lambda}(P \otimes_k P(i)) \cong \text{Hom}_A(P, (P \otimes_k P(i))_i) = \text{End}_A(P),$$

from which we see that  $P \otimes_k P(i)$  is indecomposable.

(ii) Note that  $P \otimes_k P(i) \in \text{Mon}(Q, A)$ . By (i) we know that it is an indecomposable projective object in  $\text{Mon}(Q, A)$ . On the other hand, it is clear that  $\text{Mon}(Q, A)$  is closed under taking subobjects, as a consequence any indecomposable projective object in  $\text{Mon}(Q, A)$  has this form.

(iii) Note that  $I \otimes_k P(i)$  is an indecomposable object in  $\text{Mon}(Q, A)$ . Put  $L = D(A_A) \otimes_k kQ$ , where  $D = \text{Hom}_k(-, k)$ . It suffices to prove that  $L$  is an injective object in  $\text{Mon}(Q, A)$ , by induction on  $|Q_0|$ . We write  $L = (L_i, L_{\alpha}, i \in Q_0, \alpha \in Q_1)$ .

Let  $Q'$  be the quiver obtained from  $Q$  by deleting a sink vertex 1,  $L'$  the representation in  $\text{Rep}(Q', A)$  obtained from  $L$  by deleting the branch  $L_1$ . We observe that  $L' = D(A_A) \otimes_k kQ'$ , and by inductive hypothesis  $L'$  is an injective object in  $\text{Mon}(Q', A)$ .

Let  $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$  be an exact sequence in  $\text{Mon}(Q, A)$ , with  $X = (X_i, X_{\alpha}, i \in Q_0, \alpha \in Q_1)$ , and  $h : X \rightarrow L$  a morphism in  $\text{Rep}(Q, A)$ . Let  $X'$  be the representation in  $\text{Rep}(Q', A)$  obtained from  $X$  by deleting the branch  $X_1$ , and similarly for  $Y', Z'$ . Then we have an exact sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \longrightarrow 0$$

in  $\text{Mon}(Q', A)$ , where  $f'$  is the morphism in  $\text{Rep}(Q', A)$  obtained from  $f$  by deleting the branch  $f_1$ , and similarly for  $g'$  and for  $h' : X' \rightarrow L'$ . Since  $L'$  is an injective object in  $\text{Mon}(Q', A)$ , by definition we have a morphism

$$u' = \begin{pmatrix} u_2 \\ \vdots \\ u_n \end{pmatrix} : Y' \rightarrow L'$$



in  $\text{Rep}(Q', A)$  such that  $h' = u' f'$ . It suffices to construct an  $A$ -map

$$u_1 : Y_1 \rightarrow L_1$$

such that  $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} : Y \rightarrow L$  is a morphism in  $\text{Rep}(Q, A)$ , and that  $h_1 = u_1 f_1$ .

First, we have an  $A$ -map  $u'_1 : Y_1 \rightarrow L_1$  such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ h_1 \downarrow & \swarrow u'_1 & \\ L_1 & & \end{array} .$$

commutes. Consider the  $A$ -map

$$(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} Y_{s(\alpha)} \rightarrow L_1 .$$

Since we have the exact sequence of  $A$ -modules

$$0 \longrightarrow \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} X_{s(\alpha)} \xrightarrow{\text{diag}(f_{s(\alpha)})} \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} Y_{s(\alpha)} \xrightarrow{\text{diag}(g_{s(\alpha)})} \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \longrightarrow 0,$$

and since

$$\begin{aligned} (L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} \circ \text{diag}(f_{s(\alpha)}) &= (L_\alpha u_{s(\alpha)} f_{s(\alpha)} - u'_1 Y_\alpha f_{s(\alpha)})_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} \\ &= (L_\alpha u_{s(\alpha)} f_{s(\alpha)} - u'_1 f_1 X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} \\ &= (L_\alpha h_{s(\alpha)} - h_1 X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} \\ &= 0, \end{aligned}$$

where the second equality follows from the fact that  $f : X \rightarrow Y$  is a morphism in  $\text{Rep}(Q, A)$ , it follows that  $(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}}$  factors through  $\text{diag}(g_{s(\alpha)})$ . That is, there is an  $A$ -map

$$v_1 : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \rightarrow L_1,$$

such that

$$(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} = v_1 \circ \text{diag}(g_{s(\alpha)}).$$

Since  $L_1$  is an injective  $A$ -module and

$$(Z_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \rightarrow Z_1$$

is an injective  $A$ -map, it follows that there is an  $A$ -map  $w_1 : Z_1 \rightarrow L_1$ , such that  $v_1 = w_1 \circ (Z_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}}$ . So we have

$$(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} = w_1 \circ (Z_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}} \circ \text{diag}(g_{s(\alpha)}) = (w_1 g_1 Y_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=1}},$$

where the second equality follows from the fact that  $g : Y \rightarrow Z$  is a morphism in  $\text{Rep}(Q, A)$ . This means that for each  $\alpha \in Q_1$  with  $e(\alpha) = 1$  we have

$$(2-6) \quad L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha = w_1 g_1 Y_\alpha.$$

Now put  $u_1 = u'_1 + w_1 g_1 : Y_1 \rightarrow L_1$ . Then (2-6) together with the inductive hypothesis implies that

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} : Y \rightarrow L$$

is a morphism in  $\text{Rep}(Q, A)$ . It is clear that

$$u_1 f_1 = (u'_1 + w_1 g_1) f_1 = u'_1 f_1 = h_1.$$

This completes the proof. □

**2F.** Recall from [Auslander and Reiten 1991a] that a full subcategory  $\mathcal{X}$  of  $A\text{-mod}$  is *resolving* if  $\mathcal{X}$  contains all projective  $A$ -modules and  $\mathcal{X}$  is closed under extensions, kernels of epimorphisms, and direct summands. It is straightforward to verify that  $\text{Mon}(Q, A)$  is closed under extensions, kernels of epimorphisms, and direct summands. By Proposition 2.4 we have the following.

**Corollary 2.5.** *For a finite acyclic quiver  $Q$  and a finite-dimensional algebra  $A$ ,  $\text{Mon}(Q, A)$  is a resolving subcategory of  $\text{Rep}(Q, A)$ .*

**2G.** There is another similar but different notion. Let  $A = kQ/I$  be a finite-dimensional  $k$ -algebra, where  $I$  is an admissible ideal of  $kQ$ . An  $I$ -bounded representations of  $Q$  over  $k$  is a datum  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ , where  $X_i$  is a  $k$ -space for each  $i \in Q_0$ , and  $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$  is a  $k$ -linear map for each  $\alpha \in Q_1$ , such that  $\sum_{p \in \mathcal{P}} c_p X_p = 0$  for each element  $\sum_{p \in \mathcal{P}} c_p p \in I$ , where  $l(p) \geq 2$  and  $c_p \in k$ . An  $I$ -bounded representation  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  of  $Q$  over  $k$  is a *monic representation*, if for each  $i \in Q_0$  the  $k$ -linear map

$$(X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i$$

is injective. Let  $\text{Rep}(Q, I, k)$  be the category of finite-dimensional  $I$ -bounded representations of  $Q$  over  $k$ . There is an equivalence of categories between  $A\text{-mod}$

and  $\text{Rep}(Q, I, k)$ ; see [Auslander et al. 1995, Proposition 1.7, p. 60; Ringel 1984, p. 45]. Let  $\text{Mon}(Q, I, k)$  denote the full subcategory of  $\text{Rep}(Q, I, k)$  of  $I$ -bounded monic representations  $Q$  over  $k$ . Then  $\text{Mon}(Q, 0, k) = \text{Mon}(Q, k)$ .

**Proposition 2.6.** *Let  $A = kQ/I$  be a finite-dimensional  $k$ -algebra, where  $I$  is an admissible ideal of  $kQ$ . Then  $\mathcal{P}(A) \subseteq \text{Mon}(Q, I, k)$  if and only if  $A$  is hereditary.*

*Proof.* If  $A$  is hereditary,  $I = 0$ . It is clear  $\mathcal{P}(kQ) \subseteq \text{Mon}(Q, 0, k)$ .

Conversely, if  $I \neq 0$ , take an element  $\sum_{p \in \mathcal{P}} c_p p \in I$  with  $l(p) \geq 2$  and  $c_p \in k$ . We may assume that all the paths  $p$  with  $c_p \neq 0$  have the same starting vertex  $j$  and the same ending vertex  $i$ . Consider the projective  $A$ -module  $P(j) = Ae_j$ . As an  $I$ -bounded representation of  $Q$  over  $k$ , we write  $P(j)$  as

$$P(j) = (e_i k Q e_j, f_\alpha, t \in Q_0, \alpha \in Q_1).$$

Let  $\alpha_1, \dots, \alpha_m$  be all the arrows of  $Q$  ending at  $i$ . We claim that

$$(f_{\alpha_v})_{1 \leq v \leq m} : \bigoplus_{1 \leq v \leq m} e_{s(\alpha_v)} k Q e_j \rightarrow e_i k Q e_j$$

is not injective, where  $f_{\alpha_v}$  is the  $k$ -linear map given by the left multiplication by  $\alpha_v$ . Since each path from  $j$  to  $i$  must go through some  $\alpha_v$ , and  $\sum_{p \in \mathcal{P}} c_p f_p = 0$ , it follows that

$$\sum_{1 \leq v \leq m} \dim_k(e_{s(\alpha_v)} k Q e_j) > \dim_k(e_i k Q e_j).$$

This justifies the claim, that is,  $P(j) \notin \text{Mon}(Q, I, k)$ . □

Now, let  $\Lambda = A \otimes_k kQ$  be the path algebra of  $Q$  over  $A$ . Assume that  $\Lambda$  is of the form  $\Lambda = kQ'/I'$ , where  $Q'$  is a finite quiver and  $I'$  is an admissible ideal of  $kQ'$ . We emphasize that, in general,

$$\text{Mon}(Q, A) \neq \text{Mon}(Q', I', k).$$

In fact,  $\mathcal{P}(\Lambda) \subseteq \text{Mon}(Q, A)$  (Proposition 2.4); but generally  $\mathcal{P}(\Lambda) \subseteq \text{Mon}(Q', I', k)$  is not true, as Proposition 2.6 shows. This is the reason why we do not use the notation  $\text{Mon}(\Lambda)$ .

### 3. Functorial finiteness of $\text{Mon}(Q, A)$ in $\text{Rep}(Q, A)$

The aim of this section is to prove the following.

**Theorem 3.1.** *Let  $Q$  be a finite acyclic quiver, and  $A$  a finite-dimensional algebra. Then  $\text{Mon}(Q, A)$  is functorially finite in  $\text{Rep}(Q, A)$  and  $\text{Mon}(Q, A)$  has Auslander–Reiten sequences.*

The idea of the proof given below is essentially due to Ringel and Schmidmeier [2008a] for the case of  $Q = \bullet \rightarrow \bullet$ . The same result for the case of

$$Q = \begin{matrix} \bullet & \rightarrow & \cdots & \rightarrow & \bullet \\ & & & & 1 \\ & & & & n \end{matrix}$$

has been obtained in [Moore 2010; Zhang 2011].

**3A.** Let  $Q$  be a finite acyclic quiver. Remember we label the vertices of  $Q$  as  $1, 2, \dots, n$ , such that if there is an arrow from  $j$  to  $i$ ,  $j > i$ . So vertex 1 is a sink. Denote by  $\mathcal{P}(\rightarrow i)$  the set of all the paths  $p$  with ending vertex  $e(p) = i$  and  $l(p) \geq 1$ .

For  $X \in \text{Rep}(Q, A)$  and  $i \in Q_0$ , put  $K_i$  to be the kernel of the  $A$ -map

$$(X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i.$$

Fix an injective envelope  $\delta_i : K_i \hookrightarrow IK_i$  of  $K_i$ . Then there is an  $A$ -map

$$(\varphi_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \rightarrow IK_i$$

such that the diagram

$$(3-1) \quad \begin{array}{ccc} K_i & \hookrightarrow & \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \\ \delta_i \downarrow & \swarrow (\varphi_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} & \\ IK_i & & \end{array} .$$

commutes for each  $i \in Q_0$ . We construct a representation

$$\text{rMon}(X) = (\text{rMon}(X)_i, \text{rMon}(X)_\alpha, i \in Q_0, \alpha \in Q_1) \in \text{Rep}(Q, A)$$

as follows. For each  $i \in Q_0$ , define

$$(3-2) \quad \text{rMon}(X)_i = X_i \oplus IK_i \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow i)} IK_{s(p)}.$$

(Note that if  $i$  is a source, by definition  $\text{rMon}(X)_i = X_i$ , and that if  $p_1, \dots, p_m$  are all the paths in  $\mathcal{P}(\rightarrow i)$  with the same starting vertex  $j$ , the  $\underbrace{IK_j \oplus \cdots \oplus IK_j}_m$  is a direct summand of  $\bigoplus_{p \in \mathcal{P}(\rightarrow i)} IK_{s(p)}$ .)

For each arrow  $\alpha : j \rightarrow i$ , define

$$\text{rMon}(X)_\alpha : X_j \oplus IK_j \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow j)} IK_{s(p)} \rightarrow X_i \oplus IK_i \oplus \bigoplus_{q \in \mathcal{P}(\rightarrow i)} IK_{s(q)}$$

to be the  $A$ -map given by

$$(3-3) \quad x_j + k_j + \sum_{p \in \mathcal{P}(\rightarrow j)} k_{s(p)} \mapsto X_\alpha(x_j) + \varphi_\alpha(x_j) + k_j + \sum_{p \in \mathcal{P}(\rightarrow j)} k_{s(\alpha p)},$$

where  $x_j \in X_j$ ,  $k_j \in IK_j$ ,  $k_{s(p)} \in IK_{s(p)}$ . Note that  $s(p) = s(\alpha p)$ , and that  $k_{s(\alpha p)}$  is just  $k_{s(p)}$ . Also note that at the right side of (3-3),  $k_j$  and  $\sum_{p \in \mathcal{P}(\rightarrow j)} k_{s(\alpha p)}$  belong to different direct summands of  $\bigoplus_{q \in \mathcal{P}(\rightarrow i)} IK_{s(q)}$ .

**Lemma 3.2.** *For  $X \in \text{Rep}(Q, A)$ , we have  $\text{rMon}(X) \in \text{Mon}(Q, A)$ .*

*Proof.* For each  $i \in Q_0$ , let  $\alpha_1, \dots, \alpha_m$  be all the arrows ending at  $i$ . By definition we only need to prove that the  $A$ -map

$$(\text{rMon}(X)_{\alpha_1}, \dots, \text{rMon}(X)_{\alpha_m}) : \bigoplus_{1 \leq j \leq m} \text{rMon}(X)_{s(\alpha_j)} \rightarrow \text{rMon}(X)_i$$

is injective. This is clear by (3-1)–(3-3). For completeness we include a justification.

Suppose  $z_j = x_{s(\alpha_j)} + k_{s(\alpha_j)} + (\sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(p)}) \in \text{rMon}(X)_{s(\alpha_j)}$ ,  $j = 1, \dots, m$ , and  $\sum_{1 \leq j \leq m} \text{rMon}(X)_{\alpha_j}(z_j) = 0$ . Then by (3-3) we have

$$\begin{aligned} 0 &= \sum_{1 \leq j \leq m} X_{\alpha_j}(x_{s(\alpha_j)}) + \sum_{1 \leq j \leq m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) + \sum_{1 \leq j \leq m} k_{s(\alpha_j)} + \sum_{1 \leq j \leq m} \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(\alpha_j p)} \\ &\in X_i \oplus IK_i \oplus \bigoplus_{q \in \mathcal{P}(\rightarrow i)} IK_{s(q)}. \end{aligned}$$

Thus

$$\sum_{1 \leq j \leq m} X_{\alpha_j}(x_{s(\alpha_j)}) = 0, \quad \sum_{1 \leq j \leq m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) = 0,$$

and  $k_{s(\alpha_j)} = 0 = k_{s(\alpha_j p)}$  for all  $j = 1, \dots, m$  and all  $p \in \mathcal{P}(\rightarrow s(\alpha_j))$ . Note that  $\sum_{1 \leq j \leq m} X_{\alpha_j}(x_{s(\alpha_j)}) = 0$  implies

$$\begin{pmatrix} x_{s(\alpha_1)} \\ \vdots \\ x_{s(\alpha_m)} \end{pmatrix} \in K_i.$$

By (3-1) we have

$$\delta_i \begin{pmatrix} x_{s(\alpha_1)} \\ \vdots \\ x_{s(\alpha_m)} \end{pmatrix} = \sum_{1 \leq j \leq m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) = 0.$$

Since  $\delta_i$  is injective, we have  $x_{s(\alpha_j)} = 0$  for  $j = 1, \dots, m$ . Thus  $z_j = 0$  for  $j = 1, \dots, m$ . This completes the proof.  $\square$

**3B.** Let  $\mathcal{X}$  be a full subcategory of  $A\text{-mod}$ . Recall from [Auslander and Reiten 1991a] that a *right  $\mathcal{X}$ -approximation of  $M$*  is a morphism  $f : X \rightarrow M$  with  $X \in \mathcal{X}$  such that the induced homomorphism  $\text{Hom}_A(X', X) \rightarrow \text{Hom}_A(X', M)$  is surjective for each  $X' \in \mathcal{X}$ . If every object  $M$  admits a right  $\mathcal{X}$ -approximation,  $\mathcal{X}$  is called a *contravariantly finite subcategory in  $A\text{-mod}$* . Dually one has the concept of a *covariantly finite subcategory in  $A\text{-mod}$* . If  $\mathcal{X}$  is both contravariantly and covariantly finite in  $A\text{-mod}$ ,  $\mathcal{X}$  is a *functorially finite subcategory in  $A\text{-mod}$* .

**Proposition 3.3.** *Let  $Q$  be a finite acyclic quiver, and  $A$  a finite-dimensional algebra. Then  $\text{Mon}(Q, A)$  is contravariantly finite in  $\text{Rep}(Q, A)$ .*

*More precisely, let  $X \in \text{Rep}(Q, A)$ ,  $f = (f_i, i \in Q_0) : \text{rMon}(X) \rightarrow X$ , where  $f_i : \text{rMon}(X)_i \rightarrow X_i$  is the canonical projection. Then  $f$  is a right  $\text{Mon}(Q, A)$ -approximation of  $X$ .*

*Proof.* We use induction to prove that  $f$  is a right  $\text{Mon}(Q, A)$ -approximation of  $X$ . The assertion trivially holds if  $|Q_0| = 1$ . Suppose that the assertion holds for the quivers  $Q$  with  $|Q_0| = n - 1$ . Assume that  $|Q_0| = n$  and that

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : Y \rightarrow X$$

is a morphism in  $\text{Rep}(Q, A)$  with  $Y \in \text{Mon}(Q, A)$ . We need to prove that there is a morphism

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} : Y \rightarrow \text{rMon}(X)$$

in  $\text{Rep}(Q, A)$  such that  $g = fh$ .

Let  $Q'$  be the quiver obtained from  $Q$  by deleting vertex 1,  $X'$  the representation in  $\text{Rep}(Q', A)$  obtained from  $X$  by deleting the branch  $X_1$ , and  $Y'$  the representation in  $\text{Mon}(Q', A)$  obtained from  $Y$  by deleting the branch  $Y_1$ . Then by definition  $\text{rMon}(X')$  is exactly the representation in  $\text{Mon}(Q', A)$  obtained from  $\text{rMon}(X)$  by deleting the branch  $\text{rMon}(X)_1$ . Further,

$$\begin{pmatrix} f_2 \\ \vdots \\ f_n \end{pmatrix} : \text{rMon}(X') \rightarrow X' \quad \text{and} \quad \begin{pmatrix} g_2 \\ \vdots \\ g_n \end{pmatrix} : Y' \rightarrow X'$$

are morphisms in  $\text{Rep}(Q', A)$ . By the inductive hypothesis there is a morphism

$$\begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix} : Y' \rightarrow \text{rMon}(X')$$

in  $\text{Rep}(Q', A)$ , such that

$$\begin{pmatrix} g_2 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} f_2 \\ \vdots \\ f_n \end{pmatrix} \begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix}.$$

Let  $\alpha_1, \dots, \alpha_m$  be all the arrows ending at 1. Since

$$(Y_{\alpha_1}, \dots, Y_{\alpha_m}) : \bigoplus_{1 \leq j \leq m} Y_{s(\alpha_j)} \rightarrow Y_1$$

is an injective  $A$ -map and  $IK_1 \oplus (\bigoplus_{p \in \mathcal{P}(\rightarrow 1)} IK_{s(p)})$  is an injective  $A$ -module, it follows that there is a map

$$\eta : Y_1 \rightarrow IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow 1)} IK_{s(p)}$$

such that the diagram

$$\begin{array}{ccc} \bigoplus_{1 \leq j \leq m} Y_{s(\alpha_j)} & \xrightarrow{(Y_{\alpha_1}, \dots, Y_{\alpha_m})} & Y_1 \\ \tilde{h} \downarrow & & \downarrow \eta \\ \bigoplus_{1 \leq j \leq m} \text{rMon}(X)_{s(\alpha_j)} & \xrightarrow{(B_1, \dots, B_m)} & IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow 1)} IK_{s(p)} \end{array}$$

commutes, where  $\tilde{h} = \text{diag}(h_{s(\alpha_1)}, \dots, h_{s(\alpha_m)})$  and, for each  $j = 1, \dots, m$ ,

$$B_j : \text{rMon}(X)_{s(\alpha_j)} \rightarrow IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow 1)} IK_{s(p)}$$

is the  $A$ -map given by

$$x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(p)} \mapsto \varphi_{\alpha_j}(x_{s(\alpha_j)} + k_{s(\alpha_j)}) + \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(\alpha_j p)}$$

for

$$\begin{aligned} x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(p)} &\in \text{rMon}(X)_{s(\alpha_j)} \\ &= X_{s(\alpha_j)} \oplus IK_{s(\alpha_j)} \oplus \bigoplus_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} IK_{s(p)}. \end{aligned}$$

For  $y \in Y_{s(\alpha_j)}$ , suppose

$$h_{s(\alpha_j)}(y) = x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(p)} \in \text{rMon}(X)_{s(\alpha_j)}.$$

Then we have

$$\begin{aligned}
 \text{rMon}(X)_{\alpha_j} h_{s(\alpha_j)}(y) &= X_{\alpha_j}(x_{s(\alpha_j)}) + \varphi_{\alpha_j}(x_{s(\alpha_j)}) + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\rightarrow s(\alpha_j))} k_{s(\alpha_j p)} \\
 &= X_{\alpha_j}(x_{s(\alpha_j)}) + B_j h_{s(\alpha_j)}(y) \\
 &= X_{\alpha_j}(f_{s(\alpha_j)} h_{s(\alpha_j)}(y)) + B_j h_{s(\alpha_j)}(y) \\
 &= X_{\alpha_j} g_{s(\alpha_j)}(y) + B_j h_{s(\alpha_j)}(y) \\
 &= g_1 Y_{\alpha_j}(y) + \eta Y_{\alpha_j}(y),
 \end{aligned}$$

where the last equality uses the fact that  $g : Y \rightarrow X$  is a morphism in  $\text{Rep}(Q, A)$ .

Now we define  $h_1 : Y_1 \rightarrow \text{rMon}(X)_1$  to be the  $A$ -map given by

$$h_1(y) = g_1(y) + \eta(y)$$

for each  $y \in Y_1$ . From the computation above we have  $\text{rMon}(X)_{\alpha_j} h_{s(\alpha_j)} = h_1 Y_{\alpha_j}$  for  $j = 1, \dots, m$ . It follows that

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} : Y \rightarrow \text{rMon}(X)$$

is a morphism in  $\text{Rep}(Q, A)$ . Since  $f_1 : \text{rMon}(X)_1 \rightarrow X_1$  is the canonical projection, we have  $f_1 \eta = 0$  and  $f_1 g_1 = g_1$ , and hence  $f h = g$ . This completes the proof.  $\square$

**3C. Proof of Theorem 3.1.** By Corollary 2.5 and Proposition 3.3 we know that  $\text{Mon}(Q, A)$  is a resolving, contravariantly finite subcategory of  $\text{Rep}(Q, A)$ , and hence  $\text{Mon}(Q, A)$  is functorially finite in  $\text{Rep}(Q, A)$ ; see [Krause and Solberg 2003, Corollary 2.6(i)]. It follows that  $\text{Mon}(Q, A)$  has Auslander–Reiten sequences, by [Auslander and Smalø 1981, Theorem 2.4].  $\square$

#### 4. Gorenstein-projective modules over the upper triangular matrix algebras

**4A.** Let  $A$  and  $B$  be rings,  $M$  an  $A$ - $B$ -bimodule, and  $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$  the upper triangular matrix ring, where the addition and multiplication are given by the ones of matrices. We assume that  $\Lambda$  is an Artin algebra [Auslander et al. 1995, p. 72], and consider finitely generated  $\Lambda$ -modules. A  $\Lambda$ -module can be identified with a triple  $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$ , or simply  $\begin{pmatrix} X \\ Y \end{pmatrix}$  if  $\phi$  is clear, where  $X \in A\text{-mod}$ ,  $Y \in B\text{-mod}$ , and  $\phi : M \otimes_B Y \rightarrow X$  is an  $A$ -map. A  $\Lambda$ -map  $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \rightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix}_{\phi'}$  can be identified with a pair  $\begin{pmatrix} f \\ g \end{pmatrix}$ , where  $f \in \text{Hom}_A(X, X')$ ,  $g \in \text{Hom}_B(Y, Y')$  are such that the diagram

$$\begin{array}{ccc}
 M \otimes_B Y & \xrightarrow{\phi} & X \\
 \text{id} \otimes g \downarrow & & f \downarrow \\
 M \otimes_B Y' & \xrightarrow{\phi'} & X'
 \end{array}$$



commutes. A sequence of  $\Lambda$ -maps

$$0 \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \xrightarrow{\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \rightarrow 0$$

is exact if and only if

$$0 \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow 0$$

is an exact sequence of  $A$ -maps, and

$$0 \rightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \rightarrow 0$$

is an exact sequence of  $B$ -maps. The indecomposable projective  $\Lambda$ -modules are exactly

$$\begin{pmatrix} P \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}_{\text{id}},$$

where  $P$  runs over indecomposable projective  $A$ -modules and  $Q$  runs over indecomposable projective  $B$ -modules.

Note that an algebra  $\Lambda$  is of the form above if and only if there is an idempotent decomposition  $1 = e + f$  such that  $f\Lambda e = 0$ ; and in this case

$$\Lambda = \begin{pmatrix} e\Lambda e & e\Lambda f \\ 0 & f\Lambda f \end{pmatrix}.$$

**4B.** The following result describes the Gorenstein-projective  $\Lambda$ -modules, if  ${}_A M$  and  $M_B$  are projective modules.

**Theorem 4.1.** *Let*

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

*be an Artin algebra,  $M$  an  $A$ - $B$ -bimodule such that  ${}_A M$  and  $M_B$  are projective modules. Then*

$$\begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \mathcal{GP}(\Lambda)$$

*if and only if  $\phi : M \otimes_B Y \rightarrow X$  is injective,  $\text{Coker } \phi \in \mathcal{GP}(A)$ , and  $Y \in \mathcal{GP}(B)$ . In this case,  $X \in \mathcal{GP}(A)$  if and only if  $M \otimes_B Y \in \mathcal{GP}(A)$ .*

Note that here  $\Lambda$  is not assumed to be Gorenstein: this will be important to the main result in the next section. The same result under the assumption that  $\Lambda$  is Gorenstein can be found in [Xiong and Zhang 2012, Corollary 3.3] (however, the proof there cannot be generalized to the non-Gorenstein case). The same corollary implies that, if  $\Lambda$  is Gorenstein in Theorem 4.1,  $\begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \mathcal{GP}(\Lambda)$  implies  $X \in \mathcal{GP}(A)$ .

*Proof of Theorem 4.1.* The last assertion is easy: it follows from the exact sequence

$$0 \longrightarrow M \otimes_B Y \xrightarrow{\phi} X \longrightarrow \text{Coker } \phi \longrightarrow 0$$

and the fact that  $\mathcal{GP}(A)$  is closed under extensions and the kernels of epimorphisms; see, for example, [Holm 2004].

We next prove the “if” part of the first equivalence in the theorem. We assume that  $\phi : M \otimes_B Y \rightarrow X$  is injective,  $\text{Coker } \phi \in \mathcal{GP}(A)$ , and  $Y \in \mathcal{GP}(B)$ . Then we have a complete  $B$ -projective resolution

$$(4-1) \quad Q^\bullet = \dots \longrightarrow Q^{-1} \longrightarrow Q^0 \xrightarrow{d'^0} Q^1 \longrightarrow \dots$$

with  $Y = \text{Ker } d'^0$ , and a complete  $A$ -projective resolution

$$(4-2) \quad P^\bullet = \dots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots$$

with  $\text{Coker } \phi = \text{Ker } d^0$ . Since  $M_B$  is projective, we get the following exact sequences of  $A$ -modules:

$$\begin{aligned} 0 \rightarrow M \otimes_B Y \rightarrow M \otimes_B Q^0 \rightarrow M \otimes_B Q^1 \rightarrow \dots, \\ 0 \rightarrow \text{Coker } \phi \rightarrow P^0 \rightarrow P^1 \rightarrow \dots. \end{aligned}$$

Since  ${}_A M$  is projective,  $M \otimes_B Q^i$  is a projective  $A$ -module for each  $i \geq 0$ . Since  $\text{Ext}_A^1(\text{Coker } \phi, M \otimes_B Q^0) = 0$ , it follows from the exact sequence

$$0 \rightarrow M \otimes_B Y \xrightarrow{\phi} X \rightarrow \text{Coker } \phi \rightarrow 0$$

that the map  $M \otimes_B Y \rightarrow M \otimes_B Q^0$  factors through  $\phi$ . So, by a version of the horseshoe lemma, we see that there is an exact sequence of  $A$ -modules

$$(4-3) \quad 0 \rightarrow X \rightarrow P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \rightarrow \dots$$

with

$$\partial^i = \begin{pmatrix} d^i & 0 \\ \sigma^i & \text{id} \otimes_B d'^i \end{pmatrix}, \quad \sigma^i : P^i \rightarrow M \otimes_B Q^i$$

for all  $i \in \mathbb{Z}$ , such that the diagram

$$(4-4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \xrightarrow{\text{id} \otimes_B d'^0} & M \otimes_B Q^1 & \longrightarrow & \dots \\ & & \downarrow \phi & & \downarrow \binom{0}{\text{id}} & & \downarrow \binom{0}{\text{id}} & & \\ 0 & \longrightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \xrightarrow{\partial^0} & P^1 \oplus (M \otimes_B Q^1) & \longrightarrow & \dots \end{array}$$

commutes. By the same argument we get the following commutative diagram with exact rows:

$$(4-5) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & M \otimes_B Q^{-2} & \xrightarrow{\text{id} \otimes_B d'^{-2}} & M \otimes_B Q^{-1} & \longrightarrow & M \otimes_B Y \longrightarrow 0 \\ & & \downarrow \binom{0}{\text{id}} & & \downarrow \binom{0}{\text{id}} & & \downarrow \phi \\ \cdots & \longrightarrow & P^{-2} \oplus (M \otimes_B Q^{-2}) & \xrightarrow{\partial^{-2}} & P^{-1} \oplus (M \otimes_B Q^{-1}) & \longrightarrow & X \longrightarrow 0. \end{array}$$

Putting (4-4) and (4-5) together, we get the exact sequence of projective  $\Lambda$ -modules

$$(4-6) \quad L^\bullet = \cdots \longrightarrow \begin{pmatrix} P^{-1} \oplus (M \otimes_B Q^{-1}) \\ Q^{-1} \end{pmatrix} \xrightarrow{\binom{\partial^0}{d'^0}} \begin{pmatrix} P^0 \oplus (M \otimes_B Q^0) \\ Q^0 \end{pmatrix} \xrightarrow{\binom{0}{\text{id}}} \begin{pmatrix} P^1 \oplus (M \otimes_B Q^1) \\ Q^1 \end{pmatrix} \longrightarrow \cdots$$

with  $\text{Ker} \begin{pmatrix} \partial^0 \\ d'^0 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}_\phi$ .

For each projective  $A$ -module  $P$ ,  $\text{Hom}_\Lambda(L^\bullet, \begin{pmatrix} P \\ 0 \end{pmatrix}) \cong \text{Hom}_A(P^\bullet, P)$  is exact, since  $P^\bullet$  is a complete projective resolution. For each projective  $B$ -module  $Q$ , since  $Q^\bullet$  is a complete projective resolution,  $\text{Hom}_B(Q^\bullet, Q)$  is exact. Since  $M \otimes_B Q$  is projective,  $\text{Hom}_A(P^\bullet, M \otimes_B Q)$  is exact. Note that

$$\text{Hom}_\Lambda \left( L^\bullet, \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix} \right) \cong \text{Hom}_A(P^\bullet, M \otimes_B Q) \oplus \text{Hom}_B(Q^\bullet, Q);$$

here the direct sum only means that each term of the complex at the left side is a direct sum of terms of complexes at the right side, that is, it does not mean a direct sum of complexes; in fact, the complex at the right side has differentials

$$\begin{pmatrix} \text{Hom}_A(d^i, M \otimes_B Q) & \text{Hom}_A(\sigma^i, M \otimes_B Q) \\ 0 & \text{Hom}_B(d'^i, Q) \end{pmatrix}.$$

By the canonical exact sequence of complexes

$$0 \rightarrow \text{Hom}_A(P^\bullet, M \otimes_B Q) \xrightarrow{\binom{\text{id}}{0}} \text{Hom}_\Lambda \left( L^\bullet, \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix} \right) \xrightarrow{(0 \text{ id})} \text{Hom}_B(Q^\bullet, Q) \rightarrow 0,$$

we know that

$$\text{Hom}_\Lambda \left( L^\bullet, \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix} \right)$$

is also exact. We conclude that  $L^\bullet$  is a complete  $\Lambda$ -projective resolution, and hence  $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$  is a Gorenstein-projective  $\Lambda$ -module.

Conversely, assume that  $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \in \mathcal{GP}(\Lambda)$ . Then there is a complete  $\Lambda$ -projective resolution (4-6) with

$$\text{Ker} \begin{pmatrix} \partial^0 \\ d'^0 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}_\phi.$$

Then we get an exact sequence (4-1) of projective  $B$ -modules with  $\text{Ker } d^0 = Y$ , and the exact sequence

$$(4-7) \quad V^\bullet = \dots \rightarrow P^{-1} \oplus (M \otimes_B Q^{-1}) \rightarrow P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \rightarrow \dots$$

of projective  $A$ -modules with  $\text{Ker } \partial^0 = X$ . Since  $M_B$  is projective, it follows that  $M \otimes_B Q^\bullet$  is exact. Since  $\begin{pmatrix} \partial^i \\ d^i \end{pmatrix}$  is a  $\Lambda$ -map, by (4-6) we know that  $\partial^i$  is of the form

$$\partial^i = \begin{pmatrix} d^i & 0 \\ \sigma^i & \text{id} \otimes_B d^i \end{pmatrix},$$

where  $\sigma^i : P^i \rightarrow M \otimes_B Q^i$  for all  $i \in \mathbb{Z}$ , and

$$P^\bullet = \dots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \dots$$

is a complex. By the canonical exact sequence of complexes

$$0 \rightarrow M \otimes_B Q^\bullet \xrightarrow{\begin{pmatrix} \text{id} \\ 0 \end{pmatrix}} V^\bullet \xrightarrow{(0 \text{ id})} \text{Hom}_B(Q^\bullet, Q)P^\bullet \rightarrow 0,$$

we see that  $P^\bullet$  is also exact.

From (4-6) we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \longrightarrow & M \otimes_B Q^1 \longrightarrow \dots \\
 & & \downarrow \phi & & \downarrow \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} \\
 0 & \rightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \longrightarrow & P^1 \oplus (M \otimes_B Q_1) \longrightarrow \dots \\
 & & \downarrow & & \downarrow (\text{id}, 0) & & \downarrow (\text{id}, 0) \\
 0 & \rightarrow & \text{Coker } \phi & \longrightarrow & P^0 & \longrightarrow & P^1 \longrightarrow \dots \\
 & & \downarrow & & \downarrow d^0 & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Thus  $\phi : M \otimes_B Y \rightarrow X$  is injective and  $\text{Ker } d^0 \cong \text{Coker } \phi$ . For each projective  $A$ -module  $P$ , since

$$\text{Hom}_\Lambda(L^\bullet, \begin{pmatrix} P \\ 0 \end{pmatrix}) \cong \text{Hom}_A(P^\bullet, P)$$

and  $L^\bullet$  is a complete projective resolution, it follows that  $P^\bullet$  is a complete projective resolution, and hence  $\text{Coker } \phi$  is a Gorenstein-projective  $A$ -module.

For each projective  $B$ -module  $Q$ , since  $P^\bullet$  is a complete projective resolution, it follows that  $\text{Hom}_A(P^\bullet, M \otimes_B Q)$  is exact. Since  $L^\bullet$  is a complete projective resolution, it follows that

$$\text{Hom}_\Lambda \left( L^\bullet, \left( \begin{smallmatrix} M \otimes_B Q \\ Q \end{smallmatrix} \right) \right) \cong \text{Hom}_A(P^\bullet, M \otimes_B Q) \oplus \text{Hom}_B(Q^\bullet, Q)$$

is exact (again, the direct sum does not mean a direct sum of complexes). By the same argument we know that  $\text{Hom}_B(Q^\bullet, Q)$  is exact. It follows that  $Y$  is a Gorenstein-projective  $B$ -module.  $\square$

### 5. Main result

**5A.** The aim of this section is to prove the following characterization of Gorenstein-projective  $\Lambda$ -modules, where  $\Lambda$  is the path algebra of a finite acyclic quiver over a finite-dimensional algebra. We emphasize that here  $\Lambda$  is not assumed to be Gorenstein.

**Theorem 5.1.** *Let  $Q$  be a finite acyclic quiver, and  $A$  a finite-dimensional algebra over a field  $k$ . Let  $\Lambda = A \otimes_k kQ$ , and  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  be a  $\Lambda$ -module. Then  $X \in \mathcal{GP}(\Lambda)$  if and only if  $X \in \text{Mon}(Q, A)$  and  $X$  satisfies this condition:*

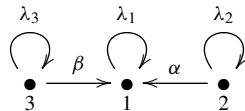
(G) for each  $i \in Q_0$ ,  $X_i$  and the quotient  $X_i / \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha$  lie in  $\mathcal{GP}(A)$ .

**Example 5.2.** (i) Taking

$$Q = \bullet_n \rightarrow \cdots \rightarrow \bullet_1$$

in Theorem 5.1, we get that a  $T_n(A)$ -module  $X = (X_i, \phi_i)$  is Gorenstein-projective if and only if each  $\phi_i$  is injective and that each  $X_i$  is a Gorenstein-projective  $A$ -module and each  $\text{Coker } \phi_i$  is a Gorenstein-projective  $A$ -module. Under the assumption that  $A$  is Gorenstein, this result has been obtained in [Zhang 2011, Corollary 4.1]; the case for  $n = 2$  was treated in [Li and Zhang 2010, Theorem 1.1(i)]; see also [Iyama et al. 2011, Proposition 3.6(i)].

(ii) Let  $\Lambda$  be the  $k$ -algebra given by quiver



with relations  $\lambda_1^2, \lambda_2^2, \lambda_3^2, \alpha\lambda_2 - \lambda_1\alpha, \beta\lambda_3 - \lambda_1\beta$ . Then

$$\Lambda = A \otimes_k kQ = \begin{pmatrix} A & A & A \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix},$$

where  $Q$  is the quiver

$$\bullet \longrightarrow \bullet \longleftarrow \bullet, \\ 3 \qquad \qquad 1 \qquad \qquad 2$$

and  $A = k[x]/\langle x^2 \rangle$ . Let  $k$  be the simple  $A$ -module, and  $\sigma : k \hookrightarrow A$  the inclusion. By Theorem 5.1, the following  $\Lambda$ -modules lie in  $\text{GP}(\Lambda)$ :

$$\begin{aligned} &(X_1 = A, X_2 = 0, X_3 = 0, X_\alpha = 0 = X_\beta), \\ &(X_1 = A, X_2 = A, X_3 = 0, X_\alpha = \text{id}, X_\beta = 0), \\ &(X_1 = A, X_2 = 0, X_3 = A, X_\alpha = 0, X_\beta = \text{id}), \\ &(X_1 = k, X_2 = 0, X_3 = 0, X_\alpha = 0 = X_\beta), \\ &(X_1 = k, X_2 = k, X_3 = 0, X_\alpha = \text{id}, X_\beta = 0), \\ &(X_1 = k, X_2 = 0, X_3 = k, X_\alpha = 0, X_\beta = \text{id}), \\ &(X_1 = A, X_2 = k, X_3 = 0, X_\alpha = \sigma, X_\beta = 0), \\ &(X_1 = A, X_2 = 0, X_3 = k, X_\alpha = 0, X_\beta = \sigma), \\ &(X_1 = A \oplus k, X_2 = k, X_3 = k, X_\alpha = \begin{pmatrix} 0 \\ \text{id} \end{pmatrix}, X_\beta = \begin{pmatrix} \sigma \\ \text{id} \end{pmatrix}). \end{aligned}$$

In fact this is the complete list of the pairwise nonisomorphic indecomposable Gorenstein-projective  $\Lambda$ -modules. Also by Theorem 5.1,

$$(Y_1 = A, Y_2 = k, Y_3 = k, Y_\alpha = \sigma = Y_\beta) \notin \mathcal{GP}(\Lambda).$$

For a description of all the pairwise nonisomorphic indecomposable Gorenstein-projective  $\Lambda$ -modules see [Ringel and Zhang 2011], where  $\Lambda$  is the path algebra of an arbitrary acyclic quiver over  $A = k[x]/\langle x^2 \rangle$ .

**5B.** We prove Theorem 5.1 by using Theorem 4.1 and induction on  $|Q_0|$ .

Remember we label  $Q_0$  as  $1, \dots, n$ , in such a way that  $j > i$  if  $\alpha : j \rightarrow i$  is in  $Q_1$ . Thus  $n$  is a source of  $Q$ . Denote by  $Q'$  the quiver obtained from  $Q$  by deleting vertex  $n$ , and  $\Lambda' = A \otimes_k kQ'$ . Let  $P(n)$  be the indecomposable projective (left)  $kQ$ -module at vertex  $n$ . Put  $P = A \otimes_k \text{rad}P(n)$ . Clearly  $P$  is a  $\Lambda'$ - $A$ -bimodule and  $\Lambda = \begin{pmatrix} \Lambda' & P \\ 0 & A \end{pmatrix}$ ; compare (2-2).

Since  $kQ$  is hereditary,  $\text{rad}P(n)$  is a projective  $kQ'$ -module, and hence  $P = A \otimes_k \text{rad}P(n)$  is a (left) projective  $\Lambda'$ -module, and a (right) projective  $A$ -module (since as a right  $A$ -module,  $P$  is a direct sum of copies of  $A_A$ ). So we can apply Theorem 4.1. For this, we write a  $\Lambda$ -module  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  as  $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$ , where  $X' = (X_i, X_\alpha, i \in Q'_0, \alpha \in Q'_1)$  is a  $\Lambda'$ -module, and

$$\phi : P \otimes_A X_n \rightarrow X'$$

is a  $\Lambda'$ -map. The explicit expression of  $\phi$  is given in the proof of Lemma 5.4. We keep all these notations of  $Q', \Lambda', P(n), P, X'$  and  $\phi$  throughout this section.

**5C.** By a direct translation from Theorem 4.1 in this special case, we have:

**Lemma 5.3.** *Let  $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$  be a  $\Lambda$ -module. Then  $X \in \mathcal{GP}(\Lambda)$  if and only if  $X$  satisfies the following conditions:*

- (i)  $X_n \in \mathcal{GP}(A)$ .
- (ii)  $\phi : P \otimes_A X_n \rightarrow X'$  is injective.
- (iii)  $\text{Coker } \phi \in \mathcal{GP}(\Lambda')$ .

For each  $i \in Q'_0$ , put  $\mathcal{A}(n \rightarrow i)$  to be the set of arrows from  $n$  to  $i$ ; and  $\mathcal{P}(n \rightarrow i)$  the set of paths from  $n$  to  $i$ . For an integer  $m \geq 0$  and a module  $M$ , let  $M^m$  denote the direct sum of  $m$  copies of  $M$ .

**Lemma 5.4.** *Let  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  be a  $\Lambda$ -module. If  $X_\beta$  is injective for each  $\beta \in Q'_1$ ,  $\phi : P \otimes_A X_n \rightarrow X'$  is injective if and only if  $X_\alpha$  is injective for all  $\alpha \in Q_1$ , and  $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$  for all  $Q'_0$ .*

*Proof.* For  $i \in Q'_0$ , set  $m_i = |\mathcal{P}(n \rightarrow i)|$ . As a  $kQ'$ -module,  $\text{rad}P(n)$  can be written as

$$\begin{pmatrix} k^{m_1} \\ \vdots \\ k^{m_{n-1}} \end{pmatrix}$$

(see (2-1) and Section 5B), hence we have isomorphisms of  $\Lambda'$ -modules

$$P \otimes_A X_n \cong (\text{rad}P(n) \otimes_k A) \otimes_A X_n \cong \text{rad}P(n) \otimes_k X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix}.$$

Let  $\mathcal{P}(n \rightarrow i) = \{p_1, \dots, p_{m_i}\}$ . Then  $\phi$  is of the form

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n-1} \end{pmatrix} : P \otimes_A X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix},$$

where  $\phi_i = (X_{p_1}, \dots, X_{p_{m_i}}) : X_n^{m_i} \rightarrow X_i$  (for the meaning of  $X_p$  see Section 2C). So  $\phi$  is injective if and only if  $\phi_i$  is injective for each  $i \in Q'_0$ , and if and only if

$$\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \quad \text{and} \quad X_p \text{ is injective for all } p \in \mathcal{P}(n \rightarrow i).$$

From this and the assumption the assertion follows. □

**Lemma 5.5.** *Let  $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$  be a monic  $\Lambda$ -module.*

- (1) *For each  $i \in Q'_0$  there holds  $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$ .*
- (2)  *$\phi : P \otimes_A X_n \rightarrow X'$  is injective.*

(3)  $\text{Coker } \phi = (X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p, \tilde{X}_\alpha, i \in Q'_0, \alpha \in Q'_1)$ , where, for each  $\alpha : j \rightarrow i$  in  $Q'_1$ ,

$$\tilde{X}_\alpha : X_j / \bigoplus_{q \in \mathcal{P}(n \rightarrow j)} \text{Im } X_q \rightarrow X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$$

is the  $A$ -map induced by  $X_\alpha$ .

*Proof.* By Lemma 5.4 and its proof, it suffices to prove (1). For each  $i \in Q'_0$ , set  $l_i = 0$  if  $\mathcal{P}(n \rightarrow i)$  is empty, and  $l_i = \max\{l(p) \mid p \in \mathcal{P}(n \rightarrow i)\}$  otherwise, where  $l(p)$  is the length of  $p$ . We use induction on  $l_i$ . If  $l_i = 0$ , (1) trivially holds. Suppose  $l_i \geq 1$ . Let  $\sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) = 0$  for  $x_{n,p} \in X_n$ . Since

$$\sum_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p = \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha \left( \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q \right),$$

we have

$$\begin{aligned} 0 &= \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} X_p(x_{n,p}) \\ &= \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{\substack{\beta \in Q'_1 \\ e(\beta)=i}} X_\beta \left( \sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) \right). \end{aligned}$$

By (m2) in Definition 2.2 we know that  $X_\alpha(x_{n,\alpha}) = 0$  for  $\alpha \in \mathcal{A}(n \rightarrow i)$ , and

$$X_\beta \left( \sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) \right) = 0$$

for  $\beta \in Q'_1$  with  $e(\beta) = i$ . So  $\sum_{q \in \mathcal{P}(n \rightarrow s(\beta))} X_q(x_{n,\beta q}) = 0$  by condition (m1) in Definition 2.2. Since  $l_{s(\beta)} < l_i$  for each  $\beta \in Q'_1$  with  $e(\beta) = i$ , it follows from the inductive hypothesis that  $X_q(x_{n,\beta q}) = 0$  for  $\beta \in Q'_1$ ,  $e(\beta) = i$ , and  $q \in \mathcal{P}(n \rightarrow s(\beta))$ . This proves (1) and the lemma.  $\square$

**Lemma 5.6.** Let  $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$  be a monic  $\Lambda$ -module. Then  $\text{Coker } \phi$  is a monic  $\Lambda'$ -module.

*Proof.* We need to prove that, for each  $i \in Q'_0$ , the  $\Lambda'$ -map

$$(\tilde{X}_\alpha)_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \left( X_{s(\alpha)} / \bigoplus_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q \right) \rightarrow X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$$

is injective. For this, assume that

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \tilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) = 0,$$



where  $\overline{x_{s(\alpha),\alpha}}$  is the image of  $x_{s(\alpha),\alpha} \in X_{s(\alpha)}$  in  $X_{s(\alpha)}/\bigoplus_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q$ . Then

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) \in \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p.$$

So there are  $x_{n,p} \in X_n$  such that

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}).$$

Thus

$$\begin{aligned} 0 &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{p \in \mathcal{P}(n \rightarrow i)} X_p(x_{n,p}) \\ &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{\beta \in \mathcal{A}(n \rightarrow i)} X_\beta(x_{n,\beta}) - \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha \left( \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q}) \right) \\ &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q}) - \sum_{\beta \in \mathcal{A}(n \rightarrow i)} X_\beta(x_{n,\beta}). \end{aligned}$$

Using the assumption on  $X$ , we get

$$x_{s(\alpha),\alpha} = \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,\alpha q}),$$

that is,  $\overline{x_{s(\alpha),\alpha}} = 0$ . □

**Lemma 5.7.** *Let  $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$  be a monic  $\Lambda$ -module satisfying (G). Then*

$$\left( X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) / \left( \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha \right)$$

*is a Gorenstein-projective  $A$ -module for all  $i \in Q'_0$ .*

*Proof.* Since

$$\bigoplus_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p \subseteq \sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta,$$

it follows that

$$(5-1) \quad \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha = \left( \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } X_\alpha + \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) / \left( \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)$$

$$\begin{aligned}
 &= \left( \sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta + \bigoplus_{p \in \mathcal{P}(n \rightarrow i) - \mathcal{A}(n \rightarrow i)} \text{Im } X_p \right) / \left( \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
 &= \left( \sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right) / \left( \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
 &= \left( \bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta \right) / \left( \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right)
 \end{aligned}$$

(the last equality following by (m2) in Definition 2.2). Hence the desired quotient is  $X_i / \bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta$ , which is Gorenstein-projective by (G).  $\square$

**Lemma 5.8.** *Let  $X = \left( \begin{smallmatrix} X' \\ X_n \end{smallmatrix} \right)_\phi$  be a monic  $\Lambda$ -module satisfying (G). Then*

$$X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow j)} \text{Im } X_p$$

*is a Gorenstein-projective  $A$ -module for each  $i \in Q'_0$ .*

*Proof.* We prove the assertion by using induction on  $l_i$ , which is defined in the proof of Lemma 5.5. If  $i \in Q'_0$  with  $l_i = 0$ , the assertion follows from (G).

Suppose  $l_i \geq 1$ . Since  $\bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \subseteq \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha$ , we have the exact sequence

$$\begin{aligned}
 0 \longrightarrow \left( \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha \right) / \left( \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) \\
 \longrightarrow X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \longrightarrow X_i / \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \text{Im } X_\alpha \longrightarrow 0,
 \end{aligned}$$

and by (G) the last term on the second line is Gorenstein-projective. It suffices to prove that the term on the first line is Gorenstein-projective. By (5-1) this term is  $\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha$ . By Lemma 5.6 each  $\tilde{X}_\alpha$  is injective, and it follows that

$$\text{Im } \tilde{X}_\alpha \cong X_j / \bigoplus_{p \in \mathcal{P}(n \rightarrow j)} \text{Im } X_p,$$

where  $j = s(\alpha)$ . Since  $l_j < l_i$ , the conclusion of the lemma follows from the inductive hypothesis.  $\square$

**Lemma 5.9.** *The sufficiency in Theorem 5.1 holds. That is, if*

$$X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$$

*is a monic  $\Lambda$ -module satisfying (G),  $X$  is Gorenstein-projective.*

*Proof.* Using induction on  $n = |Q_0|$ , the assertion clearly holds for  $n = 1$ . Suppose that the assertion holds for  $n - 1$  with  $n \geq 2$ . It suffices to prove that  $X$  satisfies Lemma 5.3(i)–(iii).

Condition (i) is contained in (G); and condition (ii) follows from Lemma 5.5(2). By Lemma 5.6  $\text{Coker } \phi$  is a monic  $\Lambda'$ -module; and by Lemmas 5.7 and 5.8 we know that  $\text{Coker } \phi$  satisfies (G). It follows from the inductive hypothesis that condition (iii) is satisfied.  $\square$

**Lemma 5.10.** *Let  $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$  be a  $\Lambda$ -module with  $X_n$  a Gorenstein-projective  $A$ -module. Then  $P \otimes_A X_n$  is a Gorenstein-projective  $\Lambda'$ -module, where  $P$  is defined in Section 5B.*

*Proof.* Let  $P(n)$  be the indecomposable projective  $kQ$ -module at vertex  $n$ . Writing  $\text{rad}P(n)$  as a representation of  $Q'$  over  $k$ , we have

$$\text{rad}P(n) = (k^{m_i}, f_\alpha, i \in Q'_0, \alpha \in Q'_1),$$

where  $m_i = |\mathcal{P}(n \rightarrow i)|$  for each  $i \in Q'_0$ . By the construction of  $P(n)$  we know that  $\text{rad}P(n)$  has the following three properties:

- (1) Each  $f_\alpha : k^{m_{e(\alpha)}} \rightarrow k^{m_{\alpha}}$  is injective.
- (2) For each  $i \in Q'_0$ ,

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha.$$

- (3) For each  $i \in Q'_0$ ,  $k^{m_i} / (\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } f_\alpha)$  and  $k^{|\mathcal{A}(n \rightarrow i)|}$  are isomorphic as  $k$ -spaces.

It follows that

$$\begin{aligned} P \otimes_A X_n &\cong (\text{rad}P(n) \otimes_k A) \otimes_A X_n \\ &\cong \text{rad}P(n) \otimes_k X_n = (X_n^{m_i}, f_\alpha \otimes_k \text{id}_{X_n}, i \in Q'_0, \alpha \in Q'_1). \end{aligned}$$

By (1), (2), and (3) we clearly see that  $P \otimes_A X_n$  is a monic  $\Lambda'$ -module satisfying (G); for example, by (3) we know that

$$X_n^{m_i} / \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im}(f_\alpha \otimes_k \text{id}_{X_n}) \cong X_n^{|\mathcal{A}(n \rightarrow i)|}$$

is a Gorenstein-projective  $A$ -module. Now the result follows from Lemma 5.9.  $\square$

**5D. Proof of Theorem 5.1.** By Lemma 5.9 it remains to prove necessity, namely, if  $X$  is a Gorenstein-projective  $\Lambda$ -module,  $X$  is a monic  $\Lambda$ -module satisfying (G). Using induction on  $n = |Q_0|$ , the assertion is clear for  $n = 1$ . Suppose that the assertion holds for  $n - 1$  with  $n \geq 2$ . We write  $X$  as  $\begin{pmatrix} X' \\ X_n \end{pmatrix}_\phi$ . Then  $X$  satisfies conclusions (i)–(iii) of Lemma 5.3.

By (i) and Lemma 5.10 we know that  $P \otimes_A X_n$  is a Gorenstein-projective  $\Lambda'$ -module. Then, by (ii) and (iii), we know that  $X' \in \mathcal{GP}(\Lambda')$ , since  $\mathcal{GP}(\Lambda')$  is closed under extensions. By the inductive hypothesis  $X'$  is a monic  $\Lambda'$ -module satisfying (G). Hence:

- (1)  $X_\beta$  is injective for each  $\beta \in Q'_1$ .
- (2)  $X_i$  is Gorenstein-projective for each  $i \in Q'_0$ .
- (3)  $X_\alpha$  is injective for each  $\alpha \in Q_1$ .
- (4)  $\sum_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p$  for all  $i \in Q'_0$ .

We get (3) and (4) from (1), condition (ii), and Lemma 5.4.

Since  $\text{Coker } \phi = (X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p, \tilde{X}_\alpha, i \in Q'_0, \alpha \in Q'_1)$  is a Gorenstein-projective  $\Lambda'$ -module, it follows from the inductive hypothesis that the following properties hold:

- (5) For each  $\alpha \in Q'_1$ ,  $\tilde{X}_\alpha$  is injective.
- (6)  $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha$ , for all  $i \in Q'_0$ .

*Claim 1:  $X$  satisfies (m2) in Definition 2.2.*

Indeed, suppose

$$(5-2) \quad \sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{S(\alpha), \alpha}) = 0.$$

Since

$$\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{S(\alpha), \alpha}) = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{S(\alpha), \alpha}) + \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{S(\alpha), \alpha}),$$

it follows that

$$\begin{aligned} \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \tilde{X}_\alpha(\overline{x_{S(\alpha), \alpha}}) &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{S(\alpha), \alpha}) + \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \\ &= - \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{S(\alpha), \alpha}) + \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p = 0, \end{aligned}$$

where we used (5-2) for the second equality.

Then by (6) we have  $\tilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) = 0$ ; and by (5) we know  $\overline{x_{s(\alpha),\alpha}} = 0$  for each  $\alpha \in Q'_1$  with  $e(\alpha) = i$ . This means that there are  $x_{n,q} \in X_n$  such that

$$x_{s(\alpha),\alpha} = \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,q}) \in \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} \text{Im } X_q$$

for each  $\alpha \in Q'_1$  with  $e(\alpha) = i$ . By (5-2) we have

$$0 = \sum_{\alpha \in \mathcal{A}(n \rightarrow i)} X_\alpha(x_{n,\alpha}) + \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha \left( \sum_{q \in \mathcal{P}(n \rightarrow s(\alpha))} X_q(x_{n,q}) \right).$$

By (4) we know that  $X_\alpha(x_{n,\alpha}) = 0$  for all  $\alpha \in \mathcal{A}(n \rightarrow i)$ , and that  $X_\alpha X_q(x_{n,q}) = 0$  for all  $\alpha \in Q'_1$  with  $e(\alpha) = i$  and  $q \in \mathcal{P}(n \rightarrow s(\alpha))$ . Thus  $X_\alpha(x_{s(\alpha),\alpha}) = 0$ , for all  $\alpha \in Q_1$  with  $e(\alpha) = i$ . This proves Claim 1.

*Claim 2:*  $X_i / \bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \text{Im } X_\beta$  is a Gorenstein-projective  $A$ -module for each  $i \in Q_0$ .

Indeed, since  $\text{Coker } \phi$  is a Gorenstein-projective  $\Lambda'$ -module, by the inductive hypothesis we know that

$$\left( X_i / \bigoplus_{p \in \mathcal{P}(n \rightarrow i)} \text{Im } X_p \right) / \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \text{Im } \tilde{X}_\alpha$$

is a Gorenstein-projective  $A$ -module: it is exactly the desired module by (5-1).

Now, (3) and Claim 1 mean that  $X$  is a monic  $\Lambda$ -module; and (2), together with conclusion (i) of Lemma 5.3 and Claim 2, means that  $X$  satisfies (G).  $\square$

## 6. Corollaries

**6A.** For the definition of a Frobenius category in the sense of [Quillen 1973], we refer to [Happel 1988, p. 11; Keller 1990, Appendix A]. As a consequence of Theorem 5.1 and Proposition 2.4, we get the following characterization of self-injectivity.

**Corollary 6.1.** *Let  $A$  be a finite-dimensional algebra, and  $Q$  a finite acyclic quiver. Then the following are equivalent:*

- (i)  $A$  is self-injective.
- (ii)  $\mathcal{GP}(A \otimes_k kQ) = \text{Mon}(Q, A)$ .
- (iii)  $\text{Mon}(Q, A)$  is a Frobenius category.

*Proof.* (i)  $\implies$  (ii): If  $A$  is self-injective, every  $A$ -module is Gorenstein-projective, and hence (ii) follows from Theorem 5.1. The implication (ii)  $\implies$  (iii) is well-known.

(iii)  $\implies$  (i): Take a sink of  $Q$ , say vertex 1, and consider  $D(A_A) \otimes_k P(1)$ . By Proposition 2.4 (iii) it is an injective object in  $\text{Mon}(Q, A)$ , and hence, by assumption, it is a projective object in  $\text{Mon}(Q, A)$ . By Proposition 2.4(ii) we know that  $D(A_A)$ , the first branch of  $D(A_A) \otimes_k P(1)$ , is a projective  $A$ -module, that is,  $A$  is self-injective.  $\square$

Let  $D^b(\Lambda)$  be the bounded derived category of  $\Lambda$ , and  $K^b(\mathcal{P}(\Lambda))$  the bounded homotopy category of  $\mathcal{P}(\Lambda)$ . By definition the singularity category  $D_{sg}^b(\Lambda)$  of  $\Lambda$  is the Verdier quotient  $D^b(\Lambda)/K^b(\mathcal{P}(\Lambda))$ . Buchweitz [1987, Theorem 4.4.1] proved that if  $\Lambda$  is Gorenstein, there is a triangle-equivalence  $D_{sg}^b(\Lambda) \cong \underline{\mathcal{GP}}(\Lambda)$ , where  $\underline{\mathcal{GP}}(\Lambda)$  is the stable category of  $\mathcal{GP}(\Lambda)$  modulo  $\mathcal{P}(\Lambda)$ ; see also [Happel 1991, Theorem 4.6]. Note that if  $A$  is Gorenstein,  $\Lambda = A \otimes_k kQ$  is Gorenstein; see [Auslander and Reiten 1991b, Proposition 2.2]. So we have the following.

**Corollary 6.2.** *Let  $A$  be a finite-dimensional Gorenstein algebra, and  $Q$  a finite acyclic quiver. Let  $\Lambda = A \otimes_k kQ$ . Then there is a triangle-equivalence  $D_{sg}^b(\Lambda) \cong \underline{\mathcal{GP}}(\Lambda)$ . In particular, if  $A$  is self-injective, then there is a triangle-equivalence  $D_{sg}^b(\Lambda) \cong \underline{\text{Mon}}(Q, A)$ .*

**6B.** Recall the tensor product  $Q \otimes Q'$  of two finite quivers  $Q$  and  $Q'$  (not necessarily acyclic). By definition  $Q \otimes Q'$  is the quiver with

$$(Q \otimes Q')_0 = Q_0 \times Q'_0 \quad \text{and} \quad (Q \otimes Q')_1 = (Q_1 \times Q'_1) \cup (Q_0 \times Q'_1).$$

More explicitly, if  $\alpha : i \rightarrow j$  is an arrow of  $Q$ , then, for each vertex  $t' \in Q'_0$ , there is an arrow  $(\alpha, t') : (i, t') \rightarrow (j, t')$  of  $Q \otimes Q'$ ; and if  $\beta' : s' \rightarrow t'$  is an arrow of  $Q'$ , then, for each vertex  $i \in Q_0$ , there is an arrow  $(i, \beta') : (i, s') \rightarrow (i, t')$  of  $Q \otimes Q'$ .

Let  $A = kQ/I$  and  $B = kQ'/I'$  be two finite-dimensional  $k$ -algebras, where  $Q$  and  $Q'$  are finite quivers (not necessarily acyclic), and  $I, I'$  are admissible ideals of  $kQ, kQ'$ , respectively. Then

$$A \otimes_k B \cong k(Q \otimes Q')/I \square I',$$

where  $I \square I'$  is the ideal of  $k(Q \otimes Q')$  generated by  $(I \times Q'_1) \cup (Q_0 \times I')$  and the elements

$$(\alpha, t')(i, \beta') - (j, \beta')(\alpha, s'),$$

where  $\alpha : i \rightarrow j$  is an arrow of  $Q$ , and  $\beta' : s' \rightarrow t'$  is an arrow of  $Q'$ . See, for example, [Leszczyński 1994]. Note that  $I \square I'$  may not be zero even if  $I = 0 = I'$ . We have proved this:

**Fact.**  *$A \otimes_k B$  is hereditary (that is,  $I \square I' = 0$ ) if and only if either  $A \cong k^{|Q_0|}$  as algebras and  $I' = 0$ , or  $B \cong k^{|Q'_0|}$  as algebras and  $I = 0$ .*

**6C.** One can describe when  $\Lambda$  is hereditary via  $\text{Mon}(Q, A)$ .

**Corollary 6.3.** *Let  $A$  be a finite-dimensional basic algebra over an algebraically closed field  $k$ ,  $Q$  a finite acyclic quiver with  $|Q_1| \neq 0$ , and  $\Lambda = A \otimes_k kQ$ . Then  $\mathcal{P}(\Lambda) = \text{Mon}(Q, A)$  if and only if  $\Lambda$  is hereditary.*

*Proof.* Without loss of generality we may assume that  $A$  is connected (an algebra is connected if it cannot be a product of two nonzero algebras).

If  $\Lambda = A \otimes_k kQ$  is hereditary, then, by the fact above and the assumption on  $Q$ , we have  $A = k$ , and hence  $\text{Mon}(Q, k) = \mathcal{GP}(kQ)$  by Theorem 5.1. It follows that

$$\text{Mon}(Q, A) = \mathcal{GP}(kQ) = \mathcal{P}(kQ) = \mathcal{P}(\Lambda).$$

Conversely, if  $A \neq k$ ,  $A$  is not semisimple since  $A$  is assumed to be connected and basic and  $k$  is assumed to be algebraically closed. It follows that there is a nonprojective  $A$ -module  $M$ . Take a sink of  $Q$ , say vertex 1, and consider  $\Lambda$ -module  $X = M \otimes_k P(1)$ , where  $P(1)$  is the simple projective  $kQ$ -module at vertex 1. Then  $X \in \text{Mon}(Q, A)$ , but  $X \notin \mathcal{P}(\Lambda)$ .  $\square$

## References

- [Arnold 2000] D. M. Arnold, *Abelian groups and representations of finite partially ordered sets*, CMS Books in Mathematics **2**, Springer, New York, 2000. MR 2001g:16030 Zbl 0959.16011
- [Auslander and Bridger 1969] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society **94**, American Mathematical Society, Providence, R.I., 1969. MR 42 #4580 Zbl 0204.36402
- [Auslander and Reiten 1991a] M. Auslander and I. Reiten, “Applications of contravariantly finite subcategories”, *Adv. Math.* **86**:1 (1991), 111–152. MR 92e:16009 Zbl 0774.16006
- [Auslander and Reiten 1991b] M. Auslander and I. Reiten, “Cohen–Macaulay and Gorenstein Artin algebras”, pp. 221–245 in *Representation theory of finite groups and finite-dimensional algebras* (Bielefeld, 1991), edited by G. O. Michler and C. M. Ringel, Progr. Math. **95**, Birkhäuser, Basel, 1991. MR 92k:16018 Zbl 0776.16003
- [Auslander and Smalø 1981] M. Auslander and S. O. Smalø, “Almost split sequences in subcategories”, *J. Algebra* **69**:2 (1981), 426–454. MR 82j:16048a Zbl 0457.16017
- [Auslander et al. 1995] M. Auslander, I. Reiten, and S. O., *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, 1995. MR 96c:16015 Zbl 0834.16001
- [Avramov and Martsinkovsky 2002] L. L. Avramov and A. Martsinkovsky, “Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension”, *Proc. London Math. Soc.* (3) **85**:2 (2002), 393–440. MR 2003g:16009 Zbl 1047.16002
- [Beligiannis 2005] A. Beligiannis, “Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras”, *J. Algebra* **288**:1 (2005), 137–211. MR 2006i:16017 Zbl 1119.16007
- [Buchweitz 1987] R.-O. Buchweitz, “Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings”, Unpub. manu., 1987, Available at <http://tinyurl.com/cohen-mac1986-pdf>.
- [Chen 2011] X.-W. Chen, “The stable monomorphism category of a Frobenius category”, *Math. Res. Lett.* **18**:1 (2011), 125–137. MR 2012e:18024 Zbl 06026606

- [Enochs and Jenda 1995] E. E. Enochs and O. M. G. Jenda, “Gorenstein injective and projective modules”, *Math. Z.* **220**:4 (1995), 611–633. MR 97c:16011 Zbl 0845.16005
- [Enochs and Jenda 2000] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics **30**, Walter de Gruyter, Berlin, 2000. MR 2001h:16013 Zbl 0952.13001
- [Gao and Zhang 2010] N. Gao and P. Zhang, “Gorenstein derived categories”, *J. Algebra* **323**:7 (2010), 2041–2057. MR 2011f:18017 Zbl 1222.18005
- [Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, 1988. MR 89e:16035 Zbl 0635.16017
- [Happel 1991] D. Happel, “On Gorenstein algebras”, pp. 389–404 in *Representation theory of finite groups and finite-dimensional algebras* (Bielefeld, 1991), edited by G. O. Michler and C. M. Ringel, Progr. Math. **95**, Birkhäuser, Basel, 1991. MR 92k:16022 Zbl 0759.16007
- [Holm 2004] H. Holm, “Gorenstein homological dimensions”, *J. Pure Appl. Algebra* **189**:1-3 (2004), 167–193. MR 2004k:16013 Zbl 1050.16003
- [Iyama et al. 2011] O. Iyama, K. Kato, and J.-I. Miyachi, “Recollement of homotopy categories and Cohen–Macaulay modules”, *J. K-Theory* **8**:3 (2011), 507–542. MR 2012k:18018 Zbl 1251.18008
- [Keller 1990] B. Keller, “Chain complexes and stable categories”, *Manuscripta Math.* **67**:4 (1990), 379–417. MR 91h:18006 Zbl 0753.18005
- [Krause and Solberg 2003] H. Krause and Ø. Solberg, “Applications of cotorsion pairs”, *J. London Math. Soc.* (2) **68**:3 (2003), 631–650. MR 2004k:16028 Zbl 1061.16006
- [Kussin et al. 2012] D. Kussin, H. Lenzing, and H. Meltzer, “Nilpotent operators and weighted projective lines”, 2012. To appear in *J. Reine Angew. Math.* arXiv 1002.3797
- [Leszczyński 1994] Z. Leszczyński, “On the representation type of tensor product algebras”, *Fund. Math.* **144**:2 (1994), 143–161. MR 95b:16013 Zbl 0817.16008
- [Li and Zhang 2010] Z.-W. Li and P. Zhang, “A construction of Gorenstein-projective modules”, *J. Algebra* **323**:6 (2010), 1802–1812. MR 2011d:16012 Zbl 1210.16011
- [Moore 2010] A. Moore, “The Auslander and Ringel–Tachikawa theorem for submodule embeddings”, *Comm. Algebra* **38**:10 (2010), 3805–3820. MR 2011k:16040 Zbl 1237.16015
- [Quillen 1973] D. Quillen, “Higher algebraic  $K$ -theory, I”, pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, Berlin, 1973. MR 49 #2895 Zbl 0292.18004
- [Ringel 1984] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics **1099**, Springer, Berlin, 1984. MR 87f:16027 Zbl 0546.16013
- [Ringel and Schmidmeier 2006] C. M. Ringel and M. Schmidmeier, “Submodule categories of wild representation type”, *J. Pure Appl. Algebra* **205**:2 (2006), 412–422. MR 2006i:16025 Zbl 1147.16019
- [Ringel and Schmidmeier 2008a] C. M. Ringel and M. Schmidmeier, “The Auslander–Reiten translation in submodule categories”, *Trans. Amer. Math. Soc.* **360**:2 (2008), 691–716. MR 2008j:16058 Zbl 1154.16011
- [Ringel and Schmidmeier 2008b] C. M. Ringel and M. Schmidmeier, “Invariant subspaces of nilpotent linear operators, I”, *J. Reine Angew. Math.* **614** (2008), 1–52. MR 2009d:16016 Zbl 1145.16005
- [Ringel and Zhang 2011] C. M. Ringel and P. Zhang, “Representations of quivers over the algebras of dual numbers”, preprint, 2011. arXiv 1112.1924
- [Simson 2007] D. Simson, “Representation types of the category of subprojective representations of a finite poset over  $K[t]/(t^m)$  and a solution of a Birkhoff type problem”, *J. Algebra* **311**:1 (2007), 1–30. MR 2009b:16040 Zbl 1123.16010



[Xiong and Zhang 2012] B.-L. Xiong and P. Zhang, “Gorenstein-projective modules over triangular matrix Artin algebras”, *J. Algebra Appl.* **11:4** (2012), Article ID 1250066, 14. MR 2959415 Zbl pre06078475

[Zhang 2011] P. Zhang, “Monomorphism categories, cotilting theory, and Gorenstein-projective modules”, *J. Algebra* **339** (2011), 181–202. MR 2012k:16034 Zbl pre06009303

Received May 10, 2012.

XIU-HUA LUO  
DEPARTMENT OF MATHEMATICS  
SHANGHAI JIAO TONG UNIVERSITY  
DONGCHUAN ROAD 800  
SHANGHAI, 200240  
CHINA  
x.h.luo@sjtu.edu.cn

PU ZHANG  
DEPARTMENT OF MATHEMATICS  
SHANGHAI JIAO TONG UNIVERSITY  
DONGCHUAN ROAD 800  
SHANGHAI, 200240  
CHINA  
pzhang@sjtu.edu.cn

# PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

## EDITORS

V. S. Varadarajan (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
pacific@math.ucla.edu

Paul Balmer  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
balmer@math.ucla.edu

Don Blasius  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
blasius@math.ucla.edu

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
chari@math.ucr.edu

Daryl Cooper  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
cooper@math.ucsb.edu

Robert Finn  
Department of Mathematics  
Stanford University  
Stanford, CA 94305-2125  
finn@math.stanford.edu

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
liu@math.ucla.edu

Jiang-Hua Lu  
Department of Mathematics  
The University of Hong Kong  
Pokfulam Rd., Hong Kong  
jhlu@maths.hku.hk

Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
popa@math.ucla.edu

Jie Qing  
Department of Mathematics  
University of California  
Santa Cruz, CA 95064  
qing@cats.ucsc.edu

Paul Yang  
Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
yang@math.princeton.edu

## PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

## SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI  
CALIFORNIA INST. OF TECHNOLOGY  
INST. DE MATEMÁTICA PURA E APLICADA  
KEIO UNIVERSITY  
MATH. SCIENCES RESEARCH INSTITUTE  
NEW MEXICO STATE UNIV.  
OREGON STATE UNIV.

STANFORD UNIVERSITY  
UNIV. OF BRITISH COLUMBIA  
UNIV. OF CALIFORNIA, BERKELEY  
UNIV. OF CALIFORNIA, DAVIS  
UNIV. OF CALIFORNIA, LOS ANGELES  
UNIV. OF CALIFORNIA, RIVERSIDE  
UNIV. OF CALIFORNIA, SAN DIEGO  
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ  
UNIV. OF MONTANA  
UNIV. OF OREGON  
UNIV. OF SOUTHERN CALIFORNIA  
UNIV. OF UTAH  
UNIV. OF WASHINGTON  
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

See inside back cover or [msp.org/pjm](http://msp.org/pjm) for submission instructions.

---

The subscription price for 2013 is US \$400/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

---

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

# PACIFIC JOURNAL OF MATHEMATICS

Volume 264 No. 1 July 2013

---

On the center of fusion categories	1
ALAIN BRUGUIÈRES and ALEXIS VIRELIZIER	
Connected quandles associated with pointed abelian groups	31
W. EDWIN CLARK, MOHAMED ELHAMDADI, XIANG-DONG HOU, MASAHICO SAITO and TIMOTHY YEATMAN	
Entropy and lowest eigenvalue on evolving manifolds	61
HONGXIN GUO, ROBERT PHILIPOWSKI and ANTON THALMAIER	
Poles of certain residual Eisenstein series of classical groups	83
DIHUA JIANG, BAIYING LIU and LEI ZHANG	
Harmonic maps on domains with piecewise Lipschitz continuous metrics	125
HAIGANG LI and CHANGYOU WANG	
$q$ -hypergeometric double sums as mock theta functions	151
JEREMY LOVEJOY and ROBERT OSBURN	
Monic representations and Gorenstein-projective modules	163
XIU-HUA LUO and PU ZHANG	
Helicoidal flat surfaces in hyperbolic 3-space	195
ANTONIO MARTÍNEZ, JOÃO PAULO DOS SANTOS and KETI TENENBLAT	
On a Galois connection between the subfield lattice and the multiplicative subgroup lattice	213
JOHN K. MCV EY	
Some characterizations of Campanato spaces via commutators on Morrey spaces	221
SHAOGUANG SHI and SHANZHEN LU	
The Siegel–Weil formula for unitary groups	235
SHUNSUKE YAMANA	



0030-8730(201307)264:1;1-4