Pacific Journal of Mathematics

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Volume 264 No. 1 July 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 264, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.264.163

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Dedicated to the memory of Hua Feng

Given a finite-dimensional algebra A over a field k, and a finite acyclic quiver Q, let $\Lambda = A \otimes_k kQ$, where kQ is the path algebra of Q over k. Then the category Λ-mod of Λ-modules is equivalent to the category $\mathrm{Rep}(Q, A)$ of representations of Q over A. This yields the notion of monic representations of Q over A. We denote the full subcategory of $\mathrm{Rep}(Q, A)$ consisting of monic representations of Q over A by $\mathrm{Mon}(Q, A)$. It is proved that $\mathrm{Mon}(Q, A)$ has Auslander–Reiten sequences.

The main result of this paper explicitly describes the Gorenstein-projective Λ-modules via the monic representations plus an extra condition. As a corollary, we prove the equivalence of three conditions: A is self-injective; Gorenstein-projective Λ-modules are exactly the monic representations of Q over A; $\mathrm{Mon}(Q, A)$ is a Frobenius category.

1. Introduction

Let A be an Artin algebra, and A-mod the category of finitely generated left A-modules. *A complete A-projective resolution* is an exact sequence of finitely generated projective A-modules

$$P^\bullet = \quad \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$$

such that $\mathrm{Hom}_A(P^\bullet, A)$ is also exact. A module $M \in A$-mod is *Gorenstein-projective* if there exists a complete A-projective resolution $P^\bullet$ such that $M \cong \mathrm{Ker}\, d^0$. Let $\mathcal{P}(A)$ be the full subcategory of A-mod of projective modules, and $\mathcal{GP}(A)$ the full subcategory of A-mod of Gorenstein-projective modules. Then

$$\mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^\perp A = \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X, A) = 0 \text{ for all } i \geq 1\}.$$

The authors were supported by the NSF of China (11271251) and the Doctoral Fund of the Ministry of Education of China (20120073110058). Pu Zhang is the corresponding author.

MSC2010: primary 16G10; secondary 16E65, 16G50, 16G60.

Keywords: representations of a quiver over an algebra, monic representations, Gorenstein-projective modules.

It is clear that $\mathcal{GP}(A) = A$-mod if and only if A is self-injective. If A is of finite global dimension, $\mathcal{GP}(A) = \mathcal{P}(A)$ (but the converse is *not* true); and if A is *a Gorenstein algebra* (that is, inj.dim ${}_AA < \infty$ and inj.dim $A_A < \infty$), then $\mathcal{GP}(A) = {}^{\perp}A$ (but the converse is *not* true); see, for example, [Enochs and Jenda 2000, Corollary 11.5.3]. This class of modules enjoys more stable properties than the usual projective modules (see [Auslander and Bridger 1969], where it was called a module of G-dimension zero); it becomes a main ingredient in the relative homological algebra [Enochs and Jenda 1995; 2000] and in the representation theory of algebras (see [Auslander and Reiten 1991a; 1991b; Beligiannis 2005; Gao and Zhang 2010; Iyama et al. 2011], for example), and plays a central role in the Tate cohomology of algebras (see [Avramov and Martsinkovsky 2002; Buchweitz 1987], for example). An important feature is that $\mathcal{GP}(A)$ is a Frobenius category with relative projective-injective objects being projective A-modules, and hence the stable category $\underline{\mathcal{GP}(A)}$ of $\mathcal{GP}(A)$ modulo $\mathcal{P}(A)$ is a triangulated category. By [Buchweitz 1987; Happel 1991], the singularity category of a Gorenstein algebra A is triangle equivalent to $\underline{\mathcal{GP}(A)}$. Thus explicitly constructing all the Gorenstein-projective modules is a fundamental problem, and is useful to all of these applications.

On the other hand, the submodule category has been extensively studied by C. M. Ringel and M. Schmidmeier [2006; 2008a; 2008b]; see also [Simson 2007]. By [Kussin et al. 2012] it is also related to the singularity category; see also [Chen 2011]. It turns out that the category of the Gorenstein-projective modules is closely related to the submodule category (see [Li and Zhang 2010; Xiong and Zhang 2012]), or, in general, to the monomorphism category [Zhang 2011]. The present paper explores such a relation in a more general set-up.

Given a finite-dimensional algebra A over a field k, and a finite acyclic quiver Q (here "acyclic" means that Q has no oriented cycles), let

$$\Lambda = A \otimes_k kQ,$$

where kQ is the path algebra of Q over k. We call Λ the path algebra of a finite quiver Q over A. As in the case of $A = k$, Λ-mod is equivalent to the category $\mathrm{Rep}(Q, A)$ of representations of Q over A. This interpretation permits us to introduce the so-called monic representations of Q over A. See Definition 2.2. Let $\mathrm{Mon}(Q, A)$ be the full subcategory of $\mathrm{Rep}(Q, A)$ consisting of monic representations of Q over A. Then $\mathrm{Mon}(Q, A)$ is a resolving, functorially finite subcategory of $\mathrm{Rep}(Q, A)$, and hence has Auslander–Reiten sequences (see Theorem 3.1). The main result of this paper, Theorem 5.1, explicitly describes all the Gorenstein-projective Λ-modules, via the monic representations of Q over A plus an extra condition. We emphasize that here Λ is not necessarily Gorenstein. By our main result, if we know all the Gorenstein-projective A-modules, we know all the Gorenstein-projective Λ-modules, and, in this way, we give an inductive construction of the Gorenstein-projective modules.

The proof of Theorem 5.1 use induction on $|Q_0|$ and a description of the Gorenstein-projective modules over the triangular extension of two algebras via a bimodule which is projective in both sides (Theorem 4.1). As a corollary, we see that A is self-injective if and only if $\mathcal{GP}(\Lambda) = \mathrm{Mon}(Q, A)$, and if and only if $\mathrm{Mon}(Q, A)$ is a Frobenius category (Corollary 6.1). As another corollary, if Q has an arrow, $\mathcal{P}(\Lambda) = \mathrm{Mon}(Q, A)$ if and only if Λ is hereditary (Corollary 6.3).

2. Monic representations of a quiver over an algebra

Throughout this section k is a field, Q a finite quiver, and A a finite-dimensional k-algebra. We consider the path algebra AQ of Q over A, describe its module category, and introduce the concept of monic representations of Q over A. In Subsections 2A–2D, Q is not assumed to be acyclic if not otherwise stated.

2A. Given a finite quiver

$$Q = (Q_0, Q_1, s, e),$$

let $\mathcal{P}$ be the set of paths of Q. We write the conjunction of paths from right to left. If $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$ with $\alpha_i \in Q_1$, $l \geq 1$, and $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq l-1$, we call l the length of p and denote it by $l(p)$, and define the starting vertex $s(p) = s(\alpha_1)$ and the ending vertex $e(p) = e(\alpha_l)$. We denote a vertex i by e_i, and regard it as a path of length 0, with $s(e_i) = i = e(e_i)$. Let kQ be the path algebra of Q over k. It is well-known that the category kQ-mod of finite-dimensional kQ-modules is equivalent to the category $\mathrm{Rep}(Q, k)$ of finite-dimensional representations of Q over k; see, for example, [Ringel 1984, p. 44].

2B. Let $\Lambda = AQ$ be the free left A-module with basis $\mathcal{P}$. An element of AQ is written as a finite sum $\sum_{p \in \mathcal{P}} a_p p$, where $a_p \in A$ and $a_p = 0$ for all but finitely many p. Then Λ is a k-algebra, with multiplication bilinearly given by

$$(a_p p)(b_q q) = (a_p b_q)(pq),$$

where $a_p b_q$ is the product in A, and pq is the product in kQ. We have isomorphisms $\Lambda \cong A \otimes_k kQ \cong kQ \otimes_k A$ of k-algebras, and we call $\Lambda = AQ$ *the path algebra of Q over A.*

For example, if $Q = \underset{n}{\bullet} \to \cdots \to \underset{1}{\bullet}$, the algebra Λ is given by the upper triangular matrix algebra of A:

$$T_n(A) = \begin{pmatrix} A & A & \cdots & A & A \\ 0 & A & \cdots & A & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A & A \\ 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

In general, if Q is acyclic and Q_0 is labeled as $1, \dots, n$ in such a way that $j > i$ whenever there is an arrow $\alpha : j \to i$ in Q_1, then

$$kQ \cong \begin{pmatrix} k & k^{m_{21}} & k^{m_{31}} & \cdots & k^{m_{n1}} \\ 0 & k & k^{m_{32}} & \cdots & k^{m_{n2}} \\ 0 & 0 & k & \cdots & k^{m_{n3}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}_{n\times n}, \tag{2-1}$$

where m_{ji} is the number of paths from j to i and $k^{m_{ji}}$ is the direct sum of m_{ji} copies of k, and hence

$$\Lambda \cong \begin{pmatrix} A & A^{m_{21}} & A^{m_{31}} & \cdots & A^{m_{n1}} \\ 0 & A & A^{m_{32}} & \cdots & A^{m_{n2}} \\ 0 & 0 & A & \cdots & A^{m_{n3}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix}_{n\times n}. \tag{2-2}$$

2C. By definition, *a representation X of Q over A* is a datum

$$X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1),$$

where X_i is an A-module for each $i \in Q_0$ and $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$ is an A-map for each $\alpha \in Q_1$. It is *a finite-dimensional representation* if each X_i is finite-dimensional. We call X_i *the i-th branch* of X. A morphism f from representation X to representation Y is a datum $(f_i, i \in Q_0)$, where $f_i : X_i \to Y_i$ is an A-map for each $i \in Q_0$, such that, for each arrow $\alpha : j \to i$, the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ \downarrow{\scriptstyle X_\alpha} & & \downarrow{\scriptstyle Y_\alpha} \\ X_i & \xrightarrow{f_i} & Y_i \end{array} \tag{2-3}$$

commutes. We call f_i *the i-th branch* of f. If $p = \alpha_l \cdots \alpha_1 \in \mathcal{P}$ with $\alpha_i \in Q_1$, $l \geq 1$, and $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq l-1$, we put X_p to be the A-map $X_{\alpha_l} \cdots X_{\alpha_1}$. Denote by $\operatorname{Rep}(Q, A)$ the category of finite-dimensional representations of Q over A. A morphism $f = (f_i, i \in Q_0)$ in $\operatorname{Rep}(Q, A)$ is a monomorphism (epimorphism, isomorphism) if and only if f_i is injective (surjective, an isomorphism) for each $i \in Q_0$.

Lemma 2.1. *Let Λ be the path algebra of Q over A. Then we have an equivalence Λ-mod $\cong \operatorname{Rep}(Q, A)$ of categories.*

We omit the proof of Lemma 2.1, which is similar to the case of $A = k$; see [Auslander et al. 1995, Theorem 1.5, p. 57; Ringel 1984, p. 44]. Throughout this paper we will identify a Λ-module with a representation of Q over A. Under this identification, a Λ-module X is a representation $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q

over A, where $X_i = (1e_i)X$, 1 is the identity of A, and the A-action on X_i is given by $a(1e_i)x = (1e_i)(ae_i)x$ for all $x \in X$ and $a \in A$; and $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$ is the A-map given by the left action by $1\alpha \in \Lambda$. On the other hand, a representation $(X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A is a Λ-module $X = \bigoplus_{i \in Q_0} X_i$, with the Λ-action on X given by

$$(ap)(x_i) = \begin{cases} 0 & \text{if } s(p) \neq i, \\ ax_i & \text{if } p = e_i, \\ aX_p(x_i) \in X_{e(p)} & \text{if } s(p) = i \text{ and } l(p) \geq 1, \end{cases}$$

for all $a \in A$, $p \in \mathcal{P}$, $x_i \in X_i$. Let $f : X \to Y$ be a morphism in $\operatorname{Rep}(Q, A)$. Then $\operatorname{Ker} f$ and $\operatorname{Coker} f$ can be explicitly written out. For example, $\operatorname{Coker} f = (\operatorname{Coker} f_i, \widetilde{Y}_\alpha, i \in Q_0, \alpha \in Q_1)$, where, for each arrow $\alpha : j \to i$,

$$\widetilde{Y}_\alpha : \operatorname{Coker} f_j \to \operatorname{Coker} f_i$$

is the A-map induced by Y_α; see (2-3). A sequence of morphisms

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\operatorname{Rep}(Q, A)$ is exact if and only if each

$$0 \longrightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \longrightarrow 0$$

is exact in A-mod, for $i \in Q_0$.

In the following, if Q_0 is labeled as $1, \ldots, n$, we also write a representation X of Q over A as

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(X_\alpha,\ \alpha \in Q_1)},$$

and a morphism in $\operatorname{Rep}(Q, A)$ as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

2D. The following is a central notion of this paper.

Definition 2.2. A representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A is *a monic representation*, or a monic Λ-module, if, for each $i \in Q_0$, the A-map

$$(X_\alpha)_{\substack{\alpha \in Q_1 \\ e(\alpha) = i}} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha) = i}} X_{s(\alpha)} \to X_i$$

is injective, or, equivalently, if the following two conditions are satisfied.

(m1) For each $\alpha \in Q_1$, the map $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$ is injective.

(m2) For each $i \in Q_0$, there holds $\sum_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} \operatorname{Im} X_\alpha = \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} \operatorname{Im} X_\alpha$.

Denote by $\operatorname{Mon}(Q, A)$ the full subcategory of $\operatorname{Rep}(Q, A)$ consisting of monic representations of Q over A. We call $\operatorname{Mon}(Q, A)$ *the monomorphism category of A over Q*.

If Q is a quiver in which, for any vertex i, there is at most one arrow ending at i, condition (m2) vanishes. For example, if $Q = \bullet \to \bullet$, then $\operatorname{Mon}(Q, A)$ is called *the submodule category of A* in [Ringel and Schmidmeier 2006; 2008a]. If

$$Q = \underset{n}{\bullet} \to \cdots \to \underset{1}{\bullet},$$

$\operatorname{Mon}(Q, A)$ is called *the filtered chain category of A* in [Arnold 2000; Simson 2007].

2E. Let Q be a finite acyclic quiver, A a finite-dimensional algebra, and $\Lambda = A \otimes_k kQ$. Throughout this paper, we label the vertices of Q as $1, 2, \dots, n$, in such a way that if there is an arrow from j to i, then $j > i$. Denote by $P(i)$ the indecomposable projective kQ-module at $i \in Q_0$. It is clear that $P(i) \in \operatorname{Mon}(Q, k)$; it follows that $M \otimes_k P(i) \in \operatorname{Mon}(Q, A)$ for $M \in A$-mod. Thus we have the functors

$$- \otimes_k P(i) : A\text{-mod} \to \operatorname{Mon}(Q, A), \quad -_i : \operatorname{Rep}(Q, A) \to A\text{-mod}$$

(by taking the i-th branch).

We also need the adjoint pair $(- \otimes_k P(i), -_i)$.

Lemma 2.3. *For each object $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1) \in \Lambda$-mod and each A-module M, we have isomorphisms of abelian groups, which are natural in both positions*

$$\operatorname{Hom}_\Lambda(M \otimes_k P(i), X) \cong \operatorname{Hom}_A(M, X_i) \tag{2-4}$$

for all $i \in Q_0$.

Proof. For $f = (f_j, j \in Q_0) \in \operatorname{Hom}_\Lambda(M \otimes_k P(i), X)$, we have $f_i \in \operatorname{Hom}_A(M, X_i)$. Since $M \otimes_k P(i) = (M \otimes_k e_j kQe_i, \mathrm{id}_M \otimes \alpha, j \in Q_0, \alpha \in Q_1)$, it follows from the commutative diagram (2-3) that

$$f_j = \begin{cases} 0 & \text{if there are no paths from } i \text{ to } j, \\ m \otimes_k p \mapsto X_p f_i(m) & \text{if there is a path } p \text{ from } i \text{ to } j. \end{cases} \tag{2-5}$$

By (2-5) we see that $f \mapsto f_i$ gives an injective map

$$\operatorname{Hom}_\Lambda(M \otimes_k P(i), X) \to \operatorname{Hom}_A(M, X_i).$$

This map is also surjective, since for a given $f_i \in \operatorname{Hom}_A(M, X_i)$, $f = (f_j, j \in Q_0)$ given by (2-5) is indeed a morphism in $\operatorname{Rep}(Q, A)$ from $M \otimes_k P(i)$ to X. □

Proposition 2.4. (i) *The indecomposable projective* Λ*-modules have the form* $P \otimes_k P(i)$, *where* P *is an indecomposable projective* A*-module, and* $P(i)$ *is the indecomposable projective* kQ*-module at* $i \in Q_0$.

(ii) *The indecomposable projective objects in* $\mathrm{Mon}(Q, A)$ *are exactly the indecomposable projective* Λ*-modules.*

(iii) *If* I *is an indecomposable injective* A*-module and* $P(i)$ *is the indecomposable projective* kQ*-module at* $i \in Q_0$, $I \otimes_k P(i)$ *is an indecomposable injective object in* $\mathrm{Mon}(Q, A)$.

Proof. (i) As a direct summand of the regular Λ-module ${}_\Lambda\Lambda$, we see that $P \otimes_k P(i)$ is a projective Λ-module, and each projective Λ-module has this form. By (2-4) we have

$$\mathrm{End}_\Lambda(P \otimes_k P(i)) \cong \mathrm{Hom}_A(P, (P \otimes_k P(i))_i) = \mathrm{End}_A(P),$$

from which we see that $P \otimes_k P(i)$ is indecomposable.

(ii) Note that $P \otimes_k P(i) \in \mathrm{Mon}(Q, A)$. By (i) we know that it is an indecomposable projective object in $\mathrm{Mon}(Q, A)$. On the other hand, it is clear that $\mathrm{Mon}(Q, A)$ is closed under taking subobjects, as a consequence any indecomposable projective object in $\mathrm{Mon}(Q, A)$ has this form.

(iii) Note that $I \otimes_k P(i)$ is an indecomposable object in $\mathrm{Mon}(Q, A)$. Put $L = D(A_A) \otimes_k kQ$, where $D = \mathrm{Hom}_k(-, k)$. It suffices to prove that L is an injective object in $\mathrm{Mon}(Q, A)$, by induction on $|Q_0|$. We write $L = (L_i, L_\alpha, i \in Q_0, \alpha \in Q_1)$.

Let Q' be the quiver obtained from Q by deleting a sink vertex 1, L' the representation in $\mathrm{Rep}(Q', A)$ obtained from L by deleting the branch L_1. We observe that $L' = D(A_A) \otimes_k kQ'$, and by inductive hypothesis L' is an injective object in $\mathrm{Mon}(Q', A)$.

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathrm{Mon}(Q, A)$, with $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$, and $h : X \to L$ a morphism in $\mathrm{Rep}(Q, A)$. Let X' be the representation in $\mathrm{Rep}(Q', A)$ obtained from X by deleting the branch X_1, and similarly for Y', Z'. Then we have an exact sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \longrightarrow 0$$

in $\mathrm{Mon}(Q', A)$, where f' is the morphism in $\mathrm{Rep}(Q', A)$ obtained from f by deleting the branch f_1, and similarly for g' and for $h' : X' \to L'$. Since L' is an injective object in $\mathrm{Mon}(Q', A)$, by definition we have a morphism

$$u' = \begin{pmatrix} u_2 \\ \vdots \\ u_n \end{pmatrix} : Y' \to L'$$

in $\operatorname{Rep}(Q', A)$ such that $h' = u' f'$. It suffices to construct an A-map

$$u_1 : Y_1 \to L_1$$

such that $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} : Y \to L$ is a morphism in $\operatorname{Rep}(Q, A)$, and that $h_1 = u_1 f_1$.

First, we have an A-map $u'_1 : Y_1 \to L_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \xhookrightarrow{f_1} & Y_1 \\ {\scriptstyle h_1}\downarrow & \swarrow {\scriptstyle u'_1} & \\ L_1 & & \end{array} .$$

commutes. Consider the A-map

$$(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} : \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} Y_{s(\alpha)} \to L_1.$$

Since we have the exact sequence of A-modules

$$0 \longrightarrow \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} X_{s(\alpha)} \xrightarrow{\operatorname{diag}(f_{s(\alpha)})} \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} Y_{s(\alpha)} \xrightarrow{\operatorname{diag}(g_{s(\alpha)})} \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \longrightarrow 0,$$

and since

$$\begin{aligned} (L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} \circ \operatorname{diag}(f_{s(\alpha)}) &= (L_\alpha u_{s(\alpha)} f_{s(\alpha)} - u'_1 Y_\alpha f_{s(\alpha)})_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} \\ &= (L_\alpha u_{s(\alpha)} f_{s(\alpha)} - u'_1 f_1 X_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} \\ &= (L_\alpha h_{s(\alpha)} - h_1 X_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} \\ &= 0, \end{aligned}$$

where the second equality follows from the fact that $f : X \to Y$ is a morphism in $\operatorname{Rep}(Q, A)$, it follows that $(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}}$ factors through $\operatorname{diag}(g_{s(\alpha)})$. That is, there is an A-map

$$v_1 : \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \to L_1,$$

such that

$$(L_\alpha u_{s(\alpha)} - u'_1 Y_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} = v_1 \circ \operatorname{diag}(g_{s(\alpha)}).$$

Since L_1 is an injective A-module and

$$(Z_\alpha)_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} : \bigoplus_{\substack{\alpha\in Q_1 \\ e(\alpha)=1}} Z_{s(\alpha)} \to Z_1$$

is an injective A-map, it follows that there is an A-map $w_1 : Z_1 \to L_1$, such that $v_1 = w_1 \circ (Z_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=1}}$. So we have

$$(L_\alpha u_{s(\alpha)} - u_1' Y_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=1}} = w_1 \circ (Z_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=1}} \circ \operatorname{diag}(g_{s(\alpha)}) = (w_1 g_1 Y_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=1}},$$

where the second equality follows from the fact that $g : Y \to Z$ is a morphism in $\operatorname{Rep}(Q, A)$. This means that for each $\alpha \in Q_1$ with $e(\alpha) = 1$ we have

$$L_\alpha u_{s(\alpha)} - u_1' Y_\alpha = w_1 g_1 Y_\alpha. \tag{2-6}$$

Now put $u_1 = u_1' + w_1 g_1 : Y_1 \to L_1$. Then (2-6) together with the inductive hypothesis implies that

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} : Y \to L$$

is a morphism in $\operatorname{Rep}(Q, A)$. It is clear that

$$u_1 f_1 = (u_1' + w_1 g_1) f_1 = u_1' f_1 = h_1.$$

This completes the proof. □

2F. Recall from [Auslander and Reiten 1991a] that a full subcategory $\mathcal{X}$ of A-mod is *resolving* if $\mathcal{X}$ contains all projective A-modules and $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and direct summands. It is straightforward to verify that $\operatorname{Mon}(Q, A)$ is closed under extensions, kernels of epimorphisms, and direct summands. By Proposition 2.4 we have the following.

Corollary 2.5. *For a finite acyclic quiver Q and a finite-dimensional algebra A,* $\operatorname{Mon}(Q, A)$ *is a resolving subcategory of* $\operatorname{Rep}(Q, A)$.

2G. There is another similar but different notion. Let $A = kQ/I$ be a finite-dimensional k-algebra, where I is an admissible ideal of kQ. An I-bounded representations of Q over k is a datum $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$, where X_i is a k-space for each $i \in Q_0$, and $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$ is a k-linear map for each $\alpha \in Q_1$, such that $\sum_{p\in\mathcal{P}} c_p X_p = 0$ for each element $\sum_{p\in\mathcal{P}} c_p p \in I$, where $l(p) \geq 2$ and $c_p \in k$. An I-bounded representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over k is *a monic representation*, if for each $i \in Q_0$ the k-linear map

$$(X_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \to X_i$$

is injective. Let $\operatorname{Rep}(Q, I, k)$ be the category of finite-dimensional I-bounded representations of Q over k. There is an equivalence of categories between A-mod and $\mathrm{Rep}(Q, I, k)$; see [Auslander et al. 1995, Proposition 1.7, p. 60; Ringel 1984, p. 45]. Let $\mathrm{Mon}(Q, I, k)$ denote the full subcategory of $\mathrm{Rep}(Q, I, k)$ of I-bounded monic representations Q over k. Then $\mathrm{Mon}(Q, 0, k) = \mathrm{Mon}(Q, k)$.

Proposition 2.6. *Let $A = kQ/I$ be a finite-dimensional k-algebra, where I is an admissible ideal of kQ. Then $\mathcal{P}(A) \subseteq \mathrm{Mon}(Q, I, k)$ if and only if A is hereditary.*

Proof. If A is hereditary, $I = 0$. It is clear $\mathcal{P}(kQ) \subseteq \mathrm{Mon}(Q, 0, k)$.

Conversely, if $I \neq 0$, take an element $\sum_{p\in\mathcal{P}} c_p p \in I$ with $l(p) \geq 2$ and $c_p \in k$. We may assume that all the paths p with $c_p \neq 0$ have the same starting vertex j and the same ending vertex i. Consider the projective A-module $P(j) = Ae_j$. As an I-bounded representation of Q over k, we write $P(j)$ as

$$P(j) = (e_t kQe_j,\ f_\alpha, t \in Q_0, \alpha \in Q_1).$$

Let $\alpha_1, \dots, \alpha_m$ be all the arrows of Q ending at i. We claim that

$$(f_{\alpha_v})_{1\leq v\leq m} : \bigoplus_{1\leq v\leq m} e_{s(\alpha_v)} kQe_j \to e_i kQe_j$$

is not injective, where f_{α_v} is the k-linear map given by the left multiplication by α_v. Since each path from j to i must go through some α_v, and $\sum_{p\in\mathcal{P}} c_p f_p = 0$, it follows that

$$\sum_{1\leq v\leq m} \dim_k(e_{s(\alpha_v)} kQe_j) > \dim_k(e_i kQe_j).$$

This justifies the claim, that is, $P(j) \notin \mathrm{Mon}(Q, I, k)$. □

Now, let $\Lambda = A \otimes_k kQ$ be the path algebra of Q over A. Assume that Λ is of the form $\Lambda = kQ'/I'$, where Q' is a finite quiver and I' is an admissible ideal of kQ'. We emphasize that, in general,

$$\mathrm{Mon}(Q, A) \neq \mathrm{Mon}(Q', I', k).$$

In fact, $\mathcal{P}(\Lambda) \subseteq \mathrm{Mon}(Q, A)$ (Proposition 2.4); but generally $\mathcal{P}(\Lambda) \subseteq \mathrm{Mon}(Q', I', k)$ is not true, as Proposition 2.6 shows. This is the reason why we do not use the notation $\mathrm{Mon}(\Lambda)$.

3. Functorial finiteness of Mon(Q, A) in Rep(Q, A)

The aim of this section is to prove the following.

Theorem 3.1. *Let Q be a finite acyclic quiver, and A a finite-dimensional algebra. Then $\mathrm{Mon}(Q, A)$ is functorially finite in $\mathrm{Rep}(Q, A)$ and $\mathrm{Mon}(Q, A)$ has Auslander–Reiten sequences.*

The idea of the proof given below is essentially due to Ringel and Schmidmeier [2008a] for the case of $Q = \bullet \to \bullet$. The same result for the case of

$$Q = \underset{n}{\bullet} \to \cdots \to \underset{1}{\bullet}$$

has been obtained in [Moore 2010; Zhang 2011].

3A. Let Q be a finite acyclic quiver. Remember we label the vertices of Q as $1, 2, \dots, n$, such that if there is an arrow from j to i, $j > i$. So vertex 1 is a sink. Denote by $\mathcal{P}(\to i)$ the set of all the paths p with ending vertex $e(p) = i$ and $l(p) \geq 1$.

For $X \in \operatorname{Rep}(Q, A)$ and $i \in Q_0$, put K_i to be the kernel of the A-map

$$(X_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \to X_i.$$

Fix an injective envelope $\delta_i : K_i \hookrightarrow IK_i$ of K_i. Then there is an A-map

$$(\varphi_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} : \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \to IK_i$$

such that the diagram

(3-1)
$$\begin{array}{ccc} K_i & \hookrightarrow & \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \\ {\scriptstyle\delta_i}\downarrow & \swarrow {\scriptstyle (\varphi_\alpha)_{\substack{\alpha\in Q_1\\ e(\alpha)=i}}} & \\ IK_i & & \end{array}.$$

commutes for each $i \in Q_0$. We construct a representation

$$\operatorname{rMon}(X) = (\operatorname{rMon}(X)_i, \operatorname{rMon}(X)_\alpha, i \in Q_0, \alpha \in Q_1) \in \operatorname{Rep}(Q, A)$$

as follows. For each $i \in Q_0$, define

$$\operatorname{rMon}(X)_i = X_i \oplus IK_i \oplus \bigoplus_{p\in\mathcal{P}(\to i)} IK_{s(p)}. \tag{3-2}$$

(Note that if i is a source, by definition $\operatorname{rMon}(X)_i = X_i$, and that if $p_1, \dots, p_m$ are all the paths in $\mathcal{P}(\to i)$ with the same starting vertex j, the $\underbrace{IK_j \oplus \cdots \oplus IK_j}_{m}$ is a direct summand of $\bigoplus_{p\in\mathcal{P}(\to i)} IK_{s(p)}$.)

For each arrow $\alpha : j \to i$, define

$$\operatorname{rMon}(X)_\alpha : X_j \oplus IK_j \oplus \bigoplus_{p\in\mathcal{P}(\to j)} IK_{s(p)} \to X_i \oplus IK_i \oplus \bigoplus_{q\in\mathcal{P}(\to i)} IK_{s(q)}$$

to be the A-map given by

$$x_j + k_j + \sum_{p\in\mathcal{P}(\to j)} k_{s(p)} \mapsto X_\alpha(x_j) + \varphi_\alpha(x_j) + k_j + \sum_{p\in\mathcal{P}(\to j)} k_{s(\alpha p)}, \tag{3-3}$$

where $x_j \in X_j$, $k_j \in IK_j$, $k_{s(p)} \in IK_{s(p)}$. Note that $s(p) = s(\alpha p)$, and that $k_{s(\alpha p)}$ is just $k_{s(p)}$. Also note that at the right side of (3-3), k_j and $\sum_{p\in\mathcal{P}(\to j)} k_{s(\alpha p)}$ belong to different direct summands of $\bigoplus_{q\in\mathcal{P}(\to i)} IK_{s(q)}$.

Lemma 3.2. *For* $X \in \operatorname{Rep}(Q, A)$, *we have* $\operatorname{rMon}(X) \in \operatorname{Mon}(Q, A)$.

Proof. For each $i \in Q_0$, let $\alpha_1, \dots, \alpha_m$ be all the arrows ending at i. By definition we only need to prove that the A-map

$$(\operatorname{rMon}(X)_{\alpha_1}, \dots, \operatorname{rMon}(X)_{\alpha_m}) : \bigoplus_{1\le j\le m} \operatorname{rMon}(X)_{s(\alpha_j)} \to \operatorname{rMon}(X)_i$$

is injective. This is clear by (3-1)–(3-3). For completeness we include a justification.

Suppose $z_j = x_{s(\alpha_j)} + k_{s(\alpha_j)} + (\sum_{p\in\mathcal{P}(\to s(\alpha_j))} k_{s(p)}) \in \operatorname{rMon}(X)_{s(\alpha_j)}$, $j = 1, \dots, m$, and $\sum_{1\le j\le m} \operatorname{rMon}(X)_{\alpha_j}(z_j) = 0$. Then by (3-3) we have

$$0 = \sum_{1\le j\le m} X_{\alpha_j}(x_{s(\alpha_j)}) + \sum_{1\le j\le m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) + \sum_{1\le j\le m} k_{s(\alpha_j)} + \sum_{1\le j\le m}\ \sum_{p\in\mathcal{P}(\to s(\alpha_j))} k_{s(\alpha_j p)} \in X_i \oplus IK_i \oplus \bigoplus_{q\in\mathcal{P}(\to i)} IK_{s(q)}.$$

Thus

$$\sum_{1\le j\le m} X_{\alpha_j}(x_{s(\alpha_j)}) = 0, \qquad \sum_{1\le j\le m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) = 0,$$

and $k_{s(\alpha_j)} = 0 = k_{s(\alpha_j p)}$ for all $j = 1, \dots, m$ and all $p \in \mathcal{P}(\to s(\alpha_j))$. Note that $\sum_{1\le j\le m} X_{\alpha_j}(x_{s(\alpha_j)}) = 0$ implies

$$\begin{pmatrix} x_{s(\alpha_1)} \\ \vdots \\ x_{s(\alpha_m)} \end{pmatrix} \in K_i.$$

By (3-1) we have

$$\delta_i \begin{pmatrix} x_{s(\alpha_1)} \\ \vdots \\ x_{s(\alpha_m)} \end{pmatrix} = \sum_{1\le j\le m} \varphi_{\alpha_j}(x_{s(\alpha_j)}) = 0.$$

Since δ_i is injective, we have $x_{s(\alpha_j)} = 0$ for $j = 1, \dots, m$. Thus $z_j = 0$ for $j = 1, \dots, m$. This completes the proof. □

3B. Let $\mathcal{X}$ be a full subcategory of A-mod. Recall from [Auslander and Reiten 1991a] that *a right $\mathcal{X}$-approximation of M* is a morphism $f : X \to M$ with $X \in \mathcal{X}$ such that the induced homomorphism $\mathrm{Hom}_A(X', X) \to \mathrm{Hom}_A(X', M)$ is surjective for each $X' \in \mathcal{X}$. If every object M admits a right $\mathcal{X}$-approximation, $\mathcal{X}$ is called *a contravariantly finite subcategory in* A-mod. Dually one has the concept of *a covariantly finite subcategory in* A-mod. If $\mathcal{X}$ is both contravariantly and covariantly finite in A-mod, $\mathcal{X}$ is *a functorially finite subcategory in* A-mod.

Proposition 3.3. *Let Q be a finite acyclic quiver, and A a finite-dimensional algebra. Then* $\mathrm{Mon}(Q, A)$ *is contravariantly finite in* $\mathrm{Rep}(Q, A)$.

More precisely, let $X \in \mathrm{Rep}(Q, A)$, $f = (f_i, i \in Q_0) : \mathrm{rMon}(X) \to X$, *where* $f_i : \mathrm{rMon}(X)_i \to X_i$ *is the canonical projection. Then f is a right* $\mathrm{Mon}(Q, A)$-*approximation of* X.

Proof. We use induction to prove that f is a right $\mathrm{Mon}(Q, A)$-approximation of X. The assertion trivially holds if $|Q_0| = 1$. Suppose that the assertion holds for the quivers Q with $|Q_0| = n - 1$. Assume that $|Q_0| = n$ and that

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : Y \to X$$

is a morphism in $\mathrm{Rep}(Q, A)$ with $Y \in \mathrm{Mon}(Q, A)$. We need to prove that there is a morphism

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} : Y \to \mathrm{rMon}(X)$$

in $\mathrm{Rep}(Q, A)$ such that $g = fh$.

Let Q' be the quiver obtained from Q by deleting vertex 1, X' the representation in $\mathrm{Rep}(Q', A)$ obtained from X by deleting the branch X_1, and Y' the representation in $\mathrm{Mon}(Q', A)$ obtained from Y by deleting the branch Y_1. Then by definition $\mathrm{rMon}(X')$ is exactly the representation in $\mathrm{Mon}(Q', A)$ obtained from $\mathrm{rMon}(X)$ by deleting the branch $\mathrm{rMon}(X)_1$. Further,

$$\begin{pmatrix} f_2 \\ \vdots \\ f_n \end{pmatrix} : \mathrm{rMon}(X') \to X' \quad \text{and} \quad \begin{pmatrix} g_2 \\ \vdots \\ g_n \end{pmatrix} : Y' \to X'$$

are morphisms in $\mathrm{Rep}(Q', A)$. By the inductive hypothesis there is a morphism

$$\begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix} : Y' \to \mathrm{rMon}(X')$$

in $\operatorname{Rep}(Q', A)$, such that

$$\begin{pmatrix} g_2 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} f_2 \\ \vdots \\ f_n \end{pmatrix} \begin{pmatrix} h_2 \\ \vdots \\ h_n \end{pmatrix}.$$

Let $\alpha_1, \ldots, \alpha_m$ be all the arrows ending at 1. Since

$$(Y_{\alpha_1}, \ldots, Y_{\alpha_m}) : \bigoplus_{1 \le j \le m} Y_{s(\alpha_j)} \to Y_1$$

is an injective A-map and $IK_1 \oplus \left(\bigoplus_{p \in \mathcal{P}(\to 1)} IK_{s(p)}\right)$ is an injective A-module, it follows that there is a map

$$\eta : Y_1 \to IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\to 1)} IK_{s(p)}$$

such that the diagram

$$\begin{array}{ccc} \bigoplus_{1 \le j \le m} Y_{s(\alpha_j)} & \xrightarrow{(Y_{\alpha_1}, \ldots, Y_{\alpha_m})} & Y_1 \\ \downarrow \tilde{h} & & \downarrow \eta \\ \bigoplus_{1 \le j \le m} \operatorname{rMon}(X)_{s(\alpha_j)} & \xrightarrow{(B_1, \ldots, B_m)} & IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\to 1)} IK_{s(p)} \end{array}$$

commutes, where $\tilde{h} = \operatorname{diag}(h_{s(\alpha_1)}, \ldots, h_{s(\alpha_m)})$ and, for each $j = 1, \ldots, m$,

$$B_j : \operatorname{rMon}(X)_{s(\alpha_j)} \to IK_1 \oplus \bigoplus_{p \in \mathcal{P}(\to 1)} IK_{s(p)}$$

is the A-map given by

$$x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\to s(\alpha_j))} k_{s(p)} \mapsto \varphi_{\alpha_j}(x_{s(\alpha_j)}) + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\to s(\alpha_j))} k_{s(\alpha_j p)}$$

for

$$x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\to s(\alpha_j))} k_{s(p)} \in \operatorname{rMon}(X)_{s(\alpha_j)} = X_{s(\alpha_j)} \oplus IK_{s(\alpha_j)} \oplus \bigoplus_{p \in \mathcal{P}(\to s(\alpha_j))} IK_{s(p)}.$$

For $y \in Y_{s(\alpha_j)}$, suppose

$$h_{s(\alpha_j)}(y) = x_{s(\alpha_j)} + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\to s(\alpha_j))} k_{s(p)} \in \operatorname{rMon}(X)_{s(\alpha_j)}.$$

Then we have

$$\begin{aligned}\mathrm{rMon}(X)_{\alpha_j} h_{s(\alpha_j)}(y) &= X_{\alpha_j}(x_{s(\alpha_j)}) + \varphi_{\alpha_j}(x_{s(\alpha_j)}) + k_{s(\alpha_j)} + \sum_{p \in \mathcal{P}(\to s(\alpha_j))} k_{s(\alpha_j p)} \\ &= X_{\alpha_j}(x_{s(\alpha_j)}) + B_j h_{s(\alpha_j)}(y) \\ &= X_{\alpha_j}(f_{s(\alpha_j)} h_{s(\alpha_j)}(y)) + B_j h_{s(\alpha_j)}(y) \\ &= X_{\alpha_j} g_{s(\alpha_j)}(y) + B_j h_{s(\alpha_j)}(y) \\ &= g_1 Y_{\alpha_j}(y) + \eta Y_{\alpha_j}(y),\end{aligned}$$

where the last equality uses the fact that $g : Y \to X$ is a morphism in $\mathrm{Rep}(Q, A)$.

Now we define $h_1 : Y_1 \to \mathrm{rMon}(X)_1$ to be the A-map given by

$$h_1(y) = g_1(y) + \eta(y)$$

for each $y \in Y_1$. From the computation above we have $\mathrm{rMon}(X)_{\alpha_j} h_{s(\alpha_j)} = h_1 Y_{\alpha_j}$ for $j = 1, \ldots, m$. It follows that

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} : Y \to \mathrm{rMon}(X)$$

is a morphism in $\mathrm{Rep}(Q, A)$. Since $f_1 : \mathrm{rMon}(X)_1 \to X_1$ is the canonical projection, we have $f_1 \eta = 0$ and $f_1 g_1 = g_1$, and hence $fh = g$. This completes the proof. □

3C. ***Proof of Theorem 3.1.*** By Corollary 2.5 and Proposition 3.3 we know that $\mathrm{Mon}(Q, A)$ is a resolving, contravariantly finite subcategory of $\mathrm{Rep}(Q, A)$, and hence $\mathrm{Mon}(Q, A)$ is functorially finite in $\mathrm{Rep}(Q, A)$; see [Krause and Solberg 2003, Corollary 2.6(i)]. It follows that $\mathrm{Mon}(Q, A)$ has Auslander–Reiten sequences, by [Auslander and Smalø 1981, Theorem 2.4]. □

4. Gorenstein-projective modules over the upper triangular matrix algebras

4A. Let A and B be rings, M an A-B-bimodule, and $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ the upper triangular matrix ring, where the addition and multiplication are given by the ones of matrices. We assume that Λ is an Artin algebra [Auslander et al. 1995, p. 72], and consider finitely generated Λ-modules. A Λ-module can be identified with a triple $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$, or simply $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ if ϕ is clear, where $X \in A\text{-mod}$, $Y \in B\text{-mod}$, and $\phi : M \otimes_B Y \to X$ is an A-map. A Λ-map $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \to \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'}$ can be identified with a pair $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$, where $f \in \mathrm{Hom}_A(X, X')$, $g \in \mathrm{Hom}_B(Y, Y')$ are such that the diagram

$$\begin{array}{ccc} M \otimes_B Y & \xrightarrow{\phi} & X \\ {\scriptstyle \mathrm{id} \otimes g} \downarrow & & \downarrow {\scriptstyle f} \\ M \otimes_B Y' & \xrightarrow{\phi'} & X' \end{array}$$

commutes. A sequence of Λ-maps

$$0 \to \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \xrightarrow{\binom{f_1}{g_1}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \xrightarrow{\binom{f_2}{g_2}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \to 0$$

is exact if and only if

$$0 \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow 0$$

is an exact sequence of A-maps, and

$$0 \longrightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \longrightarrow 0$$

is an exact sequence of B-maps. The indecomposable projective Λ-modules are exactly

$$\begin{pmatrix} P \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}_{\text{id}},$$

where P runs over indecomposable projective A-modules and Q runs over indecomposable projective B-modules.

Note that an algebra Λ is of the form above if and only if there is an idempotent decomposition $1 = e + f$ such that $f\Lambda e = 0$; and in this case

$$\Lambda = \begin{pmatrix} e\Lambda e & e\Lambda f \\ 0 & f\Lambda f \end{pmatrix}.$$

4B. The following result describes the Gorenstein-projective Λ-modules, if ${}_AM$ and M_B are projective modules.

Theorem 4.1. *Let*

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

be an Artin algebra, M an A-B-bimodule such that ${}_AM$ and M_B are projective modules. Then

$$\begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \mathcal{GP}(\Lambda)$$

if and only if $\phi : M \otimes_B Y \to X$ is injective, $\operatorname{Coker} \phi \in \mathcal{GP}(A)$, *and $Y \in \mathcal{GP}(B)$. In this case, $X \in \mathcal{GP}(A)$ if and only if $M \otimes_B Y \in \mathcal{GP}(A)$.*

Note that here Λ is not assumed to be Gorenstein: this will be important to the main result in the next section. The same result under the assumption that Λ is Gorenstein can be found in [Xiong and Zhang 2012, Corollary 3.3] (however, the proof there cannot be generalized to the non-Gorenstein case). The same corollary implies that, if Λ is Gorenstein in Theorem 4.1, $\binom{X}{Y}_{\phi} \in \mathcal{GP}(\Lambda)$ implies $X \in \mathcal{GP}(A)$.

Proof of Theorem 4.1. The last assertion is easy: it follows from the exact sequence

$$0 \longrightarrow M \otimes_B Y \xrightarrow{\phi} X \longrightarrow \operatorname{Coker} \phi \longrightarrow 0$$

and the fact that $\mathcal{GP}(A)$ is closed under extensions and the kernels of epimorphisms; see, for example, [Holm 2004].

We next prove the "if" part of the first equivalence in the theorem. We assume that $\phi : M \otimes_B Y \to X$ is injective, $\operatorname{Coker} \phi \in \mathcal{GP}(A)$, and $Y \in \mathcal{GP}(B)$. Then we have a complete B-projective resolution

$$Q^\bullet = \quad \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \xrightarrow{d'^0} Q^1 \longrightarrow \cdots \tag{4-1}$$

with $Y = \operatorname{Ker} d'^0$, and a complete A-projective resolution

$$P^\bullet = \quad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots \tag{4-2}$$

with $\operatorname{Coker} \phi = \operatorname{Ker} d^0$. Since M_B is projective, we get the following exact sequences of A-modules:

$$\begin{gathered} 0 \to M \otimes_B Y \to M \otimes_B Q^0 \to M \otimes_B Q^1 \to \cdots, \\ 0 \to \operatorname{Coker} \phi \to P^0 \to P^1 \to \cdots. \end{gathered}$$

Since ${}_AM$ is projective, $M \otimes_B Q^i$ is a projective A-module for each $i \geq 0$. Since $\operatorname{Ext}^1_A(\operatorname{Coker} \phi, M \otimes_B Q^0) = 0$, it follows from the exact sequence

$$0 \to M \otimes_B Y \xrightarrow{\phi} X \to \operatorname{Coker} \phi \to 0$$

that the map $M \otimes_B Y \to M \otimes_B Q^0$ factors through ϕ. So, by a version of the horseshoe lemma, we see that there is an exact sequence of A-modules

$$0 \to X \to P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \to \cdots \tag{4-3}$$

with

$$\partial^i = \begin{pmatrix} d^i & 0 \\ \sigma^i & \mathrm{id} \otimes_B d'^i \end{pmatrix}, \qquad \sigma^i : P^i \to M \otimes_B Q^i$$

for all $i \in \mathbb{Z}$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \xrightarrow{\mathrm{id} \otimes_B d'^0} & M \otimes_B Q^1 & \longrightarrow & \cdots \\ & & \downarrow \phi & & \downarrow \binom{0}{\mathrm{id}} & & \downarrow \binom{0}{\mathrm{id}} & & \\ 0 & \longrightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \xrightarrow{\partial^0} & P^1 \oplus (M \otimes_B Q^1) & \longrightarrow & \cdots \end{array} \tag{4-4}$$

commutes. By the same argument we get the following commutative diagram with exact rows:

$$\text{(4-5)}\qquad \begin{array}{ccccccc} \cdots \longrightarrow & M\otimes_B Q^{-2} & \xrightarrow{\mathrm{id}\otimes_B d'^{-2}} & M\otimes_B Q^{-1} & \longrightarrow & M\otimes_B Y & \longrightarrow 0 \\ & \downarrow{\scriptstyle\binom{0}{\mathrm{id}}} & & {\scriptstyle\binom{0}{\mathrm{id}}}\downarrow & & \downarrow{\scriptstyle\phi} & \\ \cdots \longrightarrow & P^{-2}\oplus(M\otimes_B Q^{-2}) & \xrightarrow{\partial^{-2}} & P^{-1}\oplus(M\otimes_B Q^{-1}) & \longrightarrow & X & \longrightarrow 0. \end{array}$$

Putting (4-4) and (4-5) together, we get the exact sequence of projective Λ-modules

$$\text{(4-6)}\quad L^\bullet = \quad \cdots\longrightarrow \begin{pmatrix} P^{-1}\oplus(M\otimes_B Q^{-1}) \\ Q^{-1}\end{pmatrix} \longrightarrow \begin{pmatrix} P^{0}\oplus(M\otimes_B Q^{0}) \\ Q^{0}\end{pmatrix}_{\binom{0}{\mathrm{id}}} \xrightarrow{\begin{pmatrix}\partial^0\\ d'^0\end{pmatrix}} \begin{pmatrix} P^{1}\oplus(M\otimes_B Q^{1}) \\ Q^{1}\end{pmatrix} \longrightarrow\cdots$$

with $\operatorname{Ker}\begin{pmatrix}\partial^0\\ d'^0\end{pmatrix} = \begin{pmatrix}X\\ Y\end{pmatrix}_\phi$.

For each projective A-module P, $\operatorname{Hom}_\Lambda\left(L^\bullet, \binom{P}{0}\right)\cong\operatorname{Hom}_A(P^\bullet, P)$ is exact, since $P^\bullet$ is a complete projective resolution. For each projective B-module Q, since $Q^\bullet$ is a complete projective resolution, $\operatorname{Hom}_B(Q^\bullet, Q)$ is exact. Since $M\otimes_B Q$ is projective, $\operatorname{Hom}_A(P^\bullet, M\otimes_B Q)$ is exact. Note that

$$\operatorname{Hom}_\Lambda\left(L^\bullet, \begin{pmatrix}M\otimes_B Q\\ Q\end{pmatrix}\right)\cong \operatorname{Hom}_A(P^\bullet, M\otimes_B Q)\oplus\operatorname{Hom}_B(Q^\bullet, Q);$$

here the direct sum only means that each term of the complex at the left side is a direct sum of terms of complexes at the right side, that is, it does not mean a direct sum of complexes; in fact, the complex at the right side has differentials

$$\begin{pmatrix}\operatorname{Hom}_A(d^i, M\otimes_B Q) & \operatorname{Hom}_A(\sigma^i, M\otimes_B Q)\\ 0 & \operatorname{Hom}_B(d'^i, Q)\end{pmatrix}.$$

By the canonical exact sequence of complexes

$$0\to\operatorname{Hom}_A(P^\bullet, M\otimes_B Q)\xrightarrow{\binom{\mathrm{id}}{0}}\operatorname{Hom}_\Lambda\left(L^\bullet,\begin{pmatrix}M\otimes_B Q\\ Q\end{pmatrix}\right)\xrightarrow{(0\ \mathrm{id})}\operatorname{Hom}_B(Q^\bullet, Q)\to 0,$$

we know that

$$\operatorname{Hom}_\Lambda\left(L^\bullet,\begin{pmatrix}M\otimes_B Q\\ Q\end{pmatrix}\right)$$

is also exact. We conclude that $L^\bullet$ is a complete Λ-projective resolution, and hence $\binom{X}{Y}_\phi$ is a Gorenstein-projective Λ-module.

Conversely, assume that $\binom{X}{Y}_\phi\in\mathcal{GP}(\Lambda)$. Then there is a complete Λ-projective resolution (4-6) with

$$\operatorname{Ker}\begin{pmatrix}\partial^0\\ d'^0\end{pmatrix}=\begin{pmatrix}X\\ Y\end{pmatrix}_\phi.$$

Then we get an exact sequence (4-1) of projective B-modules with $\operatorname{Ker} d'^0 = Y$, and the exact sequence

$$\text{(4-7)}\quad V^\bullet = \cdots \to P^{-1} \oplus (M \otimes_B Q^{-1}) \to P^0 \oplus (M \otimes_B Q^0) \xrightarrow{\partial^0} P^1 \oplus (M \otimes_B Q^1) \to \cdots$$

of projective A-modules with $\operatorname{Ker} \partial^0 = X$. Since M_B is projective, it follows that $M \otimes_B Q^\bullet$ is exact. Since $\binom{\partial^i}{d'^i}$ is a Λ-map, by (4-6) we know that ∂^i is of the form

$$\partial^i = \begin{pmatrix} d^i & 0 \\ \sigma^i & \mathrm{id} \otimes_B d'^i \end{pmatrix},$$

where $\sigma^i : P^i \to M \otimes_B Q^i$ for all $i \in \mathbb{Z}$, and

$$P^\bullet = \quad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

is a complex. By the canonical exact sequence of complexes

$$0 \longrightarrow M \otimes_B Q^\bullet \xrightarrow{\binom{\mathrm{id}}{0}} V^\bullet \xrightarrow{(0\ \mathrm{id})} \operatorname{Hom}_B(Q^\bullet, Q)P^\bullet \longrightarrow 0,$$

we see that $P^\bullet$ is also exact.

From (4-6) we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M \otimes_B Y & \longrightarrow & M \otimes_B Q^0 & \longrightarrow & M \otimes_B Q^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \binom{0}{\mathrm{id}}} & & \downarrow{\scriptstyle \binom{0}{\mathrm{id}}} & & \\
0 & \longrightarrow & X & \longrightarrow & P^0 \oplus (M \otimes_B Q^0) & \to & P^1 \oplus (M \otimes_B Q_1) & \to & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \\
0 & \to & \operatorname{Coker} \phi & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle d^0} & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus $\phi : M \otimes_B Y \longrightarrow X$ is injective and $\operatorname{Ker} d^0 \cong \operatorname{Coker} \phi$. For each projective A-module P, since

$$\operatorname{Hom}_\Lambda\left(L^\bullet, \binom{P}{0}\right) \cong \operatorname{Hom}_A(P^\bullet, P)$$

and $L^\bullet$ is a complete projective resolution, it follows that $P^\bullet$ is a complete projective resolution, and hence $\operatorname{Coker} \phi$ is a Gorenstein-projective A-module.

For each projective B-module Q, since $P^\bullet$ is a complete projective resolution, it follows that $\operatorname{Hom}_A(P^\bullet, M \otimes_B Q)$ is exact. Since $L^\bullet$ is a complete projective resolution, it follows that

$$\operatorname{Hom}_\Lambda\left(L^\bullet, \begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}\right) \cong \operatorname{Hom}_A(P^\bullet, M \otimes_B Q) \oplus \operatorname{Hom}_B(Q^\bullet, Q)$$

is exact (again, the direct sum does not mean a direct sum of complexes). By the same argument we know that $\operatorname{Hom}_B(Q^\bullet, Q)$ is exact. It follows that Y is a Gorenstein-projective B-module. □

5. Main result

5A. The aim of this section is to prove the following characterization of Gorenstein-projective Λ-modules, where Λ is the path algebra of a finite acyclic quiver over a finite-dimensional algebra. We emphasize that here Λ is not assumed to be Gorenstein.

Theorem 5.1. *Let Q be a finite acyclic quiver, and A a finite-dimensional algebra over a field k. Let $\Lambda = A \otimes_k kQ$, and $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a Λ-module. Then $X \in \mathcal{GP}(\Lambda)$ if and only if $X \in \operatorname{Mon}(Q, A)$ and X satisfies this condition*:

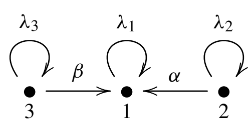
(G) *for each $i \in Q_0$, X_i and the quotient* $X_i \Big/ \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha$ *lie in* $\mathcal{GP}(A)$.

Example 5.2. (i) Taking

$$Q = \underset{n}{\bullet} \to \cdots \to \underset{1}{\bullet}$$

in Theorem 5.1, we get that a $T_n(A)$-module $X = (X_i, \phi_i)$ is Gorenstein-projective if and only if each ϕ_i is injective and that each X_i is a Gorenstein-projective A-module and each $\operatorname{Coker} \phi_i$ is a Gorenstein-projective A-module. Under the assumption that A is Gorenstein, this result has been obtained in [Zhang 2011, Corollary 4.1]; the case for $n = 2$ was treated in [Li and Zhang 2010, Theorem 1.1(i)]; see also [Iyama et al. 2011, Proposition 3.6(i)].

(ii) Let Λ be the k-algebra given by quiver



with relations $\lambda_1^2, \lambda_2^2, \lambda_3^2, \alpha\lambda_2 - \lambda_1\alpha, \beta\lambda_3 - \lambda_1\beta$. Then

$$\Lambda = A \otimes_k kQ = \begin{pmatrix} A & A & A \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix},$$

where Q is the quiver

$$\underset{3}{\bullet} \longrightarrow \underset{1}{\bullet} \longleftarrow \underset{2}{\bullet},$$

and $A = k[x]/\langle x^2 \rangle$. Let k be the simple A-module, and $\sigma : k \hookrightarrow A$ the inclusion. By Theorem 5.1, the following Λ-modules lie in $\mathrm{GP}(\Lambda)$:

$$\begin{aligned}
&(X_1 = A,\ X_2 = 0,\ X_3 = 0,\ X_\alpha = 0 = X_\beta),\\
&(X_1 = A,\ X_2 = A,\ X_3 = 0,\ X_\alpha = \mathrm{id},\ X_\beta = 0),\\
&(X_1 = A,\ X_2 = 0,\ X_3 = A,\ X_\alpha = 0,\ X_\beta = \mathrm{id}),\\
&(X_1 = k,\ X_2 = 0,\ X_3 = 0,\ X_\alpha = 0 = X_\beta),\\
&(X_1 = k,\ X_2 = k,\ X_3 = 0,\ X_\alpha = \mathrm{id},\ X_\beta = 0),\\
&(X_1 = k,\ X_2 = 0,\ X_3 = k,\ X_\alpha = 0,\ X_\beta = \mathrm{id}),\\
&(X_1 = A,\ X_2 = k,\ X_3 = 0,\ X_\alpha = \sigma,\ X_\beta = 0),\\
&(X_1 = A,\ X_2 = 0,\ X_3 = k,\ X_\alpha = 0,\ X_\beta = \sigma),\\
&\big(X_1 = A \oplus k,\ X_2 = k,\ X_3 = k,\ X_\alpha = \big(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\big),\ X_\beta = \big(\begin{smallmatrix} \sigma \\ \mathrm{id} \end{smallmatrix}\big)\big).
\end{aligned}$$

In fact this is the complete list of the pairwise nonisomorphic indecomposable Gorenstein-projective Λ-modules. Also by Theorem 5.1,

$$(Y_1 = A, Y_2 = k, Y_3 = k, Y_\alpha = \sigma = Y_\beta) \notin \mathcal{GP}(\Lambda).$$

For a description of all the pairwise nonisomorphic indecomposable Gorenstein-projective Λ-modules see [Ringel and Zhang 2011], where Λ is the path algebra of an arbitrary acyclic quiver over $A = k[x]/\langle x^2 \rangle$.

5B. We prove Theorem 5.1 by using Theorem 4.1 and induction on $|Q_0|$.

Remember we label Q_0 as $1, \dots, n$, in such a way that $j > i$ if $\alpha : j \to i$ is in Q_1. Thus n is a source of Q. Denote by Q' the quiver obtained from Q by deleting vertex n, and $\Lambda' = A \otimes_k kQ'$. Let $P(n)$ be the indecomposable projective (left) kQ-module at vertex n. Put $P = A \otimes_k \operatorname{rad} P(n)$. Clearly P is a Λ'-A-bimodule and $\Lambda = \big(\begin{smallmatrix} \Lambda' & P \\ 0 & A \end{smallmatrix}\big)$; compare (2-2).

Since kQ is hereditary, $\operatorname{rad} P(n)$ is a projective kQ'-module, and hence $P = A \otimes_k \operatorname{rad} P(n)$ is a (left) projective Λ'-module, and a (right) projective A-module (since as a right A-module, P is a direct sum of copies of A_A). So we can apply Theorem 4.1. For this, we write a Λ-module $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ as $X = \big(\begin{smallmatrix} X' \\ X_n \end{smallmatrix}\big)_\phi$, where $X' = (X_i, X_\alpha, i \in Q'_0, \alpha \in Q'_1)$ is a Λ'-module, and

$$\phi : P \otimes_A X_n \to X'$$

is a Λ'-map. The explicit expression of ϕ is given in the proof of Lemma 5.4. We keep all these notations of Q', Λ', $P(n)$, P, X' and ϕ throughout this section.

5C. By a direct translation from Theorem 4.1 in this special case, we have:

Lemma 5.3. *Let* $X = \binom{X'}{X_n}_\phi$ *be a* Λ*-module. Then* $X \in \mathcal{GP}(\Lambda)$ *if and only if* X *satisfies the following conditions*:

(i) $X_n \in \mathcal{GP}(A)$.

(ii) $\phi : P \otimes_A X_n \to X'$ *is injective.*

(iii) $\operatorname{Coker} \phi \in \mathcal{GP}(\Lambda')$.

For each $i \in Q'_0$, put $\mathcal{A}(n \to i)$ to be the set of arrows from n to i; and $\mathcal{P}(n \to i)$ the set of paths from n to i. For an integer $m \geq 0$ and a module M, let M^m denote the direct sum of m copies of M.

Lemma 5.4. *Let* $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ *be a* Λ*-module. If* X_β *is injective for each* $\beta \in Q'_1$, $\phi : P \otimes_A X_n \to X'$ *is injective if and only if* X_α *is injective for all* $\alpha \in Q_1$, *and* $\sum_{p \in \mathcal{P}(n\to i)} \operatorname{Im} X_p = \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p$ *for all* Q'_0.

Proof. For $i \in Q'_0$, set $m_i = |\mathcal{P}(n \to i)|$. As a kQ'-module, $\operatorname{rad} P(n)$ can be written as

$$\begin{pmatrix} k^{m_1} \\ \vdots \\ k^{m_{n-1}} \end{pmatrix}$$

(see (2-1) and Section 5B), hence we have isomorphisms of Λ'-modules

$$P \otimes_A X_n \cong (\operatorname{rad} P(n) \otimes_k A) \otimes_A X_n \cong \operatorname{rad} P(n) \otimes_k X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix}.$$

Let $\mathcal{P}(n \to i) = \{p_1, \dots, p_{m_i}\}$. Then ϕ is of the form

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n-1} \end{pmatrix} : P \otimes_A X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix} \to \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix},$$

where $\phi_i = (X_{p_1}, \dots, X_{p_{m_i}}) : X_n^{m_i} \to X_i$ (for the meaning of X_p see Section 2C). So ϕ is injective if and only if ϕ_i is injective for each $i \in Q'_0$, and if and only if

$$\sum_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p = \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \quad \text{and} \quad X_p \text{ is injective for all } p \in \mathcal{P}(n \to i).$$

From this and the assumption the assertion follows. □

Lemma 5.5. *Let* $X = \binom{X'}{X_n}_\phi$ *be a monic* Λ*-module.*

(1) *For each* $i \in Q'_0$ *there holds* $\sum_{p \in \mathcal{P}(n\to i)} \operatorname{Im} X_p = \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p$.

(2) $\phi : P \otimes_A X_n \to X'$ *is injective.*

(3) $\operatorname{Coker}\phi = \big(X_i/\bigoplus_{p\in\mathcal{P}(n\to i)}\operatorname{Im}X_p,\ \widetilde{X}_\alpha,\ i\in Q'_0,\ \alpha\in Q'_1\big)$, *where, for each* $\alpha: j\to i$ *in* Q'_1,

$$\widetilde{X}_\alpha : X_j\Big/\bigoplus_{q\in\mathcal{P}(n\to j)}\operatorname{Im}X_q \to X_i\Big/\bigoplus_{p\in\mathcal{P}(n\to i)}\operatorname{Im}X_p$$

is the A*-map induced by* X_α.

Proof. By Lemma 5.4 and its proof, it suffices to prove (1). For each $i\in Q'_0$, set $l_i=0$ if $\mathcal{P}(n\to i)$ is empty, and $l_i=\max\{l(p)\mid p\in\mathcal{P}(n\to i)\}$ otherwise, where $l(p)$ is the length of p. We use induction on l_i. If $l_i=0$, (1) trivially holds. Suppose $l_i\geqslant 1$. Let $\sum_{p\in\mathcal{P}(n\to i)}X_p(x_{n,p})=0$ for $x_{n,p}\in X_n$. Since

$$\sum_{p\in\mathcal{P}(n\to i)-\mathcal{A}(n\to i)}\operatorname{Im}X_p=\sum_{\substack{\alpha\in Q'_1\\ e(\alpha)=i}}X_\alpha\Big(\sum_{q\in\mathcal{P}(n\to s(\alpha))}\operatorname{Im}X_q\Big),$$

we have

$$\begin{aligned}0=\sum_{p\in\mathcal{P}(n\to i)}X_p(x_{n,p})&=\sum_{\alpha\in\mathcal{A}(n\to i)}X_\alpha(x_{n,\alpha})+\sum_{p\in\mathcal{P}(n\to i)-\mathcal{A}(n\to i)}X_p(x_{n,p})\\ &=\sum_{\alpha\in\mathcal{A}(n\to i)}X_\alpha(x_{n,\alpha})+\sum_{\substack{\beta\in Q'_1\\ e(\beta)=i}}X_\beta\Big(\sum_{q\in\mathcal{P}(n\to s(\beta))}X_q(x_{n,\beta q})\Big).\end{aligned}$$

By (m2) in Definition 2.2 we know that $X_\alpha(x_{n,\alpha})=0$ for $\alpha\in\mathcal{A}(n\to i)$, and

$$X_\beta\Big(\sum_{q\in\mathcal{P}(n\to s(\beta))}X_q(x_{n,\beta q})\Big)=0$$

for $\beta\in Q'_1$ with $e(\beta)=i$. So $\sum_{q\in\mathcal{P}(n\to s(\beta))}X_q(x_{n,\beta q})=0$ by condition (m1) in Definition 2.2. Since $l_{s(\beta)}<l_i$ for each $\beta\in Q'_1$ with $e(\beta)=i$, it follows from the inductive hypothesis that $X_q(x_{n,\beta q})=0$ for $\beta\in Q'_1$, $e(\beta)=i$, and $q\in\mathcal{P}(n\to s(\beta))$. This proves (1) and the lemma. □

Lemma 5.6. *Let* $X=\binom{X'}{X_n}_\phi$ *be a monic* Λ*-module. Then* $\operatorname{Coker}\phi$ *is a monic* Λ'*-module.*

Proof. We need to prove that, for each $i\in Q'_0$, the Λ'-map

$$(\widetilde{X}_\alpha)_{\substack{\alpha\in Q'_1\\ e(\alpha)=i}}:\bigoplus_{\substack{\alpha\in Q'_1\\ e(\alpha)=i}}\Big(X_{s(\alpha)}\Big/\bigoplus_{q\in\mathcal{P}(n\to s(\alpha))}\operatorname{Im}X_q\Big)\to X_i\Big/\bigoplus_{p\in\mathcal{P}(n\to i)}\operatorname{Im}X_p$$

is injective. For this, assume that

$$\sum_{\substack{\alpha\in Q'_1\\ e(\alpha)=i}}\widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}})=0,$$

where $\overline{x_{s(\alpha),\alpha}}$ is the image of $x_{s(\alpha),\alpha} \in X_{s(\alpha)}$ in $X_{s(\alpha)}/\bigoplus_{q\in\mathcal{P}(n\to s(\alpha))} \operatorname{Im} X_q$. Then

$$\sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) \in \bigoplus_{p\in\mathcal{P}(n\to i)} \operatorname{Im} X_p.$$

So there are $x_{n,p} \in X_n$ such that

$$\sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = \sum_{p\in\mathcal{P}(n\to i)} X_p(x_{n,p}).$$

Thus

$$\begin{aligned} 0 &= \sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{p\in\mathcal{P}(n\to i)} X_p(x_{n,p}) \\ &= \sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) - \sum_{\beta\in\mathcal{A}(n\to i)} X_\beta(x_{n,\beta}) - \sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha\Big(\sum_{q\in\mathcal{P}(n\to s(\alpha))} X_q(x_{n,\alpha q})\Big) \\ &= \sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha} - \sum_{q\in\mathcal{P}(n\to s(\alpha))} X_q(x_{n,\alpha q})) - \sum_{\beta\in\mathcal{A}(n\to i)} X_\beta(x_{n,\beta}). \end{aligned}$$

Using the assumption on X, we get

$$x_{s(\alpha),\alpha} = \sum_{q\in\mathcal{P}(n\to s(\alpha))} X_q(x_{n,\alpha q}),$$

that is, $\overline{x_{s(\alpha),\alpha}} = 0$. □

Lemma 5.7. *Let $X = \binom{X'}{X_n}_\phi$ be a monic Λ-module satisfying (G). Then*

$$\Big(X_i\Big/\bigoplus_{p\in\mathcal{P}(n\to i)} \operatorname{Im} X_p\Big)\Big/\Big(\bigoplus_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} \operatorname{Im} \widetilde{X}_\alpha\Big)$$

is a Gorenstein-projective A-module for all $i \in Q_0'$.

Proof. Since

$$\bigoplus_{p\in\mathcal{P}(n\to i)-\mathcal{A}(n\to i)} \operatorname{Im} X_p \subseteq \sum_{\substack{\beta\in Q_1\\ e(\beta)=i}} \operatorname{Im} X_\beta,$$

it follows that

$$\sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} \operatorname{Im} \widetilde{X}_\alpha = \Big(\sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} \operatorname{Im} X_\alpha + \bigoplus_{p\in\mathcal{P}(n\to i)} \operatorname{Im} X_p\Big)\Big/\Big(\bigoplus_{p\in\mathcal{P}(n\to i)} \operatorname{Im} X_p\Big) \tag{5-1}$$

$$= \Big(\sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \operatorname{Im} X_\beta + \bigoplus_{p \in \mathcal{P}(n \to i) - \mathcal{A}(n \to i)} \operatorname{Im} X_p \Big) \Big/ \Big(\bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \Big)$$

$$= \Big(\sum_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \operatorname{Im} X_\beta \Big) \Big/ \Big(\bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \Big)$$

$$= \Big(\bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \operatorname{Im} X_\beta \Big) \Big/ \Big(\bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \Big)$$

(the last equality following by (m2) in Definition 2.2). Hence the desired quotient is $X_i / \bigoplus_{\substack{\beta \in Q_1 \\ e(\beta)=i}} \operatorname{Im} X_\beta$, which is Gorenstein-projective by (G). □

Lemma 5.8. *Let $X = \binom{X'}{X_n}_\phi$ be a monic Λ-module satisfying (G). Then*

$$X_i \Big/ \bigoplus_{p \in \mathcal{P}(n \to j)} \operatorname{Im} X_p$$

is a Gorenstein-projective A-module for each $i \in Q_0'$.

Proof. We prove the assertion by using induction on l_i, which is defined in the proof of Lemma 5.5. If $i \in Q_0'$ with $l_i = 0$, the assertion follows from (G).

Suppose $l_i \geqslant 1$. Since $\bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \subseteq \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha$, we have the exact sequence

$$0 \longrightarrow \Big(\bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha \Big) \Big/ \Big(\bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \Big)$$
$$\longrightarrow X_i \Big/ \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \longrightarrow X_i \Big/ \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha \longrightarrow 0,$$

and by (G) the last term on the second line is Gorenstein-projective. It suffices to prove that the term on the first line is Gorenstein-projective. By (5-1) this term is $\bigoplus_{\substack{\alpha \in Q_1' \\ e(\alpha)=i}} \operatorname{Im} \widetilde{X}_\alpha$. By Lemma 5.6 each $\widetilde{X}_\alpha$ is injective, and it follows that

$$\operatorname{Im} \widetilde{X}_\alpha \cong X_j \Big/ \bigoplus_{p \in \mathcal{P}(n \to j)} \operatorname{Im} X_p,$$

where $j = s(\alpha)$. Since $l_j < l_i$, the conclusion of the lemma follows from the inductive hypothesis. □

Lemma 5.9. *The sufficiency in Theorem 5.1 holds. That is, if*

$$X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$$

is a monic Λ-module satisfying (G), X is Gorenstein-projective.

Proof. Using induction on $n = |Q_0|$, the assertion clearly holds for $n = 1$. Suppose that the assertion holds for $n - 1$ with $n \geq 2$. It suffices to prove that X satisfies Lemma 5.3(i)–(iii).

Condition (i) is contained in (G); and condition (ii) follows from Lemma 5.5(2). By Lemma 5.6 $\operatorname{Coker}\phi$ is a monic Λ'-module; and by Lemmas 5.7 and 5.8 we know that $\operatorname{Coker}\phi$ satisfies (G). It follows from the inductive hypothesis that condition (iii) is satisfied. □

Lemma 5.10. *Let $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a Λ-module with X_n a Gorenstein-projective A-module. Then $P \otimes_A X_n$ is a Gorenstein-projective Λ'-module, where P is defined in Section 5B.*

Proof. Let $P(n)$ be the indecomposable projective kQ-module at vertex n. Writing $\operatorname{rad} P(n)$ as a representation of Q' over k, we have

$$\operatorname{rad} P(n) = (k^{m_i}, f_\alpha, i \in Q'_0, \alpha \in Q'_1),$$

where $m_i = |\mathcal{P}(n \to i)|$ for each $i \in Q'_0$. By the construction of $P(n)$ we know that $\operatorname{rad} P(n)$ has the following three properties:

(1) Each $f_\alpha : k^{m_{s(\alpha)}} \to k^{m_{e(\alpha)}}$ is injective.

(2) For each $i \in Q'_0$,

$$\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha) = i}} \operatorname{Im} f_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha) = i}} \operatorname{Im} f_\alpha.$$

(3) For each $i \in Q'_0$, $k^{m_i} / \big(\bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha) = i}} \operatorname{Im} f_\alpha\big)$ and $k^{|\mathcal{A}(n \to i)|}$ are isomorphic as k-spaces.

It follows that

$$\begin{aligned} P \otimes_A X_n &\cong (\operatorname{rad} P(n) \otimes_k A) \otimes_A X_n \\ &\cong \operatorname{rad} P(n) \otimes_k X_n = (X_n^{m_i}, f_\alpha \otimes_k \operatorname{id}_{X_n}, i \in Q'_0, \alpha \in Q'_1). \end{aligned}$$

By (1), (2), and (3) we clearly see that $P \otimes_A X_n$ is a monic Λ'-module satisfying (G); for example, by (3) we know that

$$X_n^{m_i} \Big/ \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha) = i}} \operatorname{Im}(f_\alpha \otimes_k \operatorname{id}_{X_n}) \cong X_n^{|\mathcal{A}(n \to i)|}$$

is a Gorenstein-projective A-module. Now the result follows from Lemma 5.9. □

5D. ***Proof of Theorem 5.1.*** By Lemma 5.9 it remains to prove necessity, namely, if X is a Gorenstein-projective Λ-module, X is a monic Λ-module satisfying (G). Using induction on $n = |Q_0|$, the assertion is clear for $n = 1$. Suppose that the assertion holds for $n-1$ with $n \geq 2$. We write X as $\binom{X'}{X_n}_\phi$. Then X satisfies conclusions (i)–(iii) of Lemma 5.3.

By (i) and Lemma 5.10 we know that $P \otimes_A X_n$ is a Gorenstein-projective Λ'-module. Then, by (ii) and (iii), we know that $X' \in \mathcal{GP}(\Lambda')$, since $\mathcal{GP}(\Lambda')$ is closed under extensions. By the inductive hypothesis X' is a monic Λ'-module satisfying (G). Hence:

(1) X_β is injective for each $\beta \in Q'_1$.

(2) X_i is Gorenstein-projective for each $i \in Q'_0$.

(3) X_α is injective for each $\alpha \in Q_1$.

(4) $\sum_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p = \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p$ for all $i \in Q'_0$.

We get (3) and (4) from (1), condition (ii), and Lemma 5.4.

Since $\operatorname{Coker} \phi = \big(X_i / \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p,\ \widetilde{X}_\alpha,\ i \in Q'_0,\ \alpha \in Q'_1\big)$ is a Gorenstein-projective Λ'-module, it follows from the inductive hypothesis that the following properties hold:

(5) For each $\alpha \in Q'_1$, $\widetilde{X}_\alpha$ is injective.

(6) $\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \operatorname{Im} \widetilde{X}_\alpha = \bigoplus_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \operatorname{Im} \widetilde{X}_\alpha$, for all $i \in Q'_0$.

Claim 1: *X satisfies (m2) in Definition 2.2.*

Indeed, suppose

$$\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = 0. \tag{5-2}$$

Since

$$\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) = \sum_{\alpha \in \mathcal{A}(n \to i)} X_\alpha(x_{s(\alpha),\alpha}) + \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}),$$

it follows that

$$\begin{aligned}\sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} \widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}}) &= \sum_{\substack{\alpha \in Q'_1 \\ e(\alpha)=i}} X_\alpha(x_{s(\alpha),\alpha}) + \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p \\ &= -\sum_{\alpha \in \mathcal{A}(n \to i)} X_\alpha(x_{s(\alpha),\alpha}) + \bigoplus_{p \in \mathcal{P}(n \to i)} \operatorname{Im} X_p = 0,\end{aligned}$$

where we used (5-2) for the second equality.

Then by (6) we have $\widetilde{X}_\alpha(\overline{x_{s(\alpha),\alpha}})=0$; and by (5) we know $\overline{x_{s(\alpha),\alpha}}=0$ for each $\alpha\in Q_1'$ with $e(\alpha)=i$. This means that there are $x_{n,q}\in X_n$ such that

$$x_{s(\alpha),\alpha}=\sum_{q\in\mathcal{P}(n\to s(\alpha))} X_q(x_{n,q})\ \in \sum_{q\in\mathcal{P}(n\to s(\alpha))}\operatorname{Im} X_q$$

for each $\alpha\in Q_1'$ with $e(\alpha)=i$. By (5-2) we have

$$0=\sum_{\alpha\in\mathcal{A}(n\to i)} X_\alpha(x_{n,\alpha})+\sum_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}} X_\alpha\Bigg(\sum_{q\in\mathcal{P}(n\to s(\alpha))} X_q(x_{n,q})\Bigg).$$

By (4) we know that $X_\alpha(x_{n,\alpha})=0$ for all $\alpha\in\mathcal{A}(n\to i)$, and that $X_\alpha X_q(x_{n,q})=0$ for all $\alpha\in Q_1'$ with $e(\alpha)=i$ and $q\in\mathcal{P}(n\to s(\alpha))$. Thus $X_\alpha(x_{s(\alpha),\alpha})=0$, for all $\alpha\in Q_1$ with $e(\alpha)=i$. This proves Claim 1.

Claim 2: $X_i/\bigoplus_{\substack{\beta\in Q_1\\ e(\beta)=i}}\operatorname{Im} X_\beta$ *is a Gorenstein-projective A-module for each* $i\in Q_0$.

Indeed, since $\operatorname{Coker}\phi$ is a Gorenstein-projective Λ'-module, by the inductive hypothesis we know that

$$\Bigg(X_i\Big/\bigoplus_{p\in\mathcal{P}(n\to i)}\operatorname{Im} X_p\Bigg)\Big/\bigoplus_{\substack{\alpha\in Q_1'\\ e(\alpha)=i}}\operatorname{Im}\widetilde{X}_\alpha$$

is a Gorenstein-projective A-module: it is exactly the desired module by (5-1).

Now, (3) and Claim 1 mean that X is a monic Λ-module; and (2), together with conclusion (i) of Lemma 5.3 and Claim 2, means that X satisfies (G). □

6. Corollaries

6A. For the definition of a Frobenius category in the sense of [Quillen 1973], we refer to [Happel 1988, p. 11; Keller 1990, Appendix A]. As a consequence of Theorem 5.1 and Proposition 2.4, we get the following characterization of self-injectivity.

Corollary 6.1. *Let A be a finite-dimensional algebra, and Q a finite acyclic quiver. Then the following are equivalent*:

(i) *A is self-injective.*

(ii) $\mathcal{GP}(A\otimes_k kQ)=\operatorname{Mon}(Q,A)$.

(iii) $\operatorname{Mon}(Q,A)$ *is a Frobenius category.*

Proof. (i) $\Longrightarrow$ (ii): If A is self-injective, every A-module is Gorenstein-projective, and hence (ii) follows from Theorem 5.1. The implication (ii)$\Longrightarrow$ (iii) is well-known.

(iii) $\Longrightarrow$ (i): Take a sink of Q, say vertex 1, and consider $D(A_A) \otimes_k P(1)$. By Proposition 2.4 (iii) it is an injective object in $\mathrm{Mon}(Q, A)$, and hence, by assumption, it is a projective object in $\mathrm{Mon}(Q, A)$. By Proposition 2.4(ii) we know that $D(A_A)$, the first branch of $D(A_A) \otimes_k P(1)$, is a projective A-module, that is, A is self-injective. □

Let $D^b(\Lambda)$ be the bounded derived category of Λ, and $K^b(\mathcal{P}(\Lambda))$ the bounded homotopy category of $\mathcal{P}(\Lambda)$. By definition the singularity category $D^b_{sg}(\Lambda)$ of Λ is the Verdier quotient $D^b(\Lambda)/K^b(\mathcal{P}(\Lambda))$. Buchweitz [1987, Theorem 4.4.1] proved that if Λ is Gorenstein, there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\mathcal{GP}(\Lambda)}$, where $\underline{\mathcal{GP}(\Lambda)}$ is the stable category of $\mathcal{GP}(\Lambda)$ modulo $\mathcal{P}(\Lambda)$; see also [Happel 1991, Theorem 4.6]. Note that if A is Gorenstein, $\Lambda = A \otimes_k kQ$ is Gorenstein; see [Auslander and Reiten 1991b, Proposition 2.2]. So we have the following.

Corollary 6.2. *Let A be a finite-dimensional Gorenstein algebra, and Q a finite acyclic quiver. Let $\Lambda = A \otimes_k kQ$. Then there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\mathcal{GP}(\Lambda)}$. In particular, if A is self-injective, then there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\mathrm{Mon}(Q, A)}$.*

6B. Recall the tensor product $Q \otimes Q'$ of two finite quivers Q and Q' (not necessarily acyclic). By definition $Q \otimes Q'$ is the quiver with

$$(Q \otimes Q')_0 = Q_0 \times Q'_0 \quad \text{and} \quad (Q \otimes Q')_1 = (Q_1 \times Q'_0) \cup (Q_0 \times Q'_1).$$

More explicitly, if $\alpha : i \to j$ is an arrow of Q, then, for each vertex $t' \in Q'_0$, there is an arrow $(\alpha, t') : (i, t') \to (j, t')$ of $Q \otimes Q'$; and if $\beta' : s' \to t'$ is an arrow of Q', then, for each vertex $i \in Q_0$, there is an arrow $(i, \beta') : (i, s') \to (i, t')$ of $Q \otimes Q'$.

Let $A = kQ/I$ and $B = kQ'/I'$ be two finite-dimensional k-algebras, where Q and Q' are finite quivers (not necessarily acyclic), and I, I' are admissible ideals of kQ, kQ', respectively. Then

$$A \otimes_k B \cong k(Q \otimes Q')/I \,\square\, I',$$

where $I \,\square\, I'$ is the ideal of $k(Q \otimes Q')$ generated by $(I \times Q'_0) \cup (Q_0 \times I')$ and the elements

$$(\alpha, t')(i, \beta') - (j, \beta')(\alpha, s'),$$

where $\alpha : i \to j$ is an arrow of Q, and $\beta' : s' \to t'$ is an arrow of Q'. See, for example, [Leszczyński 1994]. Note that $I \,\square\, I'$ may not be zero even if $I = 0 = I'$. We have proved this:

Fact. *$A \otimes_k B$ is hereditary (that is, $I \,\square\, I' = 0$) if and only if either $A \cong k^{|Q_0|}$ as algebras and $I' = 0$, or $B \cong k^{|Q'_0|}$ as algebras and $I = 0$.*

6C. One can describe when Λ is hereditary via $\mathrm{Mon}(Q, A)$.

Corollary 6.3. *Let A be a finite-dimensional basic algebra over an algebraically closed field k, Q a finite acyclic quiver with $|Q_1| \neq 0$, and $\Lambda = A \otimes_k kQ$. Then $\mathcal{P}(\Lambda) = \mathrm{Mon}(Q, A)$ if and only if Λ is hereditary.*

Proof. Without loss of generality we may assume that A is connected (an algebra is connected if it cannot be a product of two nonzero algebras).

If $\Lambda = A \otimes_k kQ$ is hereditary, then, by the fact above and the assumption on Q, we have $A = k$, and hence $\mathrm{Mon}(Q, k) = \mathcal{GP}(kQ)$ by Theorem 5.1. It follows that

$$\mathrm{Mon}(Q, A) = \mathcal{GP}(kQ) = \mathcal{P}(kQ) = \mathcal{P}(\Lambda).$$

Conversely, if $A \neq k$, A is not semisimple since A is assumed to be connected and basic and k is assumed to be algebraically closed. It follows that there is a nonprojective A-module M. Take a sink of Q, say vertex 1, and consider Λ-module $X = M \otimes_k P(1)$, where $P(1)$ is the simple projective kQ-module at vertex 1. Then $X \in \mathrm{Mon}(Q, A)$, but $X \notin \mathcal{P}(\Lambda)$. □

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Received May 10, 2012.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 264 No. 1 July 2013