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We extend the Siegel–Weil formula for unitary groups of hermitian forms over a skew field with involution of the second kind.

Introduction

The Siegel–Weil formula is an identity between an Eisenstein series and an integral of a theta function. After Weil [1965] proved such an identity when both sides of the identity are absolutely convergent, Kudla and Rallis [1988a; 1988b; 1994] extended it for symplectic groups beyond the range of absolute convergence. Their results were extended to almost all classical groups by several authors, of which we mention the following sample: [Tan 1998; Ichino 2004; 2007; Gan and Takeda 2011; Yamana 2011; 2013; Gan 2000]. In this paper we discuss the last case that has to be considered in the theory of classical dual pairs over a number field, namely, unitary groups of hermitian forms over a skew field with involution of the second kind.

Let E/F be a quadratic extension of number fields and D a division algebra with center E, of dimension δ^2 over E and provided with an antiautomorphism ρ of order two under which F is the fixed subfield of E. Let $\mathbb{A}$ and $\mathbb{A}_E$ be the rings of adeles of F and E, respectively. Let $\mathcal{W}$ be a left D-vector space of dimension $2n$ with a nondegenerate skew hermitian form that has a complete polarization, and V a right D-vector space of dimension m with a nondegenerate hermitian form. Let G and H be the unitary groups of $\mathcal{W}$ and V, respectively.

Let α_E denote the standard norm of $\mathbb{A}_E^\times$. A character of $\mathbb{A}_E^\times$ is called principal if it is a complex power of α_E. We denote by P the maximal parabolic subgroup of G that stabilizes a maximal isotropic subspace of $\mathcal{W}$. Note that P has a Levi decomposition $P = MN$ with $M \simeq \mathrm{GL}_n(D)$. For any unitary character χ of $\mathbb{A}_E^\times/E^\times$ and for any $s \in \mathbb{C}$, we consider the representation $I(s,\chi) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi\alpha_E^s$ induced from the character $m \mapsto \chi(\nu(m))\alpha_E(\nu(m))^s$, where ν is the reduced norm viewed

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as a character of the algebraic group $\mathrm{GL}_n(D)$ and the induction is normalized so that $I(s,\chi)$ is naturally unitarizable when s is pure imaginary. For any holomorphic section $f^{(s)}$ of $I(s,\chi)$, the Eisenstein series

$$E(g; f^{(s)}) = \sum_{\gamma\in P(F)\backslash G(F)} f^{(s)}(\gamma g)$$

is absolutely convergent for $\Re s > \delta n/2$ and has a meromorphic continuation to the whole s-plane. We denote by χ^0 the restriction of χ to $\mathbb{A}^\times$, by $\rho(\chi)$ the character defined by $\rho(\chi)(x) = \chi(x^\rho)$ for $x \in \mathbb{A}_E^\times$, and by $\epsilon_{E/F}$ the quadratic character of $\mathbb{A}^\times/F^\times$ associated to the extension E/F. The following theorem was proven in [Tan 1999] when $\delta = 1$.

Theorem 1. *Let $f^{(s)}$ be a holomorphic section of $I(s,\chi)$.*

(1) *If $\chi\rho(\chi)$ is not principal, then $E(g; f^{(s)})$ is entire.*

(2) *If $\chi = \rho(\chi)^{-1}$, then the poles of $E(g; f^{(s)})$ in $\Re s > -\frac{1}{2}$ are at most simple and can only occur in the set*

$$\left\{ \frac{\delta(n-j)}{2} \;\middle|\; j \in \mathbb{Z},\ 0 \le j < n,\ \chi^0 = \epsilon_{E/F}^{\delta j} \right\}.$$

Fix a nontrivial additive character ψ of $\mathbb{A}/F$ and a character χ_V of $\mathbb{A}_E^\times/E^\times$ such that $\chi_V^0 = \epsilon_{E/F}^{\delta m}$. The group $G(\mathbb{A}) \times H(\mathbb{A})$ acts on the Schwartz space $\mathscr{S}(V^n(\mathbb{A}))$ of $V^n(\mathbb{A})$ via the Weil representation ω_{ψ,V,χ_V}. Let $S(V^n(\mathbb{A}))$ be the subspace of $\mathscr{S}(V^n(\mathbb{A}))$ consisting of functions that correspond to polynomials in the Fock model at every archimedean place of F.

The theta function associated to $\Phi \in S(V^n(\mathbb{A}))$ is defined by

$$\Theta(g,h;\Phi) = \sum_{x\in V^n(F)} \omega_{\psi,V,\chi_V}(g)\Phi(h^{-1}x)$$

for $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$. By Weil's criterion [1965], the integral

$$I(g;\Phi) = \int_{H(F)\backslash H(\mathbb{A})} \Theta(g,h;\Phi)\,dh$$

is absolutely convergent for all Φ either if $r = 0$ or if $m - r > n$, where r is the dimension of a maximal totally isotropic subspace of $V(F)$. When $m \le n$ and $r > 0$, the integral diverges in general, but extends uniquely to a $G(\mathbb{A})$-intertwining, $H(\mathbb{A})$-invariant map on $S(V^n(\mathbb{A}))$ in light of the regularization introduced by Kudla and Rallis [1994].

For $\Phi \in S(V^n(\mathbb{A}))$ we define a section $f_\Phi^{(s)}$ of $I(s,\chi_V)$ by

$$f_\Phi^{(s)}(g) = |a(g)|^{s-s_0}\omega_{\psi,V,\chi_V}(g)\Phi(0),$$

where $g \in G(\mathbb{A})$, $s_0 = \delta(m-n)/2$ and the quantity $|a(g)|$ is defined in the notation section below.

Theorem 2. *If $m \le n$ or if $m - r > n$, then for all $\Phi \in S(V^n(\mathbb{A}))$ the series $E(g; f_\Phi^{(s)})$ is holomorphic at $s = s_0$ and*

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \varkappa I(g; \Phi),$$

where

$$\varkappa = \begin{cases} 2 & \text{if } m \le n, \\ 1 & \text{if } m - r > n. \end{cases}$$

Theorem 2 was proven in [Weil 1965] if $m > 2n$, and in [Tan 1998; Ichino 2004; 2007; Yamana 2011] if $\delta = 1$. The proof requires only slight technical modifications once all of the necessary local facts have been established. The group $G(F_v)$ is isomorphic to the quasisplit unitary group $U(\delta n, \delta n)$ or an inner form of $\mathrm{GL}_{2\delta n}(F_v)$, depending on whether v remains prime or splits in E. The former case has already been discussed in [Kudla and Sweet 1997; Ichino 2007; Lee and Zhu 1998], and the latter case is discussed in Section 1. Coupled with the doubling method, the Siegel–Weil formula relates the theory of theta liftings to the theory of automorphic L-functions. We study the doubling zeta integral for inner forms of general linear groups in the Appendix.

Notation

Let (D, E, F, ρ) be as in the introduction. The restriction of ρ to E, which we denote also by ρ, is the nontrivial automorphism of E over F. For a matrix x with entries in D, let $x^* = {}^t x^\rho$ be the conjugate transpose of x. If x is a square matrix, then $\nu(x)$ and $\tau(x)$ stand for its reduced norm and reduced trace to E.

Fix a natural number n and put $n' = \delta n$. Let $\mathcal{W} = D^{2n}$ be a left D-vector space with the skew hermitian form

$$\langle x, y \rangle = xJy^*, \qquad J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

for $x, y \in \mathcal{W}$. Let V be a right D-vector space of dimension m equipped with a nondegenerate hermitian form $(\,,\,)$. We denote by G (resp. H) the group of all D-linear transformations of $\mathcal{W}$ (resp. V) that leave $\langle\,,\,\rangle$ (resp. $(\,,\,)$) invariant. Put $s_0 = \delta(m-n)/2$.

We write P for the stabilizer in G of the maximal isotropic subspace of $\mathcal{W}$ defined by the vanishing of all but the last n coordinates. Let

$$\mathrm{Her}_n = \{x \in \mathrm{M}_n(D) \mid x^* = x\}$$

be the F-subvariety of $n \times n$ hermitian matrices. The group G has a maximal parabolic subgroup $P = MN$ given by

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^* \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_n(D) \right\},$$

$$N = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix} \;\middle|\; b \in \mathrm{Her}_n \right\}.$$

Let K be the standard maximal compact subgroup of $G(\mathbb{A})$. For any character χ of $\mathbb{A}_E^\times / E^\times$, the representation $I(s, \chi) = I_{n'}(s, \chi)$ is realized on the space of right K-finite functions $f^{(s)} : G(\mathbb{A}) \to \mathbb{C}$ satisfying

$$f^{(s)}\big(m(a)n(b)g\big) = \chi(\nu(a))\alpha_E(\nu(a))^{s+n'/2} f^{(s)}(g)$$

for all $a \in \mathrm{GL}_n(D(\mathbb{A}))$, $b \in \mathrm{Her}_n(\mathbb{A})$ and $g \in G(\mathbb{A})$. We define $|a(g)|$ by writing $g = pk \in G(\mathbb{A})$ with $p = m(a)n(b) \in P(\mathbb{A})$ and $k \in K$, and taking $|a(g)| = \alpha_E(\nu(a))$.

1. Degenerate principal series representations

For each place v of F, let F_v be the v-completion of F and set $E_v = E \otimes_F F_v$ and $D_v = D \otimes_F F_v$. A division algebra D with center E admits an involution of the second kind if and only if D_v is isomorphic to $\mathrm{M}_\delta(E_v)$ whenever v remains prime in E, and D_v is isomorphic to a direct sum of mutually opposite simple algebras whose centers are F_v whenever v splits in E (see [Scharlau 1985, Theorem 10.2.4]).

In the local setting we will depart slightly from our previous notation. Fix a place v of F and suppress it from the notation. Thus E is a quadratic étale algebra over the local field F, D an algebra whose center is E, ρ an involution of D whose restriction to E is the nontrivial automorphism of E over F, V a free right D-module of rank m, and $(\,,\,) : V \times V \to D$ an F-bilinear map satisfying the following conditions:

- for $a, b \in D$ and $x, y \in V$,

$$(x, y)^\rho = (y, x), \quad (xa, yb) = a^\rho (x, y) b;$$

- $(x, V) = 0$ implies that $x = 0$.

Let H be the unitary group of V. Let $G = \{g \in \mathrm{GL}_{2n}(D) \mid gJg^* = J\}$. For any quasicharacter χ of $E^\times$, let $I(s, \chi)$ be the analogous local induced representation of G. By Morita context, it is enough to consider the case where the triple (D, E, ρ) belongs to the following two types:

- $D = E$ is a quadratic extension of F and ρ generates $\mathrm{Gal}(E/F)$;
- $D = \boldsymbol{D} \oplus \boldsymbol{D}^{\mathrm{op}}$, $E = F \oplus F$ and $(x, y)^\rho = (y, x)$, where $\boldsymbol{D}$ is a division algebra central over F and $\boldsymbol{D}^{\mathrm{op}}$ is its opposite algebra.

The rank of D as a module over E is a square of a natural number that will be denoted by δ. Note that $n' = \delta n$ remains intact after the change in notation.

We fix a nontrivial additive character ψ of F and a character χ_V of $E^\times$ that satisfies $\chi_V^0 = \epsilon_{E/F}^{\delta m}$. Then $G \times H$ acts on the Schwartz space $\mathscr{S}(V^n)$ via the Weil representation ω_{ψ,V,χ_V}. Note that it depends on the data ψ, $(\,,\,)$ and χ_V (compare [Kudla 1994]). When F is a p-adic field, put $S(V^n) = \mathscr{S}(V^n)$. When $F = \mathbb{R}$ or $\mathbb{C}$, let $\mathfrak{g}$ be the complexified Lie algebra of G and $S(V^n)$ the subspace of $\mathscr{S}(V^n)$ that corresponds to the space of polynomials in the Fock model of ω_{ψ,V,χ_V}. In the archimedean case we only consider admissible representations of the pair $(\mathfrak{g}, K)$, although we will allow ourselves to speak of a representation of the group G. We write $R(V, \chi_V) = R_{n'}(V, \chi_V)$ for the image of the intertwining map

$$S(V^n) \to I(s_0, \chi_V), \quad \Phi \mapsto f_\Phi^{(s_0)}(g) = \omega_{\psi,V,\chi_V}(g)\Phi(0).$$

We extend $f_\Phi^{(s_0)}$ to the standard section $f_\Phi^{(s)}$ of $I(s, \chi_V)$.

We discuss the case $E = F \oplus F$. Put

$$e_1 = (1, 0), \quad e_2 = (0, 1), \quad V_1 = Ve_1, \quad V_2 = Ve_2.$$

We regard V_1 as a right $\boldsymbol{D}$-module and V_2 as both a right $\boldsymbol{D}^{\mathrm{op}}$-module and a left $\boldsymbol{D}$-module. Since $(V_i, V_i) = 0$ for $i = 1, 2$, the spaces V_1 and V_2 are paired nondegenerately against each other by $(\,,\,)$, and so an antiisomorphism

$$\jmath : \mathrm{End}(V_1, \boldsymbol{D}) \to \mathrm{End}(V_2, \boldsymbol{D}^{\mathrm{op}})$$

is defined by

$$(ax, y) = (x, \jmath(a)y), \quad a \in \mathrm{End}(V_1, \boldsymbol{D}),\ x \in V_1,\ y \in V_2.$$

We obtain

$$H = \left\{(a, \jmath(a)^{-1}) \in \mathrm{GL}(V_1, \boldsymbol{D}) \times \mathrm{GL}(V_2, \boldsymbol{D}^{\mathrm{op}}) \,\middle|\, a \in \mathrm{GL}(V_1, \boldsymbol{D})\right\}.$$

Thus projection onto the first or second factor induces an isomorphism of H onto $\mathrm{GL}(V_1, \boldsymbol{D})$ or $\mathrm{GL}(V_2, \boldsymbol{D}^{\mathrm{op}})$, respectively. For any nonnegative integer j we write $G'_j = \mathrm{GL}_j(\boldsymbol{D})$. Observe that

$$G = \{(g, J^{-1}\,{}^t g^{-1} J) \mid g \in G'_{2n}\}.$$

Through projection onto the first factor, we identify H with G'_m, G with G'_{2n}, and $P = MN$ with

$$M = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathrm{GL}_n(\boldsymbol{D})\right\}, \quad N = \left\{\begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix} \,\middle|\, b \in \mathrm{M}_n(\boldsymbol{D})\right\}.$$

We write $\nu = \nu_j$ for the reduced norm of $\mathrm{M}_j(\boldsymbol{D})$ and τ for the reduced trace of $\mathrm{M}_j(\boldsymbol{D})$. Let $\alpha_F(x) = |x|_F$ denote the normalized absolute value of $x \in F^\times$. When we write $\chi = (\chi_1, \chi_2)$, the representation $I(s, \chi)$ is translated to

$$I(s,\chi) = \mathrm{Ind}_P^{G'_{2n}}\big((\chi_1\alpha_F^s)\circ\nu_n \boxtimes (\chi_2\alpha_F^s)^{-1}\circ\nu_n\big).$$

If $E = F \oplus F$, then since χ_V is of the form (μ, μ^{-1}), we may assume that $\chi_V = 1$ by twisting, and we write $I(s) = I(s, 1)$ and $R(V) = R(V, 1)$. The Weil representation $\omega_{j,k}$ of the dual pair (G'_j, G'_k) can be taken to be the action on $\mathscr{S}(\mathrm{M}_{k,j}(\boldsymbol{D}))$ given by

$$\omega_{j,k}(a,b)\phi(x) = \alpha_F(\nu_j(a))^{\delta k/2}\alpha_F(\nu_k(b))^{-\delta j/2}\phi(b^{-1}xa)$$

for $a \in G'_j$ and $b \in G'_k$. Note that the integral

$$(\phi,\phi') = \int_{\mathrm{M}_{k,j}(\boldsymbol{D})} \phi(u)\overline{\phi'(u)}\,du, \quad \phi,\phi' \in \mathscr{S}(\mathrm{M}_{k,j}(\boldsymbol{D}))$$

defines a $G'_j \times G'_k$ invariant positive definite hermitian form on $\omega_{j,k}$. The two models of the Weil representation $\omega_{2n,m} \simeq \omega_{\psi,V,1}$ are related by the partial Fourier transform

$$\mathscr{F}\phi(x,y) = \int_{\mathrm{M}_{m,n}(\boldsymbol{D})} \phi((x,z))\psi(-\tau(z\,{}^t y))\,dz \tag{1-1}$$

for $x \in \mathrm{M}_{m,n}(\boldsymbol{D})$ and $y \in \mathrm{M}_{m,n}(\boldsymbol{D}^{\mathrm{op}})$. In the p-adic case we write $\mathbb{O}$ for the maximal compact subring of $\boldsymbol{D}$ and put $K_n = \mathrm{GL}_n(\mathbb{O})$. In the archimedean case we set

$$K_n = \{g \in G'_n \mid {}^t\bar{g}g = \mathbf{1}_n\},$$

denoting the conjugate transpose of $x \in \mathrm{M}_n(\boldsymbol{D})$ by ${}^t\bar{x}$, where $\bar{\cdot}$ denotes the complex conjugate or the quaternion conjugate. We denote by $f_0^{(s)}$ a unique section of $I(s)$ that is identically 1 on K_{2n}.

Lemma 1.1. *If $E = F \oplus F$, then $R(V)$ contains $f_0^{(s_0)}$.*

Proof. In the p-adic case, we let $\phi_{j,k}$ be the characteristic functions of $\mathrm{M}_{j,k}(\mathbb{O})$. In the archimedean case we let

$$\phi_{j,k}(x) = e^{-\pi\,\mathrm{Tr}_{F/\mathbb{R}}(\tau({}^t\bar{x}x))},$$

assuming that $\psi(\,\cdot\,) = e^{2\pi\sqrt{-1}\mathrm{Tr}_{F/\mathbb{R}}(\,\cdot\,)}$. Put $\Phi = \mathscr{F}\phi_{2n,m}$. Then $f_\Phi^{(s_0)}$ is nonzero and right invariant under K_{2n}. □

The local intertwining operator is defined analogously by

$$M(s,\chi)f^{(s)}(g) = \int_{\mathrm{Her}_n(F)} f^{(s)}(Jn(b)g)\,db.$$

We define holomorphic sections and standard sections similarly. We write χ^0 for the restriction of χ to $F^\times$. Put

$$a(s,\chi)=a_{n'}(s,\chi)=\prod_{j=1}^{n'} L\big(2s-j+1,\,\chi^0\cdot\epsilon_{E/F}^{n'+j}\big),$$

$$b(s,\chi)=b_{n'}(s,\chi)=\prod_{j=1}^{n'} L\big(2s+j,\,\chi^0\cdot\epsilon_{E/F}^{n'+j}\big).$$

A normalized intertwining operator $M^*(s,\chi)$ is defined by setting

$$M^*(s,\chi)=a(s,\chi)^{-1}M(s,\chi).$$

Lemma 1.2. *The operator $M^*(s,\chi)$ is entire.*

Proof. When E/F is a quadratic extension of p-adic fields, Lemma 1.2 is proven in Proposition 3.2 of [Kudla and Sweet 1997]. The proof is completely analogous when $E/F=\mathbb{C}/\mathbb{R}$. Note that Proposition 3A.6 of the same work applies also to this case by a global consideration, namely, by applying (24) of [Lapid and Rallis 2005] with base field $\mathbb{Q}$ and $S=\{\infty\}$.

We suppose that $E=F\oplus F$. For $\phi\in\mathscr{S}(\mathrm{M}_n(\boldsymbol{D}))$ we define a section $f_\phi^{(s)}$ of $I(s,\chi)$ by requiring that $\mathrm{supp}(f_\phi^{(s)})\subset PJN$ and $f_\phi^{(s)}(g)=\phi(b)$ if $g=Jn(b)$ for $b\in\mathrm{Her}_n(F)$. As explained in [Piatetski-Shapiro and Rallis 1987b; Kudla and Sweet 1997], all we have to do is to show that the ratio $a(s,\chi)^{-1}M(s,\chi)f_\phi^{(s)}(J)$ is entire. One can easily observe that

$$M(s,\chi)f_\phi^{(s)}(J)=Z^{GJ}\left(2s-\frac{n'}{2},\,\phi,\,\chi^0\circ\nu_n\right),$$

where the right-hand side is the zeta integral studied in [Weil 1974; Godement and Jacquet 1972] (see the Appendix). Our claim follows at once, as the Godement–Jacquet L-factor

$$L^{GJ}\left(2s-\frac{n'-1}{2},\,\chi^0\circ\nu_n\right)$$

divided by the factor $a(s,\chi)$ is entire. □

For $\beta\in\mathrm{Her}_n(F)$, let ψ_β be the character of N defined by $\psi_\beta(n(b))=\psi(\tau(\beta b))$. Notice that $\tau(\beta b)\in F$. The Fourier transform of a Schwartz function $f\in\mathscr{S}(N)$ is defined by

$$\hat f(\beta)=\int_N f(u)\psi_\beta(u)\,du.$$

For each integer $j \le n'$, we define the subvariety Her_n^j of $\mathrm{Her}_n(F)$ by

$$(E \not\simeq F \oplus F) \qquad \mathrm{Her}_n^j = \{\beta \in \mathrm{M}_n(E) \mid {}^t\beta^\rho = \beta,\ \mathrm{rank}_E \beta \le j\},$$

$$(E = F \oplus F) \qquad \mathrm{Her}_n^j = \{(\beta, {}^t\beta) \in \mathrm{M}_n(\boldsymbol{D}) \oplus \mathrm{M}_n(\boldsymbol{D}^{\mathrm{op}}) \mid \delta(\mathrm{rank}_{\boldsymbol{D}} \beta) \le j\}.$$

Definition 1.3. We say that a representation π of G has rank at most j if $f \in \mathscr{S}(N)$ acts by zero on π whenever $\hat{f}$ vanishes on Her_n^j. We say that π is of rank j if in addition j is a multiple of δ and π does not have rank less than j.

For any H-module π, we write π_H for the maximal quotient of π on which H acts trivially. Let $\mathscr{H}_r$ be a split hermitian space of dimension $2r$, that is, $\mathscr{H}_r$ has a D-basis consisting of $2r$ elements e_i, f_i such that

$$(e_i, e_j) = (f_i, f_j) = 0, \quad (e_i, f_j) = \delta_{ij}.$$

Proposition 1.4. *Assume that $m \le n$. Let $U = V \oplus \mathscr{H}_{n-m}$.*

(1) *$R(V, \chi_V)$ is irreducible and unitarizable.*

(2) *$R(V, \chi_V)$ is isomorphic to $S(V^n)_H$.*

(3) *If E/F is a quadratic extension of p-adic fields, then $R(V, \chi_V)$ is of rank m.*

(4) *$R(U, \chi_V)$ has a unique irreducible quotient that is isomorphic to $R(V, \chi_V)$.*

(5) *$M^*(-s_0, \chi_V)$ maps $R(U, \chi_V)$ onto $R(V, \chi_V)$.*

(6) *$b(s, \chi_V) M^*(s, \chi_V) f_\Phi^{(s)}$ is holomorphic at $s = s_0$ for every $\Phi \in S(V^n)$.*

Proof. When $D = E$, these results are known (see [Li 1989; Mœglin et al. 1987; Kudla and Sweet 1997; Lee and Zhu 1998; Yamana 2011]). We may suppose that $E = F \oplus F$ and $\delta > 1$.

For $0 \le i \le k$, let $P_i^k = M_i^k N_i^k$ be the maximal parabolic subgroup of G'_k given by

$$P_i^k = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G'_k \;\middle|\; a \in G'_{k-i},\ b \in \mathrm{M}_{k-i,i}(\boldsymbol{D}),\ d \in G'_i \right\},$$

$\bar{P}_i^k$ its opposite parabolic subgroup, and r_i the representation of $G'_i \times G'_i$ on $\mathscr{S}(G'_i)$ given by

$$r_i(g_1, g_2)\phi(g) = \phi(g_2^{-1} g g_1), \quad (\phi \in \mathscr{S}(G'_i),\ g, g_1, g_2 \in G'_i).$$

In the archimedean case the representation $I(s)$ is studied extensively in [Lee 2007; Sahi 1995; Zhang 1995]. From their results we know the module structure of $I(s_0)$ and the set of K-types of each of its irreducible constituents, which combined with the technique explained in [Kudla and Rallis 1990a] prove (1), (2). We consider the nonarchimedean case. By Lemma 3.III.2 of [Mœglin et al. 1987], the representation $\omega_{2n,m}$ has a filtration

$$0 \subset S_m \subset \cdots \subset S_1 \subset S_0 = \omega_{2n,m}$$

with successive quotients

$$S_i/S_{i+1} \simeq \operatorname{Ind}_{P_i^{2n}\times \bar{P}_i^m}^{G'_{2n}\times G'_m} \mu_i,$$

where μ_i is the representation of $P_i^{2n} \times \bar{P}_i^m$ on $\mathscr{S}(G'_i)$ given by

$$\mu_i(p, p')\phi = \alpha_F\big(\nu(a)^{m-i}\nu(a')^{i-2n}\nu(d)^{m-i+2n}\nu(d')^{i-m-2n}\big)^{\delta/2} r_i(d, d')\phi,$$

where

$$p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P_i^{2n}, \quad p' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \bar{P}_i^m, \quad \phi \in \mathscr{S}(G'_i).$$

Let $\mathbb{1}_j$ denote the trivial representation of G'_j. For $0 \le i < m$ and an admissible representation π of G'_{2n}, the Frobenius reciprocity gives

$$\operatorname{Hom}_{G'_{2n}\times G'_m}\big(S_i/S_{i+1}, \pi \otimes \mathbb{1}_m\big) \simeq \operatorname{Hom}_{M_i^{2n}\times M_i^m}\Big((\pi^\vee)_{N_i^{2n}} \otimes \delta_{P_i^m}^{1/2}, \mu_i^\vee\Big),$$

where $\delta_{P_i^m}$ is the modulus function on P_i^m and $(\pi^\vee)_{N_i^{2n}}$ is the normalized Jacquet module of $\pi^\vee$ associated to P_i^{2n}. Since the quasicharacters of G'_{m-i} do not match, the space above is zero. Thus $(S_i/S_{i+1})_{G'_m} = 0$, so that the natural map $(S_m)_{G'_m} \to (\omega_{2n,m})_{G'_m}$ is surjective. If χ is a quasicharacter of G'_m and if a distribution T on $\mathscr{S}(G'_m)$ transforms according to χ under the action of $e \times G'_m$, that is,

$$T(r_m(e, h)f) = \chi(\nu(h))T(f)$$

for all $h \in G'_m$, then there is a constant $c \in \mathbb{C}$ such that

$$T(f) = c\int_{G'_m} f(h)\chi(\nu(h))\,dh, \quad f \in \mathscr{S}(G'_m)$$

(see Lemma 3.II.3 of [Mœglin et al. 1987]). It follows that

$$(S_m)_{G'_m} \simeq \operatorname{Ind}_{P_m^{2n}}^{G'_{2n}}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m).$$

Since $\operatorname{Ind}_{P_m^{2n}}^{G'_{2n}}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m)$ is irreducible as a representation of G'_{2n} induced from a unitary representation [Sécherre 2009], we have

$$(\omega_{\psi,V,1})_H \simeq \operatorname{Ind}_{P_m^{2n}}^{G'_{2n}}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m).$$

Thus the map from $(\omega_{\psi,V,1})_H$ to $R(V)$ is injective. This proves (1), (2).

In the p-adic case, Theorem 5.1 of [Mínguez 2009] tells us that $I(s_0)$ has a unique irreducible subrepresentation, which is $R(V)$, and hence $I(-s_0)$ has a unique irreducible quotient. We refer to [Lee 2007] for the archimedean analogue. From Lemma 1.1 we can infer that $f_0^{(-s_0)}$ generates $I(-s_0)$. It follows that $I(-s_0) = R(U)$. The proof of (4) is complete.

To prove (5), (6), it suffices to check that $b(s)M^*(s)f_0^{(s)}$ (resp. $M^*(s)f_0^{(s)}$) are holomorphic and nonzero at $s = s_0$ (resp. $s = -s_0$) in light of [Kudla and Rallis 1988a, Proposition 4.9]. Let $\phi_0 = \phi_{n,n} \in S(\mathrm{M}_n(\boldsymbol{D}))$ be as in the proof of Lemma 1.1. Define $\phi_1 \in S(\mathrm{M}_{n,2n}(\boldsymbol{D}))$ by $\phi_1(x, y) = \phi_0(x)\phi_0(y)$. The sections $\mathfrak{F}^{(s)}_{\phi_1}$ and $\mathfrak{F}^{(s)}_{\hat{\phi}_1}$ are defined in the Appendix. Since $\mathfrak{F}^{(s)}_{\phi_1}$ is right K-invariant, so is $\mathfrak{F}^{(s)}_{\hat{\phi}_1}$ by Lemma A.1. From Propositions 10.7 and 10.8 of [Weil 1974], we know

$$\mathfrak{F}^{(s)}_{\phi_1} = \mathfrak{F}^{(s)}_{\phi_1}(e) \cdot f_0^{(s)} = Z^{GJ}\Big(2s + \frac{n'}{2}, \phi_0, 1\Big) \cdot f_0^{(s)} = f_0^{(s)} \prod_{j=1}^{n} \xi(2s + \delta j)$$

up to multiplication by exponential factors, where $\xi(s) = \zeta(s)$ in the p-adic case, and $\xi(s) = \Gamma(s)$ in the archimedean case. Observe that

$$\begin{aligned}\mathfrak{F}^{(-s)}_{\hat{\phi}_1} &= Z^{GJ}\Big(-2s + \frac{n'}{2}, \hat{\phi}_0, 1\Big) \cdot f_0^{(s)} \\ &= (-1)^{n(\delta-1)}\gamma^{GJ}\Big(2s - \frac{n'-1}{2}, \mathbb{1}_n, \psi\Big) Z^{GJ}\Big(2s - \frac{n'}{2}, \phi_0, 1\Big) \cdot f_0^{(s)}.\end{aligned}$$

Substituting these into the equality in Lemma A.1, we get

$$M(s)f_0^{(s)} = f_0^{(-s)} \prod_{j=1}^{n} \frac{\xi(2s - \delta j + \delta)}{\xi(2s + \delta j)}. \tag{1-2}$$

Now we can easily conclude our proof. □

2. Proof of Theorem 1

Back to the global setup, we write $\mathscr{A}$ for the space of automorphic forms on $G(\mathbb{A})$. For $\beta \in \mathrm{Her}_n(F)$ and $A \in \mathscr{A}$, let

$$A_\beta(g) = \int_{\mathrm{Her}_n(F)\backslash \mathrm{Her}_n(\mathbb{A})} A(n(b)g)\psi(-\tau(\beta b))\, db, \quad g \in G(\mathbb{A})$$

denote the β-th Fourier coefficient of A. The following lemma can be proven in exactly the same way as in [Kudla and Rallis 1990b; Tan 1999].

Lemma 2.1. *Let $f^{(s)}$ be a holomorphic section of $I(s, \chi)$ and $\beta \in \mathrm{Her}_n(F)$ with $\nu(\beta) \neq 0$.*

(1) *$b(s, \chi)E_\beta(g; f^{(s)})$ is holomorphic in $\Re s > -\frac{1}{2}$.*

(2) *If $m \geq n$ and β is represented by $V(F)$, then $E_\beta(g; f_\Phi^{(s)})$ can be made nonzero at $s = s_0$ for a suitable choice of $\Phi \in S(V^n(\mathbb{A}))$.*

(3) *If $\chi\rho(\chi)$ is not principal, then $E(g; f^{(s)})$ is entire.*

(4) *If* $\chi = \rho(\chi)^{-1}$, *then the poles of* $E(g;\, f^{(s)})$ *in* $\Re s > -\frac{1}{2}$ *are at most simple and can only occur in the set*

$$\left\{ \frac{n'-j}{2} \;\middle|\; j \in \mathbb{Z},\ 0 \le j < n',\ \chi^0 = \epsilon_{E/F}^{j} \right\}.$$

(5) *If* $\chi^0 = \epsilon_{E/F}^{n'+1}$, *then* $E(g;\, f^{(s)})|_{s=0}$ *is identically zero.*

Definition 2.2. For each integer $l \le n$, we say that $A \in \mathscr{A}$ has rank δl if $A_\beta = 0$ when $\operatorname{rank}_D \beta > l$, but $A_\beta \ne 0$ for some β of rank l. When π is a representation of $G(\mathbb{A})$ realized on a subspace of $\mathscr{A}$, we say that π has rank at most δl if all functions in π have rank at most δl.

We call A singular if it has rank less than δn. The following lemma can be proven in the same way as in the proof of [Howe 1981, Lemma 2.4].

Lemma 2.3. *Let* π *be a subrepresentation of* $\mathscr{A}$. *For every integer* $l \le n$ *the following conditions are equivalent*:

- π *has rank at most* δl;
- *for every place* v, $G(F_v)$ *acts on* π *by a representation of rank at most* δl;
- *for at least one place* v, $G(F_v)$ *acts on* π *by a representation of rank at most* δl.

In particular, if $G(F_v)$ *acts on* π *by a representation of rank at most* j, *then* $G(F_v)$ *acts on* π *by a representation of rank at most* $\delta\ell$, *where* $\ell = [j/\delta]$.

For $s' \in \mathbb{C}$ with $\Re s' > -\frac{1}{2}$, the residue $\operatorname{Res}_{s=s'} E(g;\, f^{(s)})$ depends only on $f^{(s')}$, and $f^{(s')} \mapsto \operatorname{Res}_{s=s'} E(g;\, f^{(s)})$ gives a $G(\mathbb{A})$ intertwining map

$$A_{-1}(s') : I(s', \chi) \to \mathscr{A}.$$

Assume that $\chi = \rho(\chi)^{-1}$, assume that j is an integer between 0 and n', assume that $\chi^0 = \epsilon_{E/F}^{j}$, and assume that j is not divisible by δ. Let $s' = (n'-j)/2$. To complete the proof of Theorem 1, it remains to prove that $A_{-1}(s')$ is zero. Fix a finite inert place v of F. By Theorem 1.2 of [Kudla and Sweet 1997], $I_v(s', \chi_v)$ has a unique irreducible submodule R and

$$I_v(s', \chi_v)/R \simeq \bigoplus_{V_0} R(V_0, \chi_v),$$

where V_0 runs over all equivalence classes of hermitian spaces over E_v of dimension j. Since the image of $A_{-1}(s')$ lies in the space of singular automorphic forms in view of Lemma 2.1(1) and since R is nonsingular, the map $A_{-1}(s')$ factors through the quotient $\bigoplus_{V_0} R(V_0, \chi_v)$ at v. Proposition 1.4(3) shows that $G(F_v)$ acts on the image of $A_{-1}(s')$ by a representation of rank at most j. Put $\ell = [j/\delta]$. Lemma 2.3 shows that $G(F_v)$ acts on the image of $A_{-1}(s')$ by a representation of rank at most $\delta\ell$. Since $\delta\ell < j$, Proposition 1.4(3) forces $A_{-1}(s')$ to be zero.

3. Proof of Theorem 2

Lemma 3.1. *If $m = n$ or if $m - r > n$, then for all $\Phi \in S(V^n(\mathbb{A}))$ and $\beta \in \mathrm{Her}_n(F)$ with $\nu(\beta) \neq 0$,*

$$E_\beta(g;\, f_\Phi^{(s)})|_{s=s_0} = \varkappa I_\beta(g;\, \Phi).$$

Proof. The proof can be carried out by the same technique as in that of [Ichino 2004, Proposition 6.2]. We omit the details. □

First we prove Theorem 2 in the case $m - r > n$. Ichino [2007] proved the special case of this result for $\delta = 1$ (compare [Kudla and Rallis 1988b; Yamana 2013]). Many of the results there apply word for word in our general case.

If $m > 2n$, then $E(g;\, f_\Phi^{(s_0)})$ converges absolutely and the stated identity was proven by Weil [1965]. We may suppose that $m \leq 2n$. Fix $\Phi^0 = \bigotimes_v \Phi_v^0 \in S(V^n(\mathbb{A}))$. By Theorem 10.6.2 of [Scharlau 1985], there is an inert place w of F such that the Witt index r_w of V_w satisfies $r_w < \delta(r+1)$, where V_w stands for the hermitian space over E_w corresponding to $V(F_w)$. Note that

$$\delta m - r_w > \delta n.$$

We consider the $G(F_w)$-intertwining map

$$A_{-1,w} : S(V_w^{n'}) \to \mathscr{A}, \quad \Phi_w \mapsto A_{-1}(s_0)(f_\Phi^{(s_0)}),$$

where $\Phi = \Phi_w \otimes \big(\bigotimes_{v \neq w} \Phi_v^0\big)$. The invariant distribution theorem [Mœglin et al. 1987; Lee and Zhu 1998] asserts that $A_{-1,w}$ factors through the quotient $R(V_w, \chi_{V_w})$. Lemma 2.1(1) shows that $A_{-1,w}(\Phi_w)$ is singular for every $\Phi_w \in S(V_w^{n'})$. If w is finite, then $\delta m = 2r_w + 2$ and $\delta n = r_w + 1$, and hence $R(V_w, \chi_{V_w})$ is irreducible and nonsingular by [Kudla and Sweet 1997, Theorem 1.2], so that $A_{-1,w}$ must be zero. If w is real and ∇ is the element of the universal enveloping algebra of the complexified Lie algebra of $G(F_w)$ defined by (2.1) of [Ichino 2007], then $\nabla A_{-1,w}(\Phi_w) = 0$. Since Proposition 2.2 of [Ichino 2007] asserts that $\nabla f_{\Phi_w}^{(s_0)}$ generates the submodule $R(V_w, \chi_{V_w})$ for a suitable choice of Φ_w, the map $A_{-1,w}$ must be zero. Consequently, $E(g;\, f_\Phi^{(s)})$ is holomorphic at $s = s_0$ for every $\Phi \in S(V^n(\mathbb{A}))$.

Next we consider the K_w-intertwining map

$$A_w : S(V_w^{n'}) \to \mathscr{A}, \quad \Phi_w \mapsto E(g;\, f_\Phi^{(s)})|_{s=s_0} - I(g;\, \Phi),$$

where $\Phi = \Phi_w \otimes \big(\bigotimes_{v \neq w} \Phi_v^0\big)$. The image of A_w lies in the space of singular automorphic forms by Lemma 3.1. We write $\mathscr{R}_w$ for the subspace of $\mathscr{A}$ spanned by residues $\mathrm{Res}_{s=s_0} E(g;\, f^{(s)})$, where $f^{(s)}$ is a holomorphic section of $I(s, \chi_V)$ of the form

$$f^{(s)} = f_w^{(s)} \otimes \Big(\bigotimes_{v \neq w} f_{\Phi_v^0}^{(s)}\Big), \quad f_w^{(s)} \in I_w(s, \chi_{V_w}).$$

Then A_w induces a $G(F_w)$-intertwining map $R(V_w, \chi_{V_w}) \to \mathscr{A}/\mathscr{R}_w$. The remaining part of the proof continues as in Section 3 of [Ichino 2007]. □

Theorem 2 is demonstrated in [Yamana 2011], provided that $\delta = 1$ and $m \le n$. Since the proof in our general case can be done by the same technique, we shall omit most of the details. We define the functions $a(s, \chi)$ and $b(s, \chi)$ by taking the complete Hecke L-functions in place of the local abelian L-factors in the definition of $a_v(s, \chi_v)$ and $b_v(s, \chi_v)$. We define a normalized global intertwining operator by

$$M^\circ(s, \chi) = \frac{b(s, \chi)}{a(s, \chi)} M(s, \chi),$$

which is holomorphic in $\Re s > -\frac{1}{2}$ by Lemma 1.2 and (1-2).

Let $\mathscr{C} = \{W_v\}$ be a collection of local hermitian spaces of dimension m over D_v such that W_v is isometric to $V(F_v)$ for almost all v. We form a restricted tensor product $\Pi(\mathscr{C}, \chi_V) = \bigotimes'_v R_{n'}(W_v, \chi_{V_v})$, which we can regard as a subrepresentation of $I(s_0, \chi_V)$. The proof of the following result is completely analogous to that of [Kudla and Rallis 1994, Theorem 3.1].

Proposition 3.2. *Assume that $m \le n$. Then*

$$\dim \operatorname{Hom}_{G(\mathbb{A})}(\Pi(\mathscr{C}, \chi_V), \mathscr{A}) \le 1.$$

If there is no global hermitian space with W_v as its completions, then

$$\dim \operatorname{Hom}_{G(\mathbb{A})}(\Pi(\mathscr{C}, \chi_V), \mathscr{A}) = 0.$$

Next we are going to prove the special case of Theorem 2 in which $m = n$. Let $\mathscr{C} = \{V(F_v)\}$. Since Proposition 1.4(2) shows that the two intertwining maps $\Phi \mapsto E(g; f^{(s)}_\Phi)|_{s=0}$ and $\Phi \mapsto I(g; \Phi)$ define elements of the space

$$\operatorname{Hom}_{G(\mathbb{A})}(\Pi(\mathscr{C}, \chi_V), \mathscr{A}),$$

they must be proportional by Proposition 3.2. From Lemmas 2.1(2) and 3.1, they are nonvanishing, and the constant of proportionality is determined to be 2. □

We now suppose that $m < n$. Let $\mathscr{C}'$ be a collection of local hermitian spaces of dimension $2n - m$ obtained by adding a split space of suitable dimension to $\mathscr{C}$. By Proposition 1.4(4) and (5), $\Pi(\mathscr{C}', \chi_V)$ has a unique irreducible quotient $\Pi(\mathscr{C}, \chi_V)$, and $M^\circ(-s_0, \chi_V)$ induces a nonzero intertwining map $\Pi(\mathscr{C}', \chi_V) \to \Pi(\mathscr{C}, \chi_V)$. The same reasoning as in Section 4 of [Yamana 2011] implies the following result:

Proposition 3.3. *Suppose that $m < n$. Let $f^{(s)}$ be a standard section of $I(s, \chi_V)$ such that $f^{(s_0)} \in \Pi(\mathscr{C}, \chi_V)$. Put $h^{(-s)} = M^\circ(s, \chi_V) f^{(s)}$.*

(1) *$E(g; f^{(s)})$ is holomorphic at $s = s_0$.*

(2) *$h^{(s)}$ is holomorphic at $s = -s_0$, $h^{(-s_0)} \in \Pi(\mathscr{C}', \chi_V)$, and*

$$\mathrm{Res}_{s=-s_0} E(g; h^{(s)}) = -\mathrm{Res}_{s=s_0}\left[\frac{b(s, \chi_V)}{a(s, \chi_V)}\right] E(g; f^{(s)})\big|_{s=s_0}.$$

Lemma 3.4. *If $m < n$, then the image of the map $A_{-1}(-s_0)$ lies in the space of square integrable automorphic forms on $G(\mathbb{A})$.*

Proof. We use [Kudla and Sweet 1997, Proposition 6.2] and follow closely the guideline of the proof of [Kudla and Rallis 1994, Proposition 4.6]. □

Proposition 3.5. *If $m < n$, then the restriction of $A_{-1}(-s_0)$ to $\Pi(\mathscr{C}', \chi_V)$ is zero unless $\mathscr{C}$ is the set of localizations of a global space, in which case it defines a nonzero intertwining map $\Pi(\mathscr{C}, \chi_V) \to \mathscr{A}$.*

Proof. The image of $A_{-1}(-s_0)$ is completely reducible in view of Lemma 3.4. Thus the restriction of $A_{-1}(-s_0)$ to $\Pi(\mathscr{C}', \chi_V)$ must factor through the unique irreducible quotient $\Pi(\mathscr{C}, \chi_V)$. Proposition 3.2 shows that $\Pi(\mathscr{C}, \chi_V)$ makes no contribution unless $\mathscr{C}$ comes from a global space. It remains to check that $A_{-1}(-s_0)$ is nonzero on $\Pi(V, \chi_V)$. From Proposition 3.3(2) this amounts to proving that the holomorphic value $E(g; f_\Phi^{(s)})|_{s=s_0}$ is nonzero for a good choice of $\Phi \in S(V^n(\mathbb{A}))$.

Let $\beta_0 \in \mathrm{Her}_m(F)$ with $\nu(\beta_0) \neq 0$. Put

$$\beta = \begin{pmatrix} \mathbf{0} & 0 \\ 0 & \beta_0 \end{pmatrix} \in \mathrm{Her}_n(F), \qquad G_0 = \left\{ \left(\begin{array}{cc|cc} \mathbf{1}_{n-m} & & & \\ & a & & b \\ \hline & & \mathbf{1}_{n-m} & \\ & c & & d \end{array}\right) \in G \right\}.$$

Define $\Phi_0 \in S(V^m(\mathbb{A}))$ by $\Phi_0(y) = \Phi((0, y))$ for $y \in V^m(\mathbb{A})$. The nonvanishing can be proven by considering the β-th Fourier coefficient of $E(g; f_\Phi^{(s)})$ as in Section 6 of [Yamana 2011] (compare Theorem 4.9 of [Kudla and Rallis 1994]). The exponents of the $n - m + 1$ terms in this Fourier coefficient are distinct at $s = s_0$, so that there can be no cancellations among them. The first term is just the β_0-th Fourier coefficient of the central value of the Eisenstein series on $G_0(\mathbb{A})$ attached to the standard section $f_{\Phi_0}^{(s)}$. Lemma 2.1(2) now completes our proof. □

Corollary 3.6. *Suppose that $m \leq n$. Let $f^{(s)}$ be a standard section of $I(s, \chi_V)$ such that $f^{(s_0)} \in \Pi(\mathscr{C}, \chi_V)$. If $\mathscr{C}$ cannot be the set of localizations of any global space, then $E(g; f^{(s)})|_{s=s_0}$ is identically zero.*

Proof. Propositions 3.2, 3.3(2) and 3.5 prove this corollary. □

The regularized Siegel–Weil formula can be deduced from Propositions 3.2 and 3.5.

Theorem 3.7. *Assume that $m < n$. Then there is a nonzero constant c_0 such that if holomorphic sections $f^{(s)}$ of $I(s, \chi_V)$ and $\Phi \in S(V^n(\mathbb{A}))$ satisfy the relation*

$$M^\circ(-s_0, \chi_V) f^{(-s_0)} = f_\Phi^{(s_0)},$$

then we have

$$\mathrm{Res}_{s=-s_0} E(g; f^{(s)}) = c_0 I(g; \Phi).$$

Finally, we prove Theorem 2 when $m < n$. Applying Proposition 3.3(2) and Theorem 3.7 to $h^{(-s)} = M^\circ(s, \chi_V) f_\Phi^{(s)}$, we see that

$$E(g; f_\Phi^{(s)})|_{s=s_0} = cI(g; \Phi),$$

where c is independent of Φ. One can prove that $c = 2$ in exactly the same manner as in Section 6 of [Yamana 2011]. □

Appendix. Zeta integrals for $\mathrm{GL}_n(\boldsymbol{D})$

Let F be a local field of characteristic zero and $\boldsymbol{D}$ a division algebra central and of dimension δ^2 over F. We begin by reviewing the Godement–Jacquet construction of the local factors of representations of $G'_n = \mathrm{GL}_n(\boldsymbol{D})$. The Fourier transform $\hat{\phi} \in \mathscr{S}(\mathrm{M}_{ba}(\boldsymbol{D}))$ of $\phi \in \mathscr{S}(\mathrm{M}_{ab}(\boldsymbol{D}))$ is defined by

$$\hat{\phi}(x) = \int_{\mathrm{M}_{ab}(\boldsymbol{D})} \phi(y)\psi(\tau(xy))\, dy, \quad x \in \mathrm{M}_{ba}(\boldsymbol{D}),$$

where the Haar measure dy is so chosen that

$$\int_{\mathrm{M}_{ab}(\boldsymbol{D})} \hat{\phi}({}^t y)\, dy = \phi(0).$$

In the archimedean case $S(\mathrm{M}_{ab}(\boldsymbol{D}))$ is the subspace of $\mathscr{S}(\mathrm{M}_{ab}(\boldsymbol{D}))$ as defined on p. 115 of [Godement and Jacquet 1972], and in the p-adic case $S(\mathrm{M}_{ab}(\boldsymbol{D})) = \mathscr{S}(\mathrm{M}_{ab}(\boldsymbol{D}))$.

Let π be an irreducible admissible representation of G'_n. We write $\pi^\vee$ for its admissible dual and denote the standard pairing on $\pi^\vee \boxtimes \pi$ by $\langle\, ,\rangle$. For $s \in \mathbb{C}$, $\phi \in \mathscr{S}(\mathrm{M}_n(\boldsymbol{D}))$, $\xi \in \pi$ and $\xi^\vee \in \pi^\vee$ we set

$$Z^{GJ}(s, \phi, \xi \boxtimes \xi^\vee) = \int_{G'_n} \langle \pi(g)\xi, \xi^\vee \rangle \phi(g) |\nu(g)|_F^{s+n'/2}\, dg.$$

This integral converges in some half-plane and extends to a meromorphic function on the whole s-plane satisfying

$$Z^{GJ}(-s, \hat{\phi}, \xi^\vee \boxtimes \xi) = (-1)^{n(\delta-1)} \gamma^{GJ}\big(s + \tfrac{1}{2}, \pi, \psi\big) Z^{GJ}(s, \phi, \xi \boxtimes \xi^\vee).$$

Fix a pair $\chi = (\chi_1, \chi_2)$ of quasicharacters of $F^\times$. Recall $\chi^0 = \chi_1\chi_2$. We attach a section $s \mapsto \mathfrak{F}_\phi^{(s,\chi)}$ to each $\phi \in \mathscr{S}(\mathrm{M}_{n,2n}(\boldsymbol{D}))$ by setting

$$\mathfrak{F}_\phi^{(s,\chi)}(g) = \chi_1(\nu(g))|\nu(g)|_F^{s+n'/2} \int_{G'_n} \phi((0,t)g)\chi^0(\nu(t))|\nu(t)|_F^{2s+n'}\,dt.$$

This integral converges absolutely for sufficiently large $\Re s$. Observe that if ϕ belongs to $S(\mathrm{M}_{n,2n}(\boldsymbol{D}))$, then $\mathfrak{F}_\phi^{(s,\chi)} \in I(s,\chi)$ (compare (1-1)). For $\varphi \in \mathscr{S}(\mathrm{M}_{2n,n}(\boldsymbol{D}))$ we define a section $\mathfrak{F}_\varphi^{(s,\chi)}$ of $I(s,\chi)$ to be

$$\chi_2(\nu(g))^{-1}|\nu(g)|_F^{-s-n'/2} \int_{G'_n} \varphi\Big(g^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\Big)\chi^0(\nu(t))|\nu(t)|_F^{2s+n'}dt.$$

Lemma A.1. *For each* $\phi \in S(\mathrm{M}_{n,2n}(\boldsymbol{D}))$,

$$M(s,\chi)\mathfrak{F}_\phi^{(s,\chi)} = \frac{(-1)^{n(\delta-1)}\chi_1(-1)^{n'}}{\gamma^{GJ}\Big(2s - \frac{n'-1}{2}, \chi^0\circ\nu_n, \psi\Big)}\mathfrak{F}_{\hat\phi}^{(-s,\rho(\chi)^{-1})}.$$

Proof. The case $n = \delta = 1$ is discussed in Lemma 14.7.1 of [Jacquet 1972]. The proof is substantially the same. For $g \in G'_{2n}$ we put

$$\Psi_g(t) = \int_{\mathrm{M}_n(\boldsymbol{D})} \phi((t,x)g)\,dx$$

for $t \in \mathrm{M}_n(\boldsymbol{D})$. Then

$$\begin{aligned}
&M(s,\chi)\mathfrak{F}_\phi^{(s,\chi)}(g)\\
&\quad= \int_{\mathrm{M}_n(\boldsymbol{D})} \mathfrak{F}_\phi^{(s,\chi)}\bigg(\begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} g\bigg)\,dx\\
&\quad= \chi_1\big((-1)^{n'}\nu(g)\big)|\nu(g)|_F^{s+n'/2}\\
&\qquad\qquad \times \int_{\mathrm{M}_n(\boldsymbol{D})}\int_{G'_n} \phi\bigg((0,t)\begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & x \end{pmatrix} g\bigg)\chi^0(\nu(t))|\nu(t)|_F^{2s+n'}\,dt\,dx\\
&\quad= \chi_1\big((-1)^{n'}\nu(g)\big)|\nu(g)|_F^{s+n'/2}\int_{\mathrm{M}_n(\boldsymbol{D})}\int_{G'_n} \phi\big((t,x)g\big)\chi^0\big(\nu(t)\big)|\nu(t)|_F^{2s}\,dt\,dx\\
&\quad= \chi_1(-1)^{n'}\chi_1\big(\nu(g)\big)|\nu(g)|_F^{s+n'/2}Z^{GJ}\Big(2s - \frac{n'}{2}, \Psi_g, \chi^0\circ\nu_n\Big).
\end{aligned}$$

Since $\widehat{\Psi_g}(t) = |\nu(g)|_F^{-n'}\hat\phi\Big(g^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\Big)$,

$$\chi_1\big(\nu(g)\big)|\nu(g)|_F^{s+n'/2}Z^{GJ}\Big(\frac{n'}{2} - 2s, \widehat{\Psi_g}, (\chi^0\circ\nu_n)^{-1}\Big) = \mathfrak{F}_{\hat\phi}^{(-s,\rho(\chi)^{-1})}.$$

Lemma A.1 follows from the functional equation of $Z^{GJ}(s,\phi,\chi^0\circ\nu_n)$. □

Fix $A \in \mathrm{GL}_n(\boldsymbol{D})$. For a section $f^{(s)}$ of $I(s, \chi)$, the integral

$$l_A(f^{(s)}) = \int_{\mathrm{M}_n(\boldsymbol{D})} f^{(s)}\left(\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix}\right)\psi(\tau(Ax))\,dx$$

converges absolutely for $\Re s \gg 0$. In the p-adic case, Karel [1979] has proven that $l_A(f^{(s)})$ admits an entire analytic continuation to the whole s-plane and satisfies a functional equation

$$l_A \circ M(s, \chi) = \chi^0(\nu(A))^{-1}|\nu(A)|_F^{-2s} c(s, \chi, \psi) l_A$$

for some meromorphic function $c(s, \chi, \psi)$. The factor $c(s, \chi, \psi)$ is independent of the choice of A. Analogous results are proven in the archimedean case in [Wallach 1988]. The normalization $M^\dagger(s, \chi)$ of $M(s, \chi)$ is defined so that

$$l_A \circ M^\dagger(s, \chi) = \chi_2(-1)^{n'} \chi^0(\nu(A))^{-1}|\nu(A)|_F^{-2s} l_A.$$

Lemma A.2. *For each* $\Phi \in S(\mathrm{M}_{n,2n}(\boldsymbol{D}))$,

$$M^\dagger(s, \chi)\mathfrak{F}_\Phi^{(s,\chi)} = \chi_2(-1)^{n'} \mathfrak{F}_{\hat{\Phi}}^{(-s,\rho(\chi)^{-1})}.$$

Proof. It is enough to show that

$$l_A\big(\mathfrak{F}_{\hat{\Phi}}^{(-s,\rho(\chi)^{-1})}\big) = \chi^0(\nu(A))^{-1}|\nu(A)|_F^{-2s} l_A\big(\mathfrak{F}_\Phi^{(s,\chi)}\big).$$

Take $\phi_1, \phi_2 \in S(\mathrm{GL}_n(\boldsymbol{D}))$ and define $\Phi \in S(\mathrm{M}_{n,2n}(\boldsymbol{D}))$ by $\Phi(x, y) = \hat{\phi}_1(x)\phi_2(y)$. Then

$$\begin{aligned} l_A\big(\mathfrak{F}_\Phi^{(s,\chi)}\big) &= \int_{\mathrm{M}_n(\boldsymbol{D})} \mathfrak{F}_\Phi^{(s,\chi)}\left(\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix}\right)\psi(\tau(Ax))\,dx \\ &= \int_{\mathrm{M}_n(\boldsymbol{D})}\int_{\mathrm{GL}_n(\boldsymbol{D})} \Phi\left((0, t)\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix}\right)\chi^0\big(\nu(t)\big)|\nu(t)|_F^{2s+n'}\,dt\,\psi\big(\tau(Ax)\big)\,dx \\ &= \int_{\mathrm{GL}_n(\boldsymbol{D})} \phi_1(-At^{-1})\phi_2(t)\chi^0\big(\nu(t)\big)|\nu(t)|_F^{2s}\,dt. \end{aligned}$$

Similarly, $l_A\big(\mathfrak{F}_{\hat{\Phi}}^{(-s,\rho(\chi)^{-1})}\big)$ is equal to

$$\begin{aligned} \int_{\mathrm{M}_n(\boldsymbol{D})}\int_{\mathrm{GL}_n(\boldsymbol{D})} & \phi_1(-t)\hat{\phi}_2(-xt)\chi^0(\nu(t))^{-1}|\nu(t)|_F^{-2s+n'}\psi(\tau(Ax))\,dt\,dx \\ &= \int_{\mathrm{GL}_n(\boldsymbol{D})} \phi_1(-t)\phi_2(t^{-1}A)\chi^0(\nu(t))^{-1}|\nu(t)|_F^{-2s}dt \\ &= \chi^0(\nu(A))^{-1}|\nu(A)|_F^{-2s} l_A(\mathfrak{F}_\Phi^{(s,\chi)}). \end{aligned}$$

Since both $l_A\big(\mathfrak{F}_\Phi^{(s,\chi)}\big)$ and $l_A\big(\mathfrak{F}_{\hat{\Phi}}^{(-s,\rho(\chi)^{-1})}\big)$ are not identically zero for a suitable choice of ϕ_1 and ϕ_2, the proof is complete. □

The embedding i of $G'_n \times G'_n$ into G'_{2n} is given by

$$(g_1, g_2) \mapsto w_1 \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} w_1^{-1}, \quad w_1 = \begin{pmatrix} 2^{-1} \cdot \mathbf{1}_n & -2^{-1} \cdot \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{pmatrix}.$$

Let π be an irreducible admissible representation of G'_n. For $\xi \in \pi$, $\xi^\vee \in \pi^\vee$ and a section $f^{(s)}$ of $I(s, \chi)$, we define the zeta integral by

$$Z(\xi \boxtimes \xi^\vee, f^{(s)}) = \int_{G'_n} \langle \pi(g)\xi, \xi^\vee \rangle f^{(s)}(i(g, e))\, dg,$$

following [Piatetski-Shapiro and Rallis 1987a; Lapid and Rallis 2005]. This integral converges absolutely for $\Re s \gg 0$ and extends to a meromorphic function in s that satisfies the functional equation

$$Z\big(\xi \boxtimes \xi^\vee, M^\dagger(s, \chi) f^{(s)}\big) = \pi(-1)\gamma\big(s + \tfrac{1}{2}, \pi \times \chi, \psi\big) Z(\xi \boxtimes \xi^\vee, f^{(s)}).$$

Lapid and Rallis [2005] demonstrated the special case of the following result for $\delta = 1$ in a different manner. It was pointed out by Wee Teck Gan [2012] that there is a typo in [Lapid and Rallis 2005, (25)].

Proposition A.3. *For any irreducible admissible representation π of G'_n and any pair $\chi = (\chi_1, \chi_2)$ of quasicharacters of $F^\times$,*

$$\gamma(s, \pi \times \chi, \psi) = \gamma^{GJ}(s, \pi \otimes \chi_1, \psi)\gamma^{GJ}(s, \pi^\vee \otimes \chi_2, \psi).$$

Proof. Let $\mathscr{F}_\Phi^{(s,\chi)}$ be the translate of $\mathfrak{F}_\Phi^{(s,\chi)}$ by the element $w_1 \in G'_{2n}$. Then

$$\begin{aligned} Z\big(\xi \boxtimes \xi^\vee, \mathscr{F}_\Phi^{(s,\chi)}\big) &= \int_{G'_n} \langle \pi(g)\xi, \xi^\vee \rangle \chi_1(\nu(g)) |\nu(g)|_F^{s+n'/2} \\ &\quad \times \int_{G'_n} \Phi\left((0, t) w_1 \begin{pmatrix} g & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}\right) \chi^0(\nu(t)) |\nu(t)|_F^{2s+n'}\, dt\, dg \\ &= \int_{G'_n \times G'_n} \big\langle (\pi \otimes \chi_1)(g)\xi, (\pi^\vee \otimes \chi_2)(t)\xi^\vee \big\rangle |\nu(gt)|_F^{s+n'/2} \Phi(g, t)\, dg\, dt. \end{aligned}$$

If $\Phi(x, y)$ is of the form $\phi_1(x)\phi_2(y)$, then the last integral is equal to

$$\big\langle Z^{GJ}(s, \pi \otimes \chi_1, \phi_1)\xi, Z^{GJ}(s, \pi^\vee \otimes \chi_2, \phi_2)\xi^\vee \big\rangle.$$

Piatetski-Shapiro and Rallis [1987a] employ this relation to calculate the unramified local zeta integrals.

We can see by Lemma A.2 that

$$Z\big(\xi \boxtimes \xi^\vee, M^\dagger(s,\chi)\mathscr{F}_\Phi^{(s,\chi)}\big) = \chi_2(-1)^{n'} \int_{G'_n \times G'_n} \hat{\phi}_1(g)\hat{\phi}_2(t) \\ \times |\nu(gt)|_F^{-s+n'/2} \big\langle (\pi \otimes \chi_1)(g^{-1})\xi, (\pi^\vee \otimes \chi_2)(-t^{-1})\xi^\vee \big\rangle \, dg\, dt.$$

The stated relation follows upon combining these with the definitions of the gamma factors. □

Let $\chi = 1$. Put $\Delta_s(g) = f_0^{(s-n'/2)}\left(w_1\begin{pmatrix} g & \\ & \mathbf{1}_n \end{pmatrix}\right)$ for $g \in G'_n$. Note that

$$\begin{aligned}\Delta_s(k_1 g k_2) &= f_0^{(s-n'/2)}\left(w_1\begin{pmatrix} k_1 g k_2 & \\ & \mathbf{1}_n \end{pmatrix}\right) \\ &= f_0^{(s-n'/2)}\left(i(k_1,k_1) w_1\begin{pmatrix} g & \\ & \mathbf{1}_n \end{pmatrix}\begin{pmatrix} k_2 & \\ & k_1^{-1} \end{pmatrix}\right) = \Delta_s(g)\end{aligned}$$

for $k_1, k_2 \in K_n$ and $g \in G'_n$. An explicit formula for this function is obtained in [Piatetski-Shapiro and Rallis 1987a, Proposition 6.4] in the case of symplectic or split even orthogonal groups. One can deduce from their argument a formula of the same type for the unit groups of simple algebras.

Lemma A.4. (1) *If F is a p-adic field and $g = k_1 d k_2$ with elements $k_1, k_2 \in K_n$ and $d = \operatorname{diag}[\varpi^{a_1}, \dots, \varpi^{a_n}]$, where ϖ is a generator of the maximal ideal of $\mathbb{O}$, and we put $q = |\nu(\varpi)|_F^{-1}$, then*

$$\Delta_s(g) = q^{-s\sum_{i=1}^n |a_i|}.$$

(2) *Assume that $F = \mathbb{R}$ or $\mathbb{C}$. Put $t = [F : \mathbb{R}]$. If $g = k_1 d k_2$ with $k_1, k_2 \in K_n$ and $d = \operatorname{diag}[d_1, \dots, d_n]$ with positive real numbers d_i, then*

$$\Delta_s(g) = 2^{n\delta t s} \prod_{i=1}^n (d_i^{-1} + d_i)^{-\delta t s}.$$

Lemma A.5. *If $\Re s > \delta(n-1)$, then Δ_s belongs to $L^1(G'_n)$.*

Proof. Put $\sigma = \Re s$. We consider the p-adic case. Proposition 1.5.2 of [Casselman 1995] gives a positive constant c such that

$$\begin{aligned}\int_{G'_n} |\Delta_s(g)|\, dg &\le c \sum_{a_1\ge a_2\ge\cdots\ge a_n} q^{-\sigma\sum_{i=1}^n |a_i|} \prod_{j=1}^n q^{\delta(n+1-2j)a_j} \\ &\le c \prod_{j=1}^n \sum_{a_j\in\mathbb{Z}} q^{-\sigma|a_j|+\delta(n+1-2j)a_j} \\ &= c\prod_{j=1}^n \biggl(\frac{1}{1-q^{\delta(n+1-2j)-\sigma}} + \frac{q^{\delta(2j-n-1)-\sigma}}{1-q^{\delta(2j-n-1)-\sigma}}\biggr).\end{aligned}$$

The archimedean case can be proven in the same way. □

Lemma A.6. *If* $\sigma > 0$, *then the function* $z \mapsto \Delta_\sigma(zg)$ *is integrable over the center* Z *of* G'_n *for any* $g \in G'_n$. *Moreover, there exists a positive constant* A_σ *depending only on* σ *such that, for every* $g \in G'_n$,

$$\int_Z \Delta_\sigma(zg)\, dz \le A_\sigma.$$

Proof. In the p-adic case,

$$\int_Z \Delta_\sigma(zg)\, dz = \sum_{j\in\mathbb{Z}} q^{-\sigma\sum_{i=1}^n |a_i+\delta j|} \le \sum_{j\in\mathbb{Z}} q^{-\sigma|j|} = \frac{1+q^{-\sigma}}{1-q^{-\sigma}}.$$

The proof for the archimedean case is completely analogous. □

Recall that π is called square integrable if it admits a unitary central character and its matrix coefficients are square integrable modulo the center. For $(s_1, s_2) \in \mathbb{C}$, we write $I(s_1, s_2) = I(0, (\alpha_F^{s_1}, \alpha_F^{s_2}))$.

Proposition A.7. *If* π *is square integrable*, $\Re s_1, \Re s_2 > -\delta/2$ *and* $f \in I(s_1, s_2)$, *then the integral defining* $Z(\xi \boxtimes \xi^\vee, f)$ *is absolutely convergent.*

Proof. Put $\sigma = \min\{\Re s_1, \Re s_2\}$. Note that $(\alpha_F \circ \nu_{2n})^{s'} \cdot f_0^{(s)} \in I(s+s', s-s')$. By Lemma A.4, we can majorize $|f((g,e))|$ by $cf_0^{(\sigma)}((g,e))$ for some positive constant c. Our task is to check that for any $\sigma > -\delta/2$,

$$\int_{G'_n} \bigl|\langle \pi(g)\xi, \xi^\vee\rangle\bigr| \Delta_{\sigma+n'/2}(g)\, dg$$

is finite. Take a constant σ' so that $0 < \sigma' < \sigma + \delta/2$. The square of this integral is less than or equal to the product of the integrals

$$\int_{G'_n} \Delta_{2\sigma+n'-2\sigma'}(zg)\, dg$$

and

$$\int_{G'_n} \left|\langle \pi(g)\xi, \xi^\vee \rangle\right|^2 \Delta_{2\sigma'}(g)\,dg = \int_{Z\backslash G'_n} \left|\langle \pi(\dot{g})\xi, \xi^\vee \rangle\right|^2 \int_Z \Delta_{2\sigma'}(z\dot{g})\,dz\,d\dot{g}$$
$$= A_{2\sigma'} \int_{Z\backslash G'_n} \left|\langle \pi(\dot{g})\xi, \xi^\vee \rangle\right|^2 d\dot{g},$$

both of which are finite, the first by Lemma A.5 and the second by Lemma A.6. □

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