THE SIEGEL–WEIL FORMULA FOR UNITARY GROUPS

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We extend the Siegel–Weil formula for unitary groups of hermitian forms over a skew field with involution of the second kind.

Introduction

The Siegel–Weil formula is an identity between an Eisenstein series and an integral of a theta function. After Weil [1965] proved such an identity when both sides of the identity are absolutely convergent, Kudla and Rallis [1988a; 1988b; 1994] extended it for symplectic groups beyond the range of absolute convergence. Their results were extended to almost all classical groups by several authors, of which we mention the following sample: [Tan 1998; Ichino 2004; 2007; Gan and Takeda 2011; Yamana 2011; 2013; Gan 2000]. In this paper we discuss the last case that has to be considered in the theory of classical dual pairs over a number field, namely, unitary groups of hermitian forms over a skew field with involution of the second kind.

Let $E/F$ be a quadratic extension of number fields and $D$ a division algebra with center $E$, of dimension $\delta^2$ over $E$ and provided with an antiautomorphism $\rho$ of order two under which $F$ is the fixed subfield of $E$. Let $\mathbb{A}$ and $\mathbb{A}_E$ be the rings of adeles of $F$ and $E$, respectively. Let $\mathcal{W}$ be a left $D$-vector space of dimension $2n$ with a nondegenerate skew hermitian form that has a complete polarization, and $V$ a right $D$-vector space of dimension $m$ with a nondegenerate hermitian form. Let $G$ and $H$ be the unitary groups of $\mathcal{W}$ and $V$, respectively.

Let $\alpha_E$ denote the standard norm of $\mathbb{A}_E^\times$. A character of $\mathbb{A}_E^\times$ is called principal if it is a complex power of $\alpha_E$. We denote by $P$ the maximal parabolic subgroup of $G$ that stabilizes a maximal isotropic subspace of $\mathcal{W}$. Note that $P$ has a Levi decomposition $P = MN$ with $M \simeq \text{GL}_n(D)$. For any unitary character $\chi$ of $\mathbb{A}_E^\times/E_E^\times$ and for any $s \in \mathbb{C}$, we consider the representation $I(s, \chi) = \text{Ind}_{P(\mathbb{A})}^G(\chi \alpha_E^s)$ induced from the character $m \mapsto \chi(v(m))\alpha_E(v(m))^s$, where $v$ is the reduced norm viewed...
as a character of the algebraic group $GL_n(D)$ and the induction is normalized so that $I(s, \chi)$ is naturally unitarizable when $s$ is pure imaginary. For any holomorphic section $f^{(s)}$ of $I(s, \chi)$, the Eisenstein series

$$E(g; f^{(s)}) = \sum_{\gamma \in P(F) \backslash G(F)} f^{(s)}(\gamma g)$$

is absolutely convergent for $\Re s > \delta n/2$ and has a meromorphic continuation to the whole $s$-plane. We denote by $\chi^0$ the restriction of $\chi$ to $\mathbb{A}^\times$, by $\rho(\chi)$ the character defined by $\rho(\chi)(x) = \chi(x^0)$ for $x \in \mathbb{A}_E^\times$, and by $\epsilon_{E/F}$ the quadratic character of $\mathbb{A}_E^\times/F^\times$ associated to the extension $E/F$. The following theorem was proven in [Tan 1999] when $\delta = 1$.

**Theorem 1.** Let $f^{(s)}$ be a holomorphic section of $I(s, \chi)$.

1. If $\chi \rho(\chi)$ is not principal, then $E(g; f^{(s)})$ is entire.
2. If $\chi = \rho(\chi)^{-1}$, then the poles of $E(g; f^{(s)})$ in $\Re s > -\frac{1}{2}$ are at most simple and can only occur in the set

$$\left\{ \frac{\delta(n - j)}{2} \mid j \in \mathbb{Z}, 0 \leq j < n, \chi^0 = \epsilon_{E/F}^{\delta j} \right\}.$$

Fix a nontrivial additive character $\psi$ of $\mathbb{A}/F$ and a character $\chi_V$ of $\mathbb{A}^\times \times E^\times$ such that $\chi_V^0 = \epsilon_{E/F}^m$. The group $G(\mathbb{A}) \times H(\mathbb{A})$ acts on the Schwartz space $\mathcal{S}(V^n(\mathbb{A}))$ of $V^n(\mathbb{A})$ via the Weil representation $\omega_{\psi, V, \chi_V}$. Let $S(V^n(\mathbb{A}))$ be the subspace of $\mathcal{S}(V^n(\mathbb{A}))$ consisting of functions that correspond to polynomials in the Fock model at every archimedean place of $F$.

The theta function associated to $\Phi \in S(V^n(\mathbb{A}))$ is defined by

$$\Theta(g, h; \Phi) = \sum_{x \in V^n(F)} \omega_{\psi, V, \chi_V}(g) \Phi(h^{-1}x)$$

for $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$. By Weil’s criterion [1965], the integral

$$I(g; \Phi) = \int_{H(F) \backslash H(\mathbb{A})} \Theta(g, h; \Phi) \, dh$$

is absolutely convergent for all $\Phi$ either if $r = 0$ or if $m - r > n$, where $r$ is the dimension of a maximal totally isotropic subspace of $V(F)$. When $m \leq n$ and $r > 0$, the integral diverges in general, but extends uniquely to a $G(\mathbb{A})$-intertwining, $H(\mathbb{A})$-invariant map on $S(V^n(\mathbb{A}))$ in light of the regularization introduced by Kudla and Rallis [1994].

For $\Phi \in S(V^n(\mathbb{A}))$ we define a section $f^{(s)}_\Phi$ of $I(s, \chi_V)$ by

$$f^{(s)}_\Phi(g) = |a(g)|^{s-s_0} \omega_{\psi, V, \chi_V}(g) \Phi(0),$$
where \( g \in G(\mathbb{A}) \), \( s_0 = \delta(m - n)/2 \) and the quantity \( |a(g)| \) is defined in the notation section below.

**Theorem 2.** If \( m \leq n \) or if \( m - r > n \), then for all \( \Phi \in S(V^n(\mathbb{A})) \) the series \( E(g; f_{\Phi}^{(s)}) \) is holomorphic at \( s = s_0 \) and

\[
E(g; f_{\Phi}^{(s)})|_{s=s_0} = \chi I(g; \Phi),
\]

where

\[
\chi = \begin{cases} 
2 & \text{if } m \leq n, \\
1 & \text{if } m - r > n. 
\end{cases}
\]

Theorem 2 was proven in [Weil 1965] if \( m > 2n \), and in [Tan 1998; Ichino 2004; 2007; Yamana 2011] if \( \delta = 1 \). The proof requires only slight technical modifications once all of the necessary local facts have been established. The group \( G(F_v) \) is isomorphic to the quasisplit unitary group \( U(\delta n, \delta n) \) or an inner form of \( GL_{2\delta n}(F_v) \), depending on whether \( v \) remains prime or splits in \( E \). The former case has already been discussed in [Kudla and Sweet 1997; Ichino 2007; Lee and Zhu 1998], and the latter case is discussed in Section 1. Coupled with the doubling method, the Siegel–Weil formula relates the theory of theta liftings to the theory of automorphic \( L \)-functions. We study the doubling zeta integral for inner forms of general linear groups in the Appendix.

**Notation**

Let \((D, E, F, \rho)\) be as in the introduction. The restriction of \( \rho \) to \( E \), which we denote also by \( \rho \), is the nontrivial automorphism of \( E \) over \( F \). For a matrix \( x \) with entries in \( D \), let \( x^* = {}^t x^\rho \) be the conjugate transpose of \( x \). If \( x \) is a square matrix, then \( \nu(x) \) and \( \tau(x) \) stand for its reduced norm and reduced trace to \( E \).

Fix a natural number \( n \) and put \( n' = \delta n \). Let \( \mathcal{W} = D^{2n} \) be a left \( D \)-vector space with the skew hermitian form

\[
\langle x, y \rangle = x J y^*, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]

for \( x, y \in \mathcal{W} \). Let \( V \) be a right \( D \)-vector space of dimension \( m \) equipped with a nondegenerate hermitian form \( (\ , \) \). We denote by \( G \) (resp. \( H \)) the group of all \( D \)-linear transformations of \( \mathcal{W} \) (resp. \( V \)) that leave \( (\ , \) \) (resp. \( (\ , \) \)) invariant. Put \( s_0 = \delta(m - n)/2 \).

We write \( P \) for the stabilizer in \( G \) of the maximal isotropic subspace of \( \mathcal{W} \) defined by the vanishing of all but the last \( n \) coordinates. Let

\[
\text{Her}_n = \{ x \in M_n(D) \mid x^* = x \}
\]
be the $F$-subvariety of $n \times n$ hermitian matrices. The group $G$ has a maximal parabolic subgroup $P = MN$ given by

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^* \end{pmatrix} \mid a \in \text{GL}_n(D) \right\},$$

$$N = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \mid b \in \text{Her}_n \right\}.$$ 

Let $K$ be the standard maximal compact subgroup of $G(\mathbb{A})$. For any character $\chi$ of $\mathbb{A}_F^\times / E^\times$, the representation $I(s, \chi) = I_n(s, \chi)$ is realized on the space of right $K$-finite functions $f^{(s)} : G(\mathbb{A}) \to \mathbb{C}$ satisfying

$$f^{(s)}(m(a)n(b)g) = \chi(v(a))\alpha_E(v(a))^{s+n'/2}f^{(s)}(g)$$

for all $a \in \text{GL}_n(D(\mathbb{A}))$, $b \in \text{Her}_n(\mathbb{A})$ and $g \in G(\mathbb{A})$. We define $|a(g)|$ by writing $g = pk \in G(\mathbb{A})$ with $p = m(a)n(b) \in P(\mathbb{A})$ and $k \in K$, and taking $|a(g)| = \alpha_E(v(a))$.

### 1. Degenerate principal series representations

For each place $v$ of $F$, let $F_v$ be the $v$-completion of $F$ and set $E_v = E \otimes_F F_v$ and $D_v = D \otimes_F F_v$. A division algebra $D$ with center $E$ admits an involution of the second kind if and only if $D_v$ is isomorphic to $M_b(E_v)$ whenever $v$ remains prime in $E$, and $D_v$ is isomorphic to a direct sum of mutually opposite simple algebras whose centers are $F_v$ whenever $v$ splits in $E$ (see [Scharlau 1985, Theorem 10.2.4]).

In the local setting we will depart slightly from our previous notation. Fix a place $v$ of $F$ and suppress it from the notation. Thus $E$ is a quadratic étale algebra over the local field $F$, $D$ an algebra whose center is $E$, $\rho$ an involution of $D$ whose restriction to $E$ is the nontrivial automorphism of $E$ over $F$, $V$ a free right $D$-module of rank $m$, and $(\ , \ ): V \times V \to D$ an $F$-bilinear map satisfying the following conditions:

- for $a, b \in D$ and $x, y \in V$,

$$ (x, y)^\rho = (y, x), \quad (xa, yb) = a^\rho(x, y)b; $$

- $(x, V) = 0$ implies that $x = 0$.

Let $H$ be the unitary group of $V$. Let $G = \{ g \in \text{GL}_{2n}(D) \mid gJg^* = J \}$. For any quasicharacter $\chi$ of $E^\times$, let $I(s, \chi)$ be the analogous local induced representation of $G$. By Morita context, it is enough to consider the case where the triple $(D, E, \rho)$ belongs to the following two types:

- $D = E$ is a quadratic extension of $F$ and $\rho$ generates $\text{Gal}(E/F)$;

- $D = D \oplus D^{\text{op}}$, $E = F \oplus F$ and $(x, y)^\rho = (y, x)$, where $D$ is a division algebra central over $F$ and $D^{\text{op}}$ is its opposite algebra.
The rank of $D$ as a module over $E$ is a square of a natural number that will be denoted by $\delta$. Note that $n' = \delta n$ remains intact after the change in notation.

We fix a nontrivial additive character $\psi$ of $F$ and a character $\chi_V$ of $E^\times$ that satisfies $\chi_V^0 = e^{\delta m}$. Then $G \times H$ acts on the Schwartz space $\mathcal{S}(V^n)$ via the Weil representation $\omega_{\psi,V,\chi_V}$. Note that it depends on the data $\psi$, $(\cdot, \cdot)$ and $\chi_V$ (compare [Kudla 1994]). When $F$ is a $p$-adic field, put $S(V^n) = \mathcal{S}(V^n)$. When $F = \mathbb{R}$ or $\mathbb{C}$, let $g$ be the complexified Lie algebra of $G$ and $S(V^n)$ the subspace of $\mathcal{S}(V^n)$ that corresponds to the space of polynomials in the Fock model of $\omega_{\psi,V,\chi_V}$. In the archimedean case we only consider admissible representations of the pair $(g, K)$, although we will allow ourselves to speak of a representation of the group $G$. We write $R(V, \chi_V) = R_{n'}(V, \chi_V)$ for the image of the intertwining map

$$S(V^n) \rightarrow I(s_0, \chi_V), \quad \Phi \mapsto f_{\Phi}^{(s_0)}(g) = \omega_{\psi,V,\chi_V}(g)\Phi(0).$$

We extend $f_{\Phi}^{(s_0)}$ to the standard section $f_{\Phi}^{(s)}$ of $I(s, \chi_V)$.

We discuss the case $E = F \oplus F$. Put

$$e_1 = (1, 0), \quad e_2 = (0, 1), \quad V_1 = V e_1, \quad V_2 = V e_2.$$  

We regard $V_1$ as a right $D$-module and $V_2$ as both a right $D_{\text{op}}$-module and a left $D$-module. Since $(V_i, V_i) = 0$ for $i = 1, 2$, the spaces $V_1$ and $V_2$ are paired nondegenerately against each other by $(\cdot, \cdot)$, and so an antiisomorphism

$$j : \text{End}(V_1, D) \rightarrow \text{End}(V_2, D_{\text{op}})$$

is defined by

$$(ax, y) = (x, j(a)y), \quad a \in \text{End}(V_1, D), \ x \in V_1, \ y \in V_2.$$  

We obtain

$$H = \{(a, j(a)^{-1}) \in \text{GL}(V_1, D) \times \text{GL}(V_2, D_{\text{op}}) \mid a \in \text{GL}(V_1, D)\}.$$  

Thus projection onto the first or second factor induces an isomorphism of $H$ onto $\text{GL}(V_1, D)$ or $\text{GL}(V_2, D_{\text{op}})$, respectively. For any nonnegative integer $j$ we write $G'_j = \text{GL}_j(D)$. Observe that

$$G = \{(g, J^{-1} g^{-1} J) \mid g \in G'_{2n}\}.$$  

Through projection onto the first factor, we identify $H$ with $G'_m$, $G$ with $G'_{2n}$, and $P = MN$ with

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \text{GL}_n(D) \right\}, \quad N = \left\{ \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \mid b \in \text{M}_n(D) \right\}. $$
We write $\nu = v_j$ for the reduced norm of $M_j(D)$ and $\tau$ for the reduced trace of $M_j(D)$. Let $\alpha_F(x) = |x|_F$ denote the normalized absolute value of $x \in F^\times$. When we write $\chi = (\chi_1, \chi_2)$, the representation $I(s, \chi)$ is translated to

$$I(s, \chi) = \text{Ind}^{G_{2n}^2}_{\mathbf{P}}((\chi_1\alpha_F^s) \circ v_n \boxtimes (\chi_2\alpha_F^s)^{-1} \circ v_n).$$

If $E = F \oplus F$, then since $\chi_V$ is of the form $(\mu, \mu^{-1})$, we may assume that $\chi_V = 1$ by twisting, and we write $I(s) = I(s, 1)$ and $R(V) = R(V, 1)$. The Weil representation $\omega_{j,k}$ of the dual pair $(G_j', G_k')$ can be taken to be the action on $\mathcal{S}(M_{k,j}(D))$ given by

$$\omega_{j,k}(a, b)\phi(x) = \alpha_F(v_j(a))^{s/k/2} \alpha_F(v_k(b))^{-s/k/2} \phi(b^{-1}xa)$$

for $a \in G_j'$ and $b \in G_k'$. Note that the integral

$$(\phi, \phi') = \int_{M_{k,j}(D)} \phi(u)\overline{\phi'(u)} \, du, \quad \phi, \phi' \in \mathcal{S}(M_{k,j}(D))$$

defines a $G_j' \times G_k'$ invariant positive definite hermitian form on $\omega_{j,k}$. The two models of the Weil representation $\omega_{2n,m} \simeq \omega_{\psi, V, 1}$ are related by the partial Fourier transform

$$(1-1) \quad \mathcal{F}\phi(x, y) = \int_{M_{m,n}(D)} \phi((x, z))\psi(-\tau(z^t y)) \, dz$$

for $x \in M_{m,n}(D)$ and $y \in M_{m,n}(D^{op})$. In the $p$-adic case we write $\mathcal{O}$ for the maximal compact subring of $D$ and put $K_n = \text{GL}_n(\mathcal{O})$. In the archimedean case we set

$$K_n = \{ g \in G_n' \mid {}^t\overline{g}g = 1_n \},$$

denoting the conjugate transpose of $x \in M_n(D)$ by ${}^t\overline{x}$, where $\overline{\cdot}$ denotes the complex conjugate or the quaternion conjugate. We denote by $f_0^{(s)}$ a unique section of $I(s)$ that is identically 1 on $K_{2n}$.

**Lemma 1.1.** If $E = F \oplus F$, then $R(V)$ contains $f_0^{(s_0)}$.

**Proof.** In the $p$-adic case, we let $\phi_{j,k}$ be the characteristic functions of $M_{j,k}(\mathcal{O})$. In the archimedean case we let

$$\phi_{j,k}(x) = e^{-\pi \text{Tr}_{F/\mathbb{R}}(\tau({}^t\overline{x}x))},$$

assuming that $\psi(\cdot) = e^{2\pi \sqrt{-1} \text{Tr}_{F/\mathbb{R}}(\cdot)}$. Put $\Phi = \mathcal{F}\phi_{2n,m}$. Then $f_{\Phi}^{(s_0)}$ is nonzero and right invariant under $K_{2n}$.

The local intertwining operator is defined analogously by

$$M(s, \chi) f^{(s)}(g) = \int_{\text{Her}_n(F)} f^{(s)}(Jn(b)g) \, db.$$
We define holomorphic sections and standard sections similarly. We write $\chi^0$ for the restriction of $\chi$ to $F^\times$. Put

$$a(s, \chi) = a_n'(s, \chi) = \prod_{j=1}^{n'} L(2s - j + 1, \chi^0 \cdot \epsilon_{E/F}^{n' + j}),$$

$$b(s, \chi) = b_n'(s, \chi) = \prod_{j=1}^{n'} L(2s + j, \chi^0 \cdot \epsilon_{E/F}^{n' + j}).$$

A normalized intertwining operator $M^*(s, \chi)$ is defined by setting

$$M^*(s, \chi) = a(s, \chi)^{-1} M(s, \chi).$$

**Lemma 1.2.** The operator $M^*(s, \chi)$ is entire.

**Proof.** When $E/F$ is a quadratic extension of $p$-adic fields, Lemma 1.2 is proven in Proposition 3.2 of [Kudla and Sweet 1997]. The proof is completely analogous when $E/F = \mathbb{C}/\mathbb{R}$. Note that Proposition 3A.6 of the same work applies also to this case by a global consideration, namely, by applying (24) of [Lapid and Rallis 2005] with base field $\mathbb{Q}$ and $S = \{\infty\}$.

We suppose that $E = F \oplus F$. For $\phi \in \mathcal{S}(M_n(D))$ we define a section $f^{(s)}_\phi$ of $I(s, \chi)$ by requiring that $\text{supp}(f^{(s)}_\phi) \subset PJN$ and $f^{(s)}_\phi(g) = \phi(b)$ if $g = Jn(b)$ for $b \in \text{Her}_n(F)$. As explained in [Piatetski-Shapiro and Rallis 1987b; Kudla and Sweet 1997], all we have to do is to show that the ratio $a(s, \chi)^{-1} M(s, \chi) f^{(s)}_\phi(J)$ is entire. One can easily observe that

$$M(s, \chi) f^{(s)}_\phi(J) = Z^{GJ}\left(2s - n' - 1, \chi^0 \circ \nu_n\right),$$

where the right-hand side is the zeta integral studied in [Weil 1974; Godement and Jacquet 1972] (see the Appendix). Our claim follows at once, as the Godement–Jacquet $L$-factor

$$L^{GJ}\left(2s - n' - 1, \chi^0 \circ \nu_n\right)$$

divided by the factor $a(s, \chi)$ is entire. \qed

For $\beta \in \text{Her}_n(F)$, let $\psi_\beta$ be the character of $N$ defined by $\psi_\beta(n(b)) = \psi(\tau(\beta b))$. Notice that $\tau(\beta b) \in F$. The Fourier transform of a Schwartz function $f \in \mathcal{S}(N)$ is defined by

$$\hat{f}(\beta) = \int_N f(u) \psi_\beta(u) \, du.$$
For each integer \( j \leq n' \), we define the subvariety \( \text{Her}^j_n \) of \( \text{Her}_n(F) \) by
\[
(E \not\cong F \oplus F) \quad \text{Her}^j_n = \{ \beta \in M_n(E) \mid \beta^\rho = \beta, \ \text{rank}_E \beta \leq j \},
\]
\[
(E = F \oplus F) \quad \text{Her}^j_n = \{ (\beta, \beta^\rho) \in M_n(D) \oplus M_n(D^\op) \mid \delta(\text{rank}_D \beta) \leq j \}.
\]

**Definition 1.3.** We say that a representation \( \pi \) of \( G \) has rank at most \( j \) if \( f \in \mathcal{S}(N) \) acts by zero on \( \pi \) whenever \( \hat{f} \) vanishes on \( \text{Her}^j_n \). We say that \( \pi \) is of rank \( j \) if in addition \( j \) is a multiple of \( \delta \) and \( \pi \) does not have rank less than \( j \).

For any \( H \)-module \( \pi \), we write \( \pi_H \) for the maximal quotient of \( \pi \) on which \( H \) acts trivially. Let \( \mathcal{H}_r \) be a split hermitian space of dimension \( 2r \), that is, \( \mathcal{H}_r \) has a \( D \)-basis consisting of \( 2r \) elements \( e_i, f_i \) such that
\[
(e_i, e_j) = (f_i, f_j) = 0, \quad (e_i, f_j) = \delta_{ij}.
\]

**Proposition 1.4.** Assume that \( m \leq n \). Let \( U = V \oplus \mathcal{H}_{n-m} \).

1. \( R(V, \chi_V) \) is irreducible and unitarizable.
2. \( R(V, \chi_V) \) is isomorphic to \( S(V^n)_H \).
3. If \( E/F \) is a quadratic extension of \( p \)-adic fields, then \( R(V, \chi_V) \) is of rank \( m \).
4. \( R(U, \chi_V) \) has a unique irreducible quotient that is isomorphic to \( R(V, \chi_V) \).
5. \( M^*(s_0, \chi_V) \) maps \( R(U, \chi_V) \) onto \( R(V, \chi_V) \).
6. \( b(s, \chi_V)M^*(s, \chi_V)f_\Phi^{(s)} \) is holomorphic at \( s = s_0 \) for every \( \Phi \in S(V^n) \).

**Proof.** When \( D = E \), these results are known (see [Li 1989; Mœglin et al. 1987; Kudla and Sweet 1997; Lee and Zhu 1998; Yamana 2011]). We may suppose that \( E = F \oplus F \) and \( \delta > 1 \).

For \( 0 \leq i \leq k \), let \( P^k_i = M^k_i N^k_i \) be the maximal parabolic subgroup of \( G'_k \) given by
\[
P^k_i = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G'_k \mid a \in G'_{k-i}, \ b \in M_{k-i}(D), \ d \in G'_{i} \right\}.
\]
\( \bar{P}^k_i \) its opposite parabolic subgroup, and \( r_i \) the representation of \( G'_i \times G'_i \) on \( \mathcal{S}(G'_i) \) given by
\[
r_i(g_1, g_2)\phi(g) = \phi(g_2^{-1}gg_1), \quad (\phi \in \mathcal{S}(G'_i), \ g, g_1, g_2 \in G'_i).
\]
In the archimedean case the representation \( I(s) \) is studied extensively in [Lee 2007; Sahi 1995; Zhang 1995]. From their results we know the module structure of \( I(s_0) \) and the set of \( K \)-types of each of its irreducible constituents, which combined with the technique explained in [Kudla and Rallis 1990a] prove (1), (2). We consider the nonarchimedean case. By Lemma 3.III.2 of [Mœglin et al. 1987], the representation \( \omega_{2n,m} \) has a filtration
\[
0 \subset S_m \subset \cdots \subset S_1 \subset S_0 = \omega_{2n,m}
\]
with successive quotients
\[ S_i / S_{i+1} \simeq \text{Ind}_{P_i^{2n} \times \tilde{P}_i^m}^{G'_i \times G_i} \mu_i, \]
where \( \mu_i \) is the representation of \( P_i^{2n} \times \tilde{P}_i^m \) on \( \mathcal{S}(G'_i) \) given by
\[ \mu_i(p, p') \phi = \alpha_F(v(a)^m - v(a')^{-2n} \nu(d)^{m-i+2n} \nu(d')^{i-2n})^{\delta/2} r_i(d, d') \phi, \]
where
\[ p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P_i^{2n}, \quad p' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \tilde{P}_i^m, \quad \phi \in \mathcal{S}(G'_i). \]

Let \( \mathbb{1}_j \) denote the trivial representation of \( G'_j \). For \( 0 \leq i < m \) and an admissible representation \( \pi \) of \( G'_j \), the Frobenius reciprocity gives
\[ \text{Hom}_{G'_i \times G_i}^j(S_i / S_{i+1}, \pi \otimes \mathbb{1}_m) \simeq \text{Hom}_{M_i^{2n} \times M_i^m}((\pi^\vee)_{N_i^{2n}} \otimes \delta_{p_i^m}^{1/2}, \mu_i^\vee), \]
where \( \delta_{p_i^m} \) is the modulus function on \( P_i^m \) and \( (\pi^\vee)_{N_i^{2n}} \) is the normalized Jacquet module of \( \pi^\vee \) associated to \( P_i^{2n} \). Since the quasicharacters of \( G'_m - i \) do not match, the space above is zero. Thus \( (S_i / S_{i+1})_{G'_m} = 0 \), so that the natural map \( (S_m)_{G'_m} \rightarrow (\omega_{2n, m})_{G'_m} \) is surjective. If \( \chi \) is a quasicharacter of \( G'_m \) and if a distribution \( T \) on \( \mathcal{S}(G'_m) \) transforms according to \( \chi \) under the action of \( e \times G'_m \), that is,
\[ T(r_m(e, h) f) = \chi(v(h)) T(f) \]
for all \( h \in G'_m \), then there is a constant \( c \in \mathbb{C} \) such that
\[ T(f) = c \int_{G'_m} f(h) \chi(v(h)) \, dh, \quad f \in \mathcal{S}(G'_m) \]
(see Lemma 3.II.3 of [Mœglin et al. 1987]). It follows that
\[ (S_m)_{G'_m} \simeq \text{Ind}_{p_i^{2n}}^{G'_i \times G_i}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m). \]

Since \( \text{Ind}_{p_i^{2n}}^{G'_i \times G_i}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m) \) is irreducible as a representation of \( G'_i \) induced from a unitary representation [Sécherre 2009], we have
\[ (\omega_{\psi, V, 1})_H \simeq \text{Ind}_{p_i^{2n}}^{G'_i \times G_i}(\mathbb{1}_{2n-m} \otimes \mathbb{1}_m). \]
Thus the map from \( (\omega_{\psi, V, 1})_H \) to \( R(V) \) is injective. This proves (1), (2).

In the \( p \)-adic case, Theorem 5.1 of [Mínguez 2009] tells us that \( I(s_0) \) has a unique irreducible subrepresentation, which is \( R(V) \), and hence \( I(-s_0) \) has a unique irreducible quotient. We refer to [Lee 2007] for the archimedean analogue. From Lemma 1.1 we can infer that \( f_0^{(-s_0)} \) generates \( I(-s_0) \). It follows that \( I(-s_0) = R(U) \). The proof of (4) is complete.
To prove (5), (6), it suffices to check that $b(s)M^*(s)f_0^{(s)}$ (resp. $M^*(s)f_0^{(s)}$) are holomorphic and nonzero at $s = s_0$ (resp. $s = -s_0$) in light of [Kudla and Rallis 1988a, Proposition 4.9]. Let $\phi_0 = \phi_{n,n} \in S(M_n(D))$ be as in the proof of Lemma 1.1. Define $\phi_1 \in S(M_{n,2n}(D))$ by $\phi_1(x, y) = \phi_0(x)\phi_0(y)$. The sections $\tilde{\phi}_1^{(s)}$ and $\tilde{\phi}_1^{(-s)}$ are defined in the Appendix. Since $\tilde{\phi}_1^{(s)}$ is right $K$-invariant, so is $\tilde{\phi}_1^{(-s)}$ by Lemma A.1. From Propositions 10.7 and 10.8 of [Weil 1974], we know

$$\tilde{\phi}_1^{(s)} = \tilde{\phi}_1^{(-s)}(e) \cdot f_0^{(s)} = Z^{GJ}\left(2s + \frac{n'}{2}, \phi_0, 1\right) \cdot f_0^{(s)} = f_0^{(s)} = f_0^{(s)} \prod_{j=1}^{n} \xi(2s + \delta j)$$

up to multiplication by exponential factors, where $\xi(s) = \zeta(s)$ in the $p$-adic case, and $\xi(s) = \Gamma(s)$ in the archimedean case. Observe that

$$\tilde{\phi}_1^{(-s)} = Z^{GJ}\left(-2s + \frac{n'}{2}, \phi_0, 1\right) \cdot f_0^{(s)}$$

$$= (-1)^n(\delta - 1)\gamma Z^{GJ}\left(2s - \frac{n' - 1}{2}, \pi, \psi\right)Z^{GJ}\left(2s - \frac{n'}{2}, \phi_0, 1\right) \cdot f_0^{(s)}.$$

Substituting these into the equality in Lemma A.1, we get

\[(1-2) \quad M(s)f_0^{(s)} = f_0^{(-s)} \prod_{j=1}^{n} \frac{\xi(2s - \delta j + \delta)}{\xi(2s + \delta j)}.
\]

Now we can easily conclude our proof. □

2. Proof of Theorem 1

Back to the global setup, we write $\mathcal{A}$ for the space of automorphic forms on $G(\mathbb{A})$. For $\beta \in \text{Her}_n(F)$ and $A \in \mathcal{A}$, let

$$A_\beta(g) = \int_{\text{Her}_n(F) \backslash \text{Her}_n(\mathbb{A})} A(n(b)g)\psi(-\tau(\beta b)) \, db, \quad g \in G(\mathbb{A})$$

denote the $\beta$-th Fourier coefficient of $A$. The following lemma can be proven in exactly the same way as in [Kudla and Rallis 1990b; Tan 1999].

**Lemma 2.1.** Let $f$ be a holomorphic section of $I(s, \chi)$ and $\beta \in \text{Her}_n(F)$ with $v(\beta) \neq 0$.

1. $b(s, \chi)E_\beta(g; f^{(s)})$ is holomorphic in $\Re s > -\frac{1}{2}$.
2. If $m \geq n$ and $\beta$ is represented by $V(F)$, then $E_\beta(g; f^{(s)}_\phi)$ can be made nonzero at $s = s_0$ for a suitable choice of $\Phi \in S(V^n(\mathbb{A}))$.
3. If $\chi \rho(\chi)$ is not principal, then $E(g; f^{(s)})$ is entire.
(4) If $\chi = \rho(\chi)^{-1}$, then the poles of $E(g; f^{(s)})$ in $\Re s > -\frac{1}{2}$ are at most simple and can only occur in the set
$$\left\{ \frac{n' - j}{2} \bigg| j \in \mathbb{Z}, \ 0 \leq j < n', \ \chi^0 = \epsilon_{E/F}^j \right\}.$$ 

(5) If $\chi^0 = \epsilon_{E/F}^{n'+1}$, then $E(g; f^{(s)})|_{s=0}$ is identically zero.

**Definition 2.2.** For each integer $l \leq n$, we say that $A \in \mathcal{A}$ has rank $\delta l$ if $A_{\beta} = 0$ when $\text{rank}_{D_{\beta}} \beta > l$, but $A_{\beta} \neq 0$ for some $\beta$ of rank $l$. When $\pi$ is a representation of $G(\mathbb{A})$ realized on a subspace of $\mathcal{A}$, we say that $\pi$ has rank at most $\delta l$ if all functions in $\pi$ have rank at most $\delta l$.

We call $A$ singular if it has rank less than $\delta n$. The following lemma can be proven in the same way as in the proof of [Howe 1981, Lemma 2.4].

**Lemma 2.3.** Let $\pi$ be a subrepresentation of $\mathcal{A}$. For every integer $l \leq n$ the following conditions are equivalent:

- $\pi$ has rank at most $\delta l$;
- for every place $v$, $G(F_v)$ acts on $\pi$ by a representation of rank at most $\delta l$;
- for at least one place $v$, $G(F_v)$ acts on $\pi$ by a representation of rank at most $\delta l$.

In particular, if $G(F_v)$ acts on $\pi$ by a representation of rank at most $j$, then $G(F_v)$ acts on $\pi$ by a representation of rank at most $\delta \ell$, where $\ell = [j/\delta]$.

For $s' \in \mathbb{C}$ with $\Re s' > -\frac{1}{2}$, the residue $\text{Res}_{s=s'} E(g; f^{(s)})$ depends only on $f^{(s')}$, and $f^{(s')} \mapsto \text{Res}_{s=s'} E(g; f^{(s)})$ gives a $G(\mathbb{A})$ intertwining map
$$A_{-1}(s') : I(s', \chi) \to \mathcal{A}.$$ 

Assume that $\chi = \rho(\chi)^{-1}$, assume that $j$ is an integer between 0 and $n'$, assume that $\chi^0 = \epsilon_{E/F}^j$, and assume that $j$ is not divisible by $\delta$. Let $s' = (n' - j)/2$. To complete the proof of Theorem 1, it remains to prove that $A_{-1}(s')$ is zero. Fix a finite inert place $v$ of $F$. By Theorem 1.2 of [Kudla and Sweet 1997], $I_v(s', \chi_v)$ has a unique irreducible submodule $R$ and
$$I_v(s', \chi_v)/R \simeq \bigoplus_{V_0} R(V_0, \chi_v),$$ 

where $V_0$ runs over all equivalence classes of hermitian spaces over $E_v$ of dimension $j$. Since the image of $A_{-1}(s')$ lies in the space of singular automorphic forms in view of Lemma 2.1(1) and since $R$ is nonsingular, the map $A_{-1}(s')$ factors through the quotient $\bigoplus_{V_0} R(V_0, \chi_v)$ at $v$. Proposition 1.4(3) shows that $G(F_v)$ acts on the image of $A_{-1}(s')$ by a representation of rank at most $j$. Put $\ell = [j/\delta]$. Lemma 2.3 shows that $G(F_v)$ acts on the image of $A_{-1}(s')$ by a representation of rank at most $\delta \ell$. Since $\delta \ell < j$, Proposition 1.4(3) forces $A_{-1}(s')$ to be zero.
3. Proof of Theorem 2

Lemma 3.1. If $m = n$ or if $m - r > n$, then for all $\Phi \in S(V^n(\mathbb{A}))$ and $\beta \in \text{Her}_n(F)$ with $v(\beta) \neq 0$,

$$E_\beta(g; f^{(s)}_{\Phi^1})|_{s=s_0} = \kappa I_\beta(g; \Phi).$$

Proof. The proof can be carried out by the same technique as in that of [Ichino 2004, Proposition 6.2]. We omit the details. \qed

First we prove Theorem 2 in the case $m - r > n$. Ichino [2007] proved the special case of this result for $\delta = 1$ (compare [Kudla and Rallis 1988b; Yamana 2013]). Many of the results there apply word for word in our general case.

If $m > 2n$, then $E(g; f^{(s_0)}_{\Phi^0})$ converges absolutely and the stated identity was proven by Weil [1965]. We may suppose that $m \leq 2n$. Fix $\Phi^0 = \bigotimes_v \Phi^0_v \in S(V^n(\mathbb{A}))$. By Theorem 10.6.2 of [Scharlau 1985], there is an inert place $w$ of $F$ such that the Witt index $r_w$ of $V_w$ satisfies $r_w < \delta(r + 1)$, where $V_w$ stands for the hermitian space over $E_w$ corresponding to $V(F_w)$. Note that

$$\delta m - r_w > \delta n.$$

We consider the $G(F_w)$-intertwining map

$$A_{-1,w} : S(V^n_w) \to \mathcal{A}, \quad \Phi_w \mapsto A_{-1}(s_0)(f^{(s_0)}_{\Phi^0}),$$

where $\Phi = \Phi_w \otimes \left( \bigotimes_{v \neq w} \Phi^0_v \right)$. The invariant distribution theorem [Mœglin et al. 1987; Lee and Zhu 1998] asserts that $A_{-1,w}$ factors through the quotient $R(V_w, \chi_{V_w})$.

Lemma 2.1(1) shows that $A_{-1,w}(\Phi_w)$ is singular for every $\Phi_w \in S(V^n_w)$. If $w$ is finite, then $\delta m = 2r_w + 2$ and $\delta n = r_w + 1$, and hence $R(V_w, \chi_{V_w})$ is irreducible and nonsingular by [Kudla and Sweet 1997, Theorem 1.2], so that $A_{-1,w}$ must be zero. If $w$ is real and $\nabla$ is the element of the universal enveloping algebra of the complexified Lie algebra of $G(F_w)$ defined by (2.1) of [Ichino 2007], then $\nabla A_{-1,w}(\Phi_w) = 0$. Since Proposition 2.2 of [Ichino 2007] asserts that $\nabla f^{(s_0)}_{\Phi_w}$ generates the submodule $R(V_w, \chi_{V_w})$ for a suitable choice of $\Phi_w$, the map $A_{-1,w}$ must be zero. Consequently, $E(g; f^{(s)}_{\Phi^1})$ is holomorphic at $s = s_0$ for every $\Phi \in S(V^n(\mathbb{A}))$.

Next we consider the $K_w$-intertwining map

$$A_w : S(V^n_w) \to \mathcal{A}, \quad \Phi_w \mapsto E(g; f^{(s)}_{\Phi^1})|_{s=s_0} - I(g; \Phi),$$

where $\Phi = \Phi_w \otimes \left( \bigotimes_{v \neq w} \Phi^0_v \right)$. The image of $A_w$ lies in the space of singular automorphic forms by Lemma 3.1. We write $\mathcal{R}_w$ for the subspace of $\mathcal{A}$ spanned by residues $\text{Res}_{s=s_0} E(g; f^{(s)}_{\Phi^1})$, where $f^{(s)}_{\Phi^1}$ is a holomorphic section of $I(s, \chi_{V_w})$ of the form

$$f^{(s)} = f^{(s)}_w \otimes \left( \bigotimes_{v \neq w} f^{(s)}_{\Phi^0_v} \right), \quad f^{(s)}_w \in I_w(s, \chi_{V_w}).$$
Then $A_w$ induces a $G(F_w)$-intertwining map $R(V_{w}, \chi_{v_w}) \rightarrow \mathcal{A}/\mathcal{R}_w$. The remaining part of the proof continues as in Section 3 of [Ichino 2007]. □

Theorem 2 is demonstrated in [Yamana 2011], provided that $\delta = 1$ and $m \leq n$. Since the proof in our general case can be done by the same technique, we shall omit most of the details. We define the functions $a(s, \chi)$ and $b(s, \chi)$ by taking the complete Hecke $L$-functions in place of the local abelian $L$-factors in the definition of $a_v(s, \chi_v)$ and $b_v(s, \chi_v)$. We define a normalized global intertwining operator by

$$M^o(s, \chi) = \frac{b(s, \chi)}{a(s, \chi)} M(s, \chi),$$

which is holomorphic in $\Re s > -\frac{1}{2}$ by Lemma 1.2 and (1-2).

Let $\mathcal{C} = \{W_v\}$ be a collection of local hermitian spaces of dimension $m$ over $D_v$ such that $W_v$ is isometric to $V(F_v)$ for almost all $v$. We form a restricted tensor product $\Pi(\mathcal{C}, \chi_V) = \bigotimes'_v R_n'(W_v, \chi_{v_e})$, which we can regard as a subrepresentation of $I(s_0, \chi_V)$. The proof of the following result is completely analogous to that of [Kudla and Rallis 1994, Theorem 3.1].

**Proposition 3.2.** Assume that $m \leq n$. Then

$$\dim \text{Hom}_{G(\mathbb{A})}(\Pi(\mathcal{C}, \chi_V), \mathcal{A}) \leq 1.$$

If there is no global hermitian space with $W_v$ as its completions, then

$$\dim \text{Hom}_{G(\mathbb{A})}(\Pi(\mathcal{C}, \chi_V), \mathcal{A}) = 0.$$

Next we are going to prove the special case of Theorem 2 in which $m = n$. Let $\mathcal{C} = \{V(F_v)\}$. Since Proposition 1.4(2) shows that the two intertwining maps $\Phi \mapsto E(g; f_\Phi^{(s)})|_{s=0}$ and $\Phi \mapsto I(g; \Phi)$ define elements of the space

$$\text{Hom}_{G(\mathbb{A})}(\Pi(\mathcal{C}, \chi_V), \mathcal{A}),$$

they must be proportional by Proposition 3.2. From Lemmas 2.1(2) and 3.1, they are nonvanishing, and the constant of proportionality is determined to be 2. □

We now suppose that $m < n$. Let $\mathcal{C}'$ be a collection of local hermitian spaces of dimension $2n - m$ obtained by adding a split space of suitable dimension to $\mathcal{C}$. By Proposition 1.4(4) and (5), $\Pi(\mathcal{C}', \chi_V)$ has a unique irreducible quotient $\Pi(\mathcal{C}, \chi_V)$, and $M^o(-s_0, \chi_V)$ induces a nonzero intertwining map $\Pi(\mathcal{C}', \chi_V) \rightarrow \Pi(\mathcal{C}, \chi_V)$. The same reasoning as in Section 4 of [Yamana 2011] implies the following result:

**Proposition 3.3.** Suppose that $m < n$. Let $f^{(s)}$ be a standard section of $I(s, \chi_V)$ such that $f^{(s_0)} \in \Pi(\mathcal{C}, \chi_V)$. Put $h^{(-s)} = M^o(s, \chi_V)f^{(s)}$.

1. $E(g; f^{(s)})$ is holomorphic at $s = s_0$. 
(2) $h(s)$ is holomorphic at $s = -s_0$, $h(-s_0) \in \Pi(\mathcal{C}', \chi_V)$, and

$$\text{Res}_{s=-s_0} E(g; h(s)) = -\text{Res}_{s=s_0} \left[ \frac{b(s, \chi_V)}{a(s, \chi_V)} \right] E(g; f^{(s)})|_{s=s_0}.$$ 

**Lemma 3.4.** If $m < n$, then the image of the map $A^{-1}(-s_0)$ lies in the space of square integrable automorphic forms on $G(\mathbb{A})$.

**Proof.** We use [Kudla and Sweet 1997, Proposition 6.2] and follow closely the guideline of the proof of [Kudla and Rallis 1994, Proposition 4.6].

**Proposition 3.5.** If $m < n$, then the restriction of $A^{-1}(-s_0)$ to $\Pi(\mathcal{C}', \chi_V)$ is zero unless $\mathcal{C}$ is the set of localizations of a global space, in which case it defines a nonzero intertwining map $\Pi(\mathcal{C}, \chi_V) \to \mathcal{A}$.

**Proof.** The image of $A^{-1}(-s_0)$ is completely reducible in view of Lemma 3.4. Thus the restriction of $A^{-1}(-s_0)$ to $\Pi(\mathcal{C}', \chi_V)$ must factor through the unique irreducible quotient $\Pi(\mathcal{C}, \chi_V)$. Proposition 3.2 shows that $\Pi(\mathcal{C}, \chi_V)$ makes no contribution unless $\mathcal{C}$ comes from a global space. It remains to check that $A^{-1}(-s_0)$ is nonzero on $\Pi(V, \chi_V)$. From Proposition 3.3(2) this amounts to proving that the holomorphic value $E(g; f^{(s)}(s))|_{s=s_0}$ is nonzero for a good choice of $\Phi \in S(V^n(\mathbb{A}))$.

Let $\beta_0 \in \text{Her}_m(F)$ with $\nu(\beta_0) \neq 0$. Put

$$\beta = \begin{pmatrix} 0 & 0 \\ 0 & \beta_0 \end{pmatrix} \in \text{Her}_n(F), \quad G_0 = \left\{ \begin{pmatrix} 1_{n-m} & b \\ a & 1_{n-m} \\ c & d \end{pmatrix} \in G \right\}.$$ 

Define $\Phi_0 \in S(V^n(\mathbb{A}))$ by $\Phi_0(y) = \Phi((0, y))$ for $y \in V^n(\mathbb{A})$. The nonvanishing can be proven by considering the $\beta$-th Fourier coefficient of $E(g; f^{(s)}(s))$ as in Section 6 of [Yamana 2011] (compare Theorem 4.9 of [Kudla and Rallis 1994]). The exponents of the $n - m + 1$ terms in this Fourier coefficient are distinct at $s = s_0$, so that there can be no cancellations among them. The first term is just the $\beta_0$-th Fourier coefficient of the central value of the Eisenstein series on $G_0(\mathbb{A})$ attached to the standard section $f^{(s)}(s)$. Lemma 2.1(2) now completes our proof.

**Corollary 3.6.** Suppose that $m \leq n$. Let $f^{(s)}$ be a standard section of $I(s, \chi_V)$ such that $f^{(s_0)} \in \Pi(\mathcal{C}, \chi_V)$. If $\mathcal{C}$ cannot be the set of localizations of any global space, then $E(g; f^{(s)}(s))|_{s=s_0}$ is identically zero.

**Proof.** Propositions 3.2, 3.3(2) and 3.5 prove this corollary.

The regularized Siegel–Weil formula can be deduced from Propositions 3.2 and 3.5.
Theorem 3.7. Assume that $m < n$. Then there is a nonzero constant $c_0$ such that if holomorphic sections $f^{(s)}$ of $I(s, \chi_V)$ and $\Phi \in S(V^n(\mathbb{A}))$ satisfy the relation

$$M^0(-s_0, \chi_V) f^{(-s_0)} = f^{(s_0)}_\Phi,$$

then we have

$$\text{Res}_{s=-s_0} E(g; f^{(s)}) = c_0 I(g; \Phi).$$

Finally, we prove Theorem 2 when $m < n$. Applying Proposition 3.3(2) and Theorem 3.7 to $h^{(-s)} = M^0(s, \chi_V) f^{(s)}_\Phi$, we see that

$$E(g; f^{(s)}_\Phi)|_{s=s_0} = c I(g; \Phi),$$

where $c$ is independent of $\Phi$. One can prove that $c = 2$ in exactly the same manner as in Section 6 of [Yamana 2011]. □

Appendix. Zeta integrals for $\text{GL}_n(D)$

Let $F$ be a local field of characteristic zero and $D$ a division algebra central and of dimension $\delta^2$ over $F$. We begin by reviewing the Godement–Jacquet construction of the local factors of representations of $G'_n = \text{GL}_n(D)$. The Fourier transform $\hat{\phi} \in \mathcal{S}(M_{ba}(D))$ of $\phi \in \mathcal{S}(M_{ab}(D))$ is defined by

$$\hat{\phi}(x) = \int_{M_{ab}(D)} \phi(y) \psi(\tau(xy)) \, dy, \quad x \in M_{ba}(D),$$

where the Haar measure $dy$ is so chosen that

$$\int_{M_{ab}(D)} \hat{\phi}(y) \, dy = \phi(0).$$

In the archimedean case $S(M_{ab}(D))$ is the subspace of $\mathcal{S}(M_{ab}(D))$ as defined on p. 115 of [Godement and Jacquet 1972], and in the $p$-adic case $S(M_{ab}(D)) = \mathcal{S}(M_{ab}(D))$.

Let $\pi$ be an irreducible admissible representation of $G'_n$. We write $\pi^\vee$ for its admissible dual and denote the standard pairing on $\pi^\vee \otimes \pi$ by $\langle \cdot, \cdot \rangle$. For $s \in \mathbb{C}$, $\phi \in \mathcal{S}(M_n(D))$, $\xi \in \pi$ and $\xi^\vee \in \pi^\vee$ we set

$$Z_{GJ}^{GJ}(s, \phi, \xi \otimes \xi^\vee) = \int_{G'_n} \langle \pi(g)\xi, \xi^\vee \rangle \phi(g) |v(g)|_F^{s+n'/2} \, dg.$$

This integral converges in some half-plane and extends to a meromorphic function on the whole $s$-plane satisfying

$$Z_{GJ}^{GJ}(-s, \hat{\phi}, \xi^\vee \otimes \xi) = (-1)^{n(\delta-1)} \gamma^{GJ}(s + \frac{1}{2}, \pi, \psi) Z_{GJ}^{GJ}(s, \phi, \xi \otimes \xi^\vee).$$
Fix a pair \( \chi = (\chi_1, \chi_2) \) of quasicharacters of \( F^\times \). Recall \( \chi^0 = \chi_1\chi_2 \). We attach a section \( s \mapsto \tilde{\delta}^{(s, \chi)}_\phi \) to each \( \phi \in \mathcal{S}(M_{n, 2n}(D)) \) by setting
\[
\tilde{\delta}^{(s, \chi)}_\phi(g) = \chi_1(v(g))|v(g)|^{s+n'/2} \int_{G_n'} \phi((0, t)g) \chi^0(v(t))|v(t)|^{2s+n'} dt.
\]
This integral converges absolutely for sufficiently large \( \Re{s} \). Observe that if \( \phi \) belongs to \( S(M_{n, 2n}(D)) \), then \( \tilde{\delta}^{(s, \chi)}_\phi \in I(s, \chi) \) (compare (1-1)). For \( \varphi \in \mathcal{S}(M_{2n,n}(D)) \) we define a section \( \tilde{\delta}^{(s, \chi)}_\psi \) of \( I(s, \chi) \) to be
\[
\chi_2(v(g))^{-1}|v(g)|^{-s-n'/2} \int_{G_n'} \varphi\left(g^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\right) \chi^0(v(t))|v(t)|^{2s+n'} dt.
\]

**Lemma A.1.** For each \( \phi \in S(M_{n, 2n}(D)) \),
\[
M(s, \chi) \tilde{\delta}^{(s, \chi)}_\phi = \frac{(-1)^{n(\delta-1)} \chi_1(-1)^{n'}}{
u GJ\left(2s-n'-1, \frac{n'}{2}, \chi^0 \circ \nu_n, \psi\right)} \tilde{\delta}^{(-s, \rho(\chi)^{-1})}_\phi.
\]

**Proof.** The case \( n = \delta = 1 \) is discussed in Lemma 14.7.1 of [Jacquet 1972]. The proof is substantially the same. For \( g \in G_{2n}' \) we put
\[
\Psi_g(t) = \int_{M_n(D)} \phi((t, x)g) \, dx
\]
for \( t \in M_n(D) \). Then
\[
M(s, \chi) \tilde{\delta}^{(s, \chi)}_\phi(g)
\]
\[
= \int_{M_n(D)} \tilde{\delta}^{(s, \chi)}_\phi\left(g \begin{pmatrix} 0 \; 1_n \\ 1_n \; 0 \end{pmatrix} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \right) \, dx
\]
\[
= \chi_1((-1)^{n'}v(g))|v(g)|^{s+n'/2} \int_{M_n(D)} \int_{G_n'} \phi((0, t)g) \chi^0(v(t))|v(t)|^{2s+n'} dt \, dx
\]
\[
= \chi_1((-1)^{n'}v(g))|v(g)|^{s+n'/2} \int_{M_n(D)} \int_{G_n'} \phi((t, x)g) \chi^0(v(t))|v(t)|^{2s} dt \, dx
\]
\[
= \chi_1((-1)^{n'}v(g))|v(g)|^{s+n'/2} Z^{GJ}\left(2s-n', \Psi_g, \chi^0 \circ \nu_n\right).
\]
Since \( \Psi_g(t) = |v(g)|^{-n'} \hat{\phi}\left(g^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}\right) \),
\[
\chi_1(v(g))|v(g)|^{s+n'/2} Z^{GJ}\left(\frac{n'}{2} - 2s, \Psi_g, (\chi^0 \circ \nu_n)^{-1}\right) = \tilde{\delta}^{(-s, \rho(\chi)^{-1})}_\phi.
\]

Lemma A.1 follows from the functional equation of \( Z^{GJ}(s, \phi, \chi^0 \circ \nu_n) \). \( \square \)
Fix $A \in \text{GL}_n(D)$. For a section $f^{(s)}$ of $I(s, \chi)$, the integral
\[ l_A(f^{(s)}) = \int_{M_n(D)} f^{(s)} \left( \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \right) \psi(\tau(Ax)) \, dx \]
converges absolutely for $\Re s \gg 0$. In the $p$-adic case, Karel [1979] has proven that $l_A(f^{(s)})$ admits an entire analytic continuation to the whole $s$-plane and satisfies a functional equation
\[ l_A \circ M(s, \chi) = \chi^0(v(A))^{-1} |v(A)|_F^{-2s} c(s, \chi, \psi) l_A \]
for some meromorphic function $c(s, \chi, \psi)$. The factor $c(s, \chi, \psi)$ is independent of the choice of $A$. Analogous results are proven in the archimedean case in [Wallach 1988]. The normalization $M^\dagger(s, \chi)$ of $M(s, \chi)$ is defined so that
\[ l_A \circ M^\dagger(s, \chi) = \chi_2(-1)^n \chi^0(v(A))^{-1} |v(A)|_F^{-2s} l_A. \]

**Lemma A.2.** For each $\Phi \in S(M_{n,2n}(D))$,
\[ M^\dagger(s, \chi) \hat{\delta}^{(s,\chi)}_\Phi = \chi_2(-1)^n \hat{\delta}^{(-s,\rho(\chi)^{-1})}_\Phi. \]

**Proof.** It is enough to show that
\[ l_A \left( \hat{\delta}^{(-s,\rho(\chi)^{-1})}_\Phi \right) = \chi^0(v(A))^{-1} |v(A)|_F^{-2s} l_A \left( \hat{\delta}^{(s,\chi)}_\Phi \right). \]

Take $\phi_1, \phi_2 \in S(\text{GL}_n(D))$ and define $\Phi \in S(M_{n,2n}(D))$ by $\Phi(x, y) = \hat{\phi}_1(x)\phi_2(y)$. Then
\[ l_A \left( \hat{\delta}^{(s,\chi)}_\Phi \right) = \int_{M_n(D)} \hat{\delta}^{(s,\chi)}_\Phi \left( \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \right) \psi(\tau(Ax)) \, dx \]
\[ = \int_{M_n(D)} \int_{\text{GL}_n(D)} \Phi \left( \begin{pmatrix} 0, t \\ (0, t) \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} \end{pmatrix} \right) \chi^0(v(t)) |v(t)|_F^{2s+n'} \, dt \psi(\tau(Ax)) \, dx \]
\[ = \int_{\text{GL}_n(D)} \phi_1(-At^{-1}) \phi_2(t) \chi^0(v(t)) |v(t)|_F^{2s} \, dt. \]

Similarly, $l_A \left( \hat{\delta}^{(-s,\rho(\chi)^{-1})}_\Phi \right)$ is equal to
\[ \int_{M_n(D)} \int_{\text{GL}_n(D)} \phi_1(-t) \phi_2(-xt) \chi^0(v(t))^{-1} |v(t)|_F^{-2s+n'} \psi(\tau(Ax)) \, dt \, dx \]
\[ = \int_{\text{GL}_n(D)} \phi_1(-t) \phi_2(t^{-1}A) \chi^0(v(t))^{-1} |v(t)|_F^{-2s} \, dt \]
\[ = \chi^0(v(A))^{-1} |v(A)|_F^{-2s} l_A \left( \hat{\delta}^{(s,\chi)}_\Phi \right). \]

Since both $l_A \left( \hat{\delta}^{(s,\chi)}_\Phi \right)$ and $l_A \left( \hat{\delta}^{(-s,\rho(\chi)^{-1})}_\Phi \right)$ are not identically zero for a suitable choice of $\phi_1$ and $\phi_2$, the proof is complete. \qed
The embedding $i$ of $G'_n \times G'_n$ into $G'_{2n}$ is given by

$$(g_1, g_2) \mapsto w_1 \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} w_1^{-1}, \quad w_1 = \begin{pmatrix} 2^{-1} \cdot 1_n & -2^{-1} \cdot 1_n \\ 1_n & 1_n \end{pmatrix}.$$ 

Let $\pi$ be an irreducible admissible representation of $G'_n$. For $\xi \in \pi$, $\xi^\vee \in \pi^\vee$ and a section $f^{(s)}$ of $I(s, \chi)$, we define the zeta integral by

$$Z(\xi \boxtimes \xi^\vee, f^{(s)}) = \int_{G'_n} (\pi(g)\xi, \xi^\vee) f^{(s)}(i(g, e)) \, dg,$$

following [Piatetski-Shapiro and Rallis 1987a; Lapid and Rallis 2005]. This integral converges absolutely for $\Re s \gg 0$ and extends to a meromorphic function in $s$ that satisfies the functional equation

$$Z(\xi \boxtimes \xi^\vee, M^\dagger(s, \chi) f^{(s)}) = \pi(-1)^{\gamma(s + \frac{1}{2}, \pi \times \chi, \psi)} Z(\xi \boxtimes \xi^\vee, f^{(s)}).$$

Lapid and Rallis [2005] demonstrated the special case of the following result for $\delta = 1$ in a different manner. It was pointed out by Wee Teck Gan [2012] that there is a typo in [Lapid and Rallis 2005, (25)].

**Proposition A.3.** For any irreducible admissible representation $\pi$ of $G'_n$ and any pair $\chi = (\chi_1, \chi_2)$ of quasicharacters of $F^\times$,

$$\gamma(s, \pi \times \chi, \psi) = \gamma^{GJ}(s, \pi \otimes \chi_1, \psi) \gamma^{GJ}(s, \pi^\vee \otimes \chi_2, \psi).$$

**Proof.** Let $\mathcal{F}^{(s, \chi)}_\Phi$ be the translate of $\mathcal{F}^{(s, \chi)}_\Phi$ by the element $w_1 \in G'_{2n}$. Then

$$Z(\xi \boxtimes \xi^\vee, \mathcal{F}^{(s, \chi)}_\Phi)$$

$$= \int_{G'_n} (\pi(g)\xi, \xi^\vee) \chi_1(v(g)) |v(g)|_F^{s+n'/2}$$

$$\times \int_{G'_n} \Phi \begin{pmatrix} (0, t) w_1 \begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix} \end{pmatrix} \chi_0(v(t)) |v(t)|_F^{2s+n'} \, dt \, dg$$

$$= \int_{G'_n \times G'_n} (\pi \otimes \chi_1)(g)\xi, (\pi^\vee \otimes \chi_2)(t)\xi^\vee) |v(gt)|_F^{s+n'/2} \Phi(g, t) \, dg \, dt.$$ 

If $\Phi(x, y)$ is of the form $\phi_1(x)\phi_2(y)$, then the last integral is equal to

$$\{Z^{GJ}(s, \pi \otimes \chi_1, \phi_1)\xi, Z^{GJ}(s, \pi^\vee \otimes \chi_2, \phi_2)\xi^\vee \}.$$ 

Piatetski-Shapiro and Rallis [1987a] employ this relation to calculate the unramified local zeta integrals.
We can see by Lemma A.2 that

\[
Z(\xi \boxtimes \xi^\vee, M^+(s, \chi) \hat{\Phi}(s, \chi)) = \chi_2(-1)^{n'} \int_{G'_n} \hat{\phi}_1(g) \hat{\phi}_2(t) \times |\nu(gt)|_F^{-s+n'/2}((\pi \otimes \chi_1)(g^{-1})\xi, (\pi^\vee \otimes \chi_2)(-t^{-1})\xi^\vee) \, dg \, dt.
\]

The stated relation follows upon combining these with the definitions of the gamma factors. \(\square\)

Let \(\chi = 1\). Put \(\Delta_s(g) = f_0(s-n'/2) \left( w_1 \left( \begin{pmatrix} g & \mathbf{1}_n \end{pmatrix} \right) \right) \) for \(g \in G'_n\). Note that

\[
\Delta_s(k_1 g k_2) = f_0(s-n'/2) \left( w_1 \left( \begin{pmatrix} k_1 g k_2 & \mathbf{1}_n \end{pmatrix} \right) \right) = f_0(s-n'/2) \left( i(k_1, k_1) w_1 \left( \begin{pmatrix} g & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} k_2 & \mathbf{1}_n \end{pmatrix}^{-1} \right) \right) = \Delta_s(g)
\]

for \(k_1, k_2 \in K_n\) and \(g \in G'_n\). An explicit formula for this function is obtained in [Piatetski-Shapiro and Rallis 1987a, Proposition 6.4] in the case of symplectic or split even orthogonal groups. One can deduce from their argument a formula of the same type for the unit groups of simple algebras.

**Lemma A.4.** (1) If \(F\) is a \(p\)-adic field and \(g = k_1 d k_2\) with elements \(k_1, k_2 \in K_n\) and \(d = \text{diag}[\sigma^{a_1}, \ldots, \sigma^{a_n}]\), where \(\sigma\) is a generator of the maximal ideal of \(\mathbb{O}\), and we put \(q = |\nu(\sigma)|_F^{-1}\), then

\[
\Delta_s(g) = q^{-s} \sum_{i=1}^n |a_i|.
\]

(2) Assume that \(F = \mathbb{R}\) or \(\mathbb{C}\). Put \(t = [F : \mathbb{R}]\). If \(g = k_1 d k_2\) with \(k_1, k_2 \in K_n\) and \(d = \text{diag}[d_1, \ldots, d_n]\) with positive real numbers \(d_i\), then

\[
\Delta_s(g) = 2^{n \delta ts} \prod_{i=1}^n (d_i^{-1} + d_i)^{-\delta ts}.
\]

**Lemma A.5.** If \(\Re s > \delta(n - 1)\), then \(\Delta_s\) belongs to \(L^1(G'_n)\).
Proof. Put $\sigma = \Re s$. We consider the $p$-adic case. Proposition 1.5.2 of [Casselman 1995] gives a positive constant $c$ such that

$$
\int_{G'_n} |\Delta_s(g)| \, dg \leq c \sum_{a_1 \geq a_2 \geq \cdots \geq a_n} q^{-\sigma \sum_{i=1}^n |a_i|} \prod_{j=1}^n q^{\delta(n+1-2j)a_j}
$$

$$
\leq c \prod_{j=1}^n \sum_{a_j \in \mathbb{Z}} q^{-\sigma |a_j| + \delta(n+1-2j)a_j}
$$

$$
= c \prod_{j=1}^n \left( \frac{1}{1 - q^{\delta(n+1-2j) - \sigma}} + \frac{q^{\delta(2j-n-1) - \sigma}}{1 - q^{\delta(2j-n-1) - \sigma}} \right).
$$

The archimedean case can be proven in the same way. \qed

Lemma A.6. If $\sigma > 0$, then the function $z \mapsto \Delta_\sigma(zg)$ is integrable over the center $Z$ of $G'_n$ for any $g \in G'_n$. Moreover, there exists a positive constant $A_\sigma$ depending only on $\sigma$ such that, for every $g \in G'_n$,

$$
\int_Z \Delta_\sigma(zg) \, dz \leq A_\sigma.
$$

Proof. In the $p$-adic case,

$$
\int_Z \Delta_\sigma(zg) \, dz = \sum_{j \in \mathbb{Z}} q^{-\sigma \sum_{i=1}^n |a_i| + \delta j} \leq \sum_{j \in \mathbb{Z}} q^{-\sigma |j|} = \frac{1 + q^{-\sigma}}{1 - q^{-\sigma}}.
$$

The proof for the archimedean case is completely analogous. \qed

Recall that $\pi$ is called square integrable if it admits a unitary central character and its matrix coefficients are square integrable modulo the center. For $(s_1, s_2) \in \mathbb{C}$, we write $I(s_1, s_2) = I(0, (\alpha^{s_1}_F, \alpha^{s_2}_F))$.

Proposition A.7. If $\pi$ is square integrable, $\Re s_1, \Re s_2 > -\delta/2$ and $f \in I(s_1, s_2)$, then the integral defining $Z(\xi \boxtimes \xi^\vee, f)$ is absolutely convergent.

Proof. Put $\sigma = \min\{\Re s_1, \Re s_2\}$. Note that $(\alpha_F \circ v_{2n})^{s'} \cdot f_0^{(s)} \in I(s + s', s' - s)$. By Lemma A.4, we can majorize $|f((g, e))|$ by $c f_0^{(\sigma)}((g, e))$ for some positive constant $c$. Our task is to check that for any $\sigma > -\delta/2$,

$$
\int_{G'_n} |\langle \pi(g)\xi, \xi^\vee \rangle| \Delta_{\sigma+n'/2}(g) \, dg
$$

is finite. Take a constant $\sigma'$ so that $0 < \sigma' < \sigma + \delta/2$. The square of this integral is less than or equal to the product of the integrals

$$
\int_{G'_n} \Delta_{2\sigma+n'-2\sigma'}(zg) \, dg
$$

$$
\leq \frac{1 + q^{-\sigma}}{1 - q^{-\sigma}}.
$$
\[
\int_{G'_n} \left| \langle \pi(g) \xi, \xi' \rangle \right|^2 \Delta_{2\sigma'}(g) \, dg = \int_{Z \backslash G'_n} \left| \langle \pi(\hat{g}) \hat{\xi}, \hat{\xi}' \rangle \right|^2 \int_{Z} \Delta_{2\sigma'}(z \hat{g}) \, dz \, d\hat{g}
\]
both of which are finite, the first by Lemma A.5 and the second by Lemma A.6. □

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